

Article

A General Lower Bound for Average Local Discrepancy and an Application to the Farey Sequence

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Abstract

We establish a new lower bound for the average local discrepancy of the Farey sequence. First, we prove a general lower-bound formula for arbitrary sequences in the unit interval in terms of their gap distribution, namely the gaps larger than the average gap and the minimum gap. We then apply this result to the Farey sequence and obtain an explicit lower bound of logarithmic order. This complements the existing literature, which mainly focuses on upper bounds, and provides a new constraint on a quantity that is closely related to the Franel–Landau criterion for the Riemann Hypothesis. Finally, we examine the effect of gap permutations on the average local discrepancy and propose a conjecture for the corresponding mean value over all such permutations.

Keywords: Farey sequence; discrepancy; Riemann Hypothesis**MSC:** 11B57; 11K38; 11M26

1. Introduction and Statement of Main Results

The Farey sequence $F_n = \{f_1, f_2, \dots, f_N\}$ of order $n \in \mathbb{N}$ is an ascending sequence of irreducible fractions between 0 and 1 whose denominators do not exceed n , see e.g., [1–10]. Throughout this paper we exclude the fraction 0/1 from F_n . For given $n \in \mathbb{N}$ and $i \in \mathbb{N}$ with $i \leq N$, the local discrepancy $\hat{r}_{F_n}(i)$ is defined as

$$\hat{r}_{F_n}(i) = Nf_i - i.$$

The normalized local discrepancy is defined as

$$r_{F_n}(i) = \frac{\hat{r}_{F_n}(i)}{N} = f_i - \frac{i}{N}.$$

The absolute discrepancy of the Farey sequence F_n is generally defined as

$$D_n = \sup_{i \leq N} |r_{F_n}(i)|.$$

The absolute discrepancy of the Farey sequence was derived in [3] to be $D_n = 1/n$, and taking that $N = O(n^2)$, we have that $\hat{r}_{F_n}(i) = O(n)$. We define the average local discrepancy as

$$r_{F_n} = \frac{1}{N} \sum_{i \leq N} |\hat{r}_{F_n}(i)| = \sum_{i \leq N} |r_{F_n}(i)|,$$



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which is also the sum of the absolute values of the normalized local discrepancies of the elements in F_n . It is important to recall an equivalent formulation of the Riemann Hypothesis (RH). The Franel–Landau formulation [11,12] is expressed as

$$r_{F_n} = O\left(n^{\frac{1}{2}+\epsilon}\right) \quad \forall \epsilon > 0 \Leftrightarrow \text{RH}.$$

This connection makes the study of bounds for r_{F_n} a problem of intrinsic interest—see, e.g., [6], for further insights on this connection. Most of the existing literature focuses on upper bounds for the local discrepancy of Farey fractions [5,10,13,14], on the absolute discrepancy of the sequence [3], or on related estimates for partial sums [15]. By contrast, lower bounds for the average local discrepancy appear to be much less developed. The aim of the present work is to address this point by deriving a lower bound for r_{F_n} from information on the gap structure of the sequence.

Our starting point is a general question: given an increasing sequence in the unit interval, to what extent can its average local discrepancy be estimated solely from the sizes of its gaps? We show that a nontrivial lower bound can indeed be obtained from two pieces of information: the gaps larger than the average gap and the minimum gap of the sequence. In Theorem 1 we first derive a formula to compute a lower bound of the average local discrepancy of any increasing sequence in the unit interval, A_N , by only using its gap values that are larger than the average gap, stored in $\vec{g}$, and the value of its minimum gap, $g_{\min}$. This formula reads as

$$r_{A_N} \geq \frac{1}{4} \sum_{i=1}^M \frac{(g_i - g_{\min})^2}{1/N - g_{\min}} - \frac{M}{4} (1/N - g_{\min}),$$

where N is the total number of gaps and M is the number of gaps larger than $1/N$. Defining $\Delta_i = g_i - 1/N$, and $\Delta_{\min} = 1/N - g_{\min}$ we can rewrite the above formula in a more compact way avoiding the subtraction:

$$r_{A_N} \geq \frac{1}{2} \sum_{i=1}^M \Delta_i + \frac{1}{4\Delta_{\min}} \sum_{i=1}^M \Delta_i^2.$$

In general, this bound is not sharp. Applying this to F_n in Corollary 1 we obtain the following:

$$r_{F_n} \geq \frac{9}{\pi^2(\pi^2 - 3)} \left(\log\left(\frac{3n}{\pi^2}\right) + \gamma - \frac{\zeta'(2)}{\zeta(2)} - \frac{6}{\pi^2} + \frac{9}{2\pi^4} \right) + O\left(\frac{\log n}{n}\right).$$

