

Article

Boundary Value Problems for Some Quasilinear Parabolic Equations Under Minimal Conditions on the Coefficients

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Abstract

Quasilinear parabolic boundary value problems are considered. Sharp conditions on the data ensuring the existence and uniqueness of solutions in Sobolev classes $W_p^{1,2}(Q)$ ($p > (n+2)/2$) are exposed. They are smoothness and consistency conditions on the data and some inequalities ensuring the maximum principle for a solution. The proof of a priori estimates employs the method of frozen coefficients. The proof of existence relies on a priori bounds and Leray–Schauder-type continuation.

Keywords: quasilinear parabolic equation; Sobolev space; maximum principle; existence; uniqueness

MSC: 35K59; 35K61; 35K20

1. Introduction

We examine quasilinear parabolic equations of the form

$$Mu = u_t - \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(t, x, u)u_{x_j}) + b(x, t, u, \nabla u) = 0, \quad (1)$$

where $(t, x) \in Q = (0, T) \times G$, $G \subset \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$ with boundary Γ . Let $S = (0, T) \times \Gamma$. The initial and boundary conditions are written in either the form

$$u|_{t=0} = u_0, \quad u|_S = g(t, x), \quad (2)$$

or

$$u|_{t=0} = u_0, \quad \frac{\partial u}{\partial N} + \psi(t, x, u)|_S = 0, \quad \frac{\partial u}{\partial N} = \sum_{i,j=1}^n a_{ij}u_{x_j}v_i. \quad (3)$$

Equations of the parabolic type are encountered in many areas of mathematics and mathematical physics, and those encountered most frequently are linear and quasi-linear parabolic equations of second order. They are considered by many authors. First, we should refer to the monographs [1,2], where solutions to the problems (1), (2) and (1), (3) are sought in the Hölder spaces. In our opinion, the conditions on the coefficients of the equation here are far from optimal. Among the recent results devoted to solvability problems in Hölder spaces we mention the article [3], where the maximum principle approach is provided to study interior maximal C^α smoothness (i.e., local $C^{2+\alpha}$ regularity with respect to the parabolic distance) of solutions and a short survey on the regularity theory for fully nonlinear parabolic equations of second order is exhibited. Moreover, we refer to [4], where the



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oblique boundary value problem is studied for model equation $u_t - F(D^2u) = f$ in Hölder spaces. L_p -theory for fully nonlinear parabolic equations is exposed in [5]. This article is devoted mainly to the existence problems of viscosity solutions but some remarks are made on existence of $W_{p,loc}^{1,2}(Q)$ -solutions. The main conditions imposed on a nonlinear function $F(t, x, u, Du, D^2u)$ is the Lipschitz conditions on u, Du, D^2u and some growth conditions. We refer also to the article [6] devoted to L_p -estimates for the Hessians of viscosity solutions to fully nonlinear parabolic equations with oblique boundary conditions. Properties of viscosity solutions are also studied in [7] and many other articles (see [8]). The weighted and mixed-norm Sobolev estimates for fully nonlinear elliptic and parabolic equations in the whole space under a relaxed convexity condition with almost VMO dependence on space–time variables are proven in [9]. The problem (1), (2) in Sobolev spaces is considered in Theorem 4 in [8], where the functions a_{ij} meet the Lipschitz conditions in u , and the function $b(t, x, u, p)$ has at most linear growth in the variables (u, p) and satisfies the Lipschitz condition in p . The weighted and mixed-norm Sobolev estimates for fully nonlinear model elliptic and parabolic equations in the whole space under a relaxed convexity condition with almost VMO dependence on space–time variables are proven in [9]. The local (in time) solvability results in [10] for the problem (1), (3) are proven under the condition that the functions a_{ij} are of the class C^3 in all variables and the function b meets the Lipschitz condition in (u, p) . Much weaker conditions on the functions a_{ij} with almost the same conditions on b are used in the articles [11,12] (see also the references therein) in the case of second-order parabolic operators in non-divergence form. The data are Carathéodory functions, and the principal part is of VMO_x -type with respect to the variables (t, x) . The functions $a_{ij}(t, x, u, p)$ ($p = \nabla u$) of the main part are continuously differentiable with respect to the variables u, p and the derivatives a_{iju}, a_{ijp_k} meet the Lipschitz condition in u, p . Assuming the existence of a strong solution u , the authors assume that the Frechet derivative of the operator at u is invertible. Next, the implicit function theorem is applied to prove the existence of a solution for small perturbations of the data. Moreover, the Newton iteration procedure is utilized in order to construct an approximating sequence that converges to u in the corresponding Sobolev space. The local (in time) existence and uniqueness theorem is also obtained.

The semigroup approach to quasilinear evolution problems has been developed in [13–15]. Here in the quasilinear case the functions spaces used consist of functions that are Hölder continuous or continuously differentiable with respect to t . The weighted Sobolev spaces are used in [16–18] also in the abstract setting, where the approach based on maximal L_p -regularity is involved. In this case, as well as in many other articles devoted abstract problems, the main part of the operator is independent of t . Among the recent articles we mention [17]. There are a lot of articles devoted to different model cases (see, for instance [19]).

Our main results are connected with existence and uniqueness theorems for solutions to the problems (1), (2) and (1), (3) under the minimal smoothness conditions on the data of the problems, in particular, on the coefficients of the parabolic operator and the function ψ in (3). A solution is sought in the class $W_p^{1,2}(Q)$. To establish the Hölder continuity of a solution, we use the approach proposed in [1]. In contrast to the results of [1,2], to prove maximum estimates for a solution, we use a different approach, which allows to easily prove maximum estimates in the case of the boundary condition (3) for generalized solutions which are absent in [1,2]. The $W_p^{1,2}(Q)$ -estimates of a solution rely on arguments with the use of frozen coefficients. The results can be used in: establishing solvability results for a large class of inverse parabolic problems; establishing existence and uniqueness of solutions under perturbations; and providing constructive approximation methods via iterative techniques.

2. Preliminaries

Let E be a Banach space. The notations for Sobolev and Besov spaces $W_p^s(G; E)$, $B_{pq}^s(G; E)$, $W_p^s(Q; E)$, etc., are conventional (see [20,21]). If $E = \mathbb{R}$ or $E = \mathbb{R}^n$ then we denote the last space simply by $W_p^s(Q)$. The Hölder spaces $C^{\alpha,\beta}(\overline{Q})$, $C^{\alpha,\beta}(\overline{S})$ are defined, for example, in [1]. Given an interval $J = (0, T)$, denote $W_p^{s,r}(Q) = W_p^s(J; L_p(G)) \cap L_p(J; W_p^r(G))$. Similarly, $W_p^{s,r}(S) = W_p^s(J; L_p(\Gamma)) \cap L_p(J; W_p^r(\Gamma))$. Let $(u, v) = \int_G u(x)v(x) dx$. All considered spaces and coefficients of Equation (1) are assumed to be real. Denote $Q_\tau = (0, \tau) \times G$ and $S_\tau = (0, \tau) \times \Gamma$. We assume that $\Gamma \in C^2$ (see the definition in [1] (Chapter 1)).

Proceed with one auxiliary result. Consider the linear problem

$$Mu = u_t - \sum_{i,j=1}^n \partial_{x_i}(a_{ij}u_{x_j}) + \sum_{i=1}^n a_i u_{x_i} + a_0 u = f, \quad (4)$$

where $(t, x) \in Q = (0, T) \times G$. Initial and boundary conditions are written as follows: either

$$u|_{t=0} = u_0, \quad u|_S = g(t, x), \quad (5)$$

or

$$u|_{t=0} = u_0, \quad \frac{\partial u}{\partial N} + \sigma u|_S = g, \quad (6)$$

with $\vec{\nu} = (\nu_1, \dots, \nu_n)$ the outward unit normal to Γ and $\frac{\partial u}{\partial N} = \sum_{i,j=1}^n a_{ij}u_{x_j}\nu_i$. It is assumed that

$$\begin{aligned} u_0 &\in W_p^{2-2/p}(G), \quad g \in W_p^{k_0, 2k_0}(S), \quad f \in L_p(Q), \quad \Gamma \in C^2, \quad p > \frac{n+2}{2}, \\ g(0, x) &= u_0(x)|_\Gamma \text{ in the case of the condition (5),} \\ \frac{\partial u_0}{\partial N} + \sigma(0, x)u_0|_\Gamma &= g(0, x) \text{ if } p > 3 \text{ in the case of the condition (6),} \end{aligned} \quad (7)$$

where $k_0 = s_1 = 1 - 1/2p$ in the case of the conditions (5) and $k_0 = s_0 = 1/2 - 1/2p$ otherwise. Other conditions on the data are as follows: there exists a constant $\delta_0 > 0$ such that

$$\delta_0 |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}\xi_j\xi_i \leq |\xi|^2 / \delta_0 \quad \forall \xi \in \mathbb{R}^n; \quad (8)$$

$$a_{ij} \in C(\overline{Q}), \quad a_{ijx_i}, a_i \in L_{q_2}(Q) \quad (q_2 > n+2, q_2 \geq p), \quad a_0 \in L_{q_1}(Q) \quad (q_1 > \frac{n+2}{2}, q_1 \geq p); \quad (9)$$

$$a_{ij}, \sigma \in B_{q_3 p}^{s_0}(0, T; L_{q_3}(\Gamma)) \cap L_{q_3}(0, T; B_{q_3 p}^{2s_0}(\Gamma)), \quad \frac{n+1}{q_3} < 1 - \frac{1}{p} = 2s_0, \quad q_3 \geq p, \quad (10)$$

where $i, j = 1, 2, \dots, n$. The following theorem is valid (we use slightly stronger conditions than those in [22] (Theorem 3.4) or [23] (Theorem 5.3); see also [24,25] (Theorem 2.1)).

Theorem 1. Let $p > (n+2)/2$ and let conditions (7)–(9) in the case of problem (4), (5), or conditions (7)–(10) in the case of problem (4), (6) hold. In the latter case we also assume that $p \neq 3$. Then there exists a unique solution u to problems (4), (5) and (4), (6) such that $u \in W_p^{1,2}(Q)$ and u satisfies the estimate

$$\|u\|_{W_p^{1,2}(Q)} \leq C_0 (\|u_0\|_{W_p^{2-2/p}(G)} + \|f\|_{L_p(Q)} + \|g\|_{W_p^{k_0, 2k_0}(S)}), \quad (11)$$

where we can assume that constant C_0 depends only on δ_0 . It depends also on the norms of coefficients and domain G but it is bounded whenever these norms in the spaces indicated in (9), (10) are bounded.

In what follows, the constants c, C_i, c_i ($i = 0, 1, \dots$) are positive constants that can change from line to line.

Let $v^+(t, x) = \max(v(t, x), 0)$. The following theorem results from Lemma 1.2.3 in [26].

Theorem 2. Let u be a non-negative measurable function in Q . Then

$$\int_Q u \, dxdt = \int_0^\infty \mu(N_k) \, dk = \int_0^\infty \mu(L_k) \, dk,$$

$$N_k = \{(t, x) \in Q : u(t, x) \geq k\}, L_k = \{(t, x) \in Q : u(t, x) > k\},$$

where $\mu(\cdot)$ is the Lebesgue measure.

Corollary 1. Let u be a non-negative measurable function in Q . Then

$$\int_Q (u - k)^+ \, dxdt = \int_k^\infty \mu(N_\tau) \, d\tau = \int_k^\infty \mu(L_\tau) \, d\tau, \quad k > 0.$$

Proof. Indeed,

$$\int_Q (u - k)^+ \, dxdt = \int_0^\infty \mu(\tilde{N}_\xi) \, d\xi = \int_k^\infty \mu(\tilde{L}_\xi) \, d\xi,$$

where $\tilde{N}_\xi = \{(t, x) : (u - k)^+ \geq \xi\} = \{(t, x) : u(t, x) \geq k + \xi\}$. In this case, we infer $\tilde{N}_\xi = N_{k+\xi}$ and $\int_Q (u - k)^+ \, dxdt = \int_0^\infty \mu(N_{k+\xi}) \, d\xi$. Make the change of variables $k + \xi = \tau$. We obtain that $\int_Q (u - k)^+ \, dxdt = \int_k^\infty \mu(N_\tau) \, d\tau$. The case of the set L_ξ is considered by analogy. $\square$

Below, we present an inequality similar to that in Sect. 3 of Ch. 2 in [1].

Lemma 1. Let $u \in L_\infty(0, T; L_2(Q)) \cap L_2(0, T; W_2^1(G))$. Then $u \in L_r(Q)$, $r \in [2, 2(n+2)/n]$, and

$$\|u\|_{L_r(Q)} \leq c_0 \|u\|_{L_\infty(0, T; L_2(G))}^{1-s} \|u\|_{L_2(0, T; W_2^1(G))}^s \leq c_0 (\|u\|_{L_\infty(0, T; L_2(G))}^2 + \|u\|_{L_2(0, T; W_2^1(G))}^2)^{1/2} = c_0 |u|_1,$$

where $s = n(r-2)/2r$.

Proof. The inequality follows from the interpolation inequality $\|u\|_{W_p^s(G)} \leq c \|u\|_{W_p^1(G)}^s \|u\|_{L_p(G)}^{1-s}$ [20] and the embedding $W_p^s(G) \subset L_r(G)$ ($r = 2n/(n-2s)$) [20] (Sect. 4.6.1, 4.6.2). It suffices to prove the inequality for $r = 2(n+2)/n$, in this case $r = 2n/(n-2s)$ with $s = n/(n+2)$ and

$$\|u\|_{L_r(Q)} \leq \left(\int_0^T \|u\|_{L_r(G)}^r \, dt \right)^{1/r} \leq \left(\int_0^T c_1 \|u\|_{W_2^s(G)}^r \, dt \right)^{1/r} \leq (c_1 c)^{1/r} \left(\int_0^T \|u\|_{W_p^1(G)}^2 \|u\|_{L_2(G)}^{r(1-s)} \, dt \right)^{1/r} \leq (c_1 c)^{1/r} \|u\|_{L_2(0, T; W_2^1(G))}^s \|u\|_{L_\infty(0, T; L_2(G))}^{1-s}.$$

$\square$

3. Basic Results

Denote $\Gamma_T = S \cup \{(0, x) : x \in G\}$. The functions a_{ij} are assumed to be continuous in $\overline{Q} \times (-\infty, \infty)$ and

$$\sum_{i,j=1}^n a_{ij} p_i p_j \geq \delta_0 |p|^2 \quad \forall \vec{p} = (p_1, \dots, p_n) \in \mathbb{R}^n, \quad (t, x) \in Q, \quad u \in \mathbb{R}, \quad (12)$$

where $\delta_0 > 0$ is a constant, the function $b(t, x, u, \vec{p})$ meets the Caratheodory condition and

$$b(t, x, u, \vec{p})u \geq -\beta_0 a_{ij} p_i p_j - g_2 |u|^2 - g_1 |u|, \quad \forall \vec{p} \in \mathbb{R}^n, \text{ a.e. } (t, x) \in Q, |u| \geq \tilde{m}_0, p \in \mathbb{R}^n. \quad (13)$$

for some constant $\tilde{m}_0 > 0$. Here the abbreviation a.e. stands for almost everywhere. Moreover, for every $R > 0$, there exist a constant $\beta_1 > 0$ and a function g_3 such that

$$|b(t, x, u, \vec{p})| \leq \beta_1 |\vec{p}|^2 + g_3(t, x) \quad \forall (t, x) \in Q, u \in [-R, R], \vec{p} \in \mathbb{R}^n, \quad (14)$$

where $g_i \in L_p(Q)$, $i = 1, 2, 3$, $g_1, g_2, g_3 \geq 0$ a.e., β_i are non-negative constants.

We look for a solution to the problems (1), (2) and (1), (3) in the space $W_p^{1,2}(Q)$. We additionally assume that for every $R > 0$

$$a_{ijx_i} \in L_{q_3}(Q; C([-R, R])), a_{iju} \in L_\infty(Q; C([-R, R])), q_3 > n + 2, q_3 \geq p, g \in W_p^{s_1, 2s_1}(S), \quad (15)$$

where $i, j = 1, 2, \dots, n$. In the case of the problem (1), (3), we require that

$$\psi \in W_p^{s_0, 2s_0}(S; C([-R, R])), \psi_u \in L_{q_4}(S; C([-R, R])), p > \frac{n+2}{2}, q_4 > n + 1, q_4 \geq p, \quad (16)$$

$$a_{ij} \in B_{q_5 p}^{s_0}(0, T; L_{q_5}(\Gamma; C([-R, R]))) \cap L_{q_5}(0, T; B_{q_5 p}^{2s_0}(\Gamma; C([-R, R]))) \quad \forall R > 0, q_5 \geq p, \quad (17)$$

where $(n+1)/q_5 < 1 - 1/p = 2s_0$ and the derivatives are just generalized derivatives in the Sobolev sense.

Remark 1. The condition (12) is employed in almost all estimates below and ensures parabolicity of the problem. The condition (13) is involved only in the estimate of the maximum of the module of a solution. The condition (14) is used to establish the Hölder and $W_p^{1,2}(Q)$ estimates for a solution. At last, the conditions (15)–(17) are provided to obtain $W_p^{1,2}(Q)$ estimates.

By a solution to the problems (1), (2) and (1), (3), we mean a function $u \in W_p^{1,2}(Q)$, $p > (n+2)/2$ satisfying Equation (1) and the boundary condition (2), respectively, (3) a.e. in Q , respectively, on S . Note that if $u \in W_p^{1,2}(Q)$ ($p > (n+2)/2$) then the embedding theorems ensure that $u|_{t=0} = u_0 \in W_p^{2-2/p}(G) \subset C^{2\varepsilon_0}(\bar{G})$, $\varepsilon_0 = 1 - (n+2)/2p$, $u|_S = g \in C^{\varepsilon_0, 2\varepsilon_0}(\bar{S})$ ([25] (Theorems 1.19 and 1.22)).

Theorem 3. Assume that $u \in W_p^{1,2}(Q)$ ($p > (n+2)/2$) is a solution to the problem (1)–(2), the conditions (12)–(14) hold, and $g(t, x) \in L_\infty(S)$, $u_0 \in L_\infty(G)$. Then there exists a constant $M > 0$, depending on $\delta_0, \beta_0, \|g_2\|_{L_p(Q)}, \|u\|_{L_\infty(\Gamma_T)}$ such that $\|u\|_{L_\infty(Q)} \leq M$.

Proof. The idea of the proof is similar to that in Sect. 2 of Ch. 5 in [1] but we use Theorem 1 rather than Theorem 6.1 of Ch. 2 in [1]. Make the change of variables $u = e^{\lambda t} v$ ($\lambda > 0$). The function v is a solution to the equation

$$Mu = v_t - \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(t, x, v e^{\lambda t}) v_{x_j}) + e^{-\lambda t} b(x, t, v e^{\lambda t}, e^{\lambda t} \nabla v) + \lambda v = 0. \quad (18)$$

The initial and boundary conditions are written as

$$v|_{t=0} = u_0, \quad v|_S = g(t, x) e^{-\lambda t}, \quad (19)$$

Let $w = |v|^m$, $w^{(k)} = (|v|^m - k)^+$, $A_k(t) = \{x \in G : w(t, x) > k\}$. In this case $\int_0^T A_k(t) dt = \mu(L_k)$, $L_k = \{(t, x) : w(t, x) > k\}$. Multiply (18) by $|v|^{m-2} v w^{(k)}$ and integrate

over G . Choose $m_0 > \max(\|u_0\|_{L_\infty(G)}, \|ge^{-\lambda t}\|_{L_\infty(S)}, \tilde{m}_0)$. In view of this choice, $w^{(k)}|_{\Gamma_T} = 0$ for $k \geq k_0 = m_0^m$. Integrating by parts, we obtain that

$$\begin{aligned} \frac{1}{m} \frac{\partial}{\partial t} \int_G (w^{(k)})^2 dx + \int_{A_k(t)} \sum_{i,j=1}^n a_{ij} v_{x_j} v_{x_i} [(m-1)|v|^{m-2} w^{(k)} + m|v|^{2m-2}] \\ + e^{-\lambda t} b(t, x, v e^{\lambda t}, e^{\lambda t} \nabla v) |v|^{m-2} v w^{(k)} dx = 0. \end{aligned} \quad (20)$$

The condition (13) yields

$$\begin{aligned} \frac{1}{m} \frac{\partial}{\partial t} \int_G (w^{(k)})^2 dx + \int_{A_k(t)} \sum_{i,j=1}^m a_{ij} v_{x_j} v_{x_i} [(m-1)|v|^{m-2} w^{(k)} + m|v|^{2m-2}] + \lambda w w^{(k)} dx \leq \\ \int_{A_k(t)} \beta_0 a_{ij} v_{x_i} v_{x_j} |v|^{m-2} w^{(k)} + f |v|^m w^{(k)} dx, \quad f = |g_2| + \frac{1}{m_0} g_1. \end{aligned}$$

This inequality can be rewritten in the form

$$\begin{aligned} \frac{1}{m} \frac{\partial}{\partial t} \int_G (w^{(k)})^2 dx + \int_{A_k(t)} \sum_{i,j=1}^m a_{ij} v_{x_j} v_{x_i} [|v|^{m-2} w^{(k)} (m-1-\beta_0) + m|v|^{2m-2}] + \lambda w w^{(k)} dx \\ \leq \int_G f |v|^m w^{(k)} dx, \end{aligned}$$

Next, we take $m = \beta_0 + 1$. In this case we arrive at the inequality

$$\frac{1}{m} \frac{\partial}{\partial t} \int_G (w^{(k)})^2 dx + \frac{\delta_0}{2m} \int_G |\nabla w^{(k)}|^2 + \lambda w w^{(k)} dx \leq \int_G f w w^{(k)} dx. \quad (21)$$

Integrating in t , we conclude that

$$\max_{t \in (0, T)} \int_G (w^{(k)})^2 dx + \int_Q |\nabla w^{(k)}|^2 + \lambda w w^{(k)} dx dt \leq c_0 \int_{L_k} f w w^{(k)} dx dt, \quad (22)$$

where the constant c_0 depends on m, δ_0 . The Hölder inequality implies that

$$\int_{L_k} f w w^{(k)} dx dt \leq \|f\|_{L_p(Q)} \|w w^{(k)}\|_{L_{p'}(L_k)}, \quad 1/p + 1/p' = 1.$$

Since $w w^{(k)} \leq 2(w^{(k)})^2 + k^2$, we have $\|w w^{(k)}\|_{L_{p'}(L_k)} \leq c_1 (\|w^{(k)}\|_{L_{2p'}(G)}^2 + k^2 (\mu(L_k))^{1/p'})$.

This inequality, (22), and the inequality $(w^{(k)})^2 \leq w w^{(k)}$ provide the estimate

$$\max_{t \in (0, T)} \int_G (w^{(k)})^2 dx + \int_Q |\nabla w^{(k)}|^2 + \lambda (w^{(k)})^2 dx dt \leq c_2 \|w^{(k)}\|_{L_{2p'}(Q)}^2 + c_2 k^2 (\mu(L_k))^{1/p'}. \quad (23)$$

The Riesz–Thorin theorem or the results in [20] (Sect. 1.18.4) ensure that $\|w^{(k)}\|_{L_{2p'}(Q)} \leq \|w^{(k)}\|_{L_r(Q)}^\theta \|w^{(k)}\|_{L_2(Q)}^{1-\theta}$, where $r = 2(n+2)/n$, $\theta r + (1-\theta)2 = 2p'$. Since $p > (n+2)/2$, $\theta \in (0, 1)$. The inequality

$$|ab| \leq \varepsilon |a|^p / p + \varepsilon^{1-p} |b|^{p'} / p', \quad p' = p/(p-1), \quad \varepsilon > 0, \quad (24)$$

and Lemma 1, imply that

$$c_2 \|w^{(k)}\|_{L_{2p'}(Q)}^2 \leq \frac{1}{2} \|w^{(k)}\|_1^2 + c_3 \|w^{(k)}\|_{L_2(Q)}^2.$$

In this case the inequality (23) yields

$$\max_{t \in (0, T)} \int_G (w^{(k)})^2 dx + \int_Q |\nabla w^{(k)}|^2 + 2\lambda (w^{(k)})^2 dx dt \leq 2c_2 k^2 (\mu(L_k))^{1/p'} + 2c_3 \|w^{(k)}\|_{L_2(Q)}^2. \quad (25)$$

Note that the constant c_3 is independent of λ . Choosing $\lambda > \max(1, 2c_3)$, we conclude that

$$\max_{t \in (0, T)} \int_G (w^{(k)})^2 dx + \int_Q |\nabla w^{(k)}|^2 + \lambda (w^{(k)})^2 dx dt \leq 2c_2 k^2 (\mu(L_k))^{1/p'}. \quad (26)$$

Next, the Hölder inequality, Lemma 1 and (26) imply that

$$\begin{aligned} \int_Q w^{(k)} dQ &\leq \|w^{(k)}\|_{L_{2(n+2)/n}(Q)} (\mu(L_k))^{(n+4)/2(n+2)} \leq c |w^{(k)}|_1 (\mu(L_k))^{(n+4)/2(n+2)} \\ &\leq 2c \sqrt{c_2} k (\mu(L_k))^\alpha, \quad \alpha = (n+4)/2(n+2) + 1/(2p') > 1. \end{aligned}$$

Thus, we obtain the inequality

$$\int_Q w^{(k)} dQ \leq c_3 k (\mu(L_k))^\alpha, \quad \alpha = (n+4)/2(n+2) + 1/(2p') > 1. \quad (27)$$

As a consequence, we have

$$\int_Q w^{(k_0)} dQ \leq c_3 k_0 (\mu(Q))^\alpha = M_0, \quad \alpha = (n+4)/2(n+2) + 1/(2p') > 1. \quad (28)$$

By Theorem 2 $\int_Q w^{(k)} = \int_k^\infty \mu(L_\tau) d\tau = f(k)$. In this case, $f'(k) = -\mu(L_k)$ and the inequality (26) can be written in the form $f(k) \leq c_3 k ((-f'(k))^\alpha)$, or in the form $f'(k) \leq -c_4 f(k)^{1/\alpha} / k^{1/\alpha}$, $c_4 = c_3^{-1/\alpha}$. Integrating this inequality, we arrive at the relation

$$(f(k))^{1-1/\alpha} \leq (f(k_0))^{1-1/\alpha} - \frac{c_4 k^{1-1/\alpha}}{1-1/\alpha} + \frac{c_4 k_0^{1-1/\alpha}}{1-1/\alpha} \leq c_5 - \frac{c_4 k^{1-1/\alpha}}{1-1/\alpha}$$

which ensures that $f(k) \equiv 0$ for $k > k_1 = (\frac{c_5(\alpha-1)}{c_4\alpha})^{\alpha/(\alpha-1)}$, $c_5 = (M_0)^{1-1/\alpha} + \frac{c_4 k_0^{1-1/\alpha}}{1-1/\alpha}$. Hence, $|v|^m \leq k_1$ a.e. $\square$

Proceed with the problem (1), (3). We assume that $\psi \in W_p^{s_0, 2s_0}(S; C([-R, R]))$ for every $R > 0$ and

$$\psi(t, x, u) \operatorname{sgn} u \geq -g_4 - \beta_3 |u|, \quad (29)$$

where $\beta_3 \geq 0$ a constant and $g_4 \in W_p^{s_0, 2s_0}(S)$. There exists a function $\rho \in C^2(\bar{G})$ such that $\nabla \rho \cdot \nu = |\nabla \rho|$, $1/2 \leq |\nabla \rho| \leq 2$ on Γ (this function was constructed in the proof of Theorem 13.1 in [2]). Next, consider an even extension of the function g_4 for $t < 0$ and construct $\varphi(t) = 1$ for $t > -T/2$, $\varphi(t) = 0$ for $t < -3T/4$, $\varphi(t) \in C^1([-T, T])$. There exists a solution $\rho_1(t, x) \in W_p^{1,2}((-3T/4, T) \times G)$ to the problem $\rho_{1t} - \Delta \rho_1 = 0$, $\frac{\partial \rho_1}{\partial \nu}|_S = g_4 \varphi(t)$, $\rho_1(-3T/4, x) = 0$ (Theorem 1). Here ν is the outward unit normal to S .

Theorem 4. Assume that $u \in W_p^{1,2}(Q)$ ($p > (n+2)/2$) is a solution to the problem (1), (3), the conditions (12)–(14), (29) hold, and $u_0 \in L_\infty(G)$. Then there exists a constant $M > 0$, depending on β_0 , $\|g_2\|_{L_p(Q)}$, $\|u_0\|_{L_\infty(G)}$, and $\|g_4\|_{W_p^{s_0, 2s_0}(S)}$ such that $\|u\|_{L_\infty(Q)} \leq M$.

Proof. The proof is in line with that of Theorem 3. Repeating the arguments, we arrive at (20) with an additional summand $J = - \int_\Gamma e^{-\lambda t} \psi(t, x, e^{\lambda t} v) |v|^{m-2} v w^{(k)} d\Gamma$ on the right-hand side. Estimate it. In view of the conditions on ψ , we have

$$J \leq \int_{\Gamma} (g_4 e^{-\lambda t} + \beta_3 |v|) |v|^{m-1} w^{(k)} d\Gamma.$$

Next,

$$\int_G \Delta \rho \beta_3 |v| |v|^{m-1} w^{(k)} dx + \int_G \nabla \rho \cdot \nabla \beta_3 |v|^m w^{(k)} dx = \int_{\Gamma} \nabla \rho \cdot \nu \beta_3 |v|^m w^{(k)} d\Gamma.$$

Therefore, we derive the inequality

$$\begin{aligned} \int_{\Gamma} \beta_3 |v|^m w^{(k)} d\Gamma &\leq 2 \int_{\Gamma} \nabla \rho \cdot \nu \beta_3 |v|^m w^{(k)} d\Gamma \leq \\ &2 \int_G \Delta \rho \beta_3 |v|^m w^{(k)} dx + 2 \int_G \nabla \rho \cdot \nabla (\beta_3 |v|^m w^{(k)}) dx \leq \\ &c_1 \int_G \beta_3 |v|^m w^{(k)} dx + 2 \int_G |\nabla (\beta_3 |v|^m w^{(k)})| dx. \end{aligned}$$

The first summand here is estimated by the quantity

$$\int_G \beta_3 |v|^m w^{(k)} dx \leq \beta_3 \int_G w w^{(k)} dx \leq c_5 \int_G (w^{(k)})^2 dx + c_5 k^2 \mu(A_k(t)). \quad (30)$$

The estimate for the second summand is of the form

$$\int_G |\nabla \beta_3 |v|^m w^{(k)}| dx \leq \varepsilon \int_G |\nabla w^{(k)}|^2 + c(\varepsilon) ((w^{(k)})^2 dx + k^2 \mu(A_k(t))). \quad (31)$$

Next, we use the equality

$$\int_G \Delta \rho_1 |v|^{m-2} v w^{(k)} dx + \int_G \nabla \rho_1 \cdot \nabla (|v|^{m-2} v w^{(k)}) dx = \int_{\Gamma} g_4 |v|^{m-2} v w^{(k)} d\Gamma$$

which guarantees the relation

$$|\int_{\Gamma} e^{-\lambda t} g_4 |v|^{m-2} v w^{(k)} d\Gamma| \leq \int_{\Gamma} |\nabla \rho_1| |\nabla (|v|^{m-2} v w^{(k)})| d\Gamma + \frac{1}{m_0} \int_G |\Delta \rho_1| |v|^m w^{(k)} dx.$$

The former summand on the right-hand side of the previous inequality is bounded by

$$\int_G |\nabla \rho_1| |\nabla |v|^{m-2} v w^{(k)}| dx \leq \varepsilon \int_G |\nabla w^{(k)}|^2 + c(\varepsilon) (|\nabla \rho_1|^2 (w^{(k)})^2 + k^2 \chi_{A_k(t)}) dx. \quad (32)$$

where $\chi_{A_k(t)}$ is the indicator of the set $A_k(t)$. Thus, the estimates (32) validate the relation

$$\int_0^T \int_G e^{-\lambda t} g_4 |v|^{m-2} v w^{(k)} dx dt \leq \varepsilon \int_Q |\nabla w^{(k)}|^2 + c(\varepsilon) \int_Q \tilde{g}_3 ((w^{(k)})^2 + k^2 \chi_{L_k}(t, x)) dx dt,$$

where $\tilde{g}_3 = |\Delta \rho_1| + |\nabla \rho_1|^2$. By the embedding theorems, $\tilde{g}_3 \in L_p(Q)$. The same arguments as those involved in the estimate of the right-hand side in (22) ensure the inequality

$$\int_Q \tilde{g}_3 (w^{(k)})^2 dx dt \leq \varepsilon_1 |w^{(k)}|_1^2 + c(\varepsilon_1) \int_Q |w^{(k)}|^2 dx dt,$$

with ε_1 as an arbitrary positive constant. Moreover,

$$k^2 \int_Q \tilde{g}_3 \chi_{L_k(t,x)} dx dt \leq k^2 \|\tilde{g}_3\|_{L_p(Q)} (\mu(L_k))^{1/p'}, \quad p' = p/(p-1).$$

These inequalities for an appropriate ε_1 and (30), (31) validate the estimate

$$\int_0^T J dt \leq 3\varepsilon \int_Q |\nabla w^{(k)}|^2 dx dt + c(\varepsilon) \left(\int_Q |w^{(k)}|^2 dx dt + k^2 (\mu(L_k))^{1/p'} \right).$$

Thus, the inequality (26) from the previous theorem for an appropriate λ and ε is replaced with

$$\max_{t \in (0, T)} \int_G (w^{(k)})^2 dx + \int_Q |\nabla w^{(k)}|^2 + \lambda (w^{(k)})^2 dx \leq c_4 k^2 (\mu(L_k))^{1/p'}. \quad (33)$$

Further, we can repeat the arguments of the previous theorem. $\square$

Remark 2. The claims of Theorems 3 and 4 are valid also for generalized solutions to the problems (1), (2) and (1), (3). The arguments are almost the same as those in [1] (Sect. 1, 2 of Ch. 5). The only difference is that on the final step we repeat the arguments of the proofs of Theorems 3 and 4.

Theorem 5. Let $u \in W_p^{1,2}(Q)$ ($p > (n+2)/2$) be a solution to the problem (1), (2) or the problem (1), (3). Assume that the conditions (12)–(14) in the case of the problem (1), (2) or, respectively, the conditions (12)–(14), (29) in the case of the problem (1), (3) are fulfilled. Moreover, there exists $\varepsilon_0 > 0$ such that $u_0 \in C^{2\varepsilon_0}(\overline{G})$, $g \in C^{\varepsilon_0, 2\varepsilon_0}(\overline{S})$ in the case of the problem (1), (2) and $u_0 \in C^{2\varepsilon_0}(\overline{G})$ in the case of the problem (1), (3). Then there exist constants $M_1, \alpha_1 > 0$, depending on $\beta_1, \|g_3\|_{L_p(Q)}, \|u\|_{L_\infty(Q)} = M, \|u_0\|_{C^{2\varepsilon_0}(\overline{G})}$, and $\|g\|_{C^{\varepsilon_0, 2\varepsilon_0}(\overline{S})}$ in the case of problem (1), (2) and on $\beta_1, \|g_3\|_{L_p(Q)}, \|u\|_{L_\infty(Q)} = M, \|u_0\|_{C^{2\varepsilon_0}(\overline{G})}, \beta_3, \|g_4\|_{W_p^{s_0, 2s_0}(S)}$ in the case of problem (1), (3) such that $\|u\|_{C^{\alpha_1, 2\alpha_1}(\overline{Q})} \leq M_1$.

Proof. The claim in the case of problem (1), (2) results from Theorem 1.1 of Ch. 5 in [1]. Unfortunately, a similar result in the case of the problem (1), (3) is proven in [1] under more stringent conditions on the data (see Theorem 7.1 of Ch. 5 [1]). So we outline the proof which is in line with that in [1]. Let $B_r(x_0)$ be a ball of radius r centered at x_0 and $Q(\rho, \tau) = \{(t, x) : x \in B_\rho(x_0), t_0 < t < t_0 + \tau\}$. Fix $(t_0, x_0) \in \overline{Q}$ and multiply Equation (1) by $u^{(k)} \xi^2(t, x)$, $u^{(k)} = \max(u - k, 0)$, $k \geq \max_{x \in Q(\rho, \tau) \cap Q} u_0(x)$, where ξ is an arbitrary smooth function with values between 0 and 1 and is vanishing for $x \in \mathbb{R}^n \setminus B_\rho(x_0)$ for every t . Integrating the equality obtained over $B_\rho(x_0) = B_\rho$ and by parts, we infer

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \int_{B_\rho \cap G} \xi^2 (u^{(k)})^2 dx + \int_{B_\rho \cap G} \sum_{i,j=1}^n a_{ij} u_{x_j}^{(k)} u_{x_i}^{(k)} \xi^2 + 2 \sum_{i,j=1}^n a_{ij} u_{x_j} u_{x_i}^{(k)} \xi \xi_{x_i} + \\ b(t, x, u, \nabla u) u^{(k)} \xi^2 dx = - \int_{\Gamma \cap B_\rho} \xi^2 \psi(t, x, u) u^{(k)} d\Gamma + \int_{B_\rho} \xi \xi_t (u^{(k)})^2 dx, \end{aligned} \quad (34)$$

Let $A_k(t) = \{x \in B_\rho \cap G : u(t, x) > k\}$, $L_k = \{(t, x) \in Q(\rho, \tau) \cap Q : u(t, x) > k\}$. The conditions of the theorem yield

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \int_{B_\rho \cap G} \xi^2 (u^{(k)})^2 dx + \int_{B_\rho \cap G} \delta_0 \xi^2 |\nabla u^{(k)}|^2 dx \leq \int_{B_\rho \cap G} c_1 |\nabla u^{(k)}| |\nabla \xi| \xi u^{(k)} \\ + \beta_1 |\nabla u^{(k)}|^2 u^{(k)} \xi^2 + g_3 \xi^2 u^{(k)} dx + \int_{\Gamma \cap B_\rho} \xi^2 (g_4 + \beta_3 |u|) |u^{(k)}| d\Gamma + \int_{B_\rho \cap G} |\xi \xi_t| (u^{(k)})^2 dx. \end{aligned}$$

Using the inequality $|\nabla u^{(k)}| |\nabla \xi| \xi u^{(k)} \leq \varepsilon_1 |\nabla u^{(k)}|^2 \xi^2 + c(\varepsilon_1) |\nabla \xi|^2 (u^{(k)})^2$ for sufficiently small ε_1 , we validate the inequality

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \int_{B_\rho \cap G} \xi^2 (u^{(k)})^2 dx + \frac{2}{3} \int_{B_\rho \cap G} \delta_0 \xi^2 |\nabla u^{(k)}|^2 dx &\leq \int_{B_\rho \cap G} \beta_1 |\nabla u^{(k)}|^2 u^{(k)} \xi^2 + g_3 \xi^2 u^{(k)} dx \\ &\quad + \int_{\Gamma \cap B_\rho} \xi^2 (g_4 + \beta_3 |u|) |u^{(k)}| d\Gamma + \int_{B_\rho \cap G} c_1 (u^{(k)})^2 (|\nabla \xi|^2 + \xi |\xi_t|) dx. \end{aligned}$$

Integrating in t from t_0 to $t \leq t_0 + \tau$ we obtain the inequality

$$\begin{aligned} \max_{t \in [t_0, t_0 + \tau]} \int_{B_\rho \cap G} \xi^2 (u^{(k)}(t, x))^2 dx + \frac{2}{3} \int_{L_k} \delta_0 \xi^2 |\nabla u^{(k)}|^2 dx dt &\leq \\ \int_{L_k} 2\beta_1 |\nabla u^{(k)}|^2 u^{(k)} \xi^2 + 2g_3 \xi^2 |u^{(k)}| dx dt + 2 \int_{S \cap Q(\rho, \tau)} \xi^2 (g_4 + \beta_3 |u|) |u^{(k)}| dS \\ + 2 \int_{L_k} c_1 (u^{(k)})^2 (|\nabla \xi|^2 + \xi |\xi_t|) dx dt + \int_{B_\rho \cap G} \xi^2 (u^{(k)}(t_0, x))^2 dx. \quad (35) \end{aligned}$$

We have that $(g_4 + \beta_3 |u|) \leq (g_4 + \beta_3 M) = \tilde{g}_4$. Arguing as before (see the construction of the function ρ_1), we can construct a function ρ_2 such that $\rho_{2t} - \Delta \rho_2 = 0$, $\frac{\partial \rho_2}{\partial \nu}|_S = \tilde{g}_4$. In this case, we derive the equality

$$\int_{Q(\rho, \tau)} \Delta \rho_2 |u_k| \xi^2 dx dt + \int_{Q(\rho, \tau)} \nabla \rho_2 \nabla |u_k| \xi^2 dx dt = \int_{S \cap Q(\rho, \tau)} \tilde{g}_4 \xi^2 u^{(k)} dS$$

which ensures the inequality

$$\begin{aligned} \int_{S \cap Q(\rho, \tau)} \xi^2 (g_4 + \beta_3 |u|) u^{(k)} dS &\leq \int_{L_k} |\Delta \rho_2| \xi^2 |u^{(k)}| + 2 |\nabla \rho_2| |\nabla \xi| \xi |u^{(k)}| + \\ |\nabla \rho_2| \xi^2 |\nabla u^{(k)}| dx dt &\leq \varepsilon \int_{L_k} \xi^2 |\nabla u^{(k)}|^2 dx dt + c(\varepsilon) \int_{L_k} |u^{(k)}|^2 |\nabla \xi|^2 + (|\Delta \rho_2| + |\nabla \rho_2|^2) \xi^2 dx dt. \end{aligned}$$

The Hölder inequality yields

$$c(\varepsilon) \int_{L_k} (|\Delta \rho_2| + |\nabla \rho_2|^2) \xi^2 dx dt \leq c \left(\int_{L_k} \xi dx dt \right)^{1/p'}.$$

So we can write out the estimate

$$\begin{aligned} \int_{S \cap Q(\rho, \tau)} \xi^2 (g_4 + \beta_3 |u|) |u^{(k)}| dS &\leq \varepsilon \int_{L_k} \xi^2 |\nabla u^{(k)}|^2 dx dt + \\ c(\varepsilon) \int_{L_k} |u^{(k)}|^2 |\nabla \xi|^2 dx dt &+ c_1(\varepsilon) \left(\int_{L_k} \xi dx dt \right)^{1/p'}. \quad (36) \end{aligned}$$

Similarly, we infer

$$\int_{L_k} 2g_3 \xi^2 u^{(k)} dx dt \leq c_2(M) \left(\int_{L_k} \xi dx dt \right)^{1/p'} \quad (37)$$

Using (36), (37), and choosing small ε , we can rewrite (35) in the form

$$\begin{aligned} \max_{t \in [t_0, t_0 + \tau]} \int_{B_\rho \cap G} \xi^2 (u^{(k)})^2 dx + \frac{1}{3} \int_{L_k} \delta_0 \xi^2 |\nabla u^{(k)}|^2 dx dt &\leq \int_{L_k} 2\beta_1 |\nabla u^{(k)}|^2 u^{(k)} \xi^2 dx dt \\ + c_3 \int_{L_k} c_1 (u^{(k)})^2 (|\nabla \xi|^2 + \xi |\xi_t|) dx dt &+ c_4 \left(\int_{L_k} \xi dx dt \right)^{1/p'} + \int_{B_\rho \cap G} \xi^2 (u^{(k)}(t_0, x))^2 dx. \quad (38) \end{aligned}$$

In what follows, we assume that $k \geq \max_{x \in Q(\rho, \tau) \cap G} u(t, x) - \delta$, where $\delta = \delta_0 / (12\beta_1)$. This assumption proves that

$$\begin{aligned} \max_{t \in [t_0, t_0 + \tau]} \int_{B_\rho \cap G} \xi^2 (u^{(k)})^2 dx + \frac{1}{6} \int_{L_k} \delta_0 \xi^2 |\nabla u^{(k)}|^2 dx dt \leq \\ + c_3 \int_{L_k} c_1 (u^{(k)})^2 (|\nabla \xi|^2 + \xi |\xi_t|) dx dt + c_4 \left(\int_{L_k} \xi dx dt \right)^{1/p'} + \int_{B_\rho \cap G} \xi^2 (u^{(k)}(t_0, x))^2 dx. \end{aligned}$$

where $k \geq \max_{x \in Q(\rho, \tau) \cap G} u(t, x) - \delta$, $k \geq \max_{x \in Q(\rho, \tau) \cap G} u_0(x)$. Similar inequality is derived for the function $(-u)^{(k)}$. The conditions of Theorem 8.2 of Ch. 2 (see Remark 8.1 after this theorem) in [1] are fulfilled and, thus, the theorem is proven. $\square$

Lemma 2. Let the condition (15)–(17) be fulfilled. If $u \in W_p^{s_1, 2s_1}(S)$ ($s_1 = 1 - 1/2p$) and $\|u\|_{L_\infty(S)} = M$, then there exists $\varepsilon > 0$ such that

$$\|\psi(t, x, u)\|_{W_p^{s_0, 2s_0}(S)} \leq c_1(M) + c_2(M) \|u\|_{W_p^{s_1 - \varepsilon/2, 2s_1 - \varepsilon}(S)}, \quad (39)$$

If $u \in W_p^{1,2}(Q)$ and $\|u\|_{L_\infty(S)} = M$, then there exists $\varepsilon > 0$ such that

$$\|a_{ij} u_{x_j}\|_{W_p^{s_0, 2s_0}(S)} \leq c_0 \|a_{ij}\|_{L_\infty(S)} \|u\|_{W_p^{1,2}(Q)} + c_4(M) \|u\|_{W_p^{1 - \varepsilon/2, 2 - \varepsilon}(Q)}, \quad u \in W_p^{1,2}(Q), \quad (40)$$

for all i, j .

Proof. The proof of the estimate (39) is essentially simpler than that of (40). It relies on the equality

$$\varphi(t, x, v_1) - \varphi(t, x, v_2) = \int_0^1 \varphi_v(t, x, v_2 + \xi(v_1 - v_2)) d\xi (v_1 - v_2).$$

Hence, we proceed with the proof of (40). We employ the definition of the norm [20] (Subsect. 4.4.1), [21]. The norm in $W_p^{s_0, 2s_0}(S)$ is defined with the use of partition of unity and straightening of the boundary Γ . As a result, we can prove the estimate in the simplest case of $S = (0, \infty) \times \Gamma$, $\Gamma = \mathbb{R}^{n-1}$, $Q = (0, \infty) \times \mathbb{R}_+^n$, $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$. The norm in this case is defined by the equality (see [20] (Sect. 4.4.2))

$$\begin{aligned} \|v(t, x)\|_{W_p^{s_0, 2s_0}(S)}^p = \|v\|_{L_p(S)}^p + \int_0^\delta \frac{1}{|\tau|^{1+s_0 p}} \|\Delta_{\tau, t} v\|_{L_p(S)}^p d\tau + \\ \int_{|h| \leq \delta} \frac{1}{|h|^{n-1+2s_0 p}} \|\Delta_{h, x} v\|_{L_p(S)}^p dh = \|v\|_{L_p(S)}^p + J_1^p(\Delta_{\tau, t} v) + J_2^p(\Delta_{h, x} v), \quad \delta > 0, \quad (41) \end{aligned}$$

where $\Delta_{\tau, t} v = v(t + \tau, x) - v(t, x)$, $\Delta_{h, x} v = v(t, x + h) - v(t, x)$. Consider the second summand with $v = a_{ij}(t, x, u) u_{x_j}$ assuming that this function is compactly supported. We have the equality $\Delta_{\tau, t} v = a_{ij}(t + \tau, x, u(t + \tau, x)) \Delta_{\tau, t} u_{x_j} + (a_{ij}(t + \tau, x, u(t + \tau, x)) - a_{ij}(t + \tau, x, u(t, x))) u_{x_j}(t, x) + (a_{ij}(t + \tau, x, u(t, x)) - a_{ij}(t, x, u(t, x))) u_{x_j}(t, x) = I_1 + I_2 + I_3$. The triangle inequality implies that

$$J_1(\Delta_{\tau, t} a_{ij} u_{x_j}) \leq J_1(I_1) + J_1(I_2) + J_1(I_3).$$

Estimate every summand. We have (see Corollary 1.3 in [25])

$$J_1(I_1) \leq \|a_{ij}(t + \tau, x, u(t + \tau, x))\|_{L_\infty(S)} \|u_{x_j}\|_{W_p^{s_0}(0, \infty; L_p(\Gamma))} \leq c_0 \|a_{ij}(t + \tau, x, u(t + \tau, x))\|_{L_\infty(S)} \|u\|_{W_p^{1,2}(Q)}. \quad (42)$$

Next the Hölder inequality yields

$$J_1(I_3) \leq J_1((a_{ij}(t + \tau, x, u(t, x)) - a_{ij}(t, x, u(t, x)))u_{x_j}(t, x)) \leq \left(\int_0^\delta \frac{1}{|\tau|^{1+s_0p}} \|u_{x_j}\|_{L_{q_5p/(q_5-p)}(S)}^p \|a_{ij}(t + \tau, x, u) - a_{ij}(t, x, u)\|_{L_{q_5}(S; C([-M, M]))}^p d\tau \right)^{1/p} \leq \|a_{ij}(t, x, u)\|_{B_{q_5p}^{s_0, 2s_0}(S; C([-M, M]))} \|u_{x_j}\|_{L_{q_5p/(q_5-p)}(S)} \leq c_1(M) \|u_{x_j}\|_{L_{q_5p/(q_5-p)}(S)} \leq c_2(M) \|u\|_{W_p^{1/2+\varepsilon/2+1/2\tilde{q}, 1+\varepsilon+1/\tilde{q}}(Q)}^p \leq \|u\|_{W_p^{1-\varepsilon_1, 2-2\varepsilon_1}(Q)}^p, \quad \tilde{q} = q_5p/(q_5-p), \quad (43)$$

where $\varepsilon_1 > 0$, $2\varepsilon_1 \leq 1 - 1/p - (n+1)/q_5 - \varepsilon$, $0 < \varepsilon < 1 - 1/p - (n+1)/q_5$ and we employ Corollary 1.3 in [25] and the embedding $W_p^{1-\varepsilon_1, 2-2\varepsilon_1}(Q) \subset W_{q_5p/(q_5-p)}^{1/2+\varepsilon/2+1/2\tilde{q}, 1+\varepsilon+1/\tilde{q}}(Q)$ (Corollary 5.6.4 of Ch. 7 in [21]). The constant ε can be chosen to be arbitrarily small. Estimate the summand $J_1(I_2)$. We use here the inequality of the Gagliardo–Nirenberg type (Theorem 5.7.1 of Ch. 7 in [21])

$$\|u\|_{B_{q,q}^{\tilde{s}/2, \tilde{s}}(Q)} \leq c \|u\|_{B_{p,p}^{\tilde{s}_1/2, \tilde{s}_1}(Q)}^\theta \|u\|_{B_{\infty,\infty}^{\tilde{s}_0/2, \tilde{s}_0}(Q)}^{1-\theta}, \quad (44)$$

where $\tilde{s} - (n+2)/q = (1-\theta)\tilde{s}_0 + \theta(\tilde{s}_1 - (n+2)/p)$, $(\tilde{s} - \tilde{s}_0)/(\tilde{s}_1 - \tilde{s}_0) \leq \theta \leq 1$, $-\infty < \tilde{s}_0 < \tilde{s} < \tilde{s}_1 < \infty$, $1 \leq q, p \leq \infty$, $\theta > 0$. Since the function a_{ij} meets the Lipschitz condition in the variable u , we infer

$$J_1(I_2) \leq c(M) J_1(|\Delta_{\tau,t} u(t, x)| |u_{x_j}(t, x)|) \leq c(M) \|u_{x_j}\|_{L_q(S)} \|u\|_{B_{qp/(q-p)}^{s_0, 2s_0}(S)} \leq c_1(M) \|u_{x_j}\|_{W_q^{(1+1/q+\varepsilon)/2, 1+1/q+\varepsilon}(Q)} \|u\|_{W_{qp/(q-p)}^{s_0+(q-p)/2q+\varepsilon/2, 2s_0+(q-p)/qp+\varepsilon}(Q)}, \quad (45)$$

where $\varepsilon > 0$ is a small parameter, we use the Hölder inequality, Corollary 1.3 in [25], and the embedding $B_{q,q}^{s+\varepsilon} \subset B_{q,p}^s$ valid for every $\varepsilon > 0$ and $p \in [1, \infty]$. To estimate the factors in (45), we use (44). Take $\tilde{s} = 1 + 1/q + \varepsilon$, $\tilde{s}_0 = 3\varepsilon \leq \alpha_1$ (see Theorem 5), $\tilde{s}_1 = 2 - \varepsilon$, $\theta = 1/2$. We can conclude that

$$\|u_{x_j}\|_{W_q^{1/2+1/2q+\varepsilon/2, 1+1/q+\varepsilon}(G)} \leq c \|u\|_{W_p^{1-\varepsilon/2, 2-\varepsilon}(Q)}^{1/2} \|u\|_{B_{\infty,\infty}^{s_0/2, s_0}(Q)}^{1/2}, \quad (46)$$

where $q = \frac{2p(n+1)}{(n+2)}$, and, thereby, $1 + \frac{1}{q} + \varepsilon - \frac{n+2}{q} = \frac{\tilde{s}_0}{2} + \frac{1}{2}(2 - \varepsilon - (n+2)/p)$. The parameters $q, \tilde{s}, \tilde{s}_0$ are chosen so that we can apply (44) to obtain (46). Now take $\tilde{s} = 1 - 1/2q = s_0 + (q-p)/pq$, $\tilde{s}_0 = 2\varepsilon$, $\tilde{s}_1 = 2 - \varepsilon$. For this choice of q , it follows from (44) that

$$\|u\|_{W_{qp/(q-p)}^{1/2-1/2q+\varepsilon/2, 1-1/q+\varepsilon}(Q)} \leq c \|u\|_{W_p^{1-\varepsilon/2, 2-\varepsilon}(Q)}^{1/2} \|u\|_{B_{\infty,\infty}^{s_0/2, s_0}(Q)}^{1/2}, \quad (47)$$

where $1 - 1/q - (n+2)(1/p - 1/q) = \tilde{s}_0/2 + (2 - \varepsilon - (n+2)/p)/2$. In this case the inequality (45) can be written in the form

$$J_1(I_2) \leq C(M) \|u\|_{W_p^{1-\varepsilon/2, 2-\varepsilon}(Q)}. \quad (48)$$

The estimate of the expression $J_2^p(\Delta_{h,x}v)$ is proven by analogy. $\square$

Remark 3. If $p > n + 2$ then it is possible to replace the conditions (15)–(17) with more natural conditions

$$a_{ijx_i} \in L_p(Q; C([-R, R])), \quad a_{iju} \in L_\infty(Q; C([-R, R])) \quad \forall R > 0, \quad i, j = 1, 2, \dots, n, \quad (49)$$

$$a_{ij}, \psi \in W_p^{s_0, 2s_0}(S; C([-R, R])), \quad \psi_u \in L_{q_4}(S; C([-R, R])), \quad q_4 > n + 1. \quad (50)$$

In this case we can take $q_3 = q_5 = p$.

Theorem 6. Assume that $u \in W_p^{1,2}(Q)$ ($p > (n + 2)/2$) is a solution to the problem (1), (2) or the problem (1), (3), and the conditions (12)–(14), (15)–(17) hold. Then there exists a constant $M_2 > 0$ depending on the constants M, M_1 in Theorems 3–5 such that

$$\|u\|_{W_p^{1,2}(Q)} \leq M_2. \quad (51)$$

Proof. We present the proof in the case of the problem (1), (3). The problem (1), (2) is simpler and the arguments are the same. Let $N \in \mathbb{N}$ and $h = T/N$. Construct a finite covering of Γ by balls $B_h(x_i)$ of radius h , the parameter h is chosen below. Let us build it up to a covering of G . Next, we can construct the covering of Q by domains of the form $Q_{kl} = ((k - 5/4)h, (k + 1/4)h) \times B_h(x_l)$ ($k = 1, 2, \dots, N$). Construct also the corresponding partition of unity on Q : $\varphi_{kl} \in C_0^\infty(Q_{kl})$. Without loss of generality we assume that $\sum_{k,l: x_l \in \Gamma} \varphi_{kl}(t, x) = 1$ for all $x \in \Gamma, t \in [0, T]$. Since the function u satisfies the estimate from Theorem 6 and the functions a_{ij} are continuous in all variables, for every $\varepsilon > 0$, there exists $h = T/N > 0$ such that $|a_{ij}(t, x, u(t, x)) - a_{ij}(kh, x_l, u(kh, x_l))| < \varepsilon$ for all $(t, x) \in Q_{kl}$ and all i, j , where the constant h depends on M_1 but it is independent of the function u itself. Consider the family of problems

$$Mv = v_t - \sum_{i,j=1}^n a_{ij}(kh, x_l, u_{kl})v_{x_jx_i} = f, \quad u_{kl} = u(kh, x_l), \quad (52)$$

$$v|_{t=0} = v_0, \quad v|_S = g(t, x), \quad (53)$$

or

$$v|_{t=0} = u_0, \quad \sum_{i,j=1}^n a_{ij}(kh, x_l, u_{kl})v_{x_j}v_i|_S = g(t, x), \quad (54)$$

where $u_{kl} \in [-M, M]$ is a collection of constants and M is the constant defined in Theorems 3 and 4. We suppose that

$$v_0 \in W_p^{2-2/p}(G), \quad g \in W_p^{k_0, 2k_0}, \quad f \in L_p(Q), \quad \Gamma \in C^2, \quad (55)$$

where $k_0 = s_1$ in the case of the Dirichlet boundary condition and $k_0 = s_0$ otherwise. Applying Theorem 1 we can say that there exist unique solutions to the problems (52), (53) and (52), (54) and the estimates

$$\|v\|_{W_p^{1,2}(Q)} \leq C_0(\|u_0\|_{W_p^{2-2/p}(G)} + \|f\|_{L_p(Q)} + \|g\|_{W_p^{k_0, 2k_0}(S)}) \quad (56)$$

hold, where, without loss of generality, we can assume that C_0 is independent of k, l . Multiply Equation (1) and the boundary conditions (3) by φ_{kl} . The function $w = u\varphi_{kl}$ is a solution to the problem

$$Mw = w_t - \sum_{i,j=1}^n a_{ij}(t, x, u)w_{x_j x_i} + \tilde{b}_{kl} = 0, \quad \tilde{b}_{kl} = -\varphi_{kl} \left(\sum_{i,j=1}^n (a_{ijx_i} u_{x_j} + a_{ijj} u_{x_j} u_{x_i}) - \right. \\ \left. b(x, t, u, \nabla u) \right) + \sum_{i,j=1}^n a_{ij} u_{x_j} \varphi_{klx_i} + \sum_{i,j=1}^n a_{ij} u_{x_i} \varphi_{klx_j} + \sum_{i,j=1}^n a_{ij} u \varphi_{klx_i x_j}, \quad (57)$$

$$\sum_{i,j=1}^n a_{ij}(t, x, u)w_{x_j} v_i - \sum_{i,j=1}^n a_{ij}(t, x, u)u \varphi_{klx_j} v_i + \varphi_{kl} \psi(t, x, u) = 0, \quad w(0, x) = \varphi_{kl} u_0. \quad (58)$$

Rewrite this problem in the form

$$Mw = w_t - \sum_{i,j=1}^n a_{ij}(kh, x_l, u(kh, x_l))w_{x_j x_i} = - \sum_{i,j=1}^n (a_{ij}(kh, x_l, u(kh, x_l)) - \\ a_{ij}(t, x, u(t, x)))w_{x_j x_i} - \tilde{b}_{kl} = \tilde{f}_{kl}, \quad w(0, x) = \varphi_{kl} u_0. \quad (59)$$

$$\sum_{i,j=1}^n a_{ij}(kh, x_l, u(kh, x_l))w_{x_j} v_i = \sum_{i,j=1}^n (a_{ij}(kh, x_l, u(kh, x_l)) - a_{ij}(t, x, u(t, x)))w_{x_j} v_i + \\ \sum_{i,j=1}^n a_{ij}(t, x, u(t, x))u \varphi_{klx_j} v_i - \varphi_{kl} \psi(t, x, u) = \tilde{g}_{kl}, \quad w(0, x) = \varphi_{kl} u_0. \quad (60)$$

The estimate (56) yields

$$\|w\|_{W_p^{1,2}(Q)} \leq C_0 (\|\varphi_{kl} u_0\|_{W_p^{2-2/p}(G)} + \|\tilde{f}_{kl}\|_{L_p(Q)} + \|\tilde{g}\|_{W_p^{s_0, 2s_0}(S)}). \quad (61)$$

Estimate every summand on the right-hand side. We have that

$$\left\| \sum_{i,j=1}^n (a_{ij}(kh, x_l, u(kh, x_l)) - a_{ij}(t, x, u(t, x)))w_{x_j x_i} \right\|_{L_p(Q)} \leq c_0(p) \varepsilon \|w\|_{W_p^{1,2}(Q)}. \quad (62)$$

Lemma 1.21 in [25] and the conditions on the data imply that

$$\left\| \varphi_{kl} \sum_{i,j=1}^n a_{ijx_i} u_{x_j} - \sum_{i,j=1}^n a_{ij} u_{x_j} \varphi_{klx_i} - \sum_{i,j=1}^n a_{ij} u_{x_i} \varphi_{klx_j} - \sum_{i,j=1}^n a_{ij} u \varphi_{klx_i x_j} \right\|_{L_p(Q)} \leq \\ c \|u\|_{W_p^{1-\sigma, 2-2\sigma}(Q)}, \quad (63)$$

where $\sigma \in (0, 1)$ is a parameter defined in embedding theorems. Next, we have

$$\left\| \varphi_{kl} \left(\sum_{i,j=1}^n a_{ijj} u_{x_j} u_{x_i} + b(x, t, u, \nabla u) \right) \right\|_{L_p(G)}^p \leq c \|\nabla u\|_{L_{2p}(G)}^2 + \|g_3\|_{L_p(G)}^p. \quad (64)$$

We can assume that $\alpha_1 \leq 2/(n+2)$; otherwise, we assign $\alpha_1 = 2/(n+2)$. Next, we use the inequality (Lemma 1.14 in [25] or Corollary 5.7.3 of Ch. 7 in [21])

$$\|\nabla u\|_{L_{2p}(G)} \leq c \|\nabla u\|_{C^{2\alpha_1}(\bar{G})}^{1/2} \|u\|_{W_p^{2-2\alpha_1}(G)}^{1/2} \leq c(M_1) \|u\|_{W_p^{2-2\alpha_1}(G)}^{1/2}.$$

Integrating (64) in t and using (63), we conclude that

$$\left\| \varphi_{kl} \left(\sum_{i,j=1}^n a_{ijj} u_{x_j} u_{x_i} + b(x, t, u, \nabla u) \right) \right\|_{L_p(Q)} \leq c_1 \|u\|_{L_p(0,T;W_p^{2-2\alpha_1}(Q))} + c_2. \quad (65)$$

The above estimates imply that

$$\|\tilde{f}_{kl}\|_{L_p(Q)} \leq \varepsilon c_0(p) \|w\|_{W_p^{1,2}(Q)} + c(M, M_1) \|u\|_{W_p^{1-\sigma_1, 2-2\sigma_1}(Q)}, \quad (66)$$

where $\sigma_1 = \min(\sigma, \alpha_1)$. Lemma 2 ensures the inequality

$$\begin{aligned} \left\| \sum_{i,j=1}^n a_{ij}(t, x, u(t, x)) u \varphi_{ijx_j} v_i - \varphi_{ij} \psi(t, x, u) \right\|_{W_p^{s_0, 2s_0}(S)} \leq \\ c_5(M) + c_6(M) \|u\|_{W_p^{s_1-\varepsilon_0, 2s_1-2\varepsilon_0}(S)} \leq c_5(M) + c_7 \|u\|_{W_p^{1-\varepsilon_0, 2-2\varepsilon_0}(Q)}, \end{aligned} \quad (67)$$

with $\varepsilon_0 > 0$ -a constant. Moreover, Lemma 2 guarantees the estimate

$$\begin{aligned} \left\| \sum_{i,j=1}^n (a_{ij}(kh, x_l, u(kh, x_l)) - a_{ij}(t, x, u(t, x))) w_{x_j} v_i \right\|_{W_p^{s_0, 2s_0}(S)} \leq \\ c_1 \varepsilon \|w\|_{W_p^{1,2}(Q)} + c_7(M) \|w\|_{W_p^{1-\sigma_2, 2-2\sigma_2}(Q)}, \end{aligned} \quad (68)$$

where $\sigma_2 > 0$ is a constant. Thus, the estimates (67) and (68) validate the inequality

$$\|\tilde{g}_{kl}\|_{W_p^{s_0, 2s_0}(S)} \leq c_0 \varepsilon \|w\|_{W_p^{1,2}(Q)} + c_8(M, M_1) \|w\|_{W_p^{1-\sigma_3, 2-2\sigma_3}(Q)} + c_9(M, M_1), \quad (69)$$

where $\sigma_3 > 0$ is a constant. In view of (61), (66), (69), we derive that

$$\|w\|_{W_p^{1,2}(Q)} \leq (c_0(p) + c_1) C_0 \varepsilon \|w\|_{W_p^{1,2}(Q)} + c_9(M, M_1) \|u\|_{W_p^{1-\sigma_4, 2-2\sigma_4}(Q)} + C_{10}(M, M_1), \quad (70)$$

where $\sigma_4 > 0$ is a constant. Choose $\varepsilon < 1/(2C_0(c_0(p) + c_1))$ and find the corresponding parameter $h > 0$. We obtain that

$$\|w\|_{W_p^{1,2}(Q)} \leq 2c_9(M, M_1) \|u\|_{W_p^{1-\sigma_4, 2-2\sigma_4}(Q)} + 2C_{10}(M, M_1). \quad (71)$$

Write out the estimate for a function u

$$\|u\|_{W_p^{1,2}(Q)} \leq \sum_{k,l} \|\varphi_{kl} u\|_{W_p^{1,2}(Q)} \leq c_{11} \|u\|_{W_p^{1-\sigma_4, 2-2\sigma_4}(Q)} + c_{12}(M, M_1), \quad (72)$$

where the constant C_{12} depends on M, M_1 and the corresponding norms of the data, $\sigma_4 \in (0, 1)$. Next, we employ the interpolation inequality (Theorem 1.21 in [25] or [21]).

$$\|u\|_{W_p^{1-\sigma_4, 2-2\sigma_4}(Q)} \leq c_{13} \|u\|_{W_p^{1,2}(Q)}^\theta \|u\|_{L_p(Q)}^{1-\theta} \leq \varepsilon \|u\|_{W_p^{1,2}(Q)} + c_{14}(\varepsilon), \quad \theta = 1 - \sigma_4.$$

where $\varepsilon > 0$ is an arbitrary constant. Using this inequality in (72) with $\varepsilon = 1/2C_{11}$, we establish the claim. $\square$

Describe the conditions ensuring uniqueness of solutions to our problems: if $p > n + 2$ then, for every $R > 0$ there exist non-negative functions $g_1 \in L_{q_6}(Q)$, $g_2 \in L_{q_7}(Q)$ ($q_6 \geq (n + 2)/2$, $q_7 \geq n + 2$) such that

$$\begin{aligned} |b(t, x, u_1, \vec{p}_1) - b(t, x, u_2, \vec{p}_2)| \leq (|u_1 - u_2| g_1(t, x) + |\vec{p}_1 - \vec{p}_2| g_2(t, x), \\ \forall |u_1| + |\vec{p}_1| + |u_2| + |\vec{p}_2| \leq R; \end{aligned} \quad (73)$$

if $p \in ((n+1)/2, n+2]$ then, for every $R > 0$, there are non-negative functions g_1, g_2 and constants $\alpha, \beta \in [1, 2]$ such that

$$|b(t, x, u_1, \vec{p}_1) - b(t, x, u_2, \vec{p}_2)| \leq (|u_1 - u_2|g_1(t, x)(1 + (|\vec{p}_1| + |\vec{p}_2|)^\alpha) + g_2(t, x)|\vec{p}_1 - \vec{p}_2|(1 + (|\vec{p}_1| + |\vec{p}_2|)^\beta), \quad \forall |u_1| + |u_2| \leq R, \quad \vec{p}_1, \vec{p}_2 \in \mathbb{R}^n, \quad (74)$$

where $g_1 \in L_{q_8}(Q)$, $q_8 \geq (n+2)p/((2+\alpha)p - \alpha(n+2))$, $g_2 \in L_{q_9}(Q)$, $q_9 \geq p(n+2)/(p(1+\beta) - \beta(n+2))$.

Expose the consistency conditions. In the case of the problem (1), (2) we require that

$$g(0, x) = u_0(x)|_\Gamma \quad (75)$$

and in the case of problem (1), (3) that

$$\sum_{i,j=1}^n a_{ij}(0, x, u_0(x))u_{0x_j}v_i + \psi(0, x, u_0)|_S = 0, \quad \text{if } p > 3. \quad (76)$$

Theorem 7. *If the conditions (12)–(15), (75) hold then there exists a solution $u \in W_p^{1,2}(Q)$ ($p > (n+2)/2$) to the problem (1), (2). Under the conditions (12)–(17), (76), $p > (n+2)/2$, $p \neq 3$, there exists a solution $u \in W_p^{1,2}(Q)$ to the problem (1), (3). If the conditions (73), (74) are satisfied then a solution to these problems is defined uniquely.*

Proof. To prove existence theorems, we employ the conventional scheme exposed in Sect. 5, 6 of Ch. 5 in [1]. Consider the collections of problems depending on a parameter $\gamma \in [0, 1]$:

$$Mu = u_t - \sum_{i,j=1}^n \partial_{x_i}((\gamma a_{ij}(t, x, u) + (1-\gamma)\delta_{ij})u_{x_j}) + \gamma b(x, t, u, \nabla u) = 0, \quad (77)$$

In the former case the initial-boundary conditions are written in the form

$$u|_{t=0} = u_0, \quad u|_S = g(t, x), \quad (78)$$

and in the latter in the form

$$u|_{t=0} = u_0, \quad \sum_{i,j=1}^n (\gamma a_{ij}(t, x, u) + (1-\gamma)\delta_{ij})u_{x_j}v_i + \gamma\psi(t, x, u) - (1-\gamma)\sum_{j=1}^n u_{0x_j}v_j|_S = 0. \quad (79)$$

For instance, consider the family (77), (79). As is easily seen, the conditions (13)–(14) are fulfilled for the functions $\tilde{a}_{ij} = \gamma a_{ij}(t, x, u) + (1-\gamma)\delta_{ij}$ with the same constants and functions g_i and the constant δ_0 in (13) is replaced with $\min(1, \delta_0)$. The new functions $\tilde{a}_{ij}$, $\tilde{\psi} = \gamma\psi(t, x, u) - (1-\gamma)\sum_{j=1}^n u_{0x_j}v_j$ also satisfy (15)–(17) and without loss of generality we can assume that the norms of all functions in these conditions are estimated by the constants independent of γ . Thus, we can assume that the estimate

$$\|u\|_{W_p^{1,2}(Q)} \leq C(M, M_1), \quad \gamma \in [0, 1], \quad (80)$$

holds and the constant $C(M, M_1)$ is independent of $\gamma \in [0, 1]$. Construct a function $\Psi \in W_p^{1,2}(Q)$ such that $\Psi|_{t=0} = u_0(x)$ and make the change of variables $u = v + \Psi$. In order to construct this function, we can extend u_0 to the whole $\mathbb{R}^n$ preserving the class

(see Sect. 4.2.2, 4.2.3 in [20]) and find a solution to the Cauchy problem $\Psi_t - \Delta \Psi = 0$, $\Psi|_{t=0} = u_0$ (see Theorem 5.7 in [27]). We arrive at the problem

$$M_0 v = v_t - \sum_{i,j=1}^n \partial_{x_i} (\tilde{a}_{ij}(t, x, v + \Psi) v_{x_j}) - \sum_{i,j=1}^n \partial_{x_i} (\tilde{a}_{ij}(t, x, v + \Psi) \Psi_{x_j}) + b(x, t, v + \Psi, \nabla v + \Psi) + \Psi_t = 0, \quad (81)$$

$$v|_{t=0} = 0, \quad \sum_{i,j=1}^n \tilde{a}_{ij}(t, x, v + \Psi) v_{x_j} + \sum_{i,j=1}^n \tilde{a}_{ij}(t, x, v + \Psi) \Psi_{x_j} + \psi(t, x, v + \Psi)|_S = 0. \quad (82)$$

Denote by $\Phi(\gamma, w)$ a solution to the problem

$$M_0 v = v_t - \sum_{i,j=1}^n \partial_{x_i} (\tilde{a}_{ij}(t, x, w + \Psi) v_{x_j}) = \sum_{i,j=1}^n \partial_{x_i} (\tilde{a}_{ij}(t, x, w + \Psi) \Psi_{x_j}) - b(x, t, w + \Psi, \nabla w + \Psi) - \Psi_t = f(w), \quad v|_{t=0} = 0, \quad (83)$$

$$\sum_{i,j=1}^n \tilde{a}_{ij}(t, x, w + \Psi) v_{x_j} = - \sum_{i,j=1}^n \tilde{a}_{ij}(t, x, w + \Psi) \Psi_{x_j} - \psi(t, x, w + \Psi)|_S = g(w), \quad (84)$$

where $w \in H_1 = \{w \in W_p^{1-\varepsilon_0, 2-2\varepsilon_0}(Q) : w(0, x) = 0\}$ and ε_0 is a positive number less than the minimum of the constants ε in (39), (40) and the number $1/2 - (n+2)/4p$. Under this condition $W_p^{1-\varepsilon_0, 2-2\varepsilon_0}(Q) \subset W_{2p}^{1/2, 1}$ (see Corollary 5.6.4 of Ch. 7 in [21]) or Theorem 1.22 in [25]). Moreover, we can assume decreasing ε_0 if necessary that $W_p^{1-\varepsilon_0, 2-2\varepsilon_0}(Q) \subset B_{pq_4/(q_4-p)p}^{s_0+1/2p-1/2q+\varepsilon, 2s_0+1/p-1/q+2\varepsilon}(Q)$ (Corollary 5.6.4 of Ch. 7 in [21]) and thus we have the estimate

$$\|v\|_{B_{pq_4/(q_4-p)p}^{s_0+\varepsilon, 2s_0+2\varepsilon}(S)} \leq \|v\|_{B_{pq_4/(q_4-p)p}^{s_0+1/2p-1/2q+\varepsilon, 2s_0+1/p-1/q+2\varepsilon}(Q)} \leq c_1 \|v\|_{W_p^{1-\varepsilon_0, 2-2\varepsilon_0}(Q)} \quad \forall v \in H_1. \quad (85)$$

Endow the space H_1 with the norm coinciding with the norm in $W_p^{1-\varepsilon_0, 2-2\varepsilon_0}(Q)$. If $w \in H_1$ then the right-hand sides in (83), (84) belong to $L_p(Q)$, $W_p^{s_0, 2s_0}(S)$, respectively, (see Lemma 2 and the proof of Theorem 5). Note that

$$W_{2p}^{1/2, 1}(Q) \subset C^{1/2-(n+2)/4p, 1-(n+2)/2p}(\overline{Q}),$$

$$u \in W_{2p}^{1/2, 1}(Q) \Rightarrow u|_S \in W_{2p}^{1/2-1/4p, 1-1/2p}(S) \subset B_{q_3p}^{s_0}(0, T; L_{q_3}(\Gamma)) \cap L_{q_3}(0, T; B_{q_3p}^{2s_0}(\Gamma)). \quad (86)$$

The embeddings follow from Theorems 2.2.2 of Ch. 7 of [21] and Lemma 3.5 in [24]. By Theorem 1, there exists a unique solution to the problem (83), (84) of the class $W_p^{1, 2}(Q)$. The following inequality is valid:

$$\tilde{\delta}_0 |\tilde{\zeta}|^2 \leq \sum_{i,j=1}^n \tilde{a}_{ij} \tilde{\zeta}_i \tilde{\zeta}_j \leq \frac{1}{\tilde{\delta}_0} |\tilde{\zeta}|^2,$$

where $\tilde{\delta}_0 = \min(\delta_0, \delta_1)$, $1/\delta_1 = \max(1/\delta_0, \sup_{|w| \leq R_0} \sup_{|\tilde{\zeta}|=1} \sum_{i,j=1}^n \tilde{a}_{ij}(t, x, w) \tilde{\zeta}_i \tilde{\zeta}_j)$, $R_0 = \|w\|_{C(\overline{Q})}$. In this case, we infer

$$\|v\|_{W_p^{1, 2}(Q)} \leq C_0 (\|u_0\|_{W_p^{2-2/p}(G)} + \|f(w)\|_{L_p(Q)} + \|g(w)\|_{W_p^{s_0, 2s_0}(S)}), \quad (87)$$

where the constant C_0 depends on $\tilde{\delta}_0$ and the norms of $\tilde{a}_{ij}$. Since the embedding $W_p^{1, 2}(Q) \subset W_{2p}^{1/2, 1}(Q)$ [21] (Theorem 7.5.2, Ch. 7) is compact, the estimate (87) implies that

the mapping $w \rightarrow \Phi(\gamma, w)$ is compact as well. Demonstrate that it is continuous. Let a sequence $w_n \in H_1$ converges to $w \in H_1$ in the norm of the space H_1 . Consider the sequence $b_n = b(t, x, w_n + \Psi, \nabla(w_n + \Psi))$. In view of condition (14), we can conclude that

$$\|b_n\|_{L_p(Q)} \leq c_1 + c_2(\|w_n\|_{L_{2p}(Q)}^2 + \|\nabla w_n\|_{L_{2p}(Q)}^2) \leq (c_3 + c_4 R_1), \quad R_1 = \max_n \|w_n\|_{H_1}, \quad (88)$$

Assign $R_2 = \max_n \|w_n + \Psi\|_{C^{1/2-(n+2)/4p, 1-(n+2)/2p}(\overline{Q})}$. In view of (86), $R_2 < \infty$. By Lemma 3.1 of Ch. 2 in [28], there exists a subsequence w_{n_k} such that $w_{n_k} \rightarrow w$, $\nabla w_{n_k} \rightarrow \nabla w$ a.e. in Q . By Theorem 4.9 in [29], choosing one more subsequence if necessary, we can say that there exists a function $\psi \in L_{2p}(Q)$ such that $|w_{n_k}| + |\nabla w_{n_k}| \leq \psi(t, x)$ a.e. In this case $b_{n_k} - b \rightarrow 0$ a.e. and $|b_{n_k} - b|^p \leq 2g_3^p + C_2|\psi|^{2p}$ a. e. The Lebesgue dominated convergence theorem implies that $\|b_{n_k} - b\|_{L_p(Q)} \rightarrow 0$ as $k \rightarrow \infty$. Consider the second function $l(w_n) = \sum_{i,j=1}^n \partial_{x_i}(\tilde{a}_{ij}(t, x, w_n + \Psi)\Psi_{x_j})$ occurring into $f(w_n)$. We have that

$$\begin{aligned} l(w_n) &= \sum_{i,j=1}^n (\tilde{a}_{ijx_i}(t, x, w_n + \Psi)\Psi_{x_j} + \tilde{a}_{iju}(t, x, w_n + \Psi)(w_n + \Psi)_{x_i}\Psi_{x_j} \\ &\quad + \tilde{a}_{ij}(t, x, w_n + \Psi)\Psi_{x_i x_j}) = I_1(w_n) + I_2(w_n) + I_3(w_n). \end{aligned}$$

Since the functions $\tilde{a}_{ij}$ are continuous and $\|w_n - w\|_{C^{1/2-(n+2)/4p, 1-(n+2)/2p}(\overline{Q})} \rightarrow 0$ (see (86)), $\|I_3(w_n) - I_3(w)\|_{L_p(Q)} \rightarrow 0$ as $n \rightarrow \infty$. Moreover, we have that

$$\begin{aligned} I_2(w_n) - I_2(w) &= \sum_{i,j=1}^n \left(\frac{1}{2}(\tilde{a}_{iju}(t, x, w_n + \Psi) + \tilde{a}_{iju}(t, x, w + \Psi))(w_n - w)_{x_i}\Psi_{x_j} + \right. \\ &\quad \left. \frac{1}{2}(\tilde{a}_{iju}(t, x, w_n + \Psi) - \tilde{a}_{iju}(t, x, w + \Psi))(w_n + w)_{x_i}\Psi_{x_j} \right) \end{aligned}$$

This representation ensures the estimate

$$\begin{aligned} \|I_2(w_n) - I_2(w)\|_{L_p(Q)} &\leq c\|\nabla(w_n - w)\|_{L_{2p}(Q)}\|\nabla\Psi\|_{L_{2p}(Q)} + \\ &\sum_{i,j=1}^n \|(\tilde{a}_{iju}(t, x, w_n + \Psi) - \tilde{a}_{iju}(t, x, w + \Psi))\Psi_{x_j}\|_{L_{2p}(Q)}\|\nabla(w_n + w)\|_{L_{2p}(Q)} \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

where the last summand tends to 0 by the Lebesgue dominated convergence theorem. At last, we derive that

$$\begin{aligned} \|I_1(w_{n_k}) - I_1(w)\|_{L_p(Q)} &\leq c \sum_{i,j=1}^n \|(\tilde{a}_{ijx_i}(t, x, w_{n_k} + \Psi) - \tilde{a}_{ijx_i}(t, x, w + \Psi))\Psi_{x_j}\|_{L_p(Q)} \leq \\ &C \sum_{i,j=1}^n \|\tilde{a}_{ijx_i}(t, x, w_{n_k} + \Psi) - \tilde{a}_{ijx_i}(t, x, w + \Psi)\|_{L_{q_3}(Q)} \rightarrow 0 \text{ as } k \rightarrow \infty \end{aligned}$$

also in view of the Lebesgue dominated convergence theorem. Indeed, $\tilde{a}_{ijx_i}(t, x, w_{n_k} + \Psi) - \tilde{a}_{ijx_i}(t, x, w + \Psi) \rightarrow 0$ a.e. in Q . On the other hand, $|\tilde{a}_{ijx_i}(t, x, w_{n_k} + \Psi) - \tilde{a}_{ijx_i}(t, x, w + \Psi)| \leq 2\|\tilde{a}_{ijx_i}(t, x, w)\|_{C([-R_2, R_2])}$ and the last function is integrable in view of (49). Finally we can say that $\|f(w_{n_k}) - f(w)\|_{L_p(Q)} \rightarrow 0$ as $k \rightarrow \infty$. Now, we study the question of convergence of the function $g(w_n) = -\sum_{i,j=1}^n \tilde{a}_{ij}(t, x, w_n + \Psi)\Psi_{x_j} - \psi(t, x, w_n + \Psi)|_S = J_1(w_n) + J_2(w_n)$.

Estimate $\|J_2(w_{n_k}) - J_2(w)\|_{W_p^{s_0, 2s_0}(S)}$.

It suffices to establish the necessary estimate in the simplest case $S = (0, \infty) \times \Gamma$, $\Gamma = \mathbb{R}^{n-1}$, $Q = (0, \infty) \times \mathbb{R}_+^n$, $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$ assuming that all functions are compactly supported. The norm in this case is defined by the equality (41). We have that

$\Delta_{\tau,t}(J_2(w_{n_k}) - J_2(w)) = [\psi(t + \tau, x, w_{n_k}(t + \tau, x) + \Psi(t + \tau, x)) - \psi(t, x, w_{n_k}(t + \tau, x) + \Psi(t + \tau, x)) - \psi(t + \tau, x, w(t + \tau, x) + \Psi(t + \tau, x)) + \psi(t, x, w(t + \tau, x) + \Psi(t + \tau, x))] + [\psi(t, x, w_{n_k}(t + \tau, x) + \Psi(t + \tau, x)) - \psi(t, x, w_{n_k}(t, x) + \Psi(t, x)) - \psi(t, x, w(t + \tau, x) + \Psi(t + \tau, x)) + \psi(t, x, w(t, x) + \Psi(t, x))] = I_{1k} + I_{2k}$. The expression $\Delta_{h,x}(J_2(w_{n_k}) - J_2(w))$ can be represented similarly. The triangle inequality yields

$$\int_0^\delta \frac{1}{|\tau|^{1+s_0p}} \|\Delta_{\tau,t}(J_2(w_{n_k}) - J_2(w))\|_{L_p(S)}^p d\tau \leq c(p) \left(\int_0^\delta \frac{1}{|\tau|^{1+s_0p}} \|I_{1k}(\tau, t, x)\|_{L_p(S)}^p d\tau + \int_0^\delta \frac{1}{|\tau|^{1+s_0p}} \|I_{2k}(\tau, t, x)\|_{L_p(S)}^p d\tau \right) = c(p)(J_{1k} + J_{2k}).$$

We have that $I_{1k} \rightarrow 0$ as $k \rightarrow \infty$ a.e. on $S \times (0, \delta)$ and $|I_1| \leq c \|\Delta_{\tau,t}\psi(t, x, w)\|_{C([-R_2, R_2])}$ and the right-hand side here to the power p with the weight $1/|\tau|^{1+s_0p}$ is integrable. The Lebesgue dominated converge theorem ensures that $J_{1k} \rightarrow 0$ as $k \rightarrow \infty$. Next, we derive that

$$\begin{aligned} I_{2k}(\tau, t, x) = & \int_0^1 \psi_u(t, x, w_{n_k}(t, x) + \Psi(t, x) + r(w_{n_k}(t + \tau, x) - w_{n_k}(t, x) + \Psi(t + \tau, x) - \\ & \Psi(t, x))) dr (w_{n_k}(t + \tau, x) - w_{n_k}(t, x) + \Psi(t + \tau, x) - \Psi(t, x)) - \int_0^1 \psi_u(t, x, w(t, x) + \Psi(t, x) + \\ & r(w(t + \tau, x) - w(t, x) + \Psi(t + \tau, x) - \Psi(t, x))) dr (w(t + \tau, x) - w(t, x) + \Psi(t + \tau, x) - \Psi(t, x)). \end{aligned}$$

This equality can be rewritten in the form

$$\begin{aligned} I_{2k}(\tau, t, x) = & \frac{1}{2} \int_0^1 \psi_u(t, x, w_{n_k}(t, x) + \Psi(t, x) + r(\Delta_{\tau,t}(w_{n_k}(t, x) + \Psi(t, x)))) + \\ & \psi_u(t, x, w(t, x) + \Psi(t, x) + r(\Delta_{\tau,t}(w(t, x) + \Psi(t, x)))) dr \Delta_{\tau,t}(w_{n_k}(t, x) - w(t, x)) + \\ & \frac{1}{2} \int_0^1 \psi_u(t, x, w_{n_k}(t, x) + \Psi(t, x) + r(\Delta_{\tau,t}(w_{n_k}(t, x) + \Psi(t, x)))) - \psi_u(t, x, w(t, x) + \Psi(t, x) + \\ & r(\Delta_{\tau,t}(w(t, x) + \Psi(t, x)))) dr \Delta_{\tau,t}(w_{n_k}(t, x) - w(t, x) + 2\Psi(t, x)) = \\ & \frac{1}{2} \int_0^1 \tilde{I}_{1k}(\tau, t, x, r) dr \Delta_{\tau,t}(w_{n_k}(t, x) - w(t, x)) + \frac{1}{2} \int_0^1 \tilde{I}_{2k}(\tau, t, x, r) dr \Delta_{\tau,t}(w_{n_k}(t, x) + \\ & w(t, x) + 2\Psi(t, x)) = \frac{1}{2} \int_0^1 (\tilde{I}_{1k}(\tau, t, x, r) + \tilde{I}_{2k}(\tau, t, x, r)) dr \Delta_{\tau,t}(w_{n_k}(t, x) - w(t, x)) + \\ & \frac{1}{2} \int_0^1 \tilde{I}_{2k}(\tau, t, x, r) dr \Delta_{\tau,t}(2w(t, x) + 2\Psi(t, x)). \end{aligned}$$

In view of (85),

$$\|w_{n_k} - w\|_{B_{pq_4/(q_4-p)p}^{s_0}(S)} \leq c \|w_{n_k} - w\|_{H_1} \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (89)$$

The Hölder inequality implies that

$$\begin{aligned} J_{2k} \leq & c \|\psi_u(t, x, w)\|_{L_{q_4}(S; C([-R_2, R_2]))} \|w_{n_k} - w\|_{B_{pq_4/(q_4-p)p}^{s_0}(S)} + \\ & c_1 \int_0^\delta \frac{1}{\tau^{1+s_0p}} \left(\int_0^1 \int_S |\tilde{I}_{2k}|^{q_4} dS dr \right)^{p/q_4} \|\Delta_{\tau,t}(w + \Psi)\|_{L_{pq_4/(q_4-p)p}(S)}^p d\tau. \end{aligned}$$

The former summand on the right-hand side tend to zero as $k \rightarrow \infty$ in view of (89). The Lebesgue dominated converge theorem ensures that $\int_0^1 \int_S |\tilde{I}_{2k}|^{q_4} dS dr \rightarrow 0$ as $k \rightarrow \infty$. Indeed, $|\tilde{I}_{2k}|^{q_4} \leq \|\psi_u(t, x, w)\|_{C([-R_2, R_2])}^{q_4}$ and the last function is integrable. Moreover, for all $r \in (0, 1)$, $\tau > 0$ and almost all (t, x) $I_{2k} \rightarrow 0$ as $k \rightarrow \infty$. Again, the Lebesgue dominated convergence theorem and the estimate

$$\left(\int_0^1 \int_S |\tilde{I}_{2k}|^{q_4} dS dr\right)^{p/q_4} \|\Delta_{\tau, t}(w + \Psi)\|_{L^{pq_4/(q_4-p)p}(S)}^p \leq c_2 \|\psi_u(t, x, w)\|_{L^{q_4}(Q; C([-R_2, R_2]))}^p \|\Delta_{\tau, t}(w + \Psi)\|_{L^{pq_4/(q_4-p)p}(S)}^p$$

ensures that the second integral tend to 0 as $k \rightarrow \infty$. Thus, $J_{2k} \rightarrow 0$ as $k \rightarrow \infty$. We have proven that $J_2(w_{n_k}) - J_2(w) \rightarrow 0$ as $k \rightarrow \infty$ in $W_p^{s_0}(0, T; L_p(\Gamma))$. Similar arguments prove that $J_2(w_{n_k}) - J_2(w) \rightarrow 0$ as $k \rightarrow \infty$ in $L_p(0, T; W_p^{2s_0}(\Gamma))$. Finally, $\|\psi(t, x, w_{n_k} + \Psi) - \psi(t, x, w + \Psi)\|_{W_p^{s_0, 2s_0}(S)} \rightarrow 0$ as $k \rightarrow \infty$. Similar arguments are employed in the proof of the convergence $\sum_{i,j=1}^n \tilde{a}_{ij}(t, x, w_{n_k} + \Psi) \Psi_{x_j} \rightarrow \sum_{i,j=1}^n \tilde{a}_{ij}(t, x, w + \Psi) \Psi_{x_j}$ in the space $W_p^{s_0, 2s_0}(S)$.

Denote by v_n, v the solutions to the problem (83), (84) relating to the w_n, w . In this case the difference $\varphi_k = v_{n_k} - v$ is a solution to the problem

$$M_0 \varphi_n = \varphi_{kt} - \frac{1}{2} \sum_{i,j=1}^n \partial_{x_i} ((\tilde{a}_{ij}(t, x, w_{n_k} + \Psi) + \tilde{a}_{ij}(t, x, w + \Psi)) \varphi_{kx_j}) = \frac{1}{2} \sum_{i,j=1}^n \partial_{x_i} ((\tilde{a}_{ij}(t, x, w_{n_k} + \Psi) - \tilde{a}_{ij}(t, x, w + \Psi)) (v_{nx_j} + v_{x_j})) + f(w_n) - f(w), \quad (90)$$

The right-hand side can be written as $\tilde{f}(w_{n_k}) - \tilde{f}(w)$.

$$\varphi_k|_{t=0} = 0, \quad \sum_{i,j=1}^n \frac{1}{2} (\tilde{a}_{ij}(t, x, w_{n_k} + \Psi) + \tilde{a}_{ij}(t, x, w + \Psi)) \varphi_{kx_j} = - \sum_{i,j=1}^n \frac{1}{2} (\tilde{a}_{ij}(t, x, w_{n_k} + \Psi) - \tilde{a}_{ij}(t, x, w + \Psi)) (v_{nx_j} + v_{x_j}) + g(w_n) - g(w) = \tilde{g}(w_n) - \tilde{g}(w). \quad (91)$$

The functions v_n meet the estimate (87) and, hence, the norms $\|v_n\|_{W_p^{1,2}(Q)}$ are bounded uniformly in n . In this case the difference φ_k also satisfies the estimate (87) with the right-hand side $c(\|\tilde{f}(w_{n_k}) - \tilde{f}(w)\|_{L_p(Q)} + \|\tilde{g}(w_{n_k}) - \tilde{g}(w)\|_{W_p^{s_0, 2s_0}(S)})$. As in the proofs of the convergence $f(w_{n_k}) \rightarrow f(w)$, $g(w_{n_k}) \rightarrow g(w)$, we can demonstrate that $\tilde{f}(w_{n_k}) \rightarrow \tilde{f}(w)$, $\tilde{g}(w_{n_k}) \rightarrow \tilde{g}(w)$. Thus, the convergence $w_n \rightarrow w$ in H_1 implies that there exists a subsequence w_{n_k} such that $\Phi(\gamma, w_{n_k}) \rightarrow \Phi(\gamma, w)$. In this case $\Phi(\gamma, w_n) \rightarrow \Phi(\gamma, w)$ as $n \rightarrow \infty$. We have proven that the mapping $w \rightarrow \Phi(\gamma, w)$ is continuous and compact, and its continuity in the parameter $\gamma \in [0, 1]$ is obvious.

As is easily seen, v is a solution to the problem (83), (84) if and only if v is a fixed point of the mapping $\Phi(\gamma, v)$, i.e.,

$$v = \Phi(\gamma, v). \quad (92)$$

For $\gamma = 0$, a solution to the problem (79), (80) exists and $v + \Psi = u$ is a solution to the problem $u_t - \Delta u = 0$, $u|_{t=0} = u_0$, $\frac{\partial u}{\partial \nu} = \frac{\partial u_0}{\partial \nu}$. Moreover, we have estimates uniform on the parameter γ . By Theorem 37.6 in [30] Equation (92) has a solution for all $\gamma \in [0, 1]$. Now we prove the uniqueness of solutions to our problem (1), (3). Let u_1, u_2 be two solutions of the problem. Subtracting Equation (1) for u_1 and u_2 , multiplying the inequality obtained by $v = u_1 - u_2$, and integrating over G , we derive that

$$\begin{aligned} \partial_t \int_G \frac{v^2}{2} + \frac{1}{2} \sum_{i,j=1}^n (a_{ij}(t, x, u_1) + a_{ij}(t, x, u_2)) v_{x_j} v_{x_i} dx = & - \int_{\Gamma} (\psi(t, x, u_1) - \psi(t, x, u_2)) v d\Gamma \\ & + \int_G \frac{1}{2} \sum_{i,j=1}^n (a_{ij}(t, x, u_1) - \tilde{a}_{ij}(t, x, u_2)) (u_{1x_j} + u_{2x_j}) v_{x_i} - (b(t, x, u_1) - b(t, x, u_2)) v dx \end{aligned} \quad (93)$$

Fix $t \in (0, T)$. Integrating from 0 to $\tau \leq t$ and using the conditions on the coefficients, we arrive at the inequality

$$\begin{aligned} \int_G \frac{v^2}{2}(\tau, x) dx + \int_0^\tau \int_G \delta_0 |\nabla v(\xi, x)|^2 dx d\xi \leq & \int_0^\tau \int_{\Gamma} |(\psi(\xi, x, u_1) - \psi(\xi, x, u_2)) v| d\Gamma d\xi \\ & + c_1 \int_0^\tau \int_G |\nabla(u_1 + u_2)| |v| |\nabla v| + |(b(\xi, x, u_1) - b(\xi, x, u_2)) v| dG d\xi \leq \\ & \int_0^t \int_{\Gamma} |(\psi(\xi, x, u_1) - \psi(\xi, x, u_2)) v| d\Gamma d\xi \\ & + c_1 \int_0^t \int_G |\nabla(u_1 + u_2)| |v| |\nabla v| + |(b(\xi, x, u_1, \nabla u_1) - b(\xi, x, u_2, \nabla u_2)) v| dG d\xi. \end{aligned} \quad (94)$$

Taking the maximum in τ on the left-hand side, we conclude that

$$\begin{aligned} \max_{\tau \in [0, t]} \int_G \frac{v^2(\tau, x)}{2} dx + \int_0^t \int_G \delta_0 |\nabla v|^2 dx d\xi \leq & \int_0^t \int_{\Gamma} |(\psi(\xi, x, u_1) - \psi(\xi, x, u_2)) v| d\Gamma d\xi \\ & + c_1 \int_0^t \int_G |\nabla(u_1 + u_2)| |v| |\nabla v| + |(b(\xi, x, u_1, \nabla u_1) - b(\xi, x, u_2, \nabla u_2)) v| dG d\xi. \end{aligned} \quad (95)$$

Estimate every summand on the right-hand side. Let $R = \|u_1\|_{C(S)} + \|u_2\|_{C(S)}$. We have

$$\begin{aligned} \int_0^t \int_{\Gamma} |(\psi(\xi, x, u_1) - \psi(\xi, x, u_2)) v| d\Gamma d\xi = & \int_0^t \int_{\Gamma} \left| \int_0^1 \psi_u(\xi, x, u_1 + s(u_2 - u_1)) ds \right| |v|^2 d\Gamma d\xi \\ \leq & \int_0^t \|\psi(\xi, x, u)\|_{L_{q_4}(\Gamma; C([-R, R]))} \|v\|_{L_{2q'_4}(\Gamma)}^2 d\xi. \end{aligned} \quad (96)$$

The embedding theorems [25] (Lemma 1.14) and the trace theorems [25] (Lemma 1.9) validate the estimate

$$\|v\|_{L_{2q'_4}(\Gamma)}^2 \leq c \|v\|_{W_2^s(\Gamma)}^2 \leq \|v\|_{W_2^{s+1/2}(G)}^2 \leq c_2 \|v\|_{W_2^1(G)}^{2(s+1/2)} \|v\|_{L_2(G)}^{2(s-1/2)}, \quad (97)$$

where $s = (n-1)/2q_4$. Note that $s+1/2 < 1$. In this case the inequality (24) yields $\|v\|_{L_{2q'_4}(\Gamma)}^2 \leq \varepsilon \|v\|_{W_2^1(G)}^2 + c(\varepsilon) \|v\|_{L_2(G)}^2$. This inequality and (96) imply that

$$\int_0^t \int_{\Gamma} |(\psi(\xi, x, u_1) - \psi(\xi, x, u_2)) v| d\Gamma d\xi \leq \varepsilon \int_0^t \|\nabla v\|_{L_2(G)}^2 d\xi + c(\varepsilon) t \|v\|_{L_\infty(0, t; L_2(G))}^2. \quad (98)$$

Estimate the second summand in (95). The Hölder inequality ensures that

$$\int_0^t \int_G |\nabla(u_1 + u_2)| |v| |\nabla v| dx d\xi \leq \int_0^t \|\nabla(u_1 + u_2)\|_{L_q(G)} \|\nabla v\|_{L_2(G)} \|v\|_{L_{2q/(q-2)}(G)} d\xi. \quad (99)$$

Choose $q = np/(n + 2 - p)$. In this case (see Theorem 4.10.2 of Ch. III in [15] or Theorem 1.23 in [25]) $\|\nabla(u_1 + u_2)\|_{L_q(G)} \leq c\|\nabla(u_1 + u_2)(\xi, x)\|_{W_p^{1-2/p}(G)} \leq c_1\|u_1 + u_2\|_{W_p^{1,2}(Q)}$. As before,

$$\|v\|_{L_{2q/(q-2)}(G)} \leq c\|v\|_{W_2^1(G)}^{2s}\|v\|_{L_2(G)}^{2(1-s)}, \quad s = n/q < 1.$$

In accord with (99), (24), we infer

$$\begin{aligned} \int_0^t \int_G |\nabla(u_1 + u_2)| |v| |\nabla v| dx &\leq c_2 \int_0^t \|\nabla v\|_{L_2(G)}^{1+s} \|v\|_{L_{2q/(q-2)}(G)}^{1-s} d\xi \leq \\ &\varepsilon \int_0^t \|\nabla v\|_{L_2(G)}^2 d\xi + c(\varepsilon)t \|v\|_{L_\infty(0,t;L_2(G))}^2, \end{aligned} \quad (100)$$

where $\varepsilon > 0$ is arbitrary. Now take $p > n + 2$. The condition (73) implies that

$$\begin{aligned} \int_0^t \int_G |(b(\xi, x, u_1, \nabla u_1) - b(\xi, x, u_2, \nabla u_2))v| dG d\xi &\leq \\ &\int_0^t \int_G g_1(\xi, x)|v|^2 + g_2(\xi, x)|\nabla v||v| dx d\xi. \end{aligned} \quad (101)$$

For the former summand we have the estimate

$$\begin{aligned} \int_0^t \int_G g_1|v|^2 d\xi &\leq \int_0^t \|g_1\|_{L_{q_6}(G)} \|v\|_{L_{2q'_6}(G)}^2 d\xi \leq \int_0^t \|g_1\|_{L_{q_6}(G)} \|v\|_{W_2^1(G)}^{2s} \|v\|_{L_2(G)}^{2(1-s)} d\xi \leq \\ &\varepsilon \int_0^t \|\nabla v\|_{L_2(G)}^2 d\xi + c(\varepsilon) \|v\|_{L_\infty(0,t;L_2(G))}^2 (c(\varepsilon) \int_0^t \|g_1\|_{L_{q_6}(G)}^{1/(1-s)} d\xi + \varepsilon t), \quad s = n/2q_6, \end{aligned} \quad (102)$$

where we employ the equality $\|v\|_{W_2^1(G)}^2 = \|\nabla v\|_{L_2(G)}^2 + \|v\|_{L_2(G)}^2$. Note that $1/(1-s) \leq q_6$. In view of the absolute continuity of the integral, $\int_0^t \|g_1\|_{L_{q_6}(G)}^{1/(1-s)} d\xi \rightarrow 0$ as $t \rightarrow 0$. The integral $\int_0^t \int_G g_2(\xi, x)|\nabla v||v| dx d\xi$ is estimated similarly. We have

$$\begin{aligned} \int_0^t \int_G g_2(\xi, x)|\nabla v||v| dx d\xi &\leq \int_0^t \|g_2\|_{L_{q_7}(G)} \|\nabla v\|_{L_2(G)} \|v\|_{L_{2q'_7/(2-q'_7)}(G)} d\xi \leq \\ &\int_0^t \|g_2\|_{L_{q_7}(G)} \|v\|_{W_2^1(G)}^{1+s} \|v\|_{L_2(G)}^{1-s} d\xi \leq \varepsilon \int_0^t \|\nabla v\|_{L_2(G)}^2 d\xi + \\ &\|v\|_{L_\infty(0,t;L_2(G))}^2 (c(\varepsilon) \int_0^t \|g_2\|_{L_{q_7}(G)}^{2/(1-s)} d\xi + \varepsilon t), \quad s = n/q_7. \end{aligned} \quad (103)$$

Note that $2/(1-s) \leq q_7$. In this case, the relations (98), (100), (101), (102), (103) and (95), ensure the estimate

$$\max_{\tau \in [0,t]} \int_G \frac{v^2(\tau, x)}{2} dx + \int_0^t \int_G \delta_0 |\nabla v|^2 dx d\xi \leq 4\varepsilon \int_0^t \|\nabla v\|_{L_2(G)}^2 d\xi + c(\varepsilon) \|v\|_{L_\infty(0,t;L_2(G))}^2 \varphi(t), \quad (104)$$

where $\varphi(t)$ is a continuous function such that $\varphi(0) = 0$. Choose ε so that $4\varepsilon < \delta_0/2$ and t_0 so that $c(\varepsilon)\varphi(t) < 1/4$ for $t \leq t_0$. In this case the inequality (104) implies that $v(t, x) = 0$ for $t \leq t_0$. Repeating the arguments for $t \geq t_0$, we obtain that $v = 0$ on some segment $[t_0, t_1]$, and so on. It is easy to see that the length of the segments $[t_{i-1}, t_i]$ does not tend to 0 since it depends on the properties of the functions g_i and constants arising in the inequalities

are really constants from the embedding theorems and interpolation inequalities. Proceed with the case of $p \in ((n+2)/2, n+2]$. In this case we have an estimate

$$\int_0^t \int_G |(b(\xi, x, u_1, \nabla u_1) - b(\xi, x, u_2, \nabla u_2))v| dG d\xi \leq \int_0^t \int_G g_1 v^2 (1 + (|\nabla u_1| + |\nabla u_2|)^\alpha) + g_2 |\nabla v| |v| (1 + (|\nabla u_1| + |\nabla u_2|)^\beta) dx d\xi. \quad (105)$$

The summands of the form $\int_0^t \int_G g_1 v^2 + g_2 |\nabla v| |v| dx d\xi$ have been already estimated. Estimate the expressions $\int_0^t \int_G g_1 v^2 (|\nabla u_1| + |\nabla u_2|)^\alpha + g_2 |\nabla v| |v| (|\nabla u_1| + |\nabla u_2|)^\beta dx d\xi$. As in the proof of (102), we have

$$I_1 = \int_0^t \int_G g_1 v^2 (|\nabla u_1| + |\nabla u_2|)^\alpha dx d\xi \leq \varepsilon \int_0^t \|\nabla v\|_{L_2(G)}^2 d\xi + \|v\|_{L_\infty(0,t;L_2(G))}^2 (c(\varepsilon) \int_0^t \|g_1 (|\nabla u_1| + |\nabla u_2|)^\alpha\|_{L_q(G)}^{1/(1-s)} d\xi + \varepsilon t), \quad s = n/2q, \quad (106)$$

where $q = q_8 np / (np + \alpha q_8 (n+2-p)) < q_8$ for $p < n+2$. For $p = n+2$, we take $nq_8/2(q_8-1) \geq q < q_8$. Let $p < n+2$. The Hölder inequality yields

$$\int_0^t \|g_1 (|\nabla u_1| + |\nabla u_2|)^\alpha\|_{L_q(G)}^{1/(1-s)} d\xi \leq \int_0^t \|g_1\|_{L_{q_8}(G)}^{1/(1-s)} \| |\nabla u_1| + |\nabla u_2| \|_{L_{\alpha q q_8/(q_8-q)}(G)}^{\alpha/(1-s)} d\xi \leq c \int_0^t \|g_1\|_{L_{q_8}(G)}^{1/(1-s)} d\xi, \quad (107)$$

where $\alpha q q_8 / (q_8 - q) = np / (n+2-p)$ and, thereby, $\max_t \| |\nabla u_1| + |\nabla u_2| \|_{L_{\alpha q q_8/(q_8-q)}(G)} \leq c(\|u_1\|_{W_p^{1,2}(Q)} + \|u_2\|_{W_p^{1,2}(Q)})$. Let $p = n+2$. Again, we have that

$$\int_0^t \|g_1 (|\nabla u_1| + |\nabla u_2|)^\alpha\|_{L_q(G)}^{1/(1-s)} d\xi \leq \int_0^t \|g_1\|_{L_{q_8}(G)}^{1/(1-s)} \| |\nabla u_1| + |\nabla u_2| \|_{L_{\alpha q q_8/(q_8-q)}(G)}^{\alpha/(1-s)} d\xi \leq c \int_0^t \|g_1\|_{L_{q_8}(G)}^{1/(1-s)} d\xi. \quad (108)$$

We have used the fact that, for $p = n+2$, $\nabla u \in C([0, T]; L_q(G))$ for every $q \geq 1$. The condition of the parameter q_8 in (74) ensures the inequality $1/(1-s) \leq q_8$. As before, we infer

$$\int_0^t \int_G g_2 |v| |\nabla v| (|\nabla u_1| + |\nabla u_2|)^\beta dx d\xi \leq \varepsilon \int_0^t \|\nabla v\|_{L_2(G)}^2 d\xi + \|v\|_{L_\infty(0,t;L_2(G))}^2 (c(\varepsilon) \int_0^t \|g_2 (|\nabla u_1| + |\nabla u_2|)^\beta\|_{L_q(G)}^{2/(1-s)} d\xi + \varepsilon t), \quad s = n/q, \quad (109)$$

where $q = q_9 np / (np + \beta q_9 (n+2-p))$ for $p < n+2$ and $q \in (nq_9/(q_9-1), q_9)$ for $p = n+2$. Let us consider the case of $p < n+2$. We can conclude that

$$\int_0^t \|g_2 (|\nabla u_1| + |\nabla u_2|)^\beta\|_{L_q(G)}^{2/(1-s)} d\xi \leq \int_0^t \|g_2\|_{L_{q_9}(G)}^{2/(1-s)} \| |\nabla u_1| + |\nabla u_2| \|_{L_{\beta q q_9/(q_9-q)}(G)}^{2\beta/(1-s)} d\xi \leq c \int_0^t \|g_2\|_{L_{q_9}(G)}^{2/(1-s)} d\xi, \quad (110)$$

where $\beta q q_9 / (q_9 - q) = np / (n+2-p)$ and, thus, the quantity

$$\max_t \| |\nabla u_1| + |\nabla u_2| \|_{L_{2\beta q q_9/(q_9-q)}(G)} \leq c(\|u_1\|_{W_p^{1,2}(Q)} + \|u_2\|_{W_p^{1,2}(Q)})$$

is finite. The same estimate holds for $p = n + 2$. We have used the embeddings $W_p^{1,2}(Q) \subset C([0, T]; W_p^{2-2/p}(G))$ and $W_p^{2-2/p}(G) \subset W_{2\beta q q_9/(q_9-q)}^1(G)$ (see Theorem 4.10.2 of Ch. III in [15] and Sect. 4.6.2 in [21]). The conditions on q_9 in (74) ensure the inequality $2/(1-s) \leq q_9$. The inequalities (106)–(110) imply that the inequality of the form (104) holds which ensures the uniqueness of solutions. The proofs in the case of the problem (1), (2) are much simpler and the sequence of arguments is the same. So we omit the proofs in this case. $\square$

Remark 4. The conditions (9), (10), (15)–(17) on the coefficients of the operator and the coefficients of the boundary operators are analogs to those arising in the linear case (in this case the boundary condition (3) has the form $Bu = \frac{\partial u}{\partial N} + \psi(t, x)u|_S = g(t, x)$) (see [22–26,31]). In the linear case they are optimal and ensure that the function Mu belongs to the class $L_p(Q)$ and Bu to the class $W_p^{s_0, 2s_0}(S)$ in the case of the boundary condition (3) or $W_p^{s_1, 2s_1}(S)$ otherwise, whenever $u \in W_p^{1,2}(Q)$. At the same time, it is proven in [22] (Theorem 3.4) that a solution to the linear problem from the class $u \in W_p^{1,2}(Q)$ exists if and only if $u_0 \in W_p^{2-2/p}(G)$, $g \in W_p^{s_0, 2s_0}(S)$ in the case of the problem (1), (3) and $u_0 \in W_p^{2-2/p}(G)$, $g \in W_p^{s_1, 2s_1}(S)$, otherwise. The conditions (9), (10), (15)–(17) can be slightly weakened and generalized if we employ anisotropic Sobolev–Besov or Lizorkin–Triebel spaces (see [22,24,31]). Note the exponent s_0, s_1 in (15)–(17) cannot be decreased in any case. We did not find analogs of the conditions (9), (10), (15)–(17) in the nonlinear case in the literature.

State the results on the local solvability of the problems (1), (2) and (1), (3). We present some new condition on the data. Let $m \leq u_0(x) \leq M_0$ in G and there exists a constant $m_0 > 0$ such that the functions $a_{ij}(t, x, u)$ are continuous in $\bar{Q} \times (-\infty, \infty)$ and

$$a_{ij}p_i p_j \geq \delta_0 |\vec{p}|^2 \quad \forall \vec{p} = (p_1, \dots, p_n) \in \mathbb{R}^n, (t, x) \in Q, -m_0 + m \leq u \leq M_0 + m_0, \quad (111)$$

where $\delta_0 = \text{const} > 0$. Let $R_0 = \|\nabla u_0\|_{C(\bar{G})}$. Denote $B_R = \{(u, \vec{p}) : |u| + |\vec{p}| \leq R\}$. The function $b(t, x, u, \vec{p})$ satisfies the Caratheodory condition and

$$b \in L_p(Q; C(B_R)) \quad \forall R > 0. \quad (112)$$

Theorem 8. Let the conditions (15), (111), (112) hold and $p > n + 2$. Then there exists $\tau_0 > 0$ such that there exists a solution $u \in W_p^{1,2}(Q_{\tau_0})$ to the problem (1), (2). If the conditions (15)–(17), (111), (112) hold and $p > n + 2$ then there exists $\tau_0 > 0$ such that there exists a solution $u \in W_p^{1,2}(Q_{\tau_0})$ to the problem (1), (3). If additionally the condition (73) holds then a solution is defined uniquely.

Proof. Denote $R = \|\nabla u_0\|_{C(\bar{G})}$. Construct functions φ, ψ such that $\varphi(u) = u$ for $u \in [-m_0 + m, M_0 + m_0]$, $\varphi(u) = M_0 + m_0$ for $u \geq M_0 + m_0$, $\varphi(u) = -m_0 + m$ for $u \leq -m_0 + m$, $\psi(\vec{p}) = \vec{p}$ for $|\vec{p}| \leq 2R$, $\psi(\vec{p}) = 0$ for $|\vec{p}| > 3R$, $\psi \in C^1(\mathbb{R}^n)$. Define the function $\tilde{b}(t, x, u, \vec{p}) = b(t, x, \varphi(u), \psi(\vec{p}))$. Consider the problem (1), (2) or the problem (1), (3), where the function $b(t, x, u, \vec{p})$ is replaced with $\tilde{b}(t, x, u, \vec{p})$. As is easily seen, the conditions of Theorem 7 are fulfilled. In this case there exists a solution $u \in W_p^{1,2}(Q)$ to the problem (1), (2) or, respectively, to the problem (1), (3). Since $W_p^{1,2}(Q) \subset C^{1-(n+2)/2p, 2-(n+2)/p}(\bar{Q})$, there exists $\tau_0 > 0$ such that $u(t, x) \in [-m_0 + m, M_0 + m_0]$, $\nabla u(t, x) \in B_{2R}$ in Q_{τ_0} . Then $\varphi(u) = u$, $\psi(\nabla u) = \nabla u$ in Q_{τ_0} . Hence, the function u is a solution to the problem (1), (2) or, respectively, (1), (3) in Q_{τ_0} . $\square$

Remark 5. The results can be easily generalized to equations in non-divergence form and the following more general parabolic problems:

$$Mu = u_t - \sum_{i=1}^n \partial_{x_i} A_i(t, x, u, \nabla u) + b(x, t, u, \nabla u) = 0, \quad (113)$$

either

$$u|_{t=0} = u_0, \quad u|_S = g(t, x), \quad (114)$$

or

$$u|_{t=0} = u_0, \quad \frac{\partial u}{\partial N} + \psi(t, x, u)|_S = 0, \quad \frac{\partial u}{\partial N} = \sum_{i=1}^n A_i v_i. \quad (115)$$

Moreover, it is not so difficult to add the coefficient $c(t, x, u)$ in front of the derivative u_t in the equation. To some extent, the results can be generalized also to degenerate parabolic equations, i.e., to equations with non-uniform ellipticity.

4. Discussion

We examine quasilinear parabolic problems under sharp conditions on the data ensuring the existence and uniqueness of solutions in Sobolev classes. The proof relies on estimates resulting from the maximum principle and the Hölder estimates of the solution. In contrast to the results of [1,2], to prove maximum estimates for a solution, we use a different approach, which allows us to easily prove maximum estimates in the case of the boundary condition (3) for generalized solutions which are absent in [1,2]. Next, to obtain estimates for higher-order derivatives in L_p , we employ the method of frozen coefficients. The main tool is the Hölder estimates of a solution and the corresponding results for a linear problem. Next, the problem is reduced to a problem of finding a fixed point of some compact continuous operator whose solvability is proven by the method of continuation in a parameter, beginning with a linear model problem. The corresponding Krasnosel'sky theorem is applied. Despite the presence of a priori estimate, the proof is rather complicated and the difficulties are connected with the proof of continuity of the corresponding compact operator. The approach can be easily applied to some more general parabolic problems (see Remark 5). The local existence theorem is also presented. The conditions on the coefficient of the operator and the functions occurring in the boundary conditions (3) are minimal and agree with those in the linear case. They are stated in terms of the Sobolev–Besov spaces in contrast to many other articles. The smoothness exponents s_0, s_1 (see (15)–(17)) cannot be decreased. The results can be used in: establishing solvability results for a large class of inverse parabolic problems; establishing the existence and uniqueness of solutions under perturbations; and providing constructive approximation methods via iterative techniques.

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