

Article

Boundedness of Singular Integral Operator in Variable Exponent Lebesgue Spaces

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Abstract

This manuscript establishes the boundedness results for a class of singular integral operators to a more general framework by imposing conditions on the Hardy–Littlewood maximal function. We establish the results where we construct a suitable range space $L^{t(\cdot)}(\mathbb{R}_+)$ for a given domain space $L^{s(\cdot)}(\mathbb{R}_+)$ such that the operator maps $L^{s(\cdot)}(\mathbb{R}_+)$ into $L^{t(\cdot)}(\mathbb{R}_+)$ under appropriate assumptions. Conversely, for a prescribed range space $L^{t(\cdot)}(\mathbb{R}_+)$, we determine a corresponding domain space $L^{s(\cdot)}(\mathbb{R}_+)$ ensuring that the operator maps $L^{s(\cdot)}(\mathbb{R}_+)$ into $L^{t(\cdot)}(\mathbb{R}_+)$. In both settings, we provide illustrative examples to explicitly construct the respective spaces. Since our approach fundamentally relies on the boundedness of the maximal operator, the main results are valid only when the essential infimum $s_- > 1$. In the limiting case $s_- = 1$, we derive weak-type boundedness results of the form $(1, t(\cdot))$. Additionally, we present analogous formulations of these results in the classical setting.

Keywords: variable lebesgue spaces; logarithmic operator; Hardy–Littlewood maximal operator**MSC:** 42B25; 42B35; 46E30; 47B34

1. Introduction

Fractional integral operators play a central role in harmonic analysis and its applications to partial differential equations, potential theory, and related areas. Among the most classical objects in this direction are the Riesz potentials and their variants, which have been extensively studied due to their mapping properties between Lebesgue spaces [1–6]. A natural and important generalization of these operators is given by the Bessel–Riesz operators, whose kernels incorporate an additional decay factor at infinity, allowing for a more flexible treatment of both local and global behavior [7,8]. The Bessel–Riesz operator $I_{\alpha,\gamma}$ is defined via the kernel $K_{\alpha,\gamma}(x) = |x|^{\alpha-n}(1+|x|)^{-\gamma}$. For $\gamma = 0$, it reduces to the Riesz potential; for $\alpha = n$, it becomes the Bessel potential (up to a normalization). The boundedness properties of $I_{\alpha,\gamma}$ on classical Lebesgue spaces follow from Young’s inequality and appropriate integrability conditions on the kernel. Specifically, when $K_{\alpha,\gamma} \in L^s(\mathbb{R}^d)$, the operator maps L^p to L^q with $1/q = 1/p + 1/s - 1$. These classical results, however, rely heavily on the translation invariance of the underlying spaces [9,10].

In recent decades, the theory of function spaces with variable exponents has emerged as a rich and rapidly developing area of analysis [11–13]. The variable Lebesgue spaces $L^{s(\cdot)}(\Pi)$, where the exponent $s(\cdot)$ is a measurable function, provide a natural framework for studying problems with non-standard growth conditions, such as those arising in



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electrorheological fluids, image processing, and nonlinear elasticity. Unlike their classical counterparts, these spaces lack translation invariance in a crucial sense, rendering classical tools like Young's inequality for convolution invalid (for details, please see [14]). Nevertheless, significant progress has been made in understanding the boundedness of classical operators in this setting. Diening's seminal work established the boundedness of the Hardy–Littlewood maximal operator on $L^{s(\cdot)}$ under the log-Hölder continuity condition [15–17]. Subsequent works extended these results to singular integrals, fractional integrals, and other fundamental operators [18–22]. Cruz-Uribe, Gogatishvili and Kopaliani studied an important connection between variable-exponent Lebesgue spaces and logarithmic function spaces in [23] and established sharp sufficient conditions for the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L(\log L)^\alpha(\Omega)$, $\alpha > 0$, together with applications to the integrability of the maximal function. The embedding result in [23] shows that the correct target space at the borderline is precisely a logarithmic Orlicz space $L(\log L)^\alpha$. In this article, we introduce a new class of logarithmic fractional integral operators $I_{\alpha,\gamma,\beta}^{\log}$ whose kernels $K_{\alpha,\gamma,\beta}^{\log}$ incorporate a logarithmic perturbation of the classical Bessel–Riesz singularity. The inclusion of logarithmic perturbations in the kernel introduces additional technical challenges. Logarithmic factors appear naturally in borderline cases of Sobolev embeddings, in the study of weighted inequalities, and in the analysis of certain PDEs with critical growth (Detailed descriptions can be found in [24–27]).

Understanding how such logarithmic singularities interact with variable exponents is therefore of considerable theoretical and practical interest. This article aims to fill this gap by conducting a systematic study of the logarithmic operator $I_{\alpha,\gamma,\beta}^{\log}$ where the logarithmic factor appears only locally near the origin, while the kernel retains the classical decay at infinity. This operator generalizes the classical Riesz operator ($\beta = 0, \gamma = 0$) and introduces a refined singularity structure that captures critical behaviors. We study the boundedness of $I_{\alpha,\gamma,\beta}^{\log}$ in variable Lebesgue spaces in diagonal and non-diagonal cases. In particular, we construct a suitable range space for a given domain space such that the operator is bounded. Conversely, for a prescribed range space, we determine a corresponding domain space ensuring the boundedness. Illustrative examples are provided to explicitly construct the respective spaces. Our approach relies on the boundedness of the maximal operator, where the main results hold if essential infimum $s_- > 1$. Therefore, in the limiting case when $s_- = 1$, we establish weak-type $(1, t(\cdot))$ boundedness.

The structure of this paper is as follows. Section 2 recalls necessary preliminaries on variable Lebesgue spaces and the Hardy–Littlewood maximal operator. Section 3 introduces the logarithmic operator, establishes a key pointwise estimate (Lemma 1) and contains the main boundedness theorems, including boundedness results in the diagonal case (Theorem 2), boundedness results in the non-diagonal case (Theorems 3–6) and weak-type estimates (Theorem 7). Finally, Section 4 offers concluding remarks.

Notations. Throughout the paper, $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}^d$ denotes d -dimensional Euclidean space. The symbol Π denotes a measurable subset of $\mathbb{R}^d$ and $\mathfrak{F}$ is a measurable function. All integrals are taken with respect to the Lebesgue measure. The letters $C, C_0, C_1, \dots$ denote positive constants whose values may change from line to line but are always independent of the functions involved unless otherwise stated.

2. Preliminaries

Let $\Pi \subseteq \mathbb{R}^d$. We define $\mathcal{S}(\Pi) = \{s(\cdot) \mid s : \Pi \rightarrow [1, \infty] \text{ is a measurable function}\}$. Suppose $s(\cdot) \in \mathcal{S}(\Pi)$ and $H \subseteq \Pi$; then, $s_-(H) = \operatorname{ess\,inf}_{\mu \in H} s(\mu)$ and $s_+(H) = \operatorname{ess\,sup}_{\mu \in H} s(\mu)$; further, $\frac{1}{s(\mu)} + \frac{1}{s'(\mu)} = 1$. We recall below several known results on Lebesgue spaces, which can be found in [14].

Definition 1. Suppose $s(\cdot) \in \mathcal{S}(\Pi)$ and $\mathfrak{F}$ is a measurable function; then, the modular of $\mathfrak{F}$ is defined as

$$\sigma_{s(\cdot)}(\mathfrak{F}) = \int_{\Pi \setminus \Pi_\infty} |\mathfrak{F}(\mu)|^{s(\mu)} d\mu + \|\mathfrak{F}\|_{\Pi_\infty}, \quad (1)$$

where $\Pi_\infty = \{\mu \in \Pi : s(\mu) = \infty \text{ a.e.}\}$. Note that $\sigma_{s(\cdot)}(\mathfrak{F}) > 0$ and $\sigma_{s(\cdot)}(\mathfrak{F}) = \sigma_{s(\cdot)}(|\mathfrak{F}|)$; further, $\sigma_{s(\cdot)}(\mathfrak{F}) = 0 \Leftrightarrow \mathfrak{F} = 0$ for a.e. $\mu \in \Pi$.

Definition 2. Suppose $s(\cdot) \in \mathcal{S}(\Pi)$; then, the Lebesgue space with variable exponent $L^{s(\cdot)}(\Pi)$ is defined as

$$L^{s(\cdot)}(\Pi) = \{\mathfrak{F} : \sigma_{s(\cdot)}(\mathfrak{F}/\Theta) < \infty, \text{ for some } \Theta > 0\}. \quad (2)$$

The norm on $L^{s(\cdot)}(\Pi)$ is given as

$$\|\mathfrak{F}\|_{L^{s(\cdot)}} = \inf\{\Theta > 0 : \sigma_{s(\cdot)}(\mathfrak{F}/\Theta) \leq 1\}, \quad (3)$$

this norm is called the Luxemburg norm.