This lower bound grows as $\log n$ and it is within the $O\left(n^{1/2+\epsilon}\right)$ from RH. This suggests that by reordering the gaps of F_n it should be possible to construct a new sequence with lower average local discrepancy than the initial sequence F_n . This is illustrated in Figure 1 for F_{26} , achieving a reduction of $r_{F_{26}}$ by more than a factor of 2 while remaining above the analytical lower bound from Corollary 1. In this particular optimization the largest gaps are placed sequentially surrounded by the smallest available gaps to cancel their contribution to the local discrepancy. The problem of finding the gap ordering with minimal average local discrepancy seems to be hard to solve as it requires checking a large number of permutations. Corollary 1 provides a lower bound to the solution of this problem without explicitly finding a solution. This is possible thanks to the fact that the Farey gaps are well known. The gap distribution in the Farey sequence is given by [16,17]

$$h(z) = \begin{cases} 0, & \text{if } z \leq 3/\pi^2; \\ 6 \log(\pi^2 z/3)/(\pi^2 z^2), & \text{if } 3/\pi^2 < z \leq 12/\pi^2; \\ -12 \log(1/2 + \sqrt{1/4 - 3/(\pi^2 z)})/(\pi^2 z^2), & \text{if } 12/\pi^2 < z, \end{cases}$$

where z represents the gap value multiplied by the factor $3n^2/\pi^2$. Figure 2 shows an illustration of the distribution of Farey gaps together with the numerical evaluation for F_{400} . The percentage of gaps in F_n larger than the average gap is approximately given by

$$\int_1^\infty h(z) dz \approx 33.18\%.$$

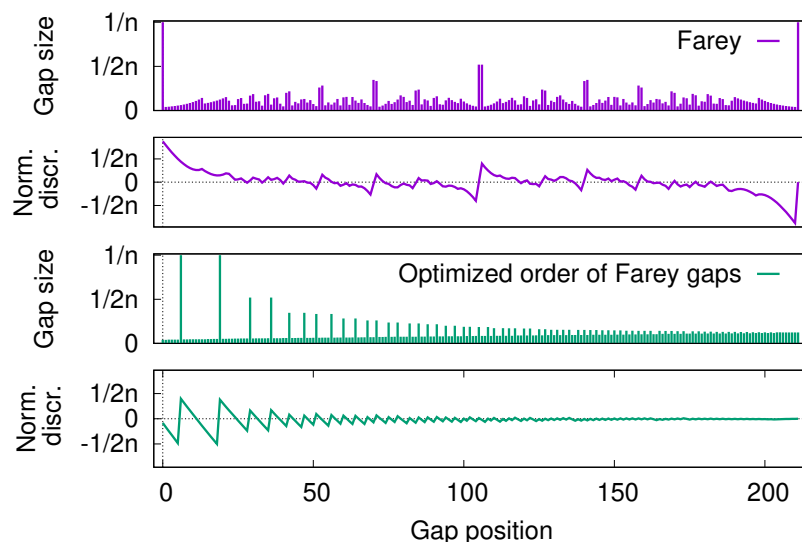


Figure 1. Gap sizes and normalized local discrepancies of the Farey sequence F_{26} (top 2 plots) and one optimized sequence obtained by reordering the gaps in F_{26} (bottom 2 plots), reducing $r_{F_{26}}$ by more than a factor 2. The number of gaps in F_{26} is $N = 212$.

