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Construction of a New Family of Baskakov Operators Based on Negative λ -Binomial Distribution

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Abstract

We propose the λ -Baskakov operator based on λ -binomial distribution, which is a new Baskakov-type operator. First, the moments and the central moments for them and their quantitative properties are computed. Second, we obtain the rate of convergence of the new operators in different function spaces using the modulus of continuity, $\mathbb{K}$ -functional and Steklov mean function. Then, weighted approximation and point-wise estimates are also discussed. Finally, theoretical results are supported by illustrative graphics and tables using Matlab software.

Keywords: λ -Baskakov operators; Voronovskaja-type theorem; rate of convergence; weighted approximation; point-wise estimates

MSC: 41A25; 41A35; 41A36

1. Introduction

Probability distributions have many importance applications in operator approximation theory. Let $\{X_k\}_{k=1}^{\infty}$ be independent Bernoulli random variables with $\mathbb{P}(X_k = 1) = p$, $\mathbb{P}(X_k = 0) = 1 - p$, $k = 1, \dots$ for some $p \in (0, 1)$. While $p = z$, let Z_r denote the number of variables with value 1 in X_k , $k = 1, \dots, r$, then Z_r has a binomial distribution $B(r, x)$ with parameters (r, x) :

$$\mathbb{P}(Z_r = i) = \binom{r}{i} z^i (1 - z)^{r-i}, \quad i = 0, 1, \dots, r.$$

Therefore, the classic Bernstein operators can be given by

$$K_r(f; z) = \sum_{i=0}^r \binom{r}{i} z^i (1 - z)^{r-i} f\left(\frac{i}{r}\right) = \sum_{i=0}^r \mathbb{P}(Z_r = i) f\left(\frac{i}{r}\right) = \mathbb{E}f\left(\frac{Z_r}{r}\right).$$

The research about Bernstein operators and their modifications can be seen in [1–14]. Let Y_r denote the number of variables with value 0 before the r th variable with value 1, then Y_r has a negative binomial distribution with parameters (r, p) :

$$\mathbb{P}(Y_r = i) = \binom{r+i-1}{i} p^r (1-p)^i, \quad i = 0, 1, 2, \dots$$

While $p = \frac{1}{1+z}$, $z \in [0, \infty)$, the classic Baskakov operators can be given by



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$$B_r(f; x) = \sum_{i=0}^{\infty} \binom{r+i-1}{i} \frac{z^i}{(1+z)^{r+i}} f\left(\frac{i}{r}\right) = \mathbb{E}f\left(\frac{Y_r}{r}\right). \quad (1)$$

The research about Baskakov operators and their modifications can be seen in [15–22]. Let $\{U_k\}_{k=1}^{\infty}$ be the independent Poisson distribution $\pi(q)$ with parameter q :

$$\mathbb{P}(X = i) = e^{-q} \frac{q^i}{i!}, i = 0, 1, 2, \dots, q > 0.$$

Then, for any $r \in \mathbb{N}_+ := 1, 2, \dots$, $V_r = U_1 + U_2 + \dots + U_r$ follows the Poisson distribution $\pi(rq)$:

$$\mathbb{P}(V_r = i) = e^{-nq} \frac{(nq)^i}{i!}, i = 0, 1, 2, \dots, q > 0.$$

While $q = z, z \in [0, \infty)$, the classic Szász–Mirakyan operators can be given by

$$S_r(f; x) = \sum_{i=0}^{\infty} e^{-nz} \frac{(nz)^i}{i!} f\left(\frac{i}{r}\right) = \mathbb{E}f\left(\frac{V_r}{r}\right). \quad (2)$$

The research about Szász–Mirakyan operators and their modifications can be seen in [23–34]. For λ being a real number, the λ -falling factorial $(r)_{i,\lambda}$ is defined by

$$(r)_{i,\lambda} = \begin{cases} r(r-\lambda)(r-2\lambda) \cdots (r-(i-1)\lambda), & i = 1, 2, \dots \\ 1, & i = 0. \end{cases}$$

In [35,36], Kim considered the following λ -binomial coefficients by

$$\binom{r}{0}_{\lambda} = 1, \quad \binom{r}{i}_{\lambda} = \frac{(r)_{i,\lambda}}{i!} = \frac{r(r-\lambda) \cdots (r-(i-1)\lambda)}{i!}, i = 1, 2, \dots.$$

The degenerate exponential function $e_{\lambda}^r(x) = (1 + \lambda x)^{\frac{r}{\lambda}}$ for $\lambda \geq 0$ is defined by

$$e_{\lambda}^r(x) = (1 + \lambda x)^{\frac{r}{\lambda}} = \sum_{i=0}^{\infty} (r)_{i,\lambda} \frac{x^i}{i!}.$$

We easily obtain that $\lim_{\lambda \rightarrow 0} e_{\lambda}^r(x) = e^{rx}$. In [37], Y. K. Ma and T. Kim constructed negative λ -binomial distribution by

$$\mathbb{P}(Y_{r,\lambda} = i) = \binom{r + (i-1)\lambda}{i}_{\lambda} (1-p)^i e_{\lambda}^r(p-1), i = 0, 1, 2, \dots.$$

where $\lambda \in (0, 1)$. In this paper, we construct a new family of Baskakov operators (named λ -Baskakov operators) with a parameter λ in term of the negative λ -binomial distribution $p = 1 - \frac{z}{1+\lambda z}$ by

$$\mathbf{B}_{r,\lambda}(f; z) := \mathbb{E}f\left(\frac{Y_{r,\lambda}}{r}\right) = \sum_{i=0}^{\infty} \binom{r + (i-1)\lambda}{i}_{\lambda} \frac{z^i}{(1+\lambda z)^{i+r/\lambda}} f\left(\frac{i}{r}\right) \quad (3)$$

where $r \in \mathbb{N}_+, z \in [0, \infty)$ and $f \in C[0, \infty)$. Meanwhile, we easily know that the operators $\mathbf{B}_{r,\lambda}$ are linear and positive. While $\lambda \rightarrow 1^-$, the operators (3) turn out to be the classical Baskakov operators (1) and $\lambda \rightarrow 0^+$, the operators (3) turn out to be the classical Szász–Mirakyan operators (2).

2. Auxiliary Lemmas

In this section, we give certain auxiliary lemmas which will be used to discuss the approximation properties for the operators (3).

Lemma 1 ([37], Theorem 2.3). For $k \in \mathbb{N}_+$, we have

$$\mathbb{E}(Y_{r,\lambda}^k) = \sum_{i=0}^k \mathbb{S}_2(k, i) \frac{(r + (i-1)\lambda)_{i,\lambda} (1-p)^i}{e_\lambda^{i\lambda} (p-1)}.$$

where $\mathbb{S}_2(k, i) = \frac{1}{i!} \sum_{j=0}^i (-1)^j \binom{i}{j} (i-j)^k$, $i = 0, 1, \dots, k$ is the Stirling number of the second kind.

Now, we define the moments for the operators (3) by $\mathbf{T}_{r,\lambda}^k(z) = \mathbf{B}_{r,\lambda}(\xi^k; z)$, $k \in \mathbb{N}$. Then, the following lemma can be established immediately.

Lemma 2. For $k \in \mathbb{N}$ and $z \in [0, \infty)$, we have

$$\begin{aligned} \mathbf{T}_{r,\lambda}^k(z) &= \sum_{i=0}^k \mathbb{S}_2(k, i) \frac{(r + (i-1)\lambda)_{i,\lambda} z^i}{r^k}, \\ \mathbf{T}_{r,\lambda}^0(z) &= 1, \mathbf{T}_{r,\lambda}^1(z) = z, \mathbf{T}_{r,\lambda}^2(z) = \left(1 + \frac{\lambda}{r}\right) z^2 + \frac{1}{r} z, \\ \mathbf{T}_{r,\lambda}^3(z) &= \left(1 + \frac{\lambda}{r}\right) \left(1 + \frac{2\lambda}{r}\right) z^3 + \frac{3}{r} \left(1 + \frac{\lambda}{r}\right) z^2 + \frac{1}{r^2} z, \\ \mathbf{T}_{r,\lambda}^4(z) &= \prod_{j=1}^3 \left(1 + \frac{j\lambda}{r}\right) z^4 + \frac{6}{r} \prod_{j=1}^2 \left(1 + \frac{j\lambda}{r}\right) z^3 + \frac{7}{r^2} \left(1 + \frac{\lambda}{r}\right) z^2 + \frac{1}{r^3} z, \\ \mathbf{T}_{r,\lambda}^5(z) &= \prod_{j=1}^4 \left(1 + \frac{j\lambda}{r}\right) z^5 + \frac{10}{r} \prod_{j=1}^3 \left(1 + \frac{j\lambda}{r}\right) z^4 + \frac{25}{r^2} \prod_{j=1}^2 \left(1 + \frac{j\lambda}{r}\right) z^3 \\ &\quad + \frac{15}{r^3} \left(1 + \frac{\lambda}{r}\right) z^2 + \frac{1}{r^4} z, \\ \mathbf{T}_{r,\lambda}^6(z) &= \prod_{j=1}^5 \left(1 + \frac{j\lambda}{r}\right) z^6 + \frac{15}{r} \prod_{j=1}^4 \left(1 + \frac{j\lambda}{r}\right) z^5 + \frac{65}{r^2} \prod_{j=1}^3 \left(1 + \frac{j\lambda}{r}\right) z^4 \\ &\quad + \frac{90}{r^3} \prod_{j=1}^2 \left(1 + \frac{j\lambda}{r}\right) z^3 + \frac{31}{r^4} \left(1 + \frac{\lambda}{r}\right) z^2 + \frac{1}{r^5} z. \end{aligned}$$

