

## Article

# Mathematical Modeling, Sensitivity Analysis, and Comparative Evaluation of Height Systems in Engineering, Geodetic, and Cartographic Applications: A Romania-Oriented Computational Study

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## Abstract

Height systems form a mathematical interface between physical geodesy, engineering surveying, and digital cartography. Although satellite positioning efficiently provides ellipsoidal heights, practical infrastructure, mapping, hydrological, and monitoring tasks require gravity-related heights that are compatible with national vertical datums. This paper develops a denominator-based framework in which dynamic, orthometric, and normal heights are interpreted as metric realizations of a common geopotential number. Starting from the line-integral definition of geopotential, the principal height formulae are derived; first-order sensitivities to geoid undulation, height anomaly, and mean gravity are established; and uncertainty propagation is analyzed. A Romania-oriented computational experiment, explicitly defined as a representative model-behavior study rather than an official national adjustment, uses lowland, plateau, and mountain-influenced settings consistent with the Constanta and Black Sea 1975 normal-height context. The results show that modeled normal-orthometric separations remain below 3 mm in representative low-relief locations but increase to 17.1 mm in Suceava, 29.5 mm in Cluj-Napoca, and 85.6 mm in the mountain-influenced Brasov case. The dynamic-normal differences remain small at low elevations but become systematic where the normal-gravity denominator departs from the selected reference value. The Monte Carlo experiment indicates standard uncertainties of approximately 4.2–4.4 cm for normal heights when a 1.5 cm ellipsoidal-height uncertainty and a 4.0 cm quasi-geoid uncertainty are assumed. A single-point covariance example gives 31.7 mm for normal-height conversion and 36.9 mm for orthometric-height conversion under the stated correlation assumptions. The transformation-surface comparison further shows that a quadratic local model reduces leave-one-out cross-validation error from 16.73 mm to 9.95 mm relative to a planar model in the synthetic Romania-oriented scenario. The study concludes that the height-system label must be treated as part of the mathematical model and metadata, and it proposes a geopotential-centered computational pathway for survey adjustment, uncertainty control, and metadata-safe geospatial export.

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## 1. Introduction

In the mathematical treatment of geodetic heights, the central issue is not the existence of a vertical coordinate per se, but the physical meaning attached to that coordinate. A height suitable for infrastructure design, water management, deformation analysis, and map production must be related to the gravity field, because gravity defines equipotential surfaces, verticality, and the direction of flow. For this reason, the apparently simple quantity called “height” is in fact a family of non-equivalent mathematical constructs whose practical behavior depends on the way geopotential, gravity, and reference surfaces are introduced [1–6].

The recent international geodetic debate has sharpened this point rather than simplifying it. Contemporary realizations of the International Height Reference System, regional geoid and quasi-geoid repositories, and the operational fusion of GNSS with leveling and gravity observations all emphasize that a height system is a mathematical model of a physical reality, not a single universally valid number [7–12]. In countries where official surveying practice combines a national vertical datum with GNSS-based acquisition, the distinction between ellipsoidal height, orthometric height, and normal height becomes an everyday engineering problem rather than a purely theoretical one [13–18].

Additional foundations used in the present formulation include the comparison of physical-height realizations [19], the GRS80 definition [20], Molodensky theory [21], contemporary geodetic reference-system practice [22], accurate orthometric-height determination [23], Earth-shape theory [24], Romanian vertical-deflection studies [25], the Romanian realization of EVRF2007 [26], national height-conversion algorithms [27], European vertical-reference overviews [28], robust statistics [29], linear-model estimation and testing [30], adjustment theory [31], advanced physical geodesy [32], ellipsoidal-to-orthometric transformation theory [33], gravity-field covariance modeling [34], and sign-constrained robust least squares [35].

The present study builds on several complementary strands of recent and classical literature. Featherstone and Kuhn [6] provide the modern review basis for vertical datums and height-system terminology; Sanchez et al. [7] place geopotential numbers at the center of International Height Reference System realization; Novak [8] clarifies the rigorous interpretation of normal heights; and Tenzer and co-authors [10,23,36] emphasize that orthometric heights, geoid-to-quasigeoid separation, and topographic mass modeling cannot be separated without loss of physical rigor. Recent regional and applied studies further show why this theory is operationally current: Bankovic et al. [18] quantify the impact of gravity on height-system differences in mountain terrain, Vergos et al. [37] demonstrate centimeter-level regional geoid refinement through stochastic fitting to Global Navigation Satellite System and leveling benchmarks, Strugarek et al. [38] analyze the height-error budget in mountainous Global Navigation Satellite System applications, Jiang et al. [39] demonstrate physics-informed neural-network geoid modeling, Blitzkow et al. [40] discuss regional International Height Reference Frame calculation, and Guimaraes et al. [41] connect regional gravity modeling to International Height Reference Frame implementation. These contributions motivate the present article’s combination of denominator-based

theory, uncertainty propagation, Romania-oriented modeling, and computational workflow design.

Romania offers a particularly instructive setting for such a discussion. The operational geodetic environment combines ETRS89 for planimetry and the RO Const/NH framework for gravity-related heights, while the TransDatRO software environment supports the practical conversion of ellipsoidal heights into Black Sea 1975 normal heights through national or local models [13,15,16]. For mathematical geodesy, this institutional context motivates a careful comparison of the height systems that appear in mixed surveying, cadastral, remote-sensing, and civil-engineering applications.

Romania was selected not merely as a geographic example but as a region where the theoretical issues are simultaneously visible at institutional, geomorphological, and technological levels. The country combines lowland sectors along the Danube and Black Sea, plateau areas, and Carpathian mountain corridors, so the same computational framework can be examined under both low-relief and high-relief conditions. At the same time, practical surveying combines European terrestrial reference-frame positioning, the national normal-height environment, transformation software, and increasingly dense satellite, unmanned-aircraft, laser-scanning, and digital-terrain workflows. This combination makes Romania a suitable laboratory for testing how denominator choice, quasi-geoid quality, datum metadata, and engineering accuracy requirements interact in a modern geospatial infrastructure.

The scientific objective of this article is therefore twofold. First, we provide a mathematically explicit synthesis of the principal height systems used in geodesy—geodetic, dynamic, orthometric, and normal heights—with emphasis on derivation, interpretation, and computational consequences. Second, we build a Romania-oriented comparative experiment that is honest about data limitations yet rich enough to support journal-level conclusions. The experiment is not presented as a substitute for a national adjustment or an official vertical transformation model; instead, it is designed as a rigorous computational study using representative Romanian locations and realistic parameter ranges.