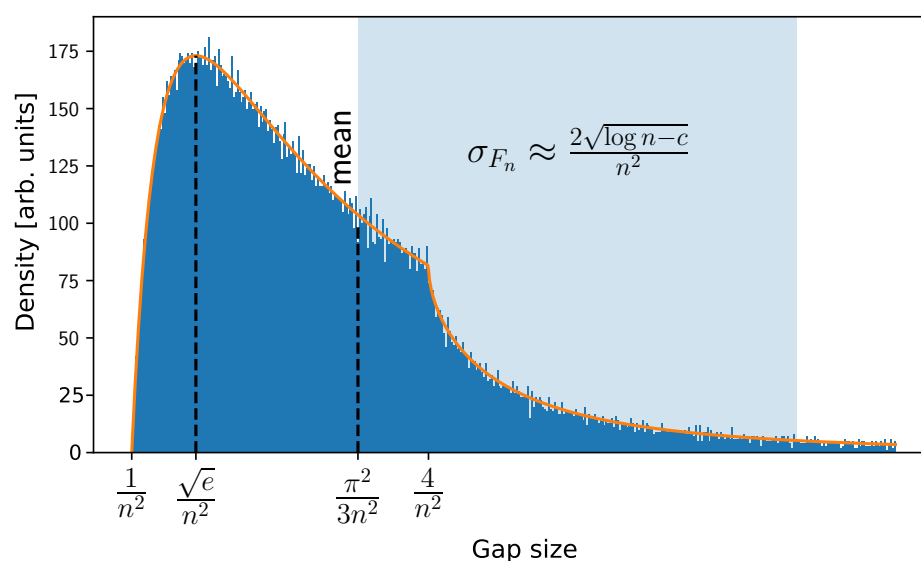


Figure 2. Distribution of the Farey gaps shown for the analytical estimate [17] in orange and an illustration for F_{400} in blue, along with the key parameters of the analytical estimate. The first dashed line marks the mode of the distribution at $\sqrt{e}/n^2$ and the second dashed line marks the mean at $\pi^2/(3n^2)$. The shaded area denotes one standard deviation starting from the mean of the distribution.

From the results in [18] we can bound the size of a gap, g , according to the minimum order of the two Farey fractions delimiting the gap, defined as k , as $g \geq 1/(kn)$. These and other bounds are illustrated in Figure 3. The Farey fractions of an order lower than or equal to $3n/\pi^2$ are surrounded by two gaps larger than the average gap. This implies that there are at least

$$\sum_{k=1}^{3n/\pi^2} \varphi(k) = \frac{27}{\pi^6} n^2 + O(n \log n) = \frac{9}{\pi^4} N + O(\sqrt{N} \log N)$$

pairs of gaps, larger than the average, that are adjacent (which represent about 18% of all gaps). This means that about 56% of the gaps larger than the average gap are placed together, which is not an optimal arrangement in terms of local discrepancy.

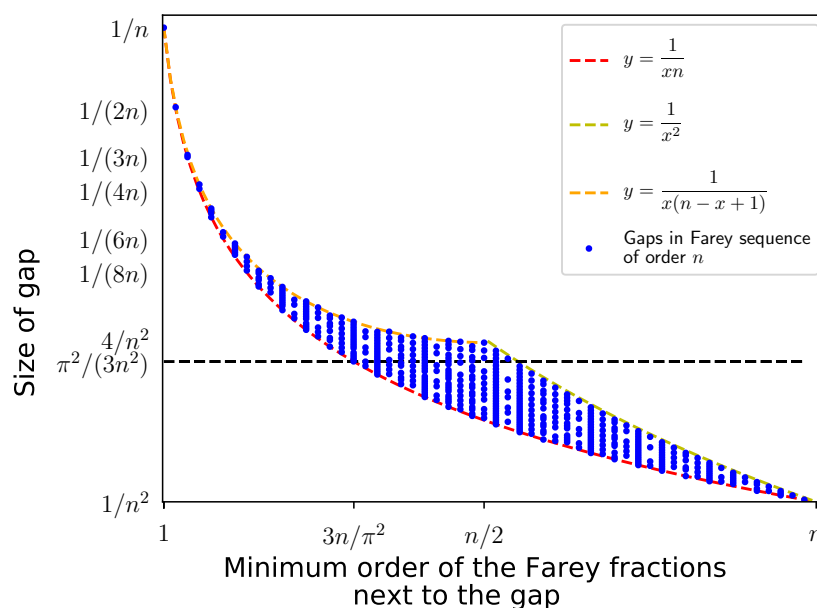


Figure 3. Gap sizes versus minimum order of the adjacent Farey fractions (blue dots correspond to Farey sequence of order $n = 56$).

Beyond the Farey application, the gap-based viewpoint suggests that the ordering of the gaps may play a substantial role in determining the size of the average local discrepancy. This motivates the study of the family of sequences obtained by permuting the gaps of a given sequence. Defining $\bar{r}_S$ as the mean of the average local discrepancies of all sequences generated as gap permutations of a given sequence S , we encounter that

$$\bar{r}_S \approx \chi \sigma_S N^{3/2},$$

where σ_S is the standard deviation of the gaps in S . Proposed bounds for χ are given in Conjecture 1. If the conjecture is true, we would have that

$$\bar{r}_{F_n} \approx \frac{3^{3/2}}{\pi^3} \chi 2n \sqrt{\log n - c},$$

where we have used that $\sigma_{F_n} = 2\sqrt{\log n - c}/n^2$, from [18], with $c \approx 1.058$. This estimate of $\bar{r}_{F_n}$ is to be compared to the RH conditional estimate $r_{F_n} = O(n^{1/2+\epsilon})$, and to the just computed lower bound as $O(\log n)$, implying that the mean over gap permutations, $\bar{r}_{F_n}$, is significantly above the expected estimate of r_{F_n} from RH and the known lower bound.