Proof. Setting $p = 1 - \frac{z}{1+\lambda z}$, we easily know

$$e_\lambda^r(p-1) = e_\lambda^r\left(-\frac{z}{1+\lambda z}\right) = \frac{1}{(1+\lambda z)^{\frac{r}{\lambda}}}.$$

Using Lemma 15 and (3), we have

$$\begin{aligned} \mathbf{T}_{r,\lambda}^k(z) &= \mathbf{B}_{r,\lambda}(\xi^k; z) = \mathbb{E}\left(\frac{Y_{r,\lambda}}{r}\right)^k = \frac{1}{r^k} \mathbb{E}(Y_{r,\lambda}^k) \\ &= \frac{1}{r^k} \sum_{i=0}^k \mathbb{S}_2(k, i) \frac{(r + (i-1)\lambda)_{i,\lambda} (1-p)^i}{e_\lambda^{i\lambda} (p-1)} \\ &= \frac{1}{r^k} \sum_{i=0}^k \mathbb{S}_2(k, i) (r + (i-1)\lambda)_{i,\lambda} z^i. \end{aligned}$$

Choosing $k = 0, 1, 2, 3, 4, 5, 6$, we can obtain the desired result. $\square$

Next, we obtain the second-order, fourth-order and sixth-order central moment computation formula and limit equalities:

Lemma 3. For $z \in [0, \infty)$, we have

$$\begin{aligned}\beta_{r,\lambda}(z) &= \mathbf{B}_{r,\lambda}((\xi - z)^2; z) = \mathbf{T}_{r,\lambda}^2(z) - 2z\mathbf{T}_{r,\lambda}^1(z) + z^2\mathbf{T}_{r,\lambda}^0(z) = \frac{\lambda z^2 + z}{r}, \\ \lim_{r \rightarrow \infty} r\beta_{r,\lambda}(z) &= \lambda z^2 + z, \lim_{r \rightarrow \infty} r^2\mathbf{B}_{r,\lambda}((\xi - z)^4; z) = 3z^2(1 + \lambda z)^2, \\ \lim_{r \rightarrow \infty} r^3\mathbf{B}_{r,\lambda}((\xi - z)^6; z) &= 15z^3(1 + \lambda z)^3.\end{aligned}$$

Proof. Using Lemma 2, we can rewrite

$$\begin{aligned}\mathbf{T}_{r,\lambda}^0(z) &= 1, \mathbf{T}_{r,\lambda}^1(z) = z, \mathbf{T}_{r,\lambda}^2(z) = z^2 + \frac{\lambda z^2 + z}{r}, \\ \mathbf{T}_{r,\lambda}^3(z) &= z^3 + \frac{3\lambda(z^3 + z^2)}{r} + \frac{2\lambda^2 z^3 + 3\lambda z^2 + z}{r^2}, \\ \mathbf{T}_{r,\lambda}^4(z) &= z^4 + \frac{6\lambda z^4 + 6z^3}{r} + \frac{11\lambda^2 z^4 + 18\lambda z^3 + 7z^2}{r^2} + \frac{6\lambda^2 z^4 + 12\lambda^2 z^3 + 7\lambda z^2 + z}{r^3}, \\ \mathbf{T}_{r,\lambda}^5(z) &= z^5 + \frac{10\lambda z^5 + 10z^4}{r} + \frac{35\lambda^2 z^5 + 60\lambda z^4 + 25z^3}{r^2} \\ &\quad + \frac{50\lambda^3 z^4 + 110\lambda^2 z^4 + 75\lambda z^3 + 15z^2}{r^3} + o\left(\frac{1}{r^3}\right), \\ \mathbf{T}_{r,\lambda}^6(z) &= z^6 + \frac{15\lambda z^6 + 15z^5}{r} + \frac{85\lambda^2 z^6 + 150\lambda z^5 + 65z^4}{r^2} \\ &\quad + \frac{225\lambda^3 z^6 + 525\lambda^2 z^5 + 390\lambda z^4 + 90z^3}{r^3} + o\left(\frac{1}{r^3}\right).\end{aligned}$$

Hence, we have

$$\beta_{r,\lambda}(z) = \mathbf{B}_{r,\lambda}((\xi - z)^2; z) = \mathbf{T}_{r,\lambda}^2(z) - 2z\mathbf{T}_{r,\lambda}^1(z) + z^2\mathbf{T}_{r,\lambda}^0(z) = \frac{\lambda z^2 + z}{r},$$

and

$$\lim_{r \rightarrow \infty} r\beta_{r,\lambda}(z) = \lambda z^2 + z.$$

Similarly, we have

$$\begin{aligned}\mathbf{B}_{r,\lambda}((\xi - z)^4; z) &= \mathbf{T}_{r,\lambda}^4(z) - 4z\mathbf{T}_{r,\lambda}^3(z) + 6z^2\mathbf{T}_{r,\lambda}^2(z) - 4z^3\mathbf{T}_{r,\lambda}^1(z) + z^4\mathbf{T}_{r,\lambda}^0(z) \\ &= \frac{3\lambda^2 z^4 + 6\lambda z^2 + 3z^2}{r^2} + \frac{6\lambda^2 z^4 + 12\lambda^2 z^3 + 7\lambda z^2 + z}{r^3} \\ &= \frac{3z^2(1 + \lambda z)^2}{r^2} + o\left(\frac{1}{r^2}\right). \\ \mathbf{B}_{r,\lambda}((\xi - z)^6; z) &= \mathbf{T}_{r,\lambda}^6(z) - 6z\mathbf{T}_{r,\lambda}^5(z) + 15z^2\mathbf{T}_{r,\lambda}^4(z) - 20z^3\mathbf{T}_{r,\lambda}^3(z) \\ &\quad + 15z^4\mathbf{T}_{r,\lambda}^2(z) - 6z^5\mathbf{T}_{r,\lambda}^1(z) + z^6\mathbf{T}_{r,\lambda}^0(z) \\ &= \frac{15\lambda^3 z^6 + 45\lambda^2 z^5 + 45\lambda z^4 + 15z^3}{r^3} + o\left(\frac{1}{r^3}\right) \\ &= \frac{15z^3(1 + \lambda z)^3}{r^3} + o\left(\frac{1}{r^3}\right).\end{aligned}$$

Hence,

$$\lim_{r \rightarrow \infty} r^2 \mathbf{B}_{r,\lambda}((\xi - z)^4; z) = 3z^2(1 + \lambda z)^2,$$

$$\lim_{r \rightarrow \infty} r^3 \mathbf{B}_{r,\lambda}((\xi - z)^6; z) = 15z^3(1 + \lambda z)^3.$$

□

Remark 1. For $\lambda \in (0, 1)$, $z \in [0, \infty)$, and sufficiently large r , we obtain

$$r\beta_{r,\lambda}(z) \leq 2(1 + z^2), \quad (4)$$

$$r^2 \mathbf{B}_{r,\lambda}((\xi - z)^4; z) \leq 6(1 + z^2)^2, \quad (5)$$

$$r^3 \mathbf{B}_{r,\lambda}((\xi - z)^6; z) \leq 45(1 + z^2)^3. \quad (6)$$

Proof. Using the property of preserving the sign under limits and Lemma 3, for sufficiently large r , we have

$$r\beta_{r,\lambda}(z) \leq 2(1 + z^2).$$

Similarly,

$$r^2 \mathbf{B}_{r,\lambda}((\xi - z)^4; z) \leq 6(1 + z^2)^2,$$

and

$$r^3 \mathbf{B}_{r,\lambda}((\xi - z)^6; z) \leq 15(1 + z^2)^{\frac{3}{2}} 2^{\frac{3}{2}} (1 + z^2)^{\frac{3}{2}} \leq 45(1 + z^2)^3.$$

□

In order to obtain the approximation of the operators (3), we need the following two lemmas.