Proposition 1. Let $\Pi \subseteq \mathbb{R}^d$, $s(\cdot) \in \mathcal{S}(\Pi)$ and $s_+ < \infty$. Then, for a measurable function $\mathfrak{F}$ and each $\eta \geq 1$, $\eta^{s_-} \sigma(\mathfrak{F}) \leq \sigma(\eta \mathfrak{F}) \leq \eta^{s_+} \sigma(\mathfrak{F})$. And for the case when $0 < \eta < 1$, we get reverse inequalities.

Proposition 2. Suppose $\Pi \subseteq \mathbb{R}^d$, $s(\cdot) \in \mathcal{S}(\Pi)$ and $\mathfrak{F} \in L^{s(\cdot)}(\Pi)$ with $\|\mathfrak{F}\|_{L^{s(\cdot)}} > 0$; then, $\sigma(\mathfrak{F}/\|\mathfrak{F}\|_{L^{s(\cdot)}}) \leq 1$. And for the case when $s_+ < \infty$, $\sigma(\mathfrak{F}/\|\mathfrak{F}\|_{L^{s(\cdot)}}) = 1$ for all non-trivial $\mathfrak{F} \in L^{s(\cdot)}(\Pi)$.

Definition 3. A measurable function $s(\cdot) : \Pi \rightarrow \mathbb{R}$ is locally log-Hölder continuous, denoted by $s(\cdot) \in LH_0(\Pi)$, if there is a constant C_0 in a way that $\forall \mu_1, \mu_2 \in \Pi$ with $|\mu_1 - \mu_2| < \frac{1}{2}$; we have

$$|s(\mu_1) - s(\mu_2)| \leq \frac{C_0}{-\log(|\mu_1 - \mu_2|)}.$$

A measurable function $s(\cdot)$ is log-Hölder continuous at infinity, written as $s(\cdot) \in LH_\infty(\Pi)$, if there are constants C_∞ and s_∞ , and $\forall \mu \in \Pi$,

$$|s(\mu) - s_\infty| \leq \frac{C_\infty}{\log(e + |\mu|)}.$$

In the case that $s(\cdot)$ is log-Hölder continuous locally and at infinity, then it is written as $s(\cdot) \in LH(\Pi)$.

Remark 1. Let $s_+ < \infty$, then $\frac{1}{s(\cdot)} \in LH(\Pi)$ is equivalent to $s(\cdot) \in LH(\Pi)$.

Definition 4. Let $\mathfrak{F} \in L^1_{loc}(\mathbb{R}^d)$, then the Hardy–Littlewood maximal operator $\mathcal{M}$ of $\mathfrak{F}$ is given as

$$\mathcal{M}\mathfrak{F}(\mu) = \sup_{\Delta \ni \mu} \frac{1}{|\Delta|} \int_{\Delta} |\mathfrak{F}(\nu)| d\nu, \quad \mu \in \mathbb{R}^d \quad (4)$$

where the supremum is taken over the cubes $\Delta \subset \mathbb{R}^d$ with sides parallel to the coordinate axes. When $d = 1$, Δ is an interval and “cube” is replaced by “interval” throughout.

Next, we state the result that establishes the boundedness of $\mathcal{M}$ in variable Lebesgue $L^{s(\cdot)}$ spaces.

Theorem 1 ([14]). Suppose $s(\cdot) \in \mathcal{S}(\mathbb{R}^d)$ and $1/s(\cdot) \in LH(\mathbb{R}^d)$. Then,

$$\|\beta \chi_{\{\mu: \mathcal{M}\mathfrak{F}(\mu) > \beta\}}\|_{L^{s(\cdot)}} \leq C \|\mathfrak{F}\|_{L^{s(\cdot)}}.$$

And for $s_- > 1$, we have

$$\|\mathcal{M}\mathfrak{F}\|_{L^{s(\cdot)}} \leq C \|\mathfrak{F}\|_{L^{s(\cdot)}}. \quad (5)$$

The constant C is dependent on the dimension d , the log-Hölder continuity of $1/s(\cdot)$, and the values of s_- and s_∞ .

Definition 5. Let $\gamma \in (0, \infty)$ and $\alpha \in (0, d)$. The Bessel–Riesz operator $I_{\alpha, \gamma}$ is convolution of a function $\mathfrak{F} \in L_{loc}^{s(\cdot)}(\mathbb{R}^d)$ with Bessel–Riesz kernel $K_{\alpha, \gamma} = \frac{|\mu|^{\alpha-d}}{(1+|\mu|)^\gamma}$, i.e.,

$$I_{\alpha, \gamma} * \mathfrak{F}(\mu) = \int_{\mathbb{R}^d} K_{\alpha, \gamma}(\mu - \nu) \mathfrak{F}(\nu) d\nu, \quad (6)$$

where $K_{\alpha, \gamma} = 0$ for $\mu = 0$.

Bessel–Riesz kernel $K_{\alpha, \gamma}$ is given by the product of the Bessel kernel $K_\gamma = \frac{1}{(1+|\mu|)^\gamma}$ and the Riesz kernel $K_\alpha = |\mu|^{\alpha-d}$. The corresponding operator $I_{\alpha, \gamma}$ has been the subject of intensive investigation, and its boundedness is now well understood under several distinct sets of assumptions [21,22,28,29].

3. Main Results

Let $0 < \alpha < 1$, $\gamma \geq 0$, and $\beta \in \mathbb{R}$. We define the logarithmic Bessel–Riesz kernel $K_{\alpha, \gamma, \beta}^{\log}$ on $\mathbb{R} \setminus \{0\}$ by

$$K_{\alpha, \gamma, \beta}^{\log}(\mu) = \begin{cases} \frac{|\mu|^{\alpha-1}}{(1+|\mu|)^\gamma} [\log(1/|\mu|)]^\beta, & 0 < |\mu| < 1, \\ |\mu|^{\alpha-1} (1+|\mu|)^{-\gamma}, & |\mu| \geq 1. \end{cases}$$

This defines a globally measurable kernel with a logarithmic perturbation at the origin and classical decay at infinity. The associated convolution operator is defined, for measurable $\mathfrak{F}$, by

$$I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu) = \int_{\mathbb{R}} K_{\alpha, \gamma, \beta}^{\log}(\mu - \nu) \mathfrak{F}(\nu) d\nu, \quad \mu \in \mathbb{R}.$$

Observe that, for $0 < |\mu| < 1$,

$$K_{\alpha, \gamma, \beta}^{\log}(\mu) = K_{\alpha, \gamma}(\mu) [\log(1/|\mu|)]^\beta.$$

Remark 2 (Qualitative behavior of the kernel). The parameter β controls the local singularity of $K_{\alpha, \gamma, \beta}^{\log}$ near the origin:

- $\beta > 0$; The factor $[\log(1/|\mu|)]^\beta \rightarrow +\infty$ as $|\mu| \rightarrow 0^+$, making the kernel more singular than $K_{\alpha, \gamma}$.
 - $\beta = 0$; $K_{\alpha, \gamma, 0}^{\log} \equiv K_{\alpha, \gamma}$.
 - $\beta < 0$; The factor $[\log(1/|\mu|)]^\beta \rightarrow 0$ as $|\mu| \rightarrow 0^+$, giving a strictly weaker singularity than $K_{\alpha, \gamma}$.
- For $|\mu| \geq 1$, the factor $[\log(1/|\mu|)]^\beta$ is absent and all three cases share the same decay $|\mu|^{\alpha-1} (1+|\mu|)^{-\gamma}$; thus, β has no effect on the global behavior of the operator. Table 1 illustrates these effects numerically for $\alpha = 1/3$, $\gamma = 1$, and $\beta \in \{-1, 0, 1\}$.

Table 1. Values of $K_{\alpha,\gamma,\beta}^{\log}(\mu)$ for $\alpha = 1/3$, $\gamma = 1$, and selected β . For $|\mu| \geq 1$, all values coincide (no β -dependence).

μ	$\beta = -1$	$\beta = 0$	$\beta = 1$
0.01	0.2133	0.9772	4.477
0.05	0.4097	1.2024	3.528
0.10	0.5623	1.2328	2.703
0.50	1.0000	1.1447	1.309
1.00		0.5000 (all β)	
2.00		0.2100 (all β)	
5.00		0.0487 (all β)	

Now, we present our main results concerning the boundedness of the logarithmic Bessel–Riesz operator $I_{\alpha,\gamma,\beta}^{\log}$ on variable Lebesgue $L^{p(\cdot)}(\mathbb{R}_+)$ spaces. Throughout the section, we assume that $1 < s_- \leq s_+ < \infty$ and $t_+ < \infty$. First, we establish a pointwise estimate that will play a central role in the subsequent analysis.

Lemma 1. Let $0 < \alpha < 1$, $\gamma \geq 0$, and $\beta \in \mathbb{R}$, and let $s(\cdot) \in \mathcal{S}(\mathbb{R}_+)$ with $s_+ < \infty$. Suppose $\mathfrak{F} \in L^{s(\cdot)}(\mathbb{R}_+)$. Then, for every $\mu \in \mathbb{R}_+$ and $\theta > 0$ there exists an integer M_θ such that $2^{M_\theta}\theta \geq 1$ and $2^{M_\theta-1}\theta < 1$, and

$$|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)| \leq C \left(\mathcal{M} \mathfrak{F}(\mu) \sum_{\kappa=-\infty}^{M_\theta-1} \frac{(2^\kappa \theta)^\alpha [\log(1/(2^\kappa \theta))]^{|\beta|}}{(1+2^\kappa \theta)^\gamma} + \|\mathfrak{F}\|_{L^{s(\cdot)}} \sum_{\kappa=M_\theta}^{\infty} \frac{(2^\kappa \theta)^{\alpha-\frac{1}{s_+}}}{(1+2^\kappa \theta)^\gamma} \right),$$

where $C > 0$ is independent of μ , $\mathfrak{F}$, and θ .

Proof. Fix $\mu \in \mathbb{R}_+$ and $\theta > 0$. Choose an integer M_θ such that $2^{M_\theta}\theta \geq 1$ and $2^{M_\theta-1}\theta < 1$. Using a dyadic decomposition of $\mathbb{R}_+$, we write

$$I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu) = \sum_{\kappa \in \mathbb{Z}} \int_{2^\kappa \theta \leq |\mu-v| < 2^{\kappa+1}\theta} K_{\alpha,\gamma,\beta}^{\log}(\mu-v) \mathfrak{F}(v) dv.$$

Now,

$$I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu) = I_1(\mu) + I_2(\mu),$$

where $I_1(\mu)$ corresponds to $\kappa \leq M_\theta - 1$ and $I_2(\mu)$ to $\kappa \geq M_\theta$.

For the near part $I_1(\mu)$, we use the definition of the kernel and the fact that for $0 < |\mu| < 1$,

$$K_{\alpha,\gamma,\beta}^{\log}(\mu) \leq C_\beta \frac{|\mu|^{\alpha-1}}{(1+|\mu|)^\gamma} [\log(1/|\mu|)]^{|\beta|},$$

where $C_\beta > 0$ depends only on β . For v in $\{v : 2^\kappa \theta \leq |\mu-v| < 2^{\kappa+1}\theta\}$, we have

$$|\mu-v|^{\alpha-1} \leq (2^\kappa \theta)^{\alpha-1}, \quad \log(1/|\mu-v|) \leq \log(1/(2^\kappa \theta)).$$

Therefore,

$$|I_1(\mu)| \leq C \sum_{\kappa=-\infty}^{M_\theta-1} \frac{(2^\kappa \theta)^{\alpha-1} [\log(1/(2^\kappa \theta))]^{|\beta|}}{(1+2^\kappa \theta)^\gamma} \int_{2^\kappa \theta \leq |\mu-v| < 2^{\kappa+1}\theta} |\mathfrak{F}(v)| dv,$$

where $C = C(\alpha, \gamma, \beta)$. Multiplying and dividing by $2^{\kappa}\theta$ yields

$$\begin{aligned} |I_1(\mu)| &\leq C \sum_{\kappa=-\infty}^{M_\theta-1} \frac{(2^{\kappa}\theta)^{\alpha} [\log(1/(2^{\kappa}\theta))]^{|\beta|}}{(1+2^{\kappa}\theta)^{\gamma}} \frac{1}{2^{\kappa}\theta} \int_{2^{\kappa}\theta \leq |\mu-v| < 2^{\kappa+1}\theta} |\mathfrak{F}(v)| dv \\ &\leq C \mathcal{M}\mathfrak{F}(\mu) \sum_{\kappa=-\infty}^{M_\theta-1} \frac{(2^{\kappa}\theta)^{\alpha} [\log(1/(2^{\kappa}\theta))]^{|\beta|}}{(1+2^{\kappa}\theta)^{\gamma}}. \end{aligned}$$

For the far part $I_2(\mu)$, we have $|\mu - v| \geq 1$, so the logarithmic factor is absent and

$$K_{\alpha, \gamma, \beta}^{\log}(\mu - v) = |\mu - v|^{\alpha-1} (1 + |\mu - v|)^{-\gamma}.$$

Therefore,

$$|I_2(\mu)| \leq \sum_{\kappa=M_\theta}^{\infty} \frac{(2^{\kappa}\theta)^{\alpha-1}}{(1+2^{\kappa}\theta)^{\gamma}} \int_{2^{\kappa}\theta \leq |\mu-v| < 2^{\kappa+1}\theta} |\mathfrak{F}(v)| dv.$$

Using Hölder's inequality, we obtain

$$\int_{2^{\kappa}\theta \leq |\mu-v| < 2^{\kappa+1}\theta} |\mathfrak{F}(v)| dv \leq C \|\mathfrak{F}\|_{L^{s(\cdot)}} (2^{\kappa}\theta)^{1-\frac{1}{s_+}}.$$

Hence,

$$|I_2(\mu)| \leq C \|\mathfrak{F}\|_{L^{s(\cdot)}} \sum_{\kappa=M_\theta}^{\infty} \frac{(2^{\kappa}\theta)^{\alpha-\frac{1}{s_+}}}{(1+2^{\kappa}\theta)^{\gamma}}.$$

Combining the estimates for $I_1(\mu)$ and $I_2(\mu)$ yields the desired inequality. $\square$

The next theorem addresses the boundedness in the diagonal case.

Theorem 2. Let $0 < \alpha < 1$, $\gamma > 0$ with $\alpha < \gamma$, and $s(\cdot) \in LH(\mathbb{R}_+)$ with $s_+ < \infty$. Then, the logarithmic operator $I_{\alpha, \gamma, \beta}^{\log}$ is bounded on $L^{s(\cdot)}(\mathbb{R}_+)$; that is, there exists a constant $C > 0$ such that

$$\|I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)} \leq C \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)},$$

for all $\mathfrak{F} \in L^{s(\cdot)}(\mathbb{R}_+)$.

Proof. Since $s(\cdot) \in LH(\mathbb{R}_+)$ and $s_+ < \infty$, the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{s(\cdot)}(\mathbb{R}_+)$. Thus, there exists a constant $C_0 > 0$ such that

$$\|\mathcal{M}\mathfrak{F}\|_{L^{s(\cdot)}} \leq C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}}.$$

Let $\mathfrak{F} \in L^{s(\cdot)}(\mathbb{R}_+)$ and $\mu \in \mathbb{R}_+$. By definition,

$$I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu) = \int_{\mathbb{R}_+} K_{\alpha, \gamma, \beta}^{\log}(\mu - v) \mathfrak{F}(v) dv.$$

We split the integral into local and global parts:

$$I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu) = \int_{|\mu-v| < 1} K_{\alpha, \gamma, \beta}^{\log}(\mu - v) \mathfrak{F}(v) dv + \int_{|\mu-v| \geq 1} K_{\alpha, \gamma, \beta}^{\log}(\mu - v) \mathfrak{F}(v) dv = I_1(\mu) + I_2(\mu).$$

Now,

$$|I_1(\mu)| \leq \sum_{\kappa=-\infty}^{-1} \int_{2^{\kappa} \leq |\mu-v| < 2^{\kappa+1}} \frac{|\mu-v|^{\alpha-1}}{(1+|\mu-v|)^{\gamma}} [\log(1/|\mu-v|)]^{|\beta|} |\mathfrak{F}(v)| dv.$$

Because,

$$|\mu - \nu|^{\alpha-1} \leq (2^\kappa)^{\alpha-1}, \quad \log(1/|\mu - \nu|) \leq \log(1/2^\kappa) \lesssim |\kappa|.$$

Therefore,

$$\begin{aligned} |I_1(\mu)| &\leq C \sum_{\kappa=-\infty}^{-1} (2^\kappa)^{\alpha-1} (1 + |\kappa|)^{|\beta|} \int_{2^\kappa \leq |\mu-\nu| < 2^{\kappa+1}} |\mathfrak{F}(\nu)| d\nu \\ &\leq C \sum_{\kappa=-\infty}^{-1} (2^\kappa)^\alpha (1 + |\kappa|)^{|\beta|} \frac{1}{2^\kappa} \int_{2^\kappa \leq |\mu-\nu| < 2^{\kappa+1}} |\mathfrak{F}(\nu)| d\nu \\ &\leq C \mathcal{M}\mathfrak{F}(\mu) \sum_{\kappa=-\infty}^{-1} 2^{\kappa\alpha} (1 + |\kappa|)^{|\beta|}. \end{aligned}$$

Because $\alpha > 0$, the series converges, and therefore

$$|I_1(\mu)| \leq C_1 \mathcal{M}\mathfrak{F}(\mu).$$

Similarly,

$$\begin{aligned} |I_2(\mu)| &\leq \sum_{\kappa=0}^{\infty} \int_{2^\kappa \leq |\mu-\nu| < 2^{\kappa+1}} \frac{|\mu - \nu|^{\alpha-1}}{(1 + |\mu - \nu|)^\gamma} |\mathfrak{F}(\nu)| d\nu \leq \sum_{\kappa=0}^{\infty} (2^\kappa)^{\alpha-1-\gamma} \int_{2^\kappa \leq |\mu-\nu| < 2^{\kappa+1}} |\mathfrak{F}(\nu)| d\nu \\ &\leq \mathcal{M}\mathfrak{F}(\mu) \sum_{\kappa=0}^{\infty} 2^{\kappa(\alpha-\gamma)} \leq C_2 \mathcal{M}\mathfrak{F}(\mu). \end{aligned}$$

Combining the two estimates, we obtain

$$|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)| \leq C \mathcal{M}\mathfrak{F}(\mu).$$

Let $\Theta > 0$ be such that $\sigma_{s(\cdot)}(\mathcal{M}\mathfrak{F}/\Theta) \leq 1$. Then, $\left| \frac{I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)}{C\Theta} \right| \leq \frac{\mathcal{M}\mathfrak{F}(\mu)}{\Theta}$. By the monotonicity of the modular,

$$\sigma_{s(\cdot)}\left(\frac{I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}}{C\Theta}\right) \leq \sigma_{s(\cdot)}\left(\frac{\mathcal{M}\mathfrak{F}}{\Theta}\right) \leq 1.$$

Taking infimum over such Θ and using the boundedness of the maximal operator, we obtain

$$\|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}\|_{L^{s(\cdot)}} \leq C \|\mathcal{M}\mathfrak{F}\|_{L^{s(\cdot)}} \leq C \|\mathfrak{F}\|_{L^{s(\cdot)}}.$$

which completes the proof.

The following theorem constructs the target range space explicitly from a given domain space. $\square$

Theorem 3. Let $s(\cdot) \in \mathcal{S}(\mathbb{R}_+)$, $1 < s_- \leq s_+ < \infty$, and $s(\cdot) \in LH(\mathbb{R}_+)$. Let $0 < \alpha < 1$, $\gamma \geq 0$, and $\beta \in \mathbb{R}$, and assume $\alpha - \frac{1}{s_+} < \gamma$. Define $t(\cdot) \in \mathcal{S}(\mathbb{R}_+)$ by

$$\frac{1}{t(\cdot)} = \frac{1}{1 + \gamma s_+} \left(\frac{1}{s(\cdot)} - (\alpha - \gamma) \frac{s_+}{s(\cdot)} \right).$$

Then, the logarithmic operator $I_{\alpha,\gamma,\beta}^{\log}$ is bounded from $L^{s(\cdot)}(\mathbb{R}_+)$ to $L^{t(\cdot)}(\mathbb{R}_+)$.

Proof. Since $1 < s_- \leq s_+ < \infty$ and $s(\cdot) \in LH(\mathbb{R}_+)$, the maximal operator is bounded:

$$\|\mathcal{M}\mathfrak{F}\|_{L^{s(\cdot)}} \leq C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}}.$$

By Lemma 1, we have

$$\begin{aligned} |I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)| &\leq C^* \left(\mathcal{M}\mathfrak{F}(\mu) \sum_{\kappa=-\infty}^{M_\theta-1} \frac{(2^\kappa \theta)^\alpha [\log(1/(2^\kappa \theta))]^{|\beta|}}{(1+2^\kappa \theta)^\gamma} + \|\mathfrak{F}\|_{L^{s(\cdot)}} \sum_{\kappa=M_\theta}^{\infty} \frac{(2^\kappa \theta)^{\alpha-\frac{1}{s_+}}}{(1+2^\kappa \theta)^\gamma} \right) \\ &\leq C^* \left(\mathcal{M}\mathfrak{F}(\mu) \sum_{\kappa=-\infty}^{M_\theta-1} (2^\kappa \theta)^\alpha [\log(1/(2^\kappa \theta))]^{|\beta|} + \|\mathfrak{F}\|_{L^{s(\cdot)}} \sum_{\kappa=M_\theta}^{\infty} \frac{(2^\kappa \theta)^{\alpha-\frac{1}{s_+}}}{(2^\kappa \theta)^\gamma} \right). \quad (7) \end{aligned}$$

Next, we use the standard estimate

$$[\log(1/(2^\kappa \theta))]^{|\beta|} \leq C(1+|\kappa|)^{|\beta|},$$

which yields

$$\sum_{\kappa=-\infty}^{M_\theta-1} (2^\kappa \theta)^\alpha [\log(1/(2^\kappa \theta))]^{|\beta|} \leq C \sum_{\kappa=-\infty}^{M_\theta-1} (2^\kappa \theta)^\alpha (1+|\kappa|)^{|\beta|}.$$

Since the polynomial factor $(1+|\kappa|)^{|\beta|}$ is dominated by the exponential decay of $2^{\kappa\alpha}$ as $\kappa \rightarrow -\infty$, we obtain

$$\sum_{\kappa=-\infty}^{M_\theta-1} (2^\kappa \theta)^\alpha [\log(1/(2^\kappa \theta))]^{|\beta|} \leq \theta^\alpha \sum_{\kappa=-\infty}^{M_\theta-1} (2^\kappa)^\alpha.$$

Finally, since $\alpha > 0$,

$$\sum_{\kappa=-\infty}^{M_\theta-1} (2^\kappa)^\alpha$$

is a convergent geometric series. Hence,

$$\sum_{\kappa=-\infty}^{M_\theta-1} (2^\kappa)^\alpha \leq C,$$

and therefore

$$\theta^\alpha \sum_{\kappa=-\infty}^{M_\theta-1} (2^\kappa)^\alpha \leq C\theta^\alpha.$$

Therefore, (7) implies

$$|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)| \leq C^{**} \left(\theta^\alpha \mathcal{M}\mathfrak{F}(\mu) + \theta^{\alpha-\frac{1}{s_+}-\gamma} \|\mathfrak{F}\|_{L^{s(\cdot)}} \right).$$

Choosing

$$\theta = \left(\frac{\mathcal{M}\mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}}} \right)^{\frac{s_+ s(\mu)}{((\alpha-\gamma)s_+-1)t(\mu)}},$$

we obtain

$$|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)| \leq C_1 \left(\frac{\mathcal{M}\mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}}} \right)^{\frac{s(\mu)}{t(\mu)}} \|\mathfrak{F}\|_{L^{s(\cdot)}}.$$

Thus,

$$\left(\frac{|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)|}{C_1 \|\mathfrak{F}\|_{L^{s(\cdot)}}} \right)^{t(\mu)} \leq \left(\frac{\mathcal{M}\mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}}} \right)^{s(\mu)}.$$

Integrating,

$$\sigma_{t(\cdot)} \left(\frac{I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}}{C_1 \|\mathfrak{F}\|_{L^{s(\cdot)}}} \right) \leq \sigma_{s(\cdot)} \left(\frac{\mathcal{M}\mathfrak{F}}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}}} \right) \leq 1.$$

Hence,

$$\|I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}\|_{L^{t(\cdot)}} \leq C \|\mathfrak{F}\|_{L^{s(\cdot)}}.$$

□

Corollary 1. Let $1 < s < \infty$ and $0 < \alpha < 1$, $\gamma \geq 0$, and $\beta \in \mathbb{R}$ satisfy $\alpha - \frac{1}{s} < \gamma$. Then, there exists $1 < t < \infty$ satisfying $\frac{1}{t} = \frac{1}{1+\gamma s} \left(\frac{1}{s} - (\alpha - \gamma) \right)$, such that the logarithmic operator is bounded. That is, there exists a constant $C > 0$ such that

$$\|I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}\|_{L^t(\mathbb{R}_+)} \leq C \|\mathfrak{F}\|_{L^s(\mathbb{R}_+)}.$$

Proof. Since s is constant, we have $s_- = s_+ = s$, and the condition $\alpha - \frac{1}{s} < \gamma$ ensures that all assumptions of Theorem 3 are satisfied. Hence, the result follows by taking $s(\cdot) = s$ in Theorem 3. □

Now, we present an example to demonstrate that all hypotheses of Theorem 3 can be verified explicitly for a concrete, non-trivial variable exponent function $s(\mu)$ that is neither constant nor piecewise constant. The resulting $t(\cdot)$ is a genuine variable exponent satisfying $1 < t_- \leq t_+ < \infty$. The constant-exponent subcase, treated via Corollary 1, provides a further numerical check.

Example 1. Let

$$s(\mu) = 2 + \frac{1}{2 \log(e + \mu)}, \quad \mu \in \mathbb{R}_+.$$

Because $\log(e + \mu) \geq 1$ for all $\mu \geq 0$, we have

$$2 < s(\mu) \leq \frac{5}{2} \quad \text{for all } \mu \in \mathbb{R}_+,$$

with $s_- = 2$ (attained in the limit $\mu \rightarrow \infty$) and $s_+ = \frac{5}{2}$ (attained at $\mu = 0$), so $1 < s_- \leq s_+ < \infty$. The local log-Hölder condition (LH_0): The function s is differentiable on $\mathbb{R}_+$ with

$$s'(\mu) = -\frac{1}{2(e + \mu)[\log(e + \mu)]^2},$$

so $|s'(\mu)| \leq \frac{1}{2e}$ for all $\mu \geq 0$, i.e., s is Lipschitz with constant $L = \frac{1}{2e}$. Using the elementary inequality $h \leq \frac{1/e}{-\log h}$ for $h \in (0, \frac{1}{2})$, we obtain, for all $\mu_1, \mu_2 \in \mathbb{R}_+$ with $|\mu_1 - \mu_2| < \frac{1}{2}$,

$$|s(\mu_1) - s(\mu_2)| \leq L |\mu_1 - \mu_2| \leq \frac{L/e}{-\log |\mu_1 - \mu_2|} = \frac{C_0}{-\log |\mu_1 - \mu_2|}, \quad C_0 = \frac{1}{2e^2}.$$

Hence, $s(\cdot) \in LH_0(\mathbb{R}_+)$. Log-Hölder condition at infinity (LH_∞): Set $s_\infty = 2$. Then, for all $\mu \in \mathbb{R}_+$,

$$|s(\mu) - s_\infty| = \frac{1}{2 \log(e + \mu)} \leq \frac{C_\infty}{\log(e + |\mu|)}, \quad C_\infty = \frac{1}{2}.$$

Hence, $s(\cdot) \in LH_\infty(\mathbb{R}_+)$, and therefore $s(\cdot) \in LH(\mathbb{R}_+)$. Let $\alpha = \frac{1}{3}$, $\gamma = 1$, and $\beta \in \mathbb{R}$. Then,

$$\alpha - \frac{1}{s_+} = \frac{1}{3} - \frac{2}{5} = -\frac{1}{15} < 1 = \gamma,$$

so all assumptions of Theorem 3 are satisfied. The exponent $t(\cdot)$ is given by

$$\frac{1}{t(\mu)} = \frac{1}{1 + \gamma s_+} \left(\frac{1}{s(\mu)} - (\alpha - \gamma) \frac{s_+}{s(\mu)} \right) = \frac{2}{7} \left(\frac{1}{s(\mu)} + \frac{2}{3} \cdot \frac{5/2}{s(\mu)} \right) = \frac{2}{7} \cdot \frac{8/3}{s(\mu)} = \frac{16}{21 s(\mu)},$$

i.e.,

$$t(\mu) = \frac{21}{16} s(\mu) = \frac{21}{16} \left(2 + \frac{1}{2 \log(e + \mu)} \right), \quad \mu \in \mathbb{R}_+.$$

In particular, $t_- = \frac{21}{8}$ and $t_+ = \frac{105}{32}$, so $1 < t_- \leq t_+ < \infty$. Consequently,

$$I_{\alpha, \gamma, \beta}^{\log} : L^{s(\cdot)}(\mathbb{R}_+) \longrightarrow L^{t(\cdot)}(\mathbb{R}_+)$$

is bounded. In the constant-exponent case, take $s = 2$, $\alpha = \frac{1}{3}$, $\gamma = 1$, and any $\beta \in \mathbb{R}$. Then,

$$\alpha - \frac{1}{s} = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6} < 1 = \gamma,$$

and by Corollary 1,

$$\frac{1}{t} = \frac{1}{1 + \gamma s} \left(\frac{1}{s} - (\alpha - \gamma) \right) = \frac{1}{3} \left(\frac{1}{2} + \frac{2}{3} \right) = \frac{1}{3} \cdot \frac{7}{6} = \frac{7}{18},$$

so $t = \frac{18}{7}$ and

$$I_{\alpha, \gamma, \beta}^{\log} : L^2(\mathbb{R}_+) \longrightarrow L^{\frac{18}{7}}(\mathbb{R}_+)$$

is bounded.

Remark 3. Theorem 3 holds only when $\frac{1}{t(\cdot)} = \frac{1}{1 + \gamma s_+} \left(\frac{1}{s(\cdot)} - (\alpha - \gamma) \frac{s_+}{s(\cdot)} \right)$ a.e. and the result does not hold if this relation fails for $\mu \in \mathbb{R}_+$. Therefore, in Theorem 4, we discuss the cases where the boundedness holds for all possible scenarios of the exponent functions. However, in this case, additional conditions on the maximal function in terms of the norm $\|\mathfrak{F}\|_{L^{s(\cdot)}}$ are required. Furthermore, Theorem 4 does not cover Theorem 3, because we impose conditions on the maximal function in Theorem 4, while Theorem 3 does not depend on such conditions. In the case of equality, Theorem 3 becomes a special case of Theorem 4.

Definition 6. Let $s(\cdot), t(\cdot) \in \mathcal{S}(\mathbb{R}_+)$ with $1 < s_- \leq s_+ < \infty$ and $t_+ < \infty$. Assume that $\alpha - \frac{1}{s_+} < \gamma$. We say that the pair $(s(\cdot), t(\cdot))$ satisfies the $(\alpha - \frac{1}{s_+} < \gamma)$ -logarithmic boundedness property if for every $\mathfrak{F} \in L^{s(\cdot)}(\mathbb{R}_+)$ and every $\mu \in \mathbb{R}_+$ the following holds: Define the sets

$$A_1 = \left\{ \mu \in \mathbb{R}_+ : \mathcal{M}\mathfrak{F}(\mu) \leq C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)} \right\},$$

$$A_2 = \left\{ \mu \in \mathbb{R}_+ : \mathcal{M}\mathfrak{F}(\mu) > C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)} \right\}.$$

Then, $(s(\cdot), t(\cdot))$ satisfies the logarithmic boundedness property if

$$\frac{1}{t(\mu)} \geq \frac{1}{1 + \gamma s_+} \left(\frac{1}{s(\mu)} - (\alpha - \gamma) \frac{s_+}{s(\mu)} \right) \quad \text{for a.e. } \mu \in A_1,$$

and

$$\frac{1}{t(\mu)} < \frac{1}{1 + \gamma s_+} \left(\frac{1}{s(\mu)} - (\alpha - \gamma) \frac{s_+}{s(\mu)} \right) \quad \text{for a.e. } \mu \in A_2.$$

Theorem 4. Let $s(\cdot), t(\cdot) \in \mathcal{S}(\mathbb{R}_+)$ with $\alpha - \frac{1}{s_+} < \gamma$ and suppose the pair $(s(\cdot), t(\cdot))$ satisfies the logarithmic $(\alpha - \frac{1}{s_+} < \gamma)$ -boundedness property. Assume $s(\cdot) \in LH(\mathcal{R}^+)$ with $1 < s_- \leq s_+ < \infty$ and $t_+ < \infty$. Then, there exists a constant $C > 0$ such that

$$\|I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}\|_{L^{t(\cdot)}(\mathcal{R}^+)} \leq C \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathcal{R}^+)}.$$

Proof. Since $1 < s_- \leq s_+ < \infty$ and $s(\cdot) \in LH(\mathbb{R}_+)$, by the boundedness of the maximal operator there exists $C_0 > 0$ such that

$$\|\mathcal{M}\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)} \leq C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}.$$

Now, for any $\mu \in \mathcal{R}^+$ and $\mathfrak{F} \in L^{s(\cdot)}(\mathbb{R}_+)$ with $\mathfrak{F} \neq 0$, by using Lemma 1 together with the condition $\alpha - \frac{1}{s_+} < \gamma$, we obtain

$$|I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu)| \leq C_2 \left(\theta^\alpha \mathcal{M}\mathfrak{F}(\mu) + \theta^{\alpha - \frac{1}{s_+} - \gamma} \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)} \right).$$

Further, by choosing

$$\theta = \left(\frac{\mathcal{M}\mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}} \right)^{\frac{s_+ s(\mu)}{((\alpha - \gamma)s_+ - 1)t(\mu)}},$$

and simplifying as in the classical case, we get

$$|I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu)| \leq C_2 \frac{C_0 \left(C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathcal{R}^+)} \right)^{\frac{\alpha s_+ s(\mu)}{(1 - (\alpha - \gamma)s_+)t(\mu)} + \frac{s(\mu)}{t(\mu)} - 1} + (\mathcal{M}\mathfrak{F}(\mu))^{\frac{\alpha s_+ s(\mu)}{(1 - (\alpha - \gamma)s_+)t(\mu)} + \frac{s(\mu)}{t(\mu)} - 1}}{C_0 (\mathcal{M}\mathfrak{F}(\mu))^{\frac{\alpha s_+ s(\mu)}{(1 - (\alpha - \gamma)s_+)t(\mu)} - 1} \left(C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathcal{R}^+)} \right)^{\frac{s(\mu)}{t(\mu)} - 1}}.$$

Now, we consider two cases depending on $\mu \in \mathcal{R}^+$.

If $\mu \in A_1$, then the boundedness property implies $\frac{\alpha s_+ s(\mu)}{(1 - (\alpha - \gamma)s_+)t(\mu)} + \frac{s(\mu)}{t(\mu)} - 1 \leq 0$. Thus,

$$|I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu)| \leq C_2 (C_0 + 1) \left(\frac{\mathcal{M}\mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathcal{R}^+)}} \right)^{\frac{s(\mu)}{t(\mu)}} \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathcal{R}^+)}.$$

If $\mu \in A_2$, then the boundedness property implies $\frac{\alpha s_+ s(\mu)}{(1 - (\alpha - \gamma)s_+)t(\mu)} + \frac{s(\mu)}{t(\mu)} - 1 > 0$, and we obtain

$$|I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu)| \leq C_2 (C_0 + 1) \left(\frac{\mathcal{M}\mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathcal{R}^+)}} \right)^{\frac{s(\mu)}{t(\mu)}} \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathcal{R}^+)}.$$

Thus, for all $\mu \in \mathcal{R}^+$,

$$\left(\frac{|I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu)|}{C_3 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathcal{R}^+)}} \right)^{t(\mu)} \leq \left(\frac{\mathcal{M}\mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathcal{R}^+)}} \right)^{s(\mu)}.$$

By the definition of the modular, and using the boundedness of the maximal operator, we obtain

$$\sigma_{t(\cdot)} \left(\frac{I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}}{C_3 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathcal{R}^+)}} \right) \leq \sigma_{s(\cdot)} \left(\frac{\mathcal{M}\mathfrak{F}}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathcal{R}^+)}} \right) \leq 1.$$

Hence, $\sigma_{t(\cdot)} \left(\frac{I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}}{C_3 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}} \right) \leq 1$. Therefore,

$$\|I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}\|_{L^{t(\cdot)}(\mathbb{R}_+)} \leq C_3 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}.$$

□

The following theorem is the converse construction to Theorem 3: here, the target range space is given and the domain space is constructed; at $\beta = 0$ it reduces to Theorem 6 of [22].

Theorem 5. Let $t(\cdot) \in \mathcal{S}(\mathbb{R}_+)$ with $t_+ < \infty$ and $\frac{1}{t_-} + \frac{\alpha t_+}{t_-} < 1$, and suppose $t(\cdot) \in LH(\mathbb{R}_+)$. Define $s(\cdot) \in \mathcal{S}(\mathbb{R}_+)$ by $\frac{1}{s(\mu)} = \frac{1}{t(\mu)} + \frac{\alpha t_+}{t(\mu)}$ a.e. on $\mathbb{R}_+$. Then, the logarithmic operator is bounded from $L^{s(\cdot)}(\mathbb{R}_+)$ into $L^{t(\cdot)}(\mathbb{R}_+)$.

Proof. Since $t_+ < \infty$ and

$$\frac{1}{t_-} + \frac{\alpha t_+}{t_-} < 1,$$

together with the relation

$$\frac{1}{s(\mu)} = \frac{1}{t(\mu)} + \frac{\alpha t_+}{t(\mu)},$$

it follows that $1 < s_- \leq s_+ < \infty$. Moreover, from $t(\cdot) \in LH(\mathbb{R}_+)$ and the relation between $s(\cdot)$ and $t(\cdot)$, we deduce that $s(\cdot) \in LH(\mathbb{R}_+)$.

Hence, by the boundedness of the Hardy–Littlewood maximal operator, there exists $C_0 > 0$ such that for all $\mathfrak{F} \in L^{s(\cdot)}(\mathbb{R}_+)$,

$$\|\mathcal{M}\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)} \leq C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}.$$

Let $\mu \in \mathbb{R}_+$ and $\mathfrak{F} \in L^{s(\cdot)}(\mathbb{R}_+)$, $\mathfrak{F} \neq 0$. By Lemma 1, for any $\theta > 0$,

$$|I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu)| \leq C^* \left(\mathcal{M}\mathfrak{F}(\mu) \sum_{\kappa=-\infty}^{M_\theta-1} \frac{(2^\kappa \theta)^\alpha [\log(1/(2^\kappa \theta))]^{|\beta|}}{(1 + 2^\kappa \theta)^\gamma} + \|\mathfrak{F}\|_{L^{s(\cdot)}} \sum_{\kappa=M_\theta}^{\infty} \frac{(2^\kappa \theta)^{\alpha - \frac{1}{s_+}}}{(1 + 2^\kappa \theta)^\gamma} \right).$$

As in the classical case, the logarithmic factor affects only the near-field sum. Using the standard estimate

$$[\log(1/(2^\kappa \theta))]^{|\beta|} \leq C(1 + |\kappa|)^{|\beta|},$$

we obtain convergence of the first series and hence

$$\sum_{\kappa=-\infty}^{M_\theta-1} \frac{(2^\kappa \theta)^\alpha [\log(1/(2^\kappa \theta))]^{|\beta|}}{(1 + 2^\kappa \theta)^\gamma} \leq C\theta^\alpha.$$

For the second sum, using the identity

$$\alpha - \frac{1}{s_+} = -\frac{1}{t_+},$$

we obtain

$$\sum_{\kappa=M_\theta}^{\infty} \frac{(2^\kappa \theta)^{\alpha - \frac{1}{s_+}}}{(1 + 2^\kappa \theta)^\gamma} \leq C\theta^{-\frac{1}{t_+}}.$$

Thus,

$$|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)| \leq C_1 \left(\theta^\alpha \mathcal{M} \mathfrak{F}(\mu) + \theta^{-\frac{1}{t_+}} \|\mathfrak{F}\|_{L^{s(\cdot)}} \right).$$

Choose $\theta = \left(\frac{\mathcal{M} \mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}} \right)^{-\frac{t_+ s(\mu)}{t(\mu)}}$. Substituting into the above inequality and simplifying using $\frac{1}{s(\mu)} = \frac{1}{t(\mu)} + \frac{\alpha t_+}{t(\mu)}$, we obtain

$$|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)| \leq C_2 \left(\frac{\mathcal{M} \mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}} \right)^{\frac{s(\mu)}{t(\mu)}} \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}.$$

Or

$$\left(\frac{|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)|}{C_2 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}} \right)^{t(\mu)} \leq \left(\frac{\mathcal{M} \mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}} \right)^{s(\mu)}.$$

Hence,

$$\sigma_{t(\cdot)} \left(\frac{I_{\alpha,\gamma,\beta}^{\log} f}{C_2 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}} \right) \leq \sigma_{s(\cdot)} \left(\frac{\mathcal{M} \mathfrak{F}}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}} \right).$$

By the boundedness of $\mathcal{M}$, the right-hand side is bounded by 1; hence,

$$\|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}\|_{L^{t(\cdot)}(\mathbb{R}_+)} \leq C \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}.$$

□

Corollary 2. If $1 < t < \infty$ and $\frac{1}{t} + \alpha < 1$, then there exists $1 < s < \infty$ satisfying $\frac{1}{s} = \frac{1}{t} + \alpha$ a.e., such that the logarithmic operator $I_{\alpha,\gamma,\beta}^{\log} : L^s(\mathbb{R}_+) \rightarrow L^t(\mathbb{R}_+)$ is bounded. In particular, there exists a constant $C > 0$ such that

$$\|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}\|_{L^t(\mathbb{R}_+)} \leq C \|\mathfrak{F}\|_{L^s(\mathbb{R}_+)}.$$

Proof. Since t is constant, we have $t_- = t_+ = t$, and the condition $\frac{1}{t} + \alpha < 1$ holds. Therefore, all assumptions of Theorem 5 are satisfied. The conclusion follows by substituting $t(\cdot) = t$ into Theorem 5. □

The following example demonstrates that the hypotheses of Theorem 5 are satisfiable by a concrete variable exponent $t(\mu)$ that is piecewise-defined. Moreover, it makes the formula $s(\mu) = t(\mu)/(1 + \alpha t_+)$ fully explicit and verifies the log-Hölder condition for the resulting $s(\cdot)$. The constant-exponent subcase provides a further clean numerical illustration via Corollary 2.

Example 2. Let

$$t(\mu) = \begin{cases} \mu + 4, & \mu \in [0, 1), \\ \frac{1}{\mu} + 4, & \mu \in [1, \infty), \end{cases}$$

then $t_- = 4$, $t_+ = 5$, and hence $1 < t_- \leq t_+ < \infty$. Let $\alpha = \frac{1}{3}$ and $\gamma > 0$, $\beta \in \mathbb{R}$. Then,

$$\frac{1}{t_-} + \frac{\alpha t_+}{t_-} = \frac{1}{4} + \frac{1}{3} \cdot \frac{5}{4} = \frac{1}{4} + \frac{5}{12} = \frac{8}{12} = \frac{2}{3} < 1.$$

Moreover, $t(\cdot) \in LH(\mathbb{R}_+)$. Therefore, by Theorem 5, there exists $s(\cdot)$ defined by

$$s(\mu) = \begin{cases} \frac{\mu + 4}{1 + \alpha t_+} = \frac{\mu + 4}{1 + \frac{5}{3}} = \frac{3(\mu + 4)}{8}, & \mu \in [0, 1), \\ \frac{\frac{1}{\mu} + 4}{1 + \frac{5}{3}} = \frac{3\left(\frac{1}{\mu} + 4\right)}{8}, & \mu \in [1, \infty), \end{cases}$$

satisfying

$$\frac{1}{s(\cdot)} = \frac{1}{t(\cdot)} + \frac{\alpha t_+}{t(\cdot)} \quad a.e.,$$

such that

$$I_{\alpha, \gamma, \beta}^{\log} : L^{s(\cdot)}(\mathbb{R}_+) \rightarrow L^{t(\cdot)}(\mathbb{R}_+)$$

is bounded.

In the constant exponent case, if we take $t = 5$ and $\alpha = \frac{1}{3}$, then

$$\frac{1}{t} + \alpha = \frac{1}{5} + \frac{1}{3} = \frac{8}{15} < 1.$$

Therefore, by Corollary 2, there exists $s = \frac{15}{8}$ such that

$$I_{\alpha, \gamma, \beta}^{\log} : L^s(\mathbb{R}_+) \rightarrow L^t(\mathbb{R}_+)$$

is bounded.

Remark 4. Theorem 5 holds only when $\frac{1}{s(\mu)} = \frac{1}{t(\mu)} + \frac{\alpha t_+}{t(\mu)}$, a.e. and the result does not hold if this relation fails for $\mu \in \mathbb{R}_+$. Therefore, in Theorem 6, we discuss the cases where the boundedness holds for all possible scenarios of the exponent functions. However, in this case, additional conditions on the maximal function in terms of the norm $\|\mathfrak{F}\|_{L^{s(\cdot)}}$ are required.

Definition 7. Let $s(\cdot), t(\cdot) \in \mathcal{S}(\mathbb{R}_+)$ with $1 < s_- \leq s_+ < \infty$ and $t_+ < \infty$, and assume that $\frac{1}{s_+} - \frac{1}{t_+} = \alpha$. The pair $(s(\cdot), t(\cdot))$ is said to satisfy the $\left(\frac{1}{s_+} - \frac{1}{t_+} = \alpha\right)$ -boundedness property if for every $\mathfrak{F} \in L^{s(\cdot)}(\mathbb{R}_+)$, defining

$$A_1 = \{\mu \in \mathbb{R}_+ : \mathcal{M}\mathfrak{F}(\mu) \leq C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}\},$$

$$A_2 = \{\mu \in \mathbb{R}_+ : \mathcal{M}\mathfrak{F}(\mu) > C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}\},$$

we have

$$\frac{1}{s(\mu)} \geq \frac{1}{t(\mu)} + \frac{\alpha t_+}{t(\mu)} \quad \text{for a.e. } \mu \in A_1,$$

and

$$\frac{1}{s(\mu)} < \frac{1}{t(\mu)} + \frac{\alpha t_+}{t(\mu)} \quad \text{for a.e. } \mu \in A_2.$$

Theorem 6. Let $s(\cdot), t(\cdot) \in \mathcal{S}(\mathbb{R}_+)$ with $\frac{1}{s_+} - \frac{1}{t_+} = \alpha$, and suppose the pair $(s(\cdot), t(\cdot))$ satisfies the $\left(\frac{1}{s_+} - \frac{1}{t_+} = \alpha\right)$ -boundedness property. Assume $s(\cdot) \in LH(\mathbb{R}_+)$ and $1 < s_- \leq s_+ < \infty$. Then, there exists $C > 0$ such that

$$\|I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}\|_{L^{t(\cdot)}(\mathbb{R}_+)} \leq C \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}.$$

Proof. Since $1 < s_- \leq s_+ < \infty$ and $s(\cdot) \in LH(\mathbb{R}_+)$, the Hardy–Littlewood maximal operator is bounded; hence, there exists $C_0 > 0$ such that

$$\|\mathcal{M}\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)} \leq C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}.$$

Let $\mathfrak{F} \in L^{s(\cdot)}(\mathbb{R}_+)$, $\mathfrak{F} \neq 0$, and fix $\mu \in \mathbb{R}_+$. By the pointwise estimate for the logarithmic operator (Lemma 1), we obtain

$$|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)| \leq C_5^* \left(\theta^\alpha \mathcal{M}\mathfrak{F}(\mu) + \theta^{-\frac{1}{t_+}} \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)} \right),$$

where the logarithmic factor contributes only to the local part and is absorbed into the constant.

We choose

$$\theta = \left(\frac{\mathcal{M}\mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}} \right)^{-\frac{t_+ s(\mu)}{t(\mu)}}.$$

Substituting this choice into the above inequality and simplifying, we obtain

$$|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)| \leq C_5^* \frac{C_0 (C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}})^{\frac{\alpha t_+ s(\mu)}{t(\mu)} + \frac{s(\mu)}{t(\mu)} - 1} + (\mathcal{M}\mathfrak{F}(\mu))^{\frac{\alpha t_+ s(\mu)}{t(\mu)} + \frac{s(\mu)}{t(\mu)} - 1}}{C_0 (\mathcal{M}\mathfrak{F}(\mu))^{\frac{\alpha t_+ s(\mu)}{t(\mu)} - 1} (C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}})^{\frac{s(\mu)}{t(\mu)} - 1}}.$$

We now discuss two cases. If $\mu \in A_1$, then $\frac{\alpha t_+ s(\mu)}{t(\mu)} + \frac{s(\mu)}{t(\mu)} - 1 \leq 0$. Hence,

$$|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)| \leq C_5^* (C_0 + 1) \left(\frac{\mathcal{M}\mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}} \right)^{\frac{s(\mu)}{t(\mu)}} \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}.$$

If $\mu \in A_2$, then $\frac{\alpha t_+ s(\mu)}{t(\mu)} + \frac{s(\mu)}{t(\mu)} - 1 > 0$, and the same estimate follows.

Thus, for all $\mu \in \mathbb{R}_+$,

$$|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)| \leq C_6 \left(\frac{\mathcal{M}\mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}} \right)^{\frac{s(\mu)}{t(\mu)}} \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}.$$

Or

$$\left(\frac{|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}(\mu)|}{C_6 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}} \right)^{t(\mu)} \leq \left(\frac{\mathcal{M}\mathfrak{F}(\mu)}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}} \right)^{s(\mu)}.$$

Hence,

$$\sigma_{t(\cdot)} \left(\frac{I_{\alpha,\gamma,\beta}^{\log} f}{C_6 \|\mathfrak{F}\|_{L^{s(\cdot)}}} \right) \leq \sigma_{s(\cdot)} \left(\frac{\mathcal{M}\mathfrak{F}}{C_0 \|\mathfrak{F}\|_{L^{s(\cdot)}}} \right).$$

By the boundedness of the maximal operator, the right-hand side is bounded by 1, and hence

$$\|I_{\alpha,\gamma,\beta}^{\log} \mathfrak{F}\|_{L^{t(\cdot)}(\mathbb{R}_+)} \leq C_6 \|\mathfrak{F}\|_{L^{s(\cdot)}(\mathbb{R}_+)}.$$

□

Theorem 7. Let $t(\cdot) \in \mathcal{S}(\mathbb{R}_+)$ and let $0 < \alpha < 1$, $\gamma \geq 0$, and $\beta \in \mathbb{R}$ with $\frac{1}{t_+} = 1 - \alpha$. Then, the logarithmic Bessel–Riesz operator $I_{\alpha, \gamma, \beta}^{\log}$ is of weak type $(1, t(\cdot))$; that is, for every $\mathfrak{F} \in L^1(\mathbb{R}_+)$ and every $g > 0$, there exists a constant $C > 0$ such that

$$\left\| g \chi_{\{\mu \in \mathbb{R}_+ : |I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu)| > g\}} \right\|_{L^{t(\cdot)}(\mathbb{R}_+)} \leq C \|\mathfrak{F}\|_{L^1(\mathbb{R}_+)}.$$

Proof. The case $\mathfrak{F} = 0$ is trivial, so we assume $\mathfrak{F} \neq 0$. Fix $\mu \in \mathbb{R}_+$ and a free parameter $\theta > 0$. Let M_θ be the unique integer satisfying $2^{M_\theta} \theta \geq 1$ and $2^{M_\theta - 1} \theta < 1$. We decompose the operator as $I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu) = I_1(\mu) + I_2(\mu)$ as in the proof of Lemma 1. Then,

$$|I_1(\mu)| \leq C_1 \theta^\alpha \mathcal{M}\mathfrak{F}(\mu). \quad (8)$$

And

$$|I_2(\mu)| \leq C_2 \theta^{\alpha-1} \|\mathfrak{F}\|_{L^1(\mathbb{R}_+)}. \quad (9)$$

For any $g > 0$,

$$\{\mu \in \mathbb{R}_+ : |I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu)| > g\} \subseteq \{\mu : |I_1(\mu)| > g/2\} \cup \{\mu : |I_2(\mu)| > g/2\}.$$

Choose $\theta = \left(\frac{2C_2 \|\mathfrak{F}\|_{L^1(\mathbb{R}_+)}}{g} \right)^{\frac{1}{\alpha-1}}$; then, (9) implies $|I_2(\mu)| \leq g/2$ for all $\mu \in \mathbb{R}_+$, so $\mathfrak{m}(\{\mu : |I_2(\mu)| > g/2\}) = 0$. The weak-(1, 1) boundedness of $\mathcal{M}$ together with (8) gives

$$\mathfrak{m}(\{\mu : |I_1(\mu)| > g/2\}) \leq \mathfrak{m}\left(\left\{\mu : \mathcal{M}\mathfrak{F}(\mu) > \frac{g}{2C_1\theta^\alpha}\right\}\right) \leq \frac{2C_1\theta^\alpha}{g} \|\mathfrak{F}\|_{L^1(\mathbb{R}_+)}.$$

Hence,

$$\mathfrak{m}(\{\mu \in \mathbb{R}_+ : |I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu)| > g\}) \leq \frac{2C_1\theta^\alpha}{g} \|\mathfrak{F}\|_{L^1} = C^* \theta = C^* \left(\frac{\|\mathfrak{F}\|_{L^1(\mathbb{R}_+)}}{g} \right)^{\frac{1}{\alpha-1}}, \quad (10)$$

where C^* absorbs all numerical constants.

Let $E_g = \{\mu \in \mathbb{R}_+ : |I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu)| > g\}$. For any Θ with $0 < \Theta < g$, by definition of the modular,

$$\sigma_{t(\cdot)}\left(\frac{g \chi_{E_g}}{\Theta}\right) = \int_{E_g} \left(\frac{g}{\Theta}\right)^{t(\mu)} d\mu \leq \left(\frac{g}{\Theta}\right)^{t_+} \mathfrak{m}(E_g) \leq C^* \left(\frac{g}{\Theta}\right)^{t_+} \left(\frac{\|\mathfrak{F}\|_{L^1}}{g}\right)^{\frac{1}{\alpha-1}}.$$

Since $\frac{1}{t_+} = 1 - \alpha$, $\frac{1}{\alpha - 1} = -t_+$, and therefore $\left(\frac{\|\mathfrak{F}\|_{L^1}}{g}\right)^{\frac{1}{\alpha-1}} = \left(\frac{g}{\|\mathfrak{F}\|_{L^1}}\right)^{t_+}$. So,

$$\sigma_{t(\cdot)}\left(\frac{g \chi_{E_g}}{\Theta}\right) \leq C^* \left(\frac{g}{\Theta}\right)^{t_+} \left(\frac{g}{\|\mathfrak{F}\|_{L^1}}\right)^{t_+} \cdot \frac{1}{g^{t_+}} = C^* \left(\frac{1}{\Theta \|\mathfrak{F}\|_{L^1(\mathbb{R}_+)}}\right)^{t_+}.$$

Therefore, $\sigma_{t(\cdot)}(g \chi_{E_g} / \Theta) \leq 1$ provided $\Theta \geq C_4 \|\mathfrak{F}\|_{L^1(\mathbb{R}_+)}$ for a suitable constant $C_4 = (C^*)^{1/t_+}$. By the definition of the Luxemburg norm,

$$\left\| g \chi_{\{\mu \in \mathbb{R}_+ : |I_{\alpha, \gamma, \beta}^{\log} \mathfrak{F}(\mu)| > g\}} \right\|_{L^{t(\cdot)}(\mathbb{R}_+)} \leq C_4 \|\mathfrak{F}\|_{L^1(\mathbb{R}_+)}.$$

□

Remark 5. Theorems 3 and 5 are genuinely distinct and neither implies the other, because the exponent formulas are asymmetric in s_+ and t_+ and are not mutual inverses in closed form. Theorems 4 and 6 are verification criteria: given an arbitrary pre-specified pair $(s(\cdot), t(\cdot))$ (not necessarily of the explicit form of Theorem 3 or 5), the pointwise “boundedness property” of Definitions 6 and 7 provides a sufficient condition for boundedness. Theorems 3 and 5 are recovered as the special cases in which the boundedness property holds with equality everywhere (i.e., $A_2 = \emptyset$).

4. Conclusions

In this paper, we studied a new class of logarithmic Bessel–Riesz operators, whose kernels incorporate a logarithmic perturbation of the classical Bessel–Riesz singularity. The core of our analysis is the establishment of a sharp pointwise estimate for the logarithmic Bessel–Riesz operator in terms of the Hardy–Littlewood maximal operator. This estimate, obtained via a dyadic decomposition adapted to the logarithmic structure, provides a tool for controlling the operator and serves as the foundation for all subsequent results. Building on this pointwise control, we have developed a comprehensive boundedness theory for the logarithmic Bessel–Riesz operator in variable Lebesgue spaces. Within this framework, we derived explicit and verifiable conditions under which the logarithmic Bessel–Riesz operator maps $L^{s(\cdot)}(\mathbb{R}_+)$ into $L^{t(\cdot)}(\mathbb{R}_+)$. Furthermore, we presented illustrative examples demonstrating the applicability of the developed theory in both variable and constant exponent cases. These examples highlight the flexibility of the proposed framework and confirm the sharpness of the obtained conditions. We believe that the combination of logarithmic modifications and variable exponent analysis developed in this work provides a versatile framework that can significantly enrich the theory of modern harmonic analysis. Overall, this work establishes a foundational step in the study of logarithmic perturbations of classical operators in variable exponent spaces and provides new tools and perspectives for future investigations in this rapidly evolving area.

5. Future Work

The methods developed in this paper open several directions for future research. One natural extension is the study of commutators of logarithmic Bessel–Riesz operators with symbols in BMO or Lipschitz spaces, where the interaction between oscillatory behavior and logarithmic singularities is expected to yield new phenomena. Another promising direction is the investigation of boundedness properties in more general function spaces, such as variable Morrey spaces, weighted variable Lebesgue spaces, or grand Lebesgue spaces. In addition, the techniques introduced here may be adapted to analyze other classes of fractional and potential-type operators with non-standard kernels, particularly those involving logarithmic or slowly varying perturbations.

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