2. Results

We define the Single-distinct-gap sequence, $SDG_{g,k} = \{a_1, a_2, \dots, a_N\}$, as a sequence of N points between 0 and 1 where all gaps but one are equal, as

$$a_j = \begin{cases} j \frac{1-g}{N-1} & \text{for } 1 \leq j < k \\ (j-1) \frac{1-g}{N-1} + g & \text{for } N \geq j \geq k \end{cases},$$

where g is the size of the unequal gap that occurs between the k th and the $(k+1)$ th elements, with $1 \leq k < N$, and all the other gaps have a size of $g_{\min} = (1-g)/(N-1)$. The first gap

of this sequence is defined to be a_1 . The same approach is taken for F_n and other sequences throughout this work. This allows for the sum of gaps to be 1, and allows us to keep N as the number of gaps. The Single-distinct-gap sequence plays the role of a building block in this work as it depicts how the small gaps cancel the contribution of a single large gap to the local discrepancy.

Lemma 1. *The average local discrepancy of the Single-distinct-gap sequence, $r_{SDG_{g,k}}$ is bounded as*

$$r_{SDG_{g,k}} \geq \frac{1}{4} \frac{(g - g_{\min})^2}{1/N - g_{\min}} - \frac{1}{4} (1/N - g_{\min}),$$

where g is the size of the single distinct gap and $g_{\min} = (1 - g)/(N - 1)$ is the size of the other $N - 1$ gaps. The inequality becomes an identity for $k = (N - 1)/2$ and N odd.

Proof. We define Δ_j as the deviation of the j th gap from the average gap:

$$\Delta_{k+1} = g - 1/N, \text{ and } \Delta_j = g_{\min} - 1/N \text{ for } j \neq k + 1.$$

It should be noted that $g + (N - 1)g_{\min} = 1$, and therefore, $N = (g - g_{\min})/\Delta_{\min}$. The average local discrepancy of $SDG_{g,k}$ can be expressed as

$$r_{SDG_{g,k}} = \sum_{i=1}^N \left| \sum_{j=1}^i \Delta_j \right|.$$

Defining $\Delta_{\min} = 1/N - g_{\min}$ and using $\sum_{i=1}^N \Delta_j = 0$ we have that the sum within the absolute value is given by

$$\left| \sum_{j=1}^i \Delta_j \right| = i\Delta_{\min}, \quad 1 \leq i \leq k,$$

$$\left| \sum_{j=1}^i \Delta_j \right| = (N - i)\Delta_{\min}, \quad k + 1 \leq i \leq N.$$

Therefore, $r_{SDG_{g,k}}$ is expressed as two sums for the elements before and after the k th element as

$$r_{SDG_{g,k}} = \Delta_{\min} \sum_{i=1}^k i + \Delta_{\min} \sum_{i=1}^{N-k-1} i,$$

$$r_{SDG_{g,k}} = \frac{1}{2} (k(k+1) + (N-k-1)(N-k)) \Delta_{\min}.$$

This expression has a minimum for $k = (N - 1)/2$,

$$r_{SDG_{g,(N-1)/2}} = \frac{1}{4} (N^2 - 1) \Delta_{\min} = \frac{1}{4} \left(\left(\frac{g - g_{\min}}{\Delta_{\min}} \right)^2 - 1 \right) \Delta_{\min},$$

where we have used the fact that $N = (g - g_{\min})/\Delta_{\min}$. $\square$

We define the M -distinct-gaps sequence, $MDG_{\vec{g}, \vec{k}} = \{b_1, b_2, \dots, b_N\}$, with N elements, $b_N = 1$, having M gaps larger than $1/N$ and $M - N$ equal gaps of size $g_{\min} < 1/N$. The vectors $\vec{g}$ and $\vec{k}$ contain the size and locations of the M gaps, respectively. We define $\Delta_{\min} = 1/N - g_{\min}$.

Lemma 2. The average local discrepancy of the M -distinct-gaps sequence, $r_{MDG_{\vec{g},\vec{k}}}$, has a lower bound given by

$$r_{MDG_{\vec{g},\vec{k}}} \geq \frac{1}{4} \sum_{i=1}^M \frac{(g_i - g_{\min})^2}{1/N - g_{\min}} - \frac{M}{4} (1/N - g_{\min}), \quad (1)$$

where g_i represents the size of the i th distinct gap and $g_{\min} = (1 - \sum_i g_i)/(N - M)$ is the size of the other $N - M$ gaps. Defining

$$N_j = \sum_{p=1}^j \frac{g_p - g_{\min}}{\Delta_{\min}}, j \in \llbracket 0, M \rrbracket,$$

the inequality (1) becomes an identity when

$$k_j = N_{j-1} + \frac{1}{2}(N_j - 1), j \in \llbracket 1, M \rrbracket, k_j \in \mathbb{N}. \quad (2)$$

Proof. Observe that $N_j - N_{j-1}$ is exactly the effective number of minimal gaps needed to compensate the excess of discrepancy of the j th large gap above the average gap. Note that $N_0 = 0$ and $N_M = N$.