Lemma 4. For $f \in C_{BO}[0, \infty)$ and $r \in \mathbb{N}_+$, we have

$$\|\mathbf{B}_{r,\lambda}(f)\| \leq \|f\|,$$

where $C_{BO}[0, \infty) := \{f \in C[0, \infty) : f \text{ be bounded function on } [0, \infty)\}$ endowed with the norm $\|f\| = \sup_{z \in [0, \infty)} |f(z)|$.

Proof. By the definition of the operators (3) and Lemma 2, for any $z \in [0, \infty)$,

$$\begin{aligned} |\mathbf{B}_{r,\lambda}(f; z)| &= \left| \sum_{i=0}^{\infty} \binom{r + (i-1)\lambda}{i}_{\lambda} \frac{z^i}{(1 + \lambda z)^{i+r/\lambda}} f\left(\frac{i}{r}\right) \right| \\ &\leq \sum_{i=0}^{\infty} \binom{r + (i-1)\lambda}{i}_{\lambda} \frac{z^i}{(1 + \lambda z)^{i+r/\lambda}} \left| f\left(\frac{i}{r}\right) \right| \\ &\leq \|f\| \sum_{i=0}^{\infty} \binom{r + (i-1)\lambda}{i}_{\lambda} \frac{z^i}{(1 + \lambda z)^{i+r/\lambda}} = \|f\| \mathbf{T}_{r,\lambda}^0(z) = \|f\|. \end{aligned}$$

Taking the supremum over all $z \in [0, \infty)$, we obtain the desired result. □

Lemma 5. For $f \in C[0, \infty)$ and $a > 0$, then the operator sequence $\{\mathbf{B}_{r,\lambda}(f; z)\}$ converges to $f(z)$ uniformly on $[0, a]$.

Proof. Using the classical Korovkin theorem [38], we only need to prove the following three limit equalities:

$$\lim_{r \rightarrow \infty} \|\mathbf{B}_{r,\lambda}(\xi^k; z) - z^k\|_{[0,a]} = 0, \quad k = 0, 1, 2.$$

By $\mathbf{B}_{r,\lambda}(\xi^0; z) = 1$ and $\mathbf{B}_{r,\lambda}(\xi^1; z) = z$, the above limit equality holds for $k = 0$ and $k = 1$. From Lemma 2, we have

$$\|\mathbf{B}_{r,\lambda}(\xi^2; z) - z^2\|_{[0,a]} = \sup_{z \in [0,a]} = \frac{\lambda a^2 + a}{r} \rightarrow 0, \quad \text{as } r \rightarrow \infty.$$

Thus, the proof of Lemma 5 is completed. $\square$

3. Approximation Properties

In this section, we discuss approximation properties for the operators $\mathbf{B}_{r,\lambda}$ in four subsections.

3.1. Voronovskaja-Type Theorem

Theorem 1. For $f \in C[0, \infty)$, $\lambda \in (0, 1)$, f'' exists at a point $z \in [0, \infty)$, then

$$\lim_{r \rightarrow \infty} r(\mathbf{B}_{r,\lambda}(f; z) - f(z)) = \frac{\lambda z^2 + z}{2} f''(z).$$

Proof. Using the two-order Taylor's formula for f , we obtain

$$f(\xi) = f(z) + f'(z)(\xi - z) + \frac{1}{2} f''(z)(\xi - z)^2 + \Theta(\xi; t)(\xi - z)^2 \quad (7)$$

where

$$\Theta(\xi; t) = \begin{cases} \frac{f(\xi) - f(z) - f'(z)(\xi - z) - \frac{1}{2} f''(z)(\xi - z)^2}{(\xi - z)^2}, & \xi \neq z, \\ 0, & \xi = z. \end{cases}$$

Using the L'Hôpital's Rule, we have

$$\lim_{\xi \rightarrow z} \Theta(\xi; t) = \frac{1}{2} \lim_{\xi \rightarrow z} \frac{f'(\xi) - f'(z)}{\xi - z} - \frac{1}{2} f''(z) = 0.$$

This means $\Theta(\cdot; t) \in C[0, \infty)$. Applying the operators to (7), we can write

$$\begin{aligned} \mathbf{B}_{r,\lambda}(f; z) - f(z) &= \mathbf{B}_{r,\lambda}(\xi - z; z) f'(z) + \frac{1}{2} \beta_{r,\lambda}(z) f''(z) + \mathbf{B}_{r,\lambda}(\Theta(\xi; t)(\xi - z)^2; z) \\ &= \frac{1}{2} \beta_{r,\lambda}(z) f''(z) + \mathbf{B}_{r,\lambda}(\Theta(\xi; t)(\xi - z)^2; z). \end{aligned}$$

Using the Cauchy–Schwarz inequality, we can obtain

$$r \left| \mathbf{B}_{r,\lambda}(\Theta(\xi; t)(\xi - z)^2; z) \right| \leq \sqrt{\mathbf{B}_{r,\lambda}(\Theta^2(\xi; t); z)} \sqrt{r^2 \mathbf{B}_{r,\lambda}((\xi - z)^4; z)}.$$

We know that $\Theta^2(t; t) = 0$ and $\Theta^2(\xi; t) \in C[0, \infty)$. Then, it follows in Lemma (5) that

$$\lim_{r \rightarrow \infty} \mathbf{B}_{r,\lambda}(\Theta^2(\xi; t); z) = \Theta^2(t; t) = 0.$$

Using the Lemma 3, we can get

$$\lim_{r \rightarrow \infty} r \mathbf{B}_{r,\lambda}(\Theta(\xi; t)(\xi - z)^2; z) = 0,$$

and we complete the proof of Theorem 1. $\square$

3.2. Rate of Convergence

For any $f \in C[0, \infty)$, we recall the following $\mathbb{K}$ -functional:

$$\mathbb{K}(f; u) = \inf_{g \in \mathbf{S}^2} \{ \|f - g\| + u \|g''\| \},$$

where $u \in [0, \infty)$ and $\mathbf{S}^2[0, \infty) = \{g : g, g'' \in C_{BO}[0, \infty)\}$. Next, the usual modulus of smoothness and the second-order modulus of smoothness of f can be defined by

$$\begin{aligned} \omega^*(f; u) &= \sup_{v \in [0, u], z \in [0, \infty)} |f(z + v) - f(z)|, \\ \omega_2^*(f; u) &= \sup_{v \in [0, u], z \in [0, \infty)} |f(z + 2v) - 2f(z + v) + f(z)|. \end{aligned}$$

We easily get the two properties of the usual modulus of smoothness as follows:

$$|f(\xi) - f(z)| \leq \omega^*(f; u) \left(1 + \frac{|\xi - z|}{u} \right), \quad (8)$$

$$|f(\xi) - f(z)| \leq \omega^*(f; u) \left(1 + \frac{(\xi - z)^2}{u^2} \right). \quad (9)$$

By DeVore–Lorentz theorem (see [38], p. 177, Theorem 2.4), there exists an absolute positive constant $C > 0$ depending only on f such that

$$\mathbb{K}(f; u) \leq C \omega_2^*(f; \sqrt{u}). \quad (10)$$

Theorem 2. For $f \in C[0, \infty)$, $\lambda \in (0, 1)$, $r \in \mathbb{N}_+$, then for any $z \in [0, \infty)$, we have

$$|\mathbf{B}_{r, \lambda}(f; z) - f(z)| \leq 2\omega^*(f; \sqrt{\beta_{r, \lambda}(z)}).$$

Proof. Applying the positive and linear properties of the operators $\mathbf{B}_{r, \lambda}$ and (9), we have

$$\begin{aligned} |\mathbf{B}_{r, \lambda}(f; z) - f(z)| &\leq \mathbf{B}_{r, \lambda}(|f(\xi) - f(z)|; z) \\ &\leq \omega^*(f; u) \left(1 + \frac{\mathbf{B}_{r, \lambda}((\xi - z)^2; z)}{u^2} \right) \\ &= \omega^*(f; u) \left(1 + \frac{\beta_{r, \lambda}(z)}{u^2} \right). \end{aligned}$$

Taking $u = \sqrt{\beta_{r, \lambda}(z)}$, we complete the proof of Theorem 2. $\square$

Theorem 3. For $f \in C[0, \infty)$, $\lambda \in (0, 1)$, $r \in \mathbb{N}_+$, then for any $z \in [0, \infty)$, we have

$$|\mathbf{B}_{r, \lambda}(f; z) - f(z)| \leq \frac{3}{2} \omega_2^* \left(f; \sqrt{\beta_{r, \lambda}(z)} \right).$$

Proof. By the result of R. Păltănea [39], for any positive linear operators L_r , the following inequality holds:

$$\begin{aligned} |L_r(f; z) - f(z)| &\leq \left| L_r(\xi^0; z) - 1 \right| \cdot |f(z)| + \frac{1}{u} |L_r(\xi - z; z)| \omega^*(f; u) \\ &\quad + \left(L_r(\xi^0; z) + \frac{1}{2u^2} L_r((\xi - z)^2; z) \right) \omega_2^*(f; u), \end{aligned}$$