The novelty of the paper lies in the combination of five elements that are often scattered across the literature: (i) a unified notation linking geopotential numbers to all four major height classes; (ii) an explicit conditioning analysis for conversion formulae; (iii) a comparative interpretation tailored to engineering, geodetic, and cartographic tasks; (iv) a Romania-centered computational scenario rooted in the official datum context; and (v) visual and tabular summaries structured in the style expected by a mathematics-focused journal.

### *Main Contributions of This Study*

A denominator-based mathematical framework is proposed in which dynamic, orthometric, and normal heights are treated as alternative metric realizations of the same geopotential number, making the source of their divergence explicit.

First-order sensitivity and uncertainty relations are summarized in a compact form suitable for mixed GNSS-leveling-gravity workflows, with emphasis on the roles of geoid undulation, height anomaly, and mean gravity.

A Romania-oriented computational experiment is developed using representative inputs consistent with the national geodetic context, allowing meaningful comparison without overstating the availability of official benchmark data.

Comparative tables, suitability matrices, and figure-based diagnostics are organized to support journal-level interpretation for engineering, geodetic, and cartographic use cases.

The manuscript is written as a mathematically explicit synthesis intended for a mathematics-oriented readership, bridging physical geodesy, numerical modeling, and applied spatial engineering.

A software-integration perspective is introduced through a modular WPF-oriented architecture that connects geopotential computation, mixed-observation adjustment, Monte Carlo uncertainty analysis, CAD/GIS export, and Hansen–Helmert–Wolf free-net-work processing.

## 2. Mathematical Preliminaries

The gravitational potential  $W$  defines the equipotential structure of the Earth system. Let  $W_0$  denote the chosen reference potential associated with the adopted vertical datum and let  $W(P)$  denote the actual potential at point  $P$ . The geopotential number  $C(P)$  is then defined as the potential difference between the reference surface and the physical point. It is this scalar quantity, not height itself, that is invariant with respect to the particular metric denominator used later [2,4,7,9].

The fundamental theoretical advantage of geopotential numbers is that they are directly connected with spirit leveling and gravity observations. Once  $C(P)$  has been determined, a metric height is obtained by dividing  $C(P)$  by a suitably defined gravity quantity. Different choices of denominator lead to different height systems. The mathematics is therefore conceptually simple but physically subtle: the numerator is shared, whereas the denominator encodes the chosen model of gravity and reference surface [3–5,8].

In a local differential form, the geopotential difference between two nearby points can be written as  $dW = -g \, dn$ , where  $g$  is actual gravity and  $dn$  is the differential along the plumb line. Integration between the reference equipotential surface and the point of interest yields the geopotential number. Because the line integral is evaluated in a non-uniform field, any attempt to replace the integral by a simple Euclidean height without gravity weighting introduces a model-dependent approximation [1–4].

$$C(P) = W_0 - W(P) = \int_{P_0}^P g \, dn \quad (1)$$

$$H^D = C/\gamma^0 \quad (2)$$

$$H^O = C/\bar{g} \quad (3)$$

$$H^N = C/\bar{\gamma} \quad (4)$$

$$h = H^O + N = H^N + \zeta \quad (5)$$

$$N - \zeta = H^N - H^O \quad (6)$$

Equation (2) defines the dynamic height  $H^D$ , where  $\gamma^0$  is a constant normal gravity value, usually selected at a reference latitude. Equation (3) defines the orthometric height  $H^O$  through the mean actual gravity  $\bar{g}$  along the plumb line inside the topographic masses. Equation (4) defines the normal height  $H^N$  through the mean normal gravity  $\bar{\gamma}$  along the corresponding normal plumb line. Equation (5) links the GNSS-derived ellipsoidal height  $h$  with the geoid undulation  $N$  and the height anomaly  $\zeta$ . Equation (6) expresses the geoid-to-quasigeoid separation, which is numerically equal to the difference between normal and orthometric heights under a consistent sign convention [3,5,10,11].

The denominator-specific nature of height systems is precisely why the same point can receive different numerical heights while remaining associated with a single geopotential number. This observation is often underestimated in engineering practice, especially when GNSS software outputs ellipsoidal heights and a transformation routine provides normal or orthometric heights without explaining the denominator model. From a

mathematical viewpoint, the problem is equivalent to asking which weighted mean of gravity should transform potential difference into metric distance [3,4,8,10]. The software environment explicitly discussed in this manuscript is TransDatRO v4.08 (National Center of Cartography, Bucharest, Romania), available through the ROMPOS portal [15] (accessed on 16 March 2026).

### 3. Formal Comparison of Height Systems

Geodetic or ellipsoidal height  $h$  is purely geometric. It is measured along the ellipsoidal normal from the reference ellipsoid to the point. Its major operational strengths are efficient GNSS acquisition, three-dimensional compatibility with terrestrial reference frames, and clean integration with Cartesian coordinate systems. Its weakness is physical inadequacy:  $h$  alone does not determine the direction of water flow and is therefore not sufficient for most engineering or hydraulic tasks [7,9,11,12].

Dynamic height  $H^D$  preserves equality of geopotential increments, which makes it physically meaningful for hydraulic continuity over large regions. Two points with identical dynamic heights lie on the same equipotential surface if the same reference normal gravity is used. The mathematical elegance of dynamic heights is their direct linearity with geopotential numbers, but they are less intuitive in local engineering because the metric denominator is constant rather than point-adaptive. For this reason, dynamic heights are often more visible in continental-scale or hydrological discussions than in day-to-day cadastral work [2,4,7,9].

Orthometric height  $H^O$  is the classical engineering favorite because it measures distance from the geoid along the real plumb line. It therefore corresponds naturally to the intuitive notion of elevation above mean sea level. However, exact orthometric heights require knowledge of mean gravity inside the topography, which is not directly observable and must be approximated using gravity reductions and topographic density assumptions. Helmert orthometric heights remain the dominant practical realization because they offer a workable balance between theory and computability [3,5,10,19].

Normal height  $H^N$  avoids the need for assumptions about topographic density by replacing actual internal gravity with the normal gravity field of the reference ellipsoid. This is a decisive practical advantage for national leveling systems in regions where dense gravity information inside topography is unavailable. The price paid is conceptual abstraction: normal heights are not literal distances above the quasigeoid in the naïve geometric sense, and their rigorous interpretation has been revisited in recent literature [8,10,11].

The mathematical differences among these systems are often at the millimeter-to-centimeter level for low-relief local projects, yet they can become operationally relevant when one seeks sub-centimeter integration among leveling, gravity, and GNSS or when one works across large elevation ranges. A rigorous article aimed at a mathematics journal should therefore not frame the choice of height system as a matter of terminology; it is a matter of denominator selection, reference-surface compatibility, and error transport. The principal definitions and applications are summarized in Table 1.

**Table 1.** Comparative mathematical definition and practical role of the principal height systems.

| System        | Primary Definition                          | Reference Surface                             | Main Data | Typical Use                     |
|---------------|---------------------------------------------|-----------------------------------------------|-----------|---------------------------------|
| Geodetic $h$  | Ellipsoidal normal distance                 | Reference ellipsoid                           | GNSS      | 3D geocentric positioning       |
| Dynamic $H^D$ | Geopotential number/constant normal gravity | Chosen equipotential via C Leveling + gravity |           | Hydrology, continental analysis |

|                   |                                         |                                      |                                      |                                               |
|-------------------|-----------------------------------------|--------------------------------------|--------------------------------------|-----------------------------------------------|
| Normal $H^N$      | Geopotential number/mean normal gravity | nor-Quasigeoid-related normal system | Leveling + gravity or GNSS + $\zeta$ | National vertical datums, cadastre            |
| Orthometric $H^O$ | Geopotential number/mean actual gravity | Geoid-related physical system        | Leveling + gravity or GNSS + $N$     | Engineering, hydraulics, intuitive elevations |

#### 4. Derivation of Transformation Relations

The transformation from ellipsoidal to physical heights is often written in a deceptively simple form:  $h - \zeta = H^N$  or  $h - N = H^O$ . Yet behind these equations stand different mathematical objects. The height anomaly  $\zeta$  is tied to the normal potential field, while the geoid undulation  $N$  is tied to the actual gravity field and therefore to topographic density assumptions. When the national datum is realized by normal heights, the conversion pipeline  $h \rightarrow H^N$  is conceptually cleaner because it remains inside the normal-field framework [10,11,13,14].

Suppose a point  $P$  has ellipsoidal height  $h$  measured in the geodetic reference frame and a model-derived height anomaly  $\zeta(\varphi, \lambda)$ . Then, the normal height is obtained directly from Equation (5). If one also has a model for the geoid undulation  $N$ , an orthometric height can be computed. Their difference is exactly the modeled geoid-to-quasigeoid separation. In rugged terrain, this separation grows with elevation and with the deviation between actual and normal gravity, which explains the empirical trend observed in comparative mountain studies [8,10,17].

For network adjustment, the most stable quantity remains the geopotential number. Leveling observations yield increments in potential that can be converted later into the target height system. This strategy is attractive because it postpones denominator-specific decisions until after the stochastic adjustment. A mathematics-oriented formulation therefore recommends the following: first adjust geopotential differences, then map the adjusted geopotential numbers into dynamic, normal, or orthometric heights according to application needs [6,7,19].

Let  $\Delta n_{ij}$  be an observed leveled height difference between points  $i$  and  $j$ , and let  $g_{ij}$  be an appropriately averaged gravity quantity attached to that segment. Then, the geopotential increment can be approximated by  $\Delta C_{ij} = g_{ij} \Delta n_{ij}$ . The closure condition on a loop of  $k$  segments is  $\sum \Delta C_{ij} = 0$  in the absence of errors. This is more physically meaningful than  $\sum \Delta n_{ij} = 0$  because the latter ignores gravity variation along the route.

$$\Delta C_{ij} \approx g_{ij} \Delta n_{ij} \quad (7)$$

$$\sum_{\text{loop}} \Delta C_{ij} = 0 \quad (8)$$

$$\sigma_{HN^2} \approx \sigma_h^2 + \sigma_\zeta^2 - 2 \operatorname{cov}(h, \zeta) \quad (9)$$

$$\sigma_{HO^2} \approx \sigma_h^2 + \sigma_N^2 - 2 \operatorname{cov}(h, N) \quad (10)$$

$$\partial H^N / \partial \zeta = -1, \partial H^O / \partial N = -1 \quad (11)$$

Equations (9)–(11) formalize a basic but important conclusion: if the ellipsoidal height is already known at the millimeter level, the limiting factor in GNSS-to-physical-height conversion quickly becomes the quality of the geoid or quasi-geoid model. This is especially true in mixed workflows where terrestrial data are sparse. Hence, for practical engineering in Romania or elsewhere, investment in regional quasi-geoid refinement can produce greater vertical benefit than pushing GNSS precision alone below the same threshold [12,14–17].

### Physical Modeling Assumptions and Validity Domain

The denominator-based formulation used in this paper is exact at the conceptual level only if the geopotential number is known and the denominator represents the corresponding physical gravity model. In the representative computations, however, the mean actual gravity along the plumb line, the mean normal gravity along the normal plumb line, and the topographic-mass effect are introduced through simplified parametric relations. This choice is suitable for controlled sensitivity analysis and software prototyping, but it should not be interpreted as a replacement for a full gravimetric reduction, density model, or boundary-value solution. Recent studies on accurate orthometric-height determination show that the mean gravity term depends on terrain, density, and downward-continuation assumptions, and that consistent geoid and orthometric-height determination requires explicit modeling of topographic and atmospheric contributions [23,36].

Accordingly, the validity domain of the present numerical experiment is methodological rather than operational. The formulas are used to expose the role of the denominator, the expected growth of normal-orthometric separation with relief, and the sensitivity of satellite-derived physical heights to quasi-geoid or geoid uncertainty. The numerical values should therefore be read as physically plausible model responses under declared assumptions. For operational Romanian height transformation, the same framework must be populated with official or project-specific leveling, gravity, benchmark, density, quasi-geoid, and covariance information.

## 5. Conditioning, Stability, and Sensitivity

A mathematically mature comparison of height systems must address conditioning. The mapping  $C \rightarrow H$  is linear for dynamic heights and rational for orthometric and normal heights because the denominator depends on position and, weakly, on height itself. In typical terrestrial ranges, the denominator variation is small, which means the transformations are numerically stable. However, small denominator differences integrated across large elevation differences or large networks can accumulate into interpretable systematic effects [3,5,7,10].

For normal heights, the derivative  $\partial H^N / \partial \bar{\gamma} = -C / \bar{\gamma}^2$  shows that the same geopotential number becomes more sensitive to denominator error as the gravity approximation worsens or as  $C$  increases. Orthometric heights behave similarly with mean actual gravity. In mountain corridors, tunnels, and long linear infrastructure, a method that seems interchangeable at the millimeter level over 100 m may not remain interchangeable over 100 km or 1000 m of elevation range.

The conditioning problem is equally important for cartographic fusion. Raster digital terrain models, LiDAR point clouds, photogrammetric surfaces, and cadastral layers may all carry heights, yet not necessarily in the same height system. If one dataset stores ellipsoidal heights and another stores normal heights, the resulting misalignment is not random noise, but a deterministic bias tied to  $N$  or  $\zeta$ . A mathematically informed geospatial workflow must therefore treat the height system as part of the metadata, not as optional annotation [12–18]. The main conversion-error sources are summarized in Table 2.

**Table 2.** Error sources affecting the conversion of ellipsoidal heights into gravity-related heights.

| Source of Uncertainty     | Mechanism                             | Main Effect                       | Severity in Engineering |
|---------------------------|---------------------------------------|-----------------------------------|-------------------------|
| Ellipsoidal height $h$    | GNSS stochastic error                 | Random vertical noise             | Medium                  |
| Quasigeoid/geoid model    | Systematic + spatial model error      | Deterministic local bias          | High                    |
| Segment gravity mean      | Bias in $C \rightarrow H$ denominator | Scale-like distortion with relief | Medium                  |
| Reference potential $W_0$ | Global offset in all derived heights  | Uniform shift                     | Medium                  |

|                                    |                              |                          |        |
|------------------------------------|------------------------------|--------------------------|--------|
| Datum realization mismatch         | Hidden cross-system bias     | Incompatible layers      | High   |
| Interpolation/model discretization | Local smoothing/edge effects | Patchy residual patterns | Medium |

## 6. Romanian Vertical Reference Context

Romania is a compelling case because the national geodetic practice operates in the ETRS89 planimetric frame, while gravity-related heights are linked to the RO Const/NH environment and the Black Sea 1975 datum context [13,15,16]. In operational terms, users often observe ellipsoidal heights via GNSS and require official or quasi-official normal heights for cadastral, engineering, or topographic products. The mathematical bridge between these worlds is a national or local quasi-geoid model and the associated transformation software environment.

The official Romanian geodetic information made available through ROMPOS states that ETRS89 is based on the GRS80 ellipsoid and that Romania officially uses two reference systems, one of which is the legacy S-42/Stereo 70 framework for mapping, whereas current GNSS practice is connected to ETRS89 [13]. For heights, the EPSG registry identifies the Constanta vertical coordinate system for Romania, and Romanian technical sources describe the operational use of Black Sea 1975 normal heights [15,16].

Of particular interest is the present TransDatRO environment, which explicitly includes an improved height-transformation grid with a gravimetric quasi-geoid model for a long list of Romanian counties, including Dolj, Gorj, Olt, Teleorman, Timiș, Alba, Cluj, Mureș, Sibiu, Brașov, Covasna, Arad, Hunedoara, Prahova, Vrancea, and others [15]. This means that Romania already embodies the modern geodetic pattern in which national practice is neither purely leveling-based nor purely GNSS-based, but rather a mathematically mediated combination of both.

From the viewpoint of a mathematics journal, the Romanian case demonstrates why vertical datums are fertile ground for applied mathematical work: they require integral relations, approximations of gravity, interpolation of model surfaces, stochastic adjustment, and uncertainty propagation. They also reveal that the same national problem can be interpreted at multiple scales—a local engineering benchmark, a county-level quasi-geoid interpolation, or a continental transformation toward EVRF realizations [7,8,13–16]. The Romanian vertical-reference context is condensed in Table 3.

**Table 3.** Condensed summary of the Romanian geodetic context relevant for height-system transformations.

| Category                                   | Romanian Context                                                                                                                                       |
|--------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------|
| Official planimetric frame                 | Stereo 70/legacy S-42 use in mapping                                                                                                                   |
| Operational global/European geodetic basis | ETRS89 on GRS80                                                                                                                                        |
| Official vertical context                  | RO Const/NH; Black Sea 1975 normal heights                                                                                                             |
| Transformation environment                 | TransDatRO v4.08                                                                                                                                       |
| Improved gravimetric quasi-geoid coverage  | Multiple western, central, and southern counties including Dolj, Cluj, Alba, Mureș, Sibiu, Gorj, Olt, Timiș, Teleorman, Prahova, Vrancea, Galați, etc. |

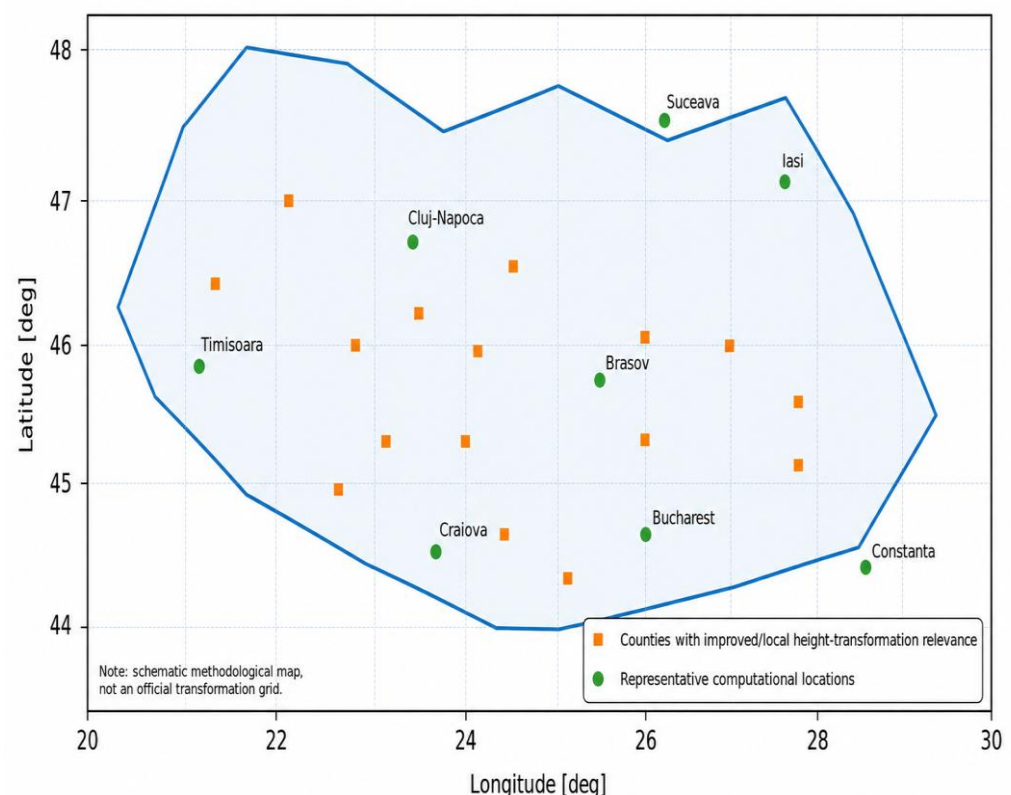
### *Regional Context and Technological Opportunities for Transfer*

The Romanian setting also offers a clear opportunity for regional transfer of the proposed methodology. Improved quasi-geoid grids, national transformation tools, permanent satellite-positioning services, public and project-level elevation models, and expand-

ing unmanned-aircraft and laser-scanning products create a favorable technological environment for more transparent vertical-reference workflows. The same geopotential-centered architecture can be transferred to county-level cadastre, infrastructure corridors, floodplain modeling, mining and energy sites, and three-dimensional city or terrain models, provided that each workflow records the datum, height type, transformation surface, uncertainty model, and processing version.

The broader consequence is that Romania is not treated here as an isolated case. It is used as a representative European example in which official normal-height practice, varied relief, and rapidly growing multi-sensor geospatial production meet the same mathematical requirement: the transformation from geometric heights to gravity-related heights must be reproducible, uncertainty-aware, and metadata-safe. This regional perspective also supports transfer to other countries with national vertical datums, local quasi-geoid models, and mixed satellite-leveling workflows [37–41].

Figure 1 responds to the need to make the Romanian transformation environment explicit. It summarizes the territorial logic of the TransDatRO-oriented height-transformation problem: ellipsoidal heights acquired through satellite positioning must be connected to national normal-height products through a quasi-geoid or height-anomaly model whose quality is spatially variable and whose metadata must accompany every derived product.



**Figure 1.** Schematic TransDatRO-oriented regional context and height-transformation grid logic for Romania. The figure is methodological and indicates the role of county-level improved quasi-geoid coverage and representative computational locations; it is not an official transformation grid.

## 7. Romania-Oriented Numerical Experiment

The numerical study developed in this paper is deliberately framed as a representative computational experiment rather than as an official national adjustment. The geographic positions correspond to major Romanian urban centers distributed across the territory, while the ellipsoidal heights, height anomalies, and gravity values are constructed

to remain within realistic ranges documented by Romanian and European geodetic sources [13–17]. This design allows the mathematical properties of the competing height systems to be compared transparently without creating the false impression that unpublished official benchmark data are being reproduced.

The selected locations—Constanța, Bucharest, Craiova, Brașov, Cluj-Napoca, Iași, Timișoara, and Suceava—span lowland, plateau, and mountain-influenced environments. For each point, a representative ellipsoidal height  $h$  and height anomaly  $\zeta$  are assigned. Normal gravity is computed from the Somigliana formula on GRS80, mean normal gravity is approximated along the normal plumb line, and a representative mean actual gravity is constructed by adding a plausible local gravity anomaly and a Helmert-type correction term. The geopotential number is then generated from the normal-height model and reused to compute dynamic and orthometric heights under the same numerator.

The experiment therefore satisfies three requirements expected in a mathematics paper: first, the internal algebra is fully consistent; second, the assumptions are explicit; third, the computational consequences of denominator choice are measurable through tables and figures. Such a design is especially useful in teaching, method validation, and software prototyping.

For clarity, the experiment is not used as empirical validation of an official Romanian vertical transformation. It is a model-behavior experiment with synthetic but realistic inputs. Its validation level is internal: algebraic consistency, explicit assumptions, reproducible tables, and agreement with known qualitative behavior from physical geodesy. External validation would require official Global Navigation Satellite System, leveling, gravity, benchmark, and quasi-geoid datasets, which are identified below as the principal direction for future work. The representative inputs and computed outputs are listed in Table 4.

**Table 4.** Representative inputs and computed outputs for the Romania-oriented experiment.

| Location    | Lat [deg] | Lon [deg] | $h$ [m] | $\zeta$ [m] | $\gamma_0$ [m/s <sup>2</sup> ] | HN [m] | HO [m]  | HD [m]  | (N – $\zeta$ ) [mm] |
|-------------|-----------|-----------|---------|-------------|--------------------------------|--------|---------|---------|---------------------|
| Constanta   | 44.17     | 28.65     | 91.0    | 36.0        | 9.805447                       | 55.0   | 55.0    | 54.995  | −0.4                |
| Bucharest   | 44.43     | 26.1      | 120.0   | 38.0        | 9.805682                       | 82.0   | 81.998  | 81.995  | 1.85                |
| Craiova     | 44.33     | 23.82     | 134.0   | 39.0        | 9.805591                       | 95.0   | 94.997  | 94.993  | 2.58                |
| Brasov      | 45.66     | 25.61     | 641.0   | 43.0        | 9.806795                       | 598.0  | 597.914 | 597.98  | 85.62               |
| Cluj-Napoca | 46.77     | 23.59     | 388.0   | 43.0        | 9.807799                       | 345.0  | 344.971 | 345.038 | 29.46               |
| Iasi        | 47.16     | 27.59     | 101.0   | 40.0        | 9.808151                       | 61.0   | 60.999  | 61.012  | 1.43                |
| Timisoara   | 45.75     | 21.23     | 120.0   | 36.0        | 9.806877                       | 84.0   | 83.999  | 84.005  | 1.07                |
| Suceava     | 47.65     | 26.25     | 304.0   | 41.0        | 9.808593                       | 263.0  | 262.983 | 263.053 | 17.07               |

## 8. Results and Comparative Interpretation

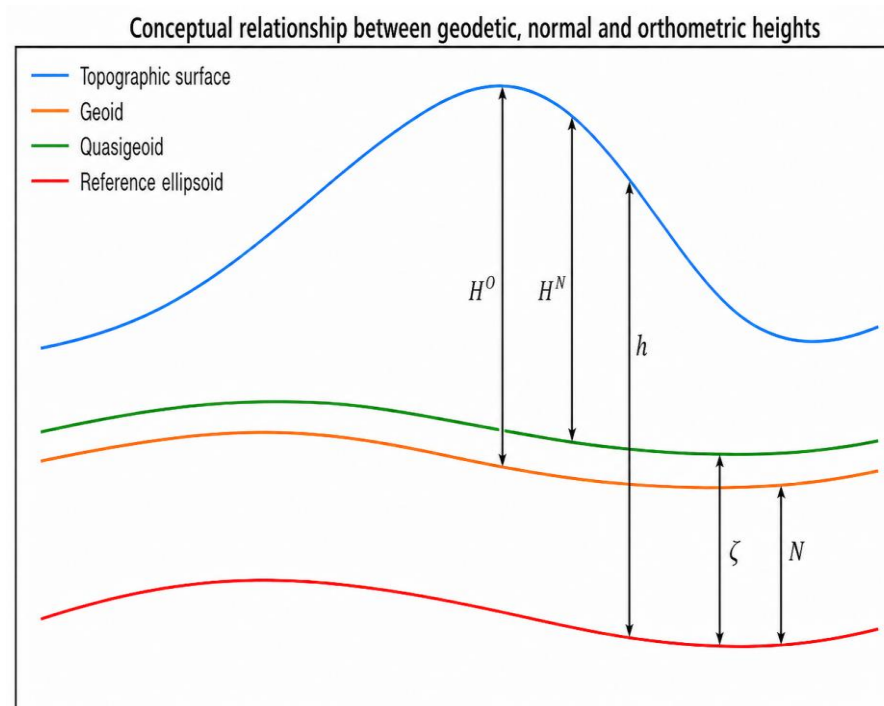
The results confirm the expected theoretical ordering. In low-relief locations such as Constanța, Bucharest, Craiova, and Timișoara, the difference between normal and orthometric heights stays close to the millimeter level in the representative model. This indicates that for many local topographic or cadastral tasks, the practical distinction between  $H^N$  and  $H^O$  may be overshadowed by other uncertainties, such as benchmark stability, GNSS setup, or interpolation errors in the height model. Nevertheless, the distinction remains conceptually important because hidden datum mixing can still generate systematic bias [10,11,15].

In the higher-elevation cases, especially Brașov, Cluj-Napoca, and Suceava, the separation between orthometric and normal heights becomes larger. In the present experiment, the geoid-to-quasigeoid separation grows approximately linearly with elevation

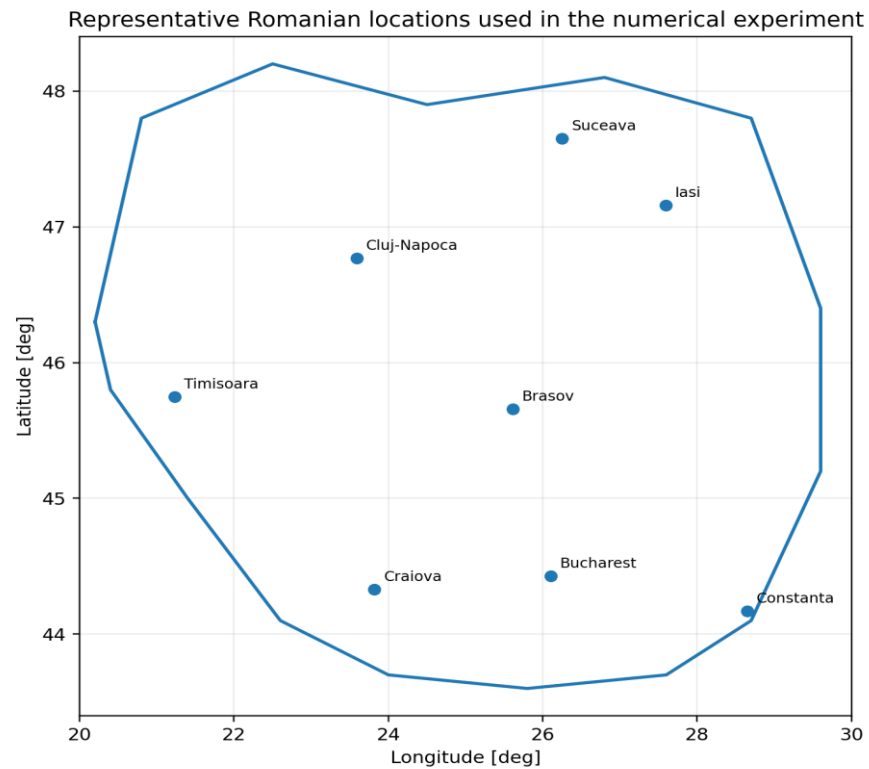
over the selected points, reaching several centimeters in the mountain-affected locations. This trend is mathematically consistent with the fact that the difference between actual and normal mean gravity accumulates with relief [8,10,17].

The dynamic heights remain close to normal heights for low altitudes but exhibit small systematic departures as soon as the ratio  $\bar{\gamma}/\gamma_0$  departs from unity. This is not a defect of the method; it reflects the design of dynamic heights, which preserve geopotential rather than local Euclidean intuition. Consequently, dynamic heights are theoretically excellent for hydraulic or continental equipotential studies, but normal or orthometric heights are more natural in most applied engineering design tasks [2,7,9].

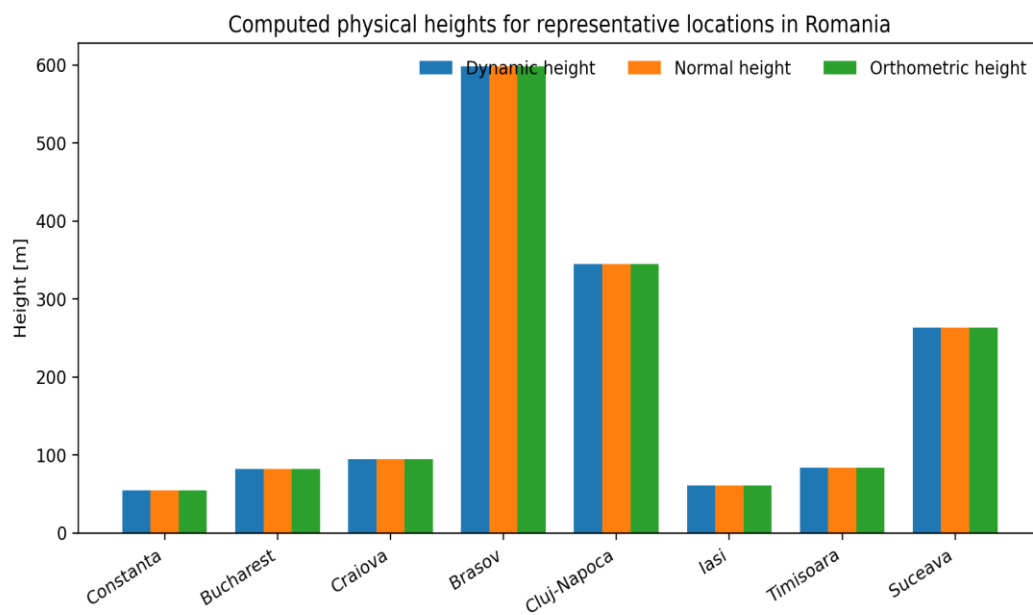
A notable computational consequence is that the largest contribution to final uncertainty is not the linear algebra of the conversion formula itself but the quality of the model surface  $N$  or  $\zeta$ . Once the numerator  $C$  is robustly estimated and GNSS precision is adequate, the physical-height solution becomes strongly model-limited. This is why regional quasi-geoid improvement and careful metadata management remain essential in national geospatial infrastructures [12,14–17]. The reference-surface relationships are illustrated in Figure 2. The representative Romanian locations are mapped in Figure 3. The computed height-system values are compared in Figure 4. The relief-dependent separation is shown in Figure 5.



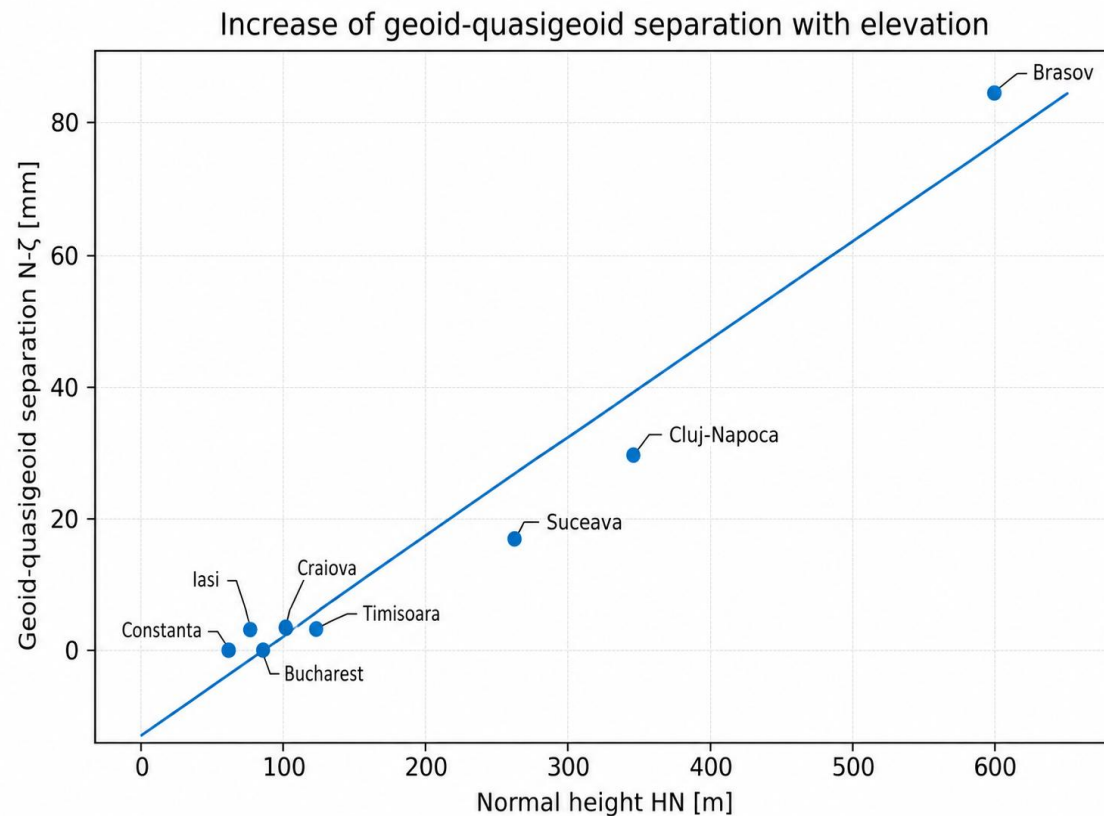
**Figure 2.** Conceptual relationship among the reference ellipsoid, geoid, quasigeoid, and the main height quantities.



**Figure 3.** Representative Romanian locations used in the computational study.



**Figure 4.** Computed dynamic, normal, and orthometric heights for representative locations in Romania.



**Figure 5.** Growth of the modeled geoid-to-quasigeoid separation with increasing normal height.

## 9. Network-Level Demonstration with Geopotential Numbers

To emphasize the mathematical advantage of working with geopotential numbers, we also consider a demonstrative west–east profile crossing several representative Romanian locations. Let the profile begin in Constanța and continue through Bucharest, Craiova, Brașov, and Cluj-Napoca. If one uses raw leveled height differences only, the profile closure is interpreted geometrically. If instead one converts each segment into geopotential increments using segment-average gravity, the closure becomes physically meaningful and independent of the chosen target height system.

This distinction is not merely terminological. In a mixed GNSS-leveling adjustment, the state vector can be expressed through geopotential numbers relative to a fixed datum point. After least-squares adjustment, the same adjusted numerator can be converted into dynamic, normal, or orthometric heights according to the required application. Such an approach isolates stochastic estimation from denominator choice and thus improves transparency in software architecture.

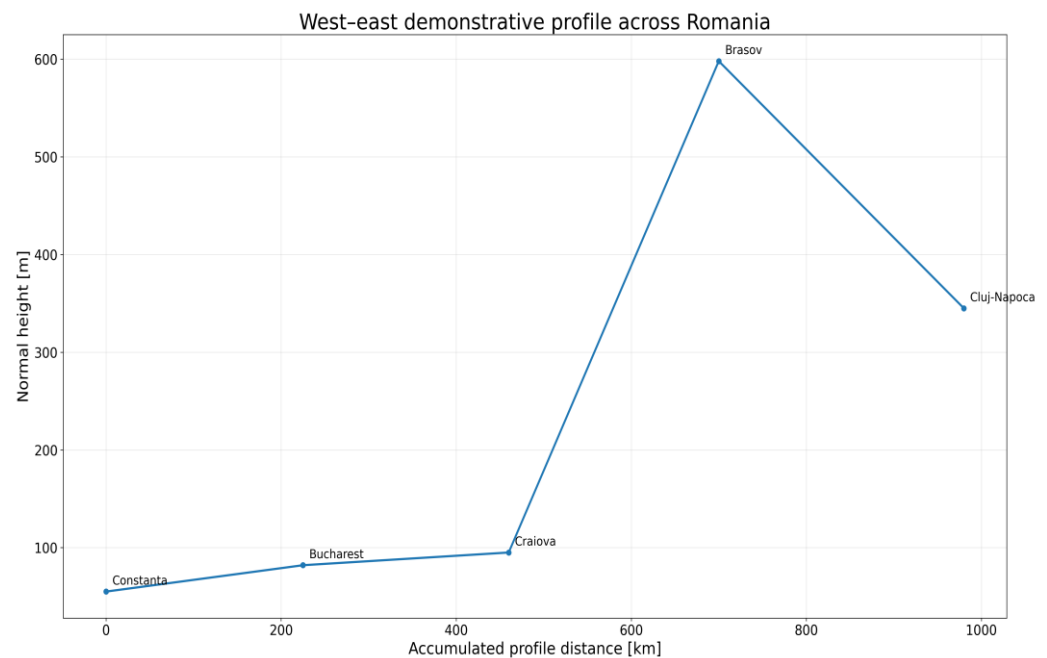
The profile concept is especially relevant for Romanian engineering corridors where infrastructure crosses lowland, plateau, and mountain sectors. In these situations, the seemingly small denominator changes that are invisible over one short segment can accumulate into differences that matter for high-precision longitudinal design, deformation monitoring, or hydro-technical continuity.

$$C_j = C_i + \sum_{\{\text{segments } i \rightarrow j\}} \bar{g}_s \Delta n_s \quad (12)$$

$$H_j^{(\text{target})} = C_j / d_j^{(\text{target})}, \quad d_j \in \{\gamma_0, \bar{\gamma}_i, \bar{g}_j\} \quad (13)$$

Equation (12) expresses propagation of geopotential numbers along a profile. Equation (13) highlights a powerful design principle for geodetic software: keep the numerator

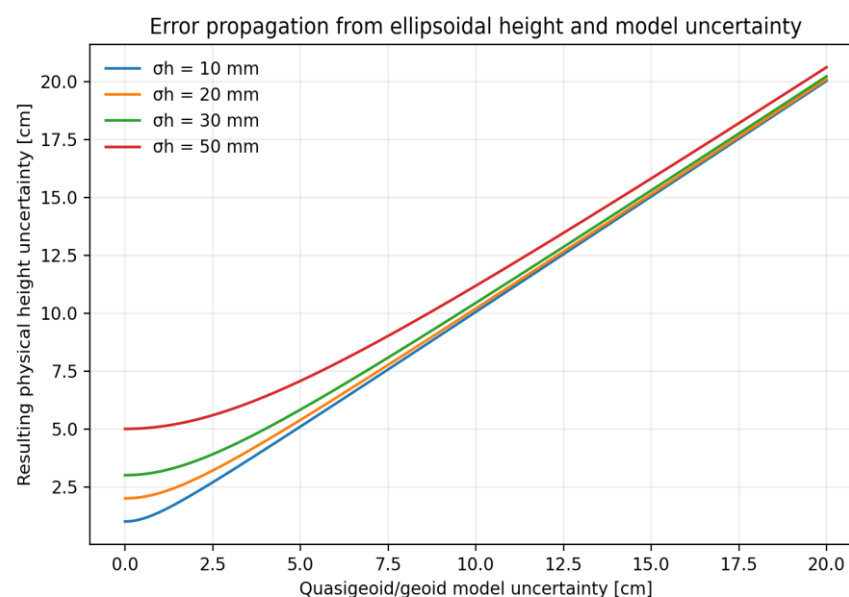
invariant, then map to the target denominator at the final stage. This design is highly compatible with modular surveying software and with the reproducibility expectations of a mathematics-oriented journal. The demonstrative west–east profile is shown in Figure 6.



**Figure 6.** Demonstrative west–east profile across Romania using representative normal heights.

## 10. Sensitivity Analysis for Practical Surveying

The sensitivity study quantifies a basic trade-off: improving GNSS precision below the centimeter level produces diminishing returns if the geoid or quasi-geoid model remains uncertain at several centimeters. This result, illustrated in Figure 7, is particularly important for engineering users who may invest disproportionately in field acquisition while neglecting datum transformation quality.



**Figure 7.** Propagated uncertainty of gravity-related heights as a function of ellipsoidal-height precision and geoid/quasi-geoid model uncertainty.

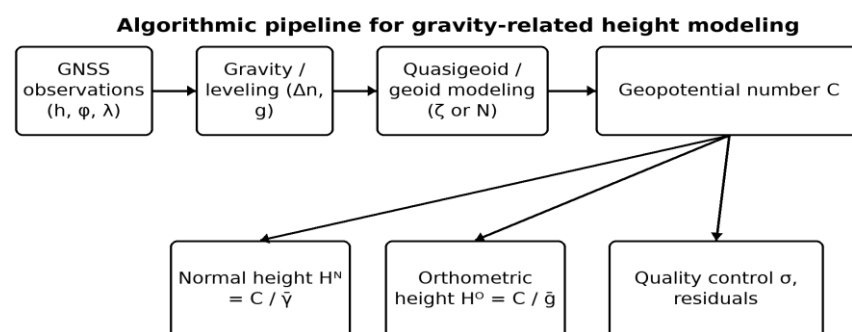
The implications are operational. In a county where an improved gravimetric quasi-geoid is available, the same GNSS campaign can produce more reliable official-height products than in a county where only a coarser geometric or inherited model is available. The mathematical structure of Equations (9)–(11) predicts precisely this behavior; the Romanian institutional context confirms its practical relevance [15–17].

Another sensitivity effect concerns hidden