Similarly to the proof of Lemma 1, the normalized local discrepancy of the $MDG_{\vec{g},\vec{k}}$ sequence, defined as $r_{MDG_{\vec{g},\vec{k}}}(i)$, is given by piece-wise functions $y_j(i)$ defined in between the j th and $(j + 1)$ th large gaps as

$$y_j(i) = \Delta_{\min} |i - N_j|, \text{ with } i \in [k_j + 1, k_{j+1}].$$

This is seen in Figure 4, which shows an illustration of a part of an M -distinct-gap sequence with the piece-wise functions in the bottom plot. Therefore, we express $r_{MDG_{\vec{g},\vec{k}}}$ as

$$\begin{aligned} r_{MDG_{\vec{g},\vec{k}}} &= \frac{\Delta_{\min}}{2} (k_1(k_1 + 1) + (N - k_M - 1)(N - k_M)) \\ &+ \sum_{j=1}^{M-1} \sum_{i=k_j+1}^{k_{j+1}} y_j(i). \end{aligned}$$

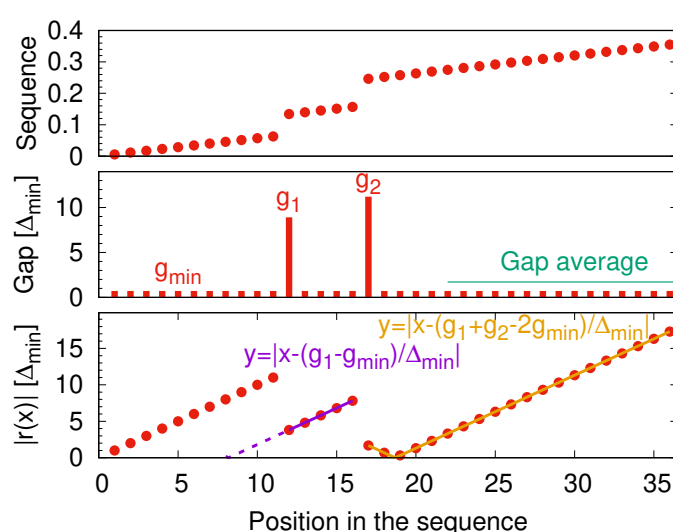


Figure 4. Illustration of part of an M -distinct-gaps sequence (**top**), its gaps in units of $\Delta_{\min}$ (**middle plot**), and the normalized local discrepancy (also in units of $\Delta_{\min}$) along with the piece-wise functions in between the large gaps (**bottom**).

We define R_j , the contribution of the j th distinct gap to $r_{MDG_{\vec{g}, \vec{k}}}$, as the sum between the $\lfloor N_{j-1} \rfloor + 1$ and $\lfloor N_j \rfloor$ locations:

$$R_j = \sum_{i=\lfloor N_{j-1} \rfloor + 1}^{\lfloor N_j \rfloor} |r_{MDG_{\vec{g}, \vec{k}}}(i)|$$

Looking at R_1 it is evident that the first necessary conditions to minimize this contribution are $k_1 < N_1$ and $k_2 > N_1$. The contribution of the first gap under these conditions is given by

$$\begin{aligned} R_1 &= \frac{\Delta_{\min}}{2} k_1(k_1 + 1) + \Delta_{\min} \sum_{j=k_1+1}^{\lfloor N_1 \rfloor} |j - N_1| \\ &= \Delta_{\min} \left(\frac{k_1}{2} (k_1 + 1) + \{N_1\} (\lfloor N_1 \rfloor - k_1) + \frac{\lfloor N_1 \rfloor - k_1}{2} (\lfloor N_1 \rfloor - k_1 - 1) \right) \\ &= \Delta_{\min} \left(\frac{k_1}{2} (k_1 + 1) + \frac{\lfloor N_1 \rfloor - k_1}{2} (N_1 - k_1 - 1 + \{N_1\}) \right) \\ &\geq \Delta_{\min} \left(\frac{k_1}{2} (k_1 + 1) + \frac{N_1 - k_1}{2} (N_1 - k_1 - 1) \right) \geq \Delta_{\min} \frac{1}{4} (N_1^2 - 1) \\ &= \frac{1}{4} \frac{(g_1 - g_{\min})^2}{\Delta_{\min}} - \frac{1}{4} \Delta_{\min}, \end{aligned}$$