Assuming that $L_r := \mathbf{B}_{r,\lambda}$ and using Lemma 3, we obtain

$$|\mathbf{B}_{r,\lambda}(f; z) - f(z)| \leq \left(1 + \frac{\mathbf{B}_{r,\lambda}((\xi - z)^2; z)}{2u^2}\right) \omega_2^*(\varphi; u),$$

Taking $u = \sqrt{\beta_{r,\lambda}(z)}$, we complete the proof of Theorem 3. $\square$

Theorem 4. For $f \in C^1[0, \infty)$, $\lambda \in (0, 1)$, $r \in \mathbb{N}_+$, then for any $z \in [0, \infty)$, we have

$$|\mathbf{B}_{r,\lambda}(f; z) - f(z)| \leq 2\sqrt{\beta_{r,\lambda}(z)} \omega^*\left(f'; \sqrt{\beta_{r,\lambda}(z)}\right).$$

Proof. Applying the operators $\mathbf{B}_{r,\lambda}$ on both sides of the below equality

$$\begin{aligned} f(\xi) &= f(z) + f'(z)(\xi - z) + f(\xi) - f(z) - f'(z)(\xi - z) + f(\xi) \\ &= f(z) + f'(z)(\xi - z) + \int_z^\xi (f'(\eta) - f'(z)) d\eta, \end{aligned}$$

and using (8), the Cauchy–Bunyakovsky–Schwarz inequality and Lemma 3, we have

$$\begin{aligned} |\mathbf{B}_{r,\lambda}(f; z) - f(z)| &\leq \mathbf{B}_{r,\lambda}(|f(\xi) - f(z)|; z) \\ &\leq |f'(z)| |\mathbf{B}_{r,\lambda}(\xi - z; z)| + \left| \mathbf{B}_{r,\lambda}\left(\int_z^\xi (f'(\eta) - f'(z)) d\eta; z\right) \right| \\ &\leq \omega^*(f'; u) \mathbf{B}_{r,\lambda}\left(|\xi - z| + \frac{(\xi - z)^2}{u}; z\right) \\ &\leq \omega^*(f'; u) \sqrt{\mathbf{B}_{r,\lambda}((\xi - z)^2; z)} \left(1 + \frac{\sqrt{\mathbf{B}_{r,\lambda}((\xi - z)^2; z)}}{u}\right). \end{aligned}$$

Taking $u = \sqrt{\beta_{r,\lambda}(z)}$, we obtain the desired inequality. $\square$

Theorem 5. For $f \in C_{BO}[0, \infty)$, $\lambda \in (0, 1)$, $r \in \mathbb{N}_+$, then for any $z \in [0, \infty)$, we have

$$|\mathbf{B}_{r,\lambda}(f; z) - f(z)| \leq 2C\omega_2^*\left(f; \sqrt{\beta_{r,\lambda}(z)}\right).$$

Proof. For any $g \in \mathbf{S}^2[0, \infty)$, by the second-order Taylor expansion with integral remainder for $z \in [0, \infty)$, we have

$$g(\xi) = g(z) + g'(z)(\xi - z) + \int_z^\xi (\xi - \eta) g''(\eta) d\eta. \quad (11)$$

Applying the operator $\mathbf{B}_{r,\lambda}$ to both sides of the equality (11) and Lemma 3, we obtain

$$\begin{aligned} |\mathbf{B}_{r,\lambda}(g; z) - g(z)| &= \left| \mathbf{B}_{r,\lambda}\left(\int_z^\xi (\xi - \eta) g''(\eta) d\eta; z\right) \right| \leq \mathbf{B}_{r,\lambda}\left(\left|\int_z^\xi (\xi - \eta) g''(\eta) d\eta\right|; z\right) \\ &\leq \mathbf{B}_{r,\lambda}\left(\|g''\|(\xi - z)^2; z\right) = \beta_{r,\lambda}(z) \|g''\|. \end{aligned}$$

By Lemma 4, we have

$$\begin{aligned} |\mathbf{B}_{r,\lambda}(f; z) - f(z)| &= |\mathbf{B}_{r,\lambda}(f - g; z) - (f - g)(z) + \mathbf{B}_{r,\lambda}(g; z) - g(z)| \\ &\leq |\mathbf{B}_{r,\lambda}(f - g; z) - (f - g)(z)| + |\mathbf{B}_{r,\lambda}(g; z) - g(z)| \\ &\leq 2\|f - g\| + \beta_{r,\lambda}(z) \|g''\|. \end{aligned}$$

Taking the infimum on both sides of the inequality above over any $g \in \mathbf{S}^2[0, \infty)$ we have

$$|\mathbf{B}_{r,\lambda}(f; z) - f(z)| \leq 2\mathbb{K}(f; \beta_{r,\lambda}(z))$$

By (11), we complete the proof of Theorem 5. $\square$

For any $f \in C_{BO}[0, \infty)$ and $\chi > 0$, the second-order Steklov mean is denoted as

$$f_\chi(z) = \frac{4}{\chi^2} \int_0^{\frac{\chi}{2}} \int_0^{\frac{\chi}{2}} (2f(z + \sigma + \rho) - f(z + 2(\sigma + \rho))) d\rho d\sigma.$$

By direct calculation, we can yield

$$\|f_\chi - f\| \leq \omega_2^*(f; \chi). \quad (12)$$

If $f \in C_{BO}[0, \infty)$, then $f'_\chi(z), f''_\chi(z) \in C_{BO}[0, \infty)$ and

$$\|f''_\chi\| \leq \frac{9}{\chi^2} \omega_2^*(f; \chi). \quad (13)$$

Theorem 6. For any $f \in C_{BO}[0, \infty)$ and $\lambda \in (0, 1)$, $r \in \mathbb{N}_+$, then for any $z \in [0, \infty)$, we have

$$|\mathbf{B}_{r,\lambda}(f; z) - f(z)| \leq \frac{13}{2} \omega_2^*\left(f; \sqrt{\beta_{r,\lambda}(z)}\right).$$

Proof. For $z \in [0, \infty)$, $\chi > 0$, by the definition of the second-order Steklov mean, we can get

$$|\mathbf{B}_{r,\lambda}(f; z) - f(z)| \leq \mathbf{B}_{r,\lambda}(|f_\chi - f|; z) + |\mathbf{B}_{r,\lambda}(f_\chi; z) - f_\chi(z)| + |f_\chi(z) - f(z)|.$$

Using Lemma 4 and (12), we have

$$\mathbf{B}_{r,\lambda}(|f_\chi - f|; z) \leq \|\mathbf{B}_{r,\lambda}(|f_\chi - f|; z)\| \leq \|f_\chi - f\| \leq \omega_2^*(f; \chi).$$

By the second-order Taylor expansion with integral remainder for f_χ , we have

$$f_\chi(\xi) = f_\chi(z) - f'_\chi(z)(\xi - z) + \int_z^\xi (\sigma - z) f''_\chi(\sigma) d\sigma.$$

Using Lemma 3 and (13), we can have

$$\begin{aligned} |\mathbf{B}_{r,\lambda}(f_\chi; z) - f_\chi(z)| &\leq \left| \mathbf{B}_{r,\lambda}\left(f'_\chi(z)(\xi - z); z\right) \right| + \left| \mathbf{B}_{r,\lambda}\left(\int_z^\xi (\sigma - z) f''_\chi(\sigma) d\sigma; z\right) \right| \\ &\leq \|f''_\chi\| \left| \mathbf{B}_{r,\lambda}\left(\int_z^\xi (\sigma - z) d\sigma; z\right) \right| = \frac{1}{2} \|f''_\chi\| \beta_{r,\lambda}(z) \leq \frac{9}{2\chi^2} \beta_{r,\lambda}(z) \omega_2^*(f; \chi). \end{aligned}$$

Hence,

$$|\mathbf{B}_{r,\lambda}(f; z) - f(z)| \leq \left(\frac{9}{2\chi^2} \beta_{r,\lambda}(z) + 2 \right) \omega_2^*(f; \chi)$$

for $z \in [0, \infty)$. Choosing $\chi = \sqrt{\beta_{r,\lambda}(z)}$, we get the desired result. $\square$