where, for the first inequality, we have used that $(a - b)(a - 1 + b) \geq a(a - 1)$ for any $a \in \mathbb{R}$ and $0 \leq b \leq 1$. The second inequality is established by finding the maximum over k_1 . The bound on R_1 is already in the desired form. It should be noted that the contribution of the last distinct gap, R_M , follows an identical development. The contribution of the second large gap, R_2 , is representative of all inner gaps, and it is given by

$$\begin{aligned} R_2 &= \Delta_{\min} \left(\frac{k_2 - \lfloor N_1 \rfloor}{2} (k_2 - \lfloor N_1 \rfloor + 1) - \{N_1\} (k_2 - \lfloor N_1 \rfloor) \right. \\ &\quad \left. + \frac{\lfloor N_2 \rfloor - k_2}{2} (\lfloor N_2 \rfloor - k_2 - 1) + \{N_2\} (\lfloor N_2 \rfloor - k_2) \right) \\ &= \Delta_{\min} \left(\frac{k_2 - \lfloor N_1 \rfloor}{2} (k_2 - N_1 + 1 - \{N_1\}) \right. \\ &\quad \left. + \frac{\lfloor N_2 \rfloor - k_2}{2} (N_2 - k_2 - 1 + \{N_2\}) \right) \\ &\geq \Delta_{\min} \left(\frac{k_2 - N_1}{2} (k_2 - N_1 + 1) + \frac{N_2 - k_2}{2} (N_2 - k_2 - 1) \right) \\ &\geq \Delta_{\min} \frac{1}{4} ((N_2 - N_1)^2 - 1) = \frac{1}{4} \frac{(g_2 - g_{\min})^2}{\Delta_{\min}} - \frac{1}{4} \Delta_{\min}. \end{aligned}$$

Since this derivation holds equally for all inner gaps, and

$$r_{MDG_{\vec{g}, \vec{k}}} = \sum_{j=1}^M R_j,$$

we have demonstrated the first statement of the lemma.

When the condition (2) is met, the corresponding M-distinct-gaps sequence is a succession of Single-distinct-gap subsequences with odd numbers of gaps and the distinct gap in the center. Hence, the average local discrepancy is given by

$$r_{MDG_{\vec{g}, \vec{k}}} = \frac{1}{4} \sum_{i=1}^M \frac{(g_i - g_{\min})^2}{1/N - g_{\min}} - \frac{M}{4} (1/N - g_{\min}),$$

as per Lemma 1. $\square$

Theorem 1. Let $A_N = \{a_1, a_2, \dots, a_N\}$ be a sequence of points in the unit interval with $a_N = 1$. Let $\vec{g}$ contain the $M < N$ gaps of A_N that are larger than the average gap, i.e., $1/N$, and let $g_{\min}$ be the minimum gap in A_N . Then, the average local discrepancy of A_N , denoted by r_{A_N} , has a lower bound given by

$$r_{A_N} \geq \frac{1}{4} \sum_{i=1}^M \frac{(g_i - g_{\min})^2}{1/N - g_{\min}} - \frac{M}{4} (1/N - g_{\min}).$$

Proof. Let $A_N = \{a_1, \dots, a_N\}$ be a sequence in the unit interval with $a_N = 1$, and let $g_1, \dots, g_M$ be the gaps of A_N that are larger than the average gap $1/N$. Denote by $g_{\min}$ the minimum gap in A_N . The remaining $N - M$ gaps therefore all belong to the interval $[g_{\min}, 1/N]$. Let $\vec{g}^-$ and $\vec{k}^-$ contain the $N - M$ gap sizes and locations of the gaps below the average, respectively, in the same order as in A_N . We define the contribution of each large gap, R_j , following the same concept as in Lemma 2, while the formulation here is more tedious,

$$R_j = \sum_{i=N_{j-1}+1}^{N_j} |r_{A_N}(i)|, \quad j \in \llbracket 1, M \rrbracket,$$

where $N_0 = 0$, $N_j = k_{p_j}^-$ and p_j is defined to fulfill the following condition,

$$\sum_{i=1}^{p_j+1} (1/N - g_i^-) > \sum_{i=1}^j (g_i - 1/N) \geq \sum_{i=1}^{p_j} (1/N - g_i^-). \quad (3)$$

In the case that all $g_i^- = g_{\min}$, we have that R_j and N_j are identical to those defined in Lemma 2. When not all g_i^- are equal to $g_{\min}$, we have that $(1/N - g_i^-) < (1/N - g_{\min})$ for at least one j . Therefore, to fulfill the inequality (3), p_j and N_j cannot decrease when one or more g_i^- increase with respect to $g_{\min}$. This implies that

$$N_j \geq \sum_{i=1}^j \frac{g_i - g_{\min}}{1/N - g_{\min}},$$

for at least one j . As R_j is a sum of non-negative numbers up to N_j , we have

$$r_{A_N} \geq \frac{1}{4} \sum_{i=1}^M \frac{(g_i - g_{\min})^2}{1/N - g_{\min}} - \frac{M}{4} \left(\frac{1}{N} - g_{\min} \right).$$

$\square$

Corollary 1. The average local discrepancy of F_n has a lower bound given by

$$r_{F_n} \geq \frac{9}{\pi^2(\pi^2 - 3)} \left(\log(3n/\pi^2) + \gamma - \frac{\zeta'(2)}{\zeta(2)} - \frac{6}{\pi^2} + \frac{9}{2\pi^4} \right) + O\left(\frac{\log n}{n}\right).$$

Proof. By Figure 3, all the Farey fractions of an order lower than or equal to $3n/\pi^2$ are surrounded by two gaps larger than the average gap. There are $\varphi(k)$ Farey fractions of order k , each surrounded by two gaps of size larger than $1/(kn)$. For F_n we have that $g_{\min} = 1/n^2$. Therefore, using Theorem 1, the average local discrepancy of F_n is bounded as

$$\begin{aligned} r_{F_n} &\geq \sum_{k=1}^{3n/\pi^2} \frac{2\varphi(k)}{4} \frac{(1/(kn) - 1/n^2)^2}{\pi^2/(3n^2) - 1/n^2} = \sum_{k=1}^{3n/\pi^2} \frac{\varphi(k)}{2} \frac{(1/k - 1/n)^2}{\pi^2/3 - 1} \\ &= \frac{1}{2(\pi^2/3 - 1)} \sum_{k=1}^{3n/\pi^2} \varphi(k) \left(\frac{1}{k^2} - \frac{2}{kn} + \frac{1}{n^2} \right) \\ &= \frac{9}{\pi^2(\pi^2 - 3)} \left(\log(3n/\pi^2) + \gamma - \frac{\zeta'(2)}{\zeta(2)} - \frac{6}{\pi^2} + \frac{9}{2\pi^4} \right) + O\left(\frac{\log n}{n}\right), \end{aligned}$$

where we have used two known estimates [19,20],

$$\begin{aligned} \sum_{k=1}^n \varphi(k) &= \frac{3}{\pi^2} n^2 + O(n \log n), \\ \sum_{k=1}^n \frac{\varphi(k)}{k} &= \frac{6}{\pi^2} n + O\left((\log n)^{\frac{2}{3}} (\log \log n)^{\frac{4}{3}}\right), \end{aligned}$$

and we present the derivation of the estimate of the remaining sum:

$$\begin{aligned} \sum_{k \leq n} \frac{\varphi(k)}{k^2} &= \sum_{k \leq n} \frac{1}{k} \sum_{d|k} \frac{\mu(d)}{d} = \sum_{d \leq n} \sum_{m \leq n/d} \frac{\mu(d)}{d^2 m} = \sum_{d \leq n} \frac{\mu(d)}{d^2} H_{\lfloor n/d \rfloor} \\ &= \sum_{d \leq N} \frac{\mu(d)}{d^2} (\log n - \log d + \gamma) + O\left(\frac{\log n}{n}\right), \end{aligned}$$

where H_k are the harmonic numbers, fulfilling $H_k = \log k + \gamma + O(1/k)$. Because both series converge absolutely, we may use the limit

$$\begin{aligned} \sum_{d \geq 1} \frac{\mu(d)}{d^2} &= \frac{1}{\zeta(2)}, \\ \sum_{d \geq 1} \frac{\mu(d) \log d}{d^2} &= \frac{\zeta'(2)}{\zeta(2)^2}, \end{aligned}$$

obtaining:

$$\sum_{k \leq n} \frac{\varphi(k)}{k^2} = \frac{\log n + \gamma}{\zeta(2)} - \frac{\zeta'(2)}{\zeta(2)^2} + O\left(\frac{\log n}{n}\right).$$

We recall that $\zeta(2) = \pi^2/6$. $\square$

3. Conjectural Behavior Under Gap Permutations

The lower bound established in Theorem 1 depends only on the gap structure of the sequence and not on any special arithmetic property of the Farey sequence. This naturally leads to the question of how much of the average local discrepancy is determined by the multiset of gaps alone, and how much depends on their ordering.

Let $S = \{a_1, \dots, a_N\} \subset (0, 1]$ be an increasing sequence with $a_N = 1$, and let

$$g_i = a_i - a_{i-1}, \quad 1 \leq i \leq N,$$

with the convention $a_0 = 0$. For a fixed multiset of gaps $\{g_1, \dots, g_N\}$, consider the family of all sequences obtained by permuting these gaps and forming the corresponding partial sums. We denote this family by $\mathcal{P}(S)$ and define

$$\bar{r}_S = \frac{1}{|\mathcal{P}(S)|} \sum_{T \in \mathcal{P}(S)} r_T,$$

where r_T denotes the average local discrepancy of the sequence T .

The quantity $\bar{r}_S$ measures the typical size of the average local discrepancy among all sequences with the same gap multiset as S . Numerical experiments suggest that $\bar{r}_S$ is governed, to the first order, by the standard deviation of the gaps.

Let σ_S denote the standard deviation of the gaps of S . The data indicate that, for broad classes of sequences,

$$\bar{r}_S \asymp \sigma_S N^{3/2}.$$

More precisely, the ratio $\bar{r}_S / (\sigma_S N^{3/2})$ appears to remain bounded away from both 0 and ∞ and to vary in a relatively narrow interval.

Conjecture 1. *There exist absolute constants $c_1, c_2 > 0$ such that, for every sequence S in the unit interval,*

$$c_1 \sigma_S N^{3/2} \leq \bar{r}_S \leq c_2 \sigma_S N^{3/2}.$$

On the basis of the numerical evidence currently available, one may tentatively take

$$c_1 = \frac{1}{3\sqrt{2}}, \quad c_2 = \frac{4}{9}\sqrt{\frac{2}{3}}.$$

These values should, however, be regarded as empirical and provisional.

A heuristic explanation for the scaling $N^{3/2}$ is as follows. The local discrepancy is obtained by summing partial sums of the centered gaps:

$$g_i - \frac{1}{N}.$$

Under random gap permutations, these partial sums behave qualitatively like a centered random walk with a step size of order σ_S . One therefore expects a typical fluctuation of order $\sigma_S \sqrt{N}$ at a given position, and summing such contributions over N positions leads to the scale $\sigma_S N^{3/2}$. The lower bound, c_1 , has been tentatively observed to appear for an extreme type of sequence of N elements with two gaps of sizes $1/N + x$ and $1/N - x$, with $x < 1/N$, and all other gap sizes being exactly $1/N$. We may name this sequence the plus-minus sequence, PM_{x, k_-, k_+} , where $k_{\pm}$ are the locations of the plus and minus gaps, respectively. Analytically, this sequence has $\bar{r}_{PM} \sim \sigma N^{3/2} / (3\sqrt{2})$. The upper bound, c_2 , is tentatively observed to appear for the Single-distinct-gap sequence with $N = 3$, that has $\bar{r}_{SDG} = \sigma_{SDG} \sqrt{N-1} (N+1)/3$. For the random sequence, $\bar{r}_{\text{rand}} = \sigma_{\text{rand}} N^{3/2} \sqrt{\pi/2}/4$ is computed in [18], which lies in between the previous proposed bounds.

For the Farey sequence, the gap standard deviation satisfies [16,18]

$$\sigma_{F_n} = \frac{2\sqrt{\log n - c}}{n^2},$$

with $c \approx 1.058$. Since

$$N \sim \frac{3}{\pi^2} n^2,$$

the conjecture would imply

$$\bar{r}_{F_n} \asymp n \sqrt{\log n}.$$

This is much larger than the conditional scale $O(n^{1/2+\varepsilon})$ appearing in the Franel–Landau criterion. If it is the case, the Farey ordering would be highly non-generic among the permutations of its own gap multiset, in the sense that its average local discrepancy would be substantially smaller than the permutation-averaged value.

This perspective suggests that the arithmetic structure of the Farey sequence is reflected not only in the size of its gaps, but also in their highly non-random arrangement.

4. Conclusions

In this work we established a general lower-bound formula for the average local discrepancy of sequences in the unit interval in terms of their gap distribution, and we applied it to the Farey sequence to obtain an explicit logarithmic lower bound. This provides a new lower estimate for a quantity of classical interest and complements previous work focused mainly on upper bounds.

These results also highlight the importance of gap ordering in the size of the average local discrepancy. This suggests that sequences with the same gap multiset may exhibit substantially different behavior, and it motivates the conjectural description proposed here for the mean value over all gap permutations.

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