3.3. Weighted Approximation

Let $\mathbf{S}_2[0, \infty)$ be the function space of all functions f defined on $[0, \infty)$ satisfying the condition $|f(z)| \leq C_f(1 + z^2)$ with an absolute positive constant C_f depending only on f . Let $C_2[0, \infty)$ denote the subspace of all continuous functions in $\mathbf{S}_2[0, \infty)$. Let $C_2^*[0, \infty)$

denote the subspace of the functions $f \in C_2[0, \infty)$ with $\lim_{z \rightarrow +\infty} \frac{|f(z)|}{1+z^2} = C_f^* < \infty$. The norm of the subspace $C_2^*[0, \infty)$ is defined by $\|f\|_2 = \sup_{z \in [0, \infty)} \frac{|f(z)|}{1+z^2}$. Meantime, we recall the modulus of continuity of f on the closed interval $[0, \kappa]$, $\kappa > 0$ by

$$\omega_\kappa^*(f; u) = \sup_{u > 0, z+u, z \in [0, \kappa]} |f(z+u) - f(z)|.$$

Theorem 7. For $f \in C_2[0, \infty)$, $\lambda \in (0, 1)$, we have

$$\|B_{r,\lambda}(f) - f\|_{C[0,\kappa]} \leq 4C_f(1 + \kappa^2)\beta_{r,\lambda}(\kappa) + 2\omega_{\kappa+1}^*\left(f; \sqrt{\beta_{r,\lambda}(\kappa)}\right).$$

Proof. For any $z \in [0, \kappa]$ and $\xi \geq \kappa + 1$, we easily get $(\xi - z)^2 \geq (\xi - \kappa)^2 > 1$, thus,

$$\begin{aligned} |f(\xi) - f(z)| &\leq |f(\xi)| + |f(z)| \leq C_f(2 + \xi^2 + z^2) \\ &\leq C_f(2 + z^2 + (\xi - z + z)^2) \leq C_f(2 + 3z^2 + 2(\xi - z)^2) \\ &\leq C_f(4 + 3z^2)(\xi - z)^2 \leq 4C_f(1 + \kappa^2)(\xi - z)^2. \end{aligned}$$

And for any $z \in [0, \kappa]$, $0 \leq \xi < \kappa + 1$ and $u > 0$, we have

$$|f(\xi) - f(z)| \leq \omega_\kappa(f; |\xi - z|) \leq \left(1 + \frac{|\xi - z|}{u}\right) \omega_\kappa(f; u).$$

Combining the two inequalities above, we can derive

$$|f(\xi) - f(z)| \leq 4C_f(1 + \kappa^2)(\xi - z)^2 + \left(1 + \frac{|\xi - z|}{u}\right) \omega_\kappa(f; u).$$

By Cauchy–Bunyakovsky–Schwarz inequality and Lemma 3, for any $z \in [0, \kappa]$, we have

$$\begin{aligned} |B_{r,\lambda}(f; z) - f(z)| &\leq B_{r,\lambda}(|f(\xi) - f(z)|; z) \\ &\leq 4C_f(1 + \kappa^2)B_{r,\lambda}((\xi - z)^2; z) + B_{r,\lambda}\left(\left(1 + \frac{|\xi - z|}{u}\right); z\right) \omega_\kappa(f; u) \\ &\leq 4C_f(1 + \kappa^2)\beta_{r,\lambda}(\kappa) + \left(1 + \frac{1}{u}\sqrt{B_{r,\lambda}((\xi - z)^2; z)}\right) \omega_\kappa(f; u) \\ &\leq 4C_f(1 + \kappa^2)\beta_{r,\lambda}(\kappa) + \left(1 + \frac{1}{u}\sqrt{\beta_{r,\lambda}(z)}\right) \omega_\kappa(f; u). \end{aligned}$$

By choosing $u = \sqrt{\beta_{r,\lambda}(z)}$ and the supremum over all $z \in [0, \kappa]$, we have the desired results. $\square$

As is well known, if the function $f(z)$ is not uniformly continuous on the infinite interval $[0, \infty)$, the usual modulus of smoothness $\omega^*(f; u)$ does not tend to 0 as $u \rightarrow 0^+$. In [40], Yüksel and Ispir constructed the following weighted modulus of continuity in $C_2^*[0, \infty)$:

$$\Omega^*(f; u) = \sup_{z \in [0, \infty), u > 0} \frac{|f(z+u) - f(z)|}{(1+z^2)(1+u^2)}, \text{ for } f \in C_2^*[0, \infty).$$

Lemma 6 ([40]). For $f \in C_2^*[0, \infty)$, we have:

- (1) With respect to u , $\Omega^*(f; u)$ is a monotonically non-decreasing function;
- (2) $\lim_{u \rightarrow 0^+} \Omega^*(f; u) = 0$;

- (3) For any $s \in \mathbb{N}_+$, it holds that $\Omega^*(f; su) \leq s\Omega^*(f; u)$;
 (4) For any $\rho > 0$, $\Omega^*(f; \rho u) \leq 2(1 + \rho)(1 + u^2)\Omega^*(f; u)$.

Theorem 8. For $f \in C_2^*[0, \infty)$, $\lambda \in (0, 1)$, and sufficiently large r , we obtain

$$\frac{|\mathbf{B}_{r,\lambda}(f; z) - f(z)|}{(1 + z^2)^{\frac{5}{2}}} \leq 40\Omega^*\left(f; \frac{1}{\sqrt{r}}\right).$$

Proof. For $\xi, z \in [0, \infty)$, $\chi > 0$, applying the definition of $\Omega^*(f; u)$ and the property (4) in Lemma 6, we obtain

$$\begin{aligned} |f(\xi) - f(z)| &\leq (1 + (\xi - z)^2)(1 + z^2)\Omega^*(f; |\xi - z|) \\ &\leq 2\left(1 + \frac{|\xi - z|}{\chi}\right)(1 + \chi^2)\Omega^*(f; \chi)(1 + (\xi - z)^2)(1 + z^2) \\ &\leq \begin{cases} 4(1 + \chi^2)^2(1 + z^2)\Omega^*(f; \chi), & |\xi - z| \leq \chi, \\ 4(1 + \chi^2)^2(1 + z^2)\frac{|\xi - z|^3}{\chi^3}\Omega^*(f; \chi) & |\xi - z| > \chi. \end{cases} \end{aligned} \quad (14)$$

For any $\chi \in (0, \sqrt{2} - 1)$, we have

$$\begin{aligned} |f(\xi) - f(z)| &\leq 4(1 + \chi^2)^2(1 + z^2)\left(1 + \frac{|\xi - z|^3}{\chi^3}\right)\Omega^*(f; \chi) \\ &\leq 8(1 + z^2)\left(1 + \frac{|\xi - z|^3}{\chi^3}\right)\Omega^*(f; \chi). \end{aligned}$$

By the Cauchy–Bunyakovsky–Schwarz inequality, we have

$$\begin{aligned} |\mathbf{B}_{r,\lambda}(f; z) - f(z)| &\leq 8(1 + z^2)\left(1 + \frac{\mathbf{B}_{r,\lambda}(|\xi - z|^3; z)}{\chi^3}\right)\Omega^*(f; \chi) \\ &\leq 8(1 + z^2)\left(1 + \frac{\sqrt{\mathbf{B}_{r,\lambda}(|\xi - z|^2; z) \cdot \mathbf{B}_{r,\lambda}(|\xi - z|^4; z)}}{\chi^3}\right)\Omega^*(f; \chi). \end{aligned}$$

Using (4) and (5) and taking $\chi = \frac{1}{\sqrt{r}}$, we can obtain

$$\begin{aligned} |\mathbf{B}_{r,\lambda}(f; z) - f(z)| &\leq 8(1 + z^2)\left(1 + \sqrt{12}(1 + z^2)^{\frac{3}{2}}\right)\Omega^*\left(f; \frac{1}{\sqrt{r}}\right) \\ &\leq 8(1 + \sqrt{12})(1 + z^2)^{\frac{5}{2}}\Omega^*\left(f; \frac{1}{\sqrt{r}}\right) \\ &\leq 40(1 + z^2)^{\frac{5}{2}}\Omega^*\left(f; \frac{1}{\sqrt{r}}\right), \end{aligned}$$

and Theorem 8 is proved. $\square$

Theorem 9. For $f \in C_2^*[0, \infty)$, $f'' \in C_2^*[0, \infty)$, $\lambda \in (0, 1)$, and sufficiently large r , we have

$$r\left|\mathbf{B}_{r,\lambda}(f; z) - f(z) - \frac{f''(z)}{2!}\beta_{r,\lambda}(z)\right| \leq 35(1 + z^2)^{\frac{7}{2}}\Omega^*\left(f''; \frac{1}{\sqrt{r}}\right).$$

Proof. By the second-order Taylor expansion for $f \in C_2^*[0, \infty)$, we have

$$\begin{aligned} f(\xi) &= f(z) + f'(z)(\xi - z) + \frac{f''(\alpha)}{2!}(\xi - z)^2 \\ &= f(z) + f'(z)(\xi - z) + \frac{f''(z)}{2!}(\xi - z)^2 + \Theta_2(\xi, z), \end{aligned}$$

where $|\alpha - z| \leq |\xi - z|$ and $\Theta_2(\xi, z) = \frac{f''(\alpha) - f''(z)}{2!}$. Applying (14) and choosing $\chi \in (0, 1)$, we have

$$\Theta_2(\xi, z) \leq 2(1 + \chi^2)^2(1 + z^2)\Omega^*(f''; \chi) \left(1 + \frac{|\xi - z|^3}{\chi^3}\right)(\xi - z)^2.$$

Using the operators $\mathbf{B}_{r,\lambda}$ on both sides of the above inequality and applying the Cauchy–Bunyakovsky–Schwarz inequality and Remark 1, we can get

$$\begin{aligned} \mathbf{B}_{r,\lambda}(\Theta_2(\xi, z); z) &\leq 2(1 + \chi^2)^2(1 + z^2)\Omega^*(f''; \chi) \left(\mathbf{B}_{r,\lambda}((\xi - z)^2; z) + \frac{\mathbf{B}_{r,\lambda}(|\xi - z|^3; z)}{\chi^3} \right) \\ &\leq 2(1 + \chi^2)^2(1 + z^2)\Omega^*(f''; \chi) \left(\beta_{r,\lambda}(z) + \frac{\sqrt{\mathbf{B}_{r,\lambda}((\xi - z)^4; z) \cdot \mathbf{B}_{r,\lambda}((\xi - z)^6; z)}}{\chi^3} \right) \\ &\leq 2(1 + \chi^2)^2(1 + z^2)\Omega^*(f''; \chi) \left(\frac{2(1 + z^2)}{r} + \frac{\sqrt{225}(1 + z^2)^{\frac{5}{2}}}{\chi^3 r^{\frac{5}{2}}} \right). \end{aligned}$$

Taking $\chi = \frac{1}{\sqrt{r}}$, we have

$$\begin{aligned} r \left| \mathbf{B}_{r,\lambda}(f; z) - f(z) - \frac{f''(z)}{2!} \beta_{r,\lambda}(z) \right| &\leq r \mathbf{B}_{r,\lambda}(\Theta_2(\xi, z); z) \\ &\leq 2(1 + \chi^2)^2(1 + z^2)\Omega^*\left(f''; \frac{1}{\sqrt{r}}\right) (2(1 + z^2) + 15(1 + z^2)^{\frac{5}{2}}) \\ &\leq 34(1 + \chi^2)^2(1 + z^2)^{\frac{7}{2}}\Omega^*\left(f''; \frac{1}{\sqrt{r}}\right) \leq 35(1 + z^2)^{\frac{7}{2}}\Omega^*\left(f''; \frac{1}{\sqrt{r}}\right). \end{aligned}$$

Hence, we obtain the desired result. $\square$

Theorem 10. For $f \in C_2^*[0, \infty)$, $\lambda \in (0, 1)$, we have

$$\lim_{r \rightarrow \infty} \|\mathbf{B}_{r,\lambda}(f) - f\|_2 = 0.$$

Proof. Using the Korovkin theorem [41], we only need to prove the following three limit equalities:

$$\lim_{r \rightarrow \infty} \|\mathbf{B}_{r,\lambda}(\xi^k; z) - z^k\|_2 = 0, k = 0, 1, 2. \quad (15)$$

By $\mathbf{B}_{r,\lambda}(\xi^0; z) = 1$, the limit equality (15) holds for $k = 0$. Similarly, the limit equality (15) holds for $k = 1$. From Lemma (3), we have

$$\|\mathbf{B}_{r,\lambda}(\xi^2; z) - z^2\|_2 = \sup_{z \in [0, \infty)} \frac{1}{1 + z^2} \left| \frac{\lambda}{r} z^2 + \frac{1}{r} z \right| = \frac{2\lambda + 1}{2r} \rightarrow 0, \text{ as } r \rightarrow \infty.$$

Thus, the proof of Theorem 10 is completed. $\square$

Theorem 11. For $f \in C_2^*[0, \infty)$, $\lambda \in (0, 1)$ and $\kappa > 0$, we have

$$\lim_{r \rightarrow \infty} \sup_{z \in [0, \infty)} \frac{|\mathbf{B}_{r,\lambda}(f; z) - f(z)|}{(1 + z^2)^{1+\kappa}} = 0.$$

Proof. For any fixed $z_0 \in (0, \infty)$, the later will be provided. Then

$$\begin{aligned} \sup_{z \in [0, \infty)} \frac{|\mathbf{B}_{r,\lambda}(f; z) - f(z)|}{(1+z^2)^{1+\kappa}} &\leq \sup_{z \in [0, z_0]} \frac{|\mathbf{B}_{r,\lambda}(f; z) - f(z)|}{(1+z^2)^{1+\kappa}} + \sup_{z \in [z_0, \infty)} \frac{|\mathbf{B}_{r,\lambda}(f; z) - f(z)|}{(1+z^2)^{1+\kappa}} \\ &\leq \|\mathbf{B}_{r,\lambda}(f) - f\|_{[0, z_0]} + \|f\|_2 \sup_{z \in [z_0, \infty)} \frac{\mathbf{B}_{r,\lambda}(1+\xi^2; z)}{(1+z^2)^{1+\kappa}} + \sup_{z \in [z_0, \infty)} \frac{|f(z)|}{(1+z^2)^{1+\kappa}} \\ &:= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 \end{aligned} \quad (16)$$

Using $|f(z)| \leq \mathbf{C}_f(1+z^2) \leq \|f\|_2(1+z^2)$, we have

$$\mathbf{I}_3 = \sup_{z \in [z_0, \infty)} \frac{\|f\|_2(1+z^2)}{(1+z^2)^{1+\kappa}} \leq \frac{\|f\|_2}{(1+z_0^2)^\kappa}$$

for $\forall \epsilon > 0$. Applying $\lim_{r \rightarrow \infty} \sup_{z \in [z_0, \infty)} \frac{\mathbf{B}_{r,\lambda}(1+\xi^2; z)}{1+z^2} = 1$, there exists $R_1 \in \mathbb{N}_+$, such that for all $r > R_1$,

$$\sup_{z \in [z_0, \infty)} \frac{\mathbf{B}_{r,\lambda}(1+\xi^2; z)}{1+z^2} \leq \frac{(1+z_0^2)^\kappa}{\|f\|_2} \cdot \frac{\epsilon}{3} + 1.$$

Thus,

$$\mathbf{I}_2 = \|f\|_2 \sup_{z \in [z_0, \infty)} \frac{\mathbf{B}_{r,\lambda}(1+\xi^2; z)}{(1+z^2)^{1+\kappa}} \leq \frac{\|f\|_2}{(1+z_0^2)^\kappa} + \frac{\epsilon}{3}, \forall r > R_1.$$

Hence,

$$\mathbf{I}_2 + \mathbf{I}_3 \leq \frac{2\|f\|_2}{(1+z_0^2)^\kappa} + \frac{\epsilon}{3}, \forall r > R_1. \quad (17)$$

Next, in view of Lemma 5, for the given ϵ , there exists $R_2 \in \mathbb{N}_+$ such that

$$\mathbf{I}_1 = \|\mathbf{B}_{r,\lambda}(f) - f\|_{[0, z_0]} < \frac{\epsilon}{3}, \forall r > R_2. \quad (18)$$

Sufficiently large z_0 is chosen such that

$$\frac{\|f\|_2}{(1+z_0^2)^\kappa} < \frac{\epsilon}{6}. \quad (19)$$

Let $R = \max\{R_1, R_2\}$. Combining (15), (16), (17), (18), and (19), we have

$$\sup_{z \in [0, \infty)} \frac{|\mathbf{B}_{r,\lambda}(f; z) - f(z)|}{(1+z^2)^{1+\kappa}} < \epsilon, \forall r > R.$$

Hence, the proof of Theorem 11 is completed. $\square$

3.4. Point-Wise Estimates

In this subsection, we derive three point-wise estimates of the operators (3). First, we recall that a function $f \in \text{Lip}_{\mathbf{M}}(\gamma, \mathbf{D})$, $\gamma \in (0, 1]$, $\mathbf{D} \subset [0, \infty)$ if it satisfies the following inequality:

$$|f(\xi) - f(z)| \leq \mathbf{M}|\xi - z|^\gamma, \xi \in \mathbf{D}, z \in [0, \infty),$$

where $\mathbf{M}$ is an absolute positive constant depending only on γ and f .

Theorem 12. For $f \in C_2[0, \infty) \cap \text{Lip}_{\mathbf{M}}(\gamma, \mathbf{D})$, $\lambda \in (0, 1)$, then for any $z \in [0, \infty)$, we have

$$|\mathbf{B}_{r,\lambda}(f; z) - f(z)| \leq \mathbf{M} \left((\beta_{r,\lambda}(z))^{\frac{\gamma}{2}} + 2d^\gamma(z; \mathbf{D}) \right),$$

where $d(z; \mathbf{D}) = \inf\{|\xi - z| : \xi \in \mathbf{D}\}$ denotes the distance between z and $\mathbf{D}$.

Proof. Assuming $\overline{\mathbf{D}}$ be that closure of $\mathbf{D}$ and utilizing the properties of infimum, there exists a point $\xi_0 \in \overline{\mathbf{D}}$ such that $d(z; \mathbf{D}) = |\xi_0 - z|$. Using the triangle inequality

$$|f(\xi) - f(z)| \leq |f(\xi) - f(\xi_0)| + |f(\xi_0) - f(z)|,$$

we have

$$\begin{aligned} |\mathbf{B}_{r,\lambda}(f; z) - f(z)| &\leq \mathbf{B}_{r,\lambda}(|f(\xi) - f(\xi_0)|; z) + \mathbf{B}_{r,\lambda}(|f(\xi_0) - f(z)|; z) \\ &\leq \mathbf{M}\{\mathbf{B}_{r,\lambda}(|\xi - \xi_0|^\gamma; z) + |\xi_0 - z|^\gamma\} \\ &\leq \mathbf{M}\{\mathbf{B}_{r,\lambda}(|\xi - z|^\gamma + |\xi_0 - z|^\gamma; z) + |\xi_0 - z|^\gamma\} \\ &\leq \mathbf{M}\{\mathbf{B}_{r,\lambda}(|\xi - z|^\gamma; z) + 2|\xi_0 - z|^\gamma\}. \end{aligned}$$

Choosing $s_1 = \frac{2}{\gamma}$ and $s_2 = \frac{2}{2-\gamma}$ and utilizing the well-known Hölder inequality, we obtain

$$\begin{aligned} |\mathbf{B}_{r,\lambda}(f; z) - f(z)| &\leq \mathbf{M} \left\{ (\mathbf{B}_{r,\lambda}(|\xi - z|^{s_1\gamma}; z))^{\frac{1}{s_1}} (\mathbf{B}_{r,\lambda}(1^{s_2}; z))^{\frac{1}{s_2}} + 2d^\gamma(z; \mathbf{D}) \right\} \\ &\leq \mathbf{M} \left\{ \left(\mathbf{B}_{r,\lambda}((\xi - z)^2; z) \right)^{\frac{1}{s_1}} + 2d^\gamma(z; \mathbf{D}) \right\} \\ &= \mathbf{M} \left((\beta_{r,\lambda}(z))^{\frac{\gamma}{2}} + 2d^\gamma(z; \mathbf{D}) \right), \end{aligned}$$

and we get the desired result. $\square$

Next, Lenze introduced the following Lipschitz-type maximal function of the order γ in [42]:

$$\tilde{\omega}_\gamma^*(f; z) = \sup_{\xi \in [0, \infty), \xi \neq z} \frac{|f(\xi) - f(z)|}{|\xi - z|^\gamma}, \quad \gamma \in (0, 1]. \quad (20)$$

Theorem 13. For $f \in C_2[0, \infty)$, $\lambda \in (0, 1)$, then for any $z \in (0, \infty)$, we have

$$|\mathbf{B}_{r,\lambda}(f; z) - f(z)| \leq \tilde{\omega}_\gamma^*(f; z)(\beta_{r,\lambda}(z))^{\frac{\gamma}{2}}.$$

Proof. Using the equation (20), we have

$$|\mathbf{B}_{r,\lambda}(f; z) - f(z)| \leq \tilde{\omega}_\gamma^*(f; z) \mathbf{B}_{r,\lambda}(|\xi - z|^\gamma; z).$$

By the well-known Hölder inequality, we get

$$|\mathbf{B}_{r,\lambda}(f; z) - f(z)| \leq \tilde{\omega}_\gamma^*(f; z) (\mathbf{B}_{r,\lambda}((\xi - z)^\gamma; z))^{\frac{\gamma}{2}} = \tilde{\omega}_\gamma^*(f; z) (\beta_{r,\lambda}(z))^{\frac{\gamma}{2}},$$

and we get the desired relation. $\square$

Finally, Özarslan and Aktuğlu considered the following Lipschitz-type space with two distinct parameters $\kappa_1 \geq 0, \kappa_2 > 0$ in [43]:

$$\text{Lip}_{\mathbf{M}}^{(\kappa_1, \kappa_2)}(\gamma) := \left\{ f \in C_{BO}[0, \infty) : |f(\xi) - f(z)| \leq \mathbf{M} \frac{|\xi - z|^\gamma}{(1 + \kappa_1 z^2 + \kappa_2 z)^{\frac{\gamma}{2}}}, \xi \in [0, \infty), z \in (0, \infty), \right\}$$

where $\gamma \in (0, 1]$ and $\mathbf{M}$ is an absolute positive constant depending only on $\gamma, \kappa_1, \kappa_2$ and f .

Theorem 14. For $f \in \text{Lip}_{\mathbf{M}}^{(\kappa_1, \kappa_2)}(\gamma)$, $\lambda \in (0, 1)$, then for any $z \in (0, \infty)$, we have

$$|\mathbf{B}_{r,\lambda}(f; z) - f(z)| \leq \mathbf{M} \left(\frac{\beta_{r,\lambda}(z)}{\kappa_1 z^2 + \kappa_2 z} \right)^{\frac{\gamma}{2}}.$$

Proof. Utilizing the well-known Hölder inequality with $s_1 = \frac{2}{\gamma}$ and $s_2 = \frac{2}{2-\gamma}$, we obtain

$$\begin{aligned} |\mathbf{B}_{r,\lambda}(f; z) - f(z)| &\leq \mathbf{B}_{r,\lambda}(|f(\xi) - f(z)|; z) \leq \mathbf{B}_{r,\lambda} \left(\mathbf{M} \frac{|\xi - z|^\gamma}{(1 + \kappa_1 z^2 + \kappa_2 z)^{\frac{\gamma}{2}}}; z \right) \\ &\leq \frac{\mathbf{M}}{(\kappa_1 z^2 + \kappa_2 z)^{\frac{\gamma}{2}}} \mathbf{B}_{r,\lambda}(|\xi - z|^\gamma; z) \\ &\leq \frac{\mathbf{M}}{(\kappa_1 z^2 + \kappa_2 z)^{\frac{\gamma}{2}}} (\mathbf{B}_{r,\lambda}(|\xi - z|^{\gamma s_1}; z))^{\frac{1}{s_1}} (\mathbf{B}_{r,\lambda}(1^{s_2}; z))^{\frac{1}{s_2}} \\ &= \mathbf{M} \left(\frac{\beta_{r,\lambda}(z)}{\kappa_1 z^2 + \kappa_2 z} \right)^{\frac{\gamma}{2}}. \end{aligned}$$

This completes the proof of Theorem 14. $\square$

4. Numerical Examples

Note $E_{r,\lambda}(f; z) = |\mathbf{B}_{r,\lambda}(f; z) - f(z)|$, the error function of approximation by the operators $\mathbf{B}_{r,\lambda}(f; z)$ to the target function $f(z)$.

Example 1. Let $f(z) = z^4 - 6z^3 + 13z^2 - 12z + 4$, $z \in [0, 1]$, $\lambda = 0.45$ and $r = 5, 10, 20, 40$.

Figure 1 shows the convergence of the operators $\mathbf{B}_{r,\lambda}(f; z)$ to the target function $f(z)$ while we choose different r . We can clearly see that when the values of r are increasing, the graph of the operators $\mathbf{B}_{r,\lambda}(f; z)$ are going to the graph of the target function $f(z)$ under λ given. Table 1 shows the error of approximation for $E_{5,0.45}(f; z)$, $E_{10,0.45}(f; z)$, $E_{20,0.45}(f; z)$, and $E_{40,0.45}(f; z)$ at the points $\{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$. Combining with Figure 2, we can clearly see that when the values of r are increasing, the error function of the approximation $E_{r,\lambda}(f; z)$ becomes smaller.

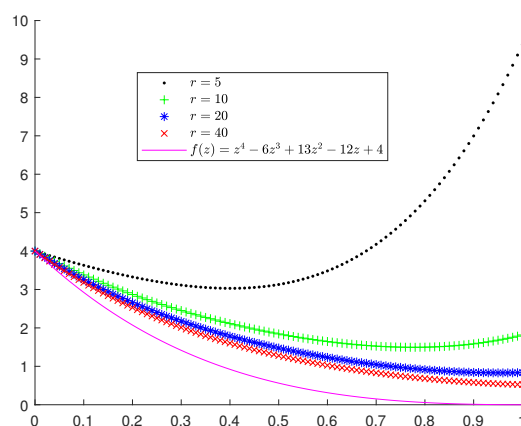


Figure 1. The operators $\mathbf{B}_{r,0.45}(f; z)$ and the target function $f(z)$.

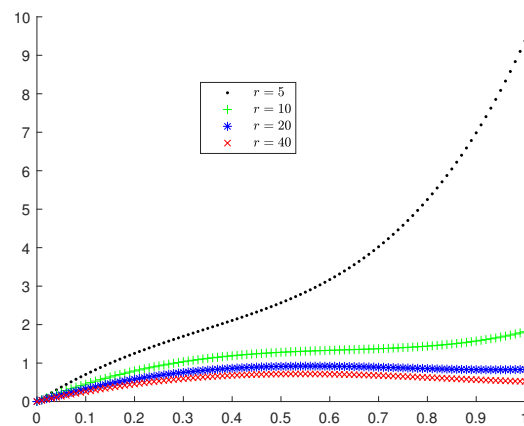


Figure 2. Approximation error function $E_{r,0.45}(f; z)$.

Table 1. Error of approximation $E_{r,0.45}(f; z)$ for $r = 5, 10, 20, 40$.

z	$E_{5,0.45}(f; z)$	$E_{10,0.45}(f; z)$	$E_{20,0.45}(f; z)$	$E_{40,0.45}(f; z)$
0.1	0.7055	0.4538	0.3300	0.2584
0.2	1.2528	0.7953	0.5800	0.4561
0.3	1.6973	1.0356	0.7561	0.5951
0.4	2.1081	1.1905	0.8638	0.6797
0.5	2.5671	1.2808	0.9134	0.7159
0.6	3.1695	1.3317	0.9179	0.7118
0.7	4.0234	1.3734	0.8926	0.6775
0.8	5.2503	1.4407	0.8560	0.6251
0.9	6.9847	1.5732	0.8291	0.5685

Example 2. Let $f(z) = z^3 - 2z^2 - z + 2$, $z \in [0.5, 2.5]$, $r = 20$.

Figure 3 shows the graphs of the operators $\mathbf{B}_{20,\lambda}(f; z)$, the Baskakov operators $B_{20}(f; x)$ and the Szász–Mirakyan operators $S_{20}(f; x)$. Combining with Figure 4 and Table 2, we can clearly see that when the values of λ are monotonically increasing towards 1, the error between the operators $\mathbf{B}_{20,\lambda}(f; z)$ and the Baskakov operators becomes smaller. Combining with Figure 5 and and Table 3, we can clearly see that when the values of λ are monotonically decreasing towards 0, the error between the operators $\mathbf{S}_{20,\lambda}(f; z)$ and the Szász–Mirakyan operators becomes smaller.

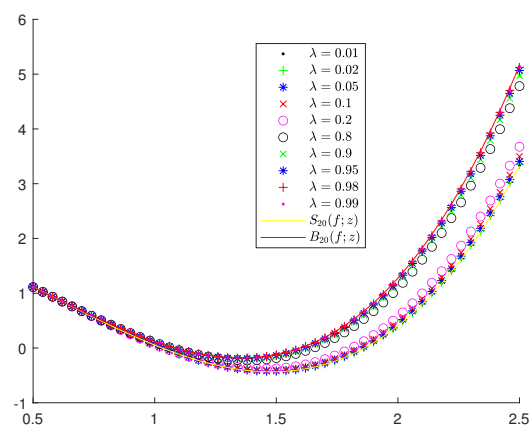


Figure 3. The operators $\mathbf{B}_{20,\lambda}(f; z)$, the Baskakov operators $B_{20}(f; x)$ and the Szász–Mirakyan operators $S_{20}(f; x)$.

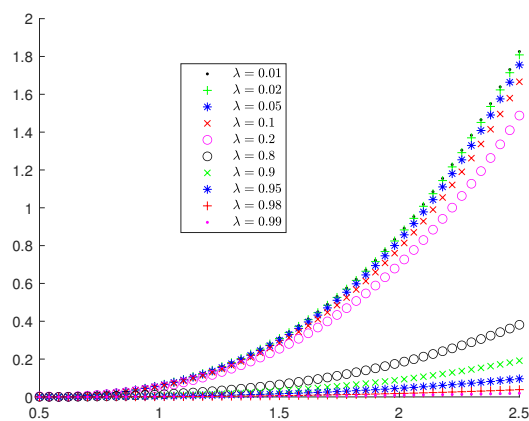


Figure 4. Error between the operators $\mathbf{B}_{20,\lambda}(f; z)$ and the Baskakov operators $B_{20}(f; x)$.

Table 2. Error of approximation $|\mathbf{B}_{20,\lambda}(f; z) - B_{20}(f; x)|$ at points $z = \{0.5, 0.9, 1.3, 1.7, 2.1, 2.5\}$.

λ	0.5	0.9	1.3	1.7	2.1	2.5
0.01	0.0037	0.0299	0.1617	0.4498	0.9533	1.7310
0.02	0.0037	0.0296	0.1601	0.4455	0.9441	1.7142
0.05	0.0035	0.0288	0.1555	0.4325	0.9164	1.6639
0.10	0.0033	0.0274	0.1478	0.4108	0.8701	1.5796
0.20	0.0029	0.0246	0.1322	0.3670	0.7770	1.4101
0.80	0.0007	0.0065	0.0342	0.0945	0.1995	0.3614
0.90	0.0003	0.0033	0.0172	0.0475	0.1002	0.1815
0.95	0.0002	0.0017	0.0086	0.0238	0.0502	0.0909
0.98	0.0001	0.0007	0.0035	0.0095	0.0201	0.0364
0.99	0.0000	0.0003	0.0017	0.0048	0.0101	0.0182

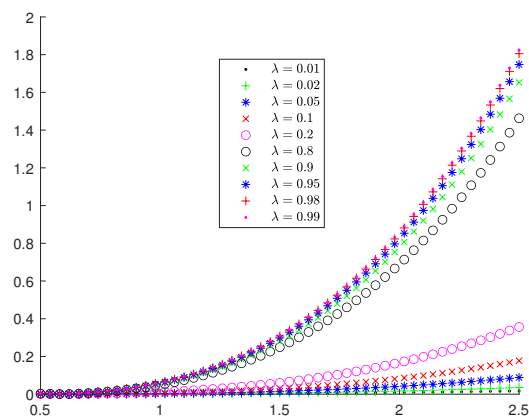


Figure 5. Error between the operators $\mathbf{B}_{20,\lambda}(f; z)$ and the Szász–Mirakyan operators $S_{20}(f; x)$.

Table 3. Error of approximation $|\mathbf{B}_{20,\lambda}(f; z) - S_{20}(f; x)|$ at points $z = \{0.5, 0.9, 1.3, 1.7, 2.1, 2.5\}$.

λ	0.5	0.9	1.3	1.7	2.1	2.5
0.01	0.0000	0.0003	0.0015	0.0043	0.0092	0.0167
0.02	0.0001	0.0005	0.0031	0.0086	0.0184	0.0335
0.05	0.0002	0.0014	0.0077	0.0216	0.0460	0.0838
0.10	0.0004	0.0027	0.0154	0.0434	0.0923	0.1681
0.20	0.0009	0.0055	0.0310	0.0872	0.1855	0.3376
0.80	0.0031	0.0236	0.1290	0.3596	0.7630	1.3863
0.90	0.0034	0.0269	0.1460	0.4067	0.8623	1.5662
0.95	0.0036	0.0285	0.1546	0.4303	0.9122	1.6568
0.98	0.0037	0.0295	0.1597	0.4446	0.9423	1.7113
0.99	0.0037	0.0298	0.1615	0.4494	0.9524	1.7295

5. Conclusions

In our manuscript, we construct λ -Baskakov operators based on negative λ -binomial distribution. We yield many approximation properties of the λ -Baskakov operators (3). First, we obtain a Voronovskaja-type theorem of them. Then, we yield the rate of convergence of the new operators in different function spaces based on the modulus of continuity, $\mathbb{K}$ -functional and Steklov mean function. Then, weighted approximation and point-wise estimates are also discussed. Finally, theoretical results are supported by illustrative graphics and tables using Matlab software.

We remark that the new operators have the following properties:

- The operators are linear and positive.
- While $\lambda \rightarrow 1^-$, the operators (3) turn out to be the classical Baskakov operators (1).
- While $\lambda \rightarrow 0^+$, the operators (3) turn out to be the classical Szász–Mirakyan operators (2).

In the future, we will construct the following modification of the new operators:

- Kantorovich-type modification.
- Durrmeyer-type modification.
- Based on the methods of Usta, F.
- Higher order operators.
- Riemann–Liouville fractional integral.
- q - and (p, q) -analogue.

Also, we will consider the following open problems:

- Approximation properties in L^p and Orlicz spaces of the new operators.
- Application in CAGD or Pattern Recognition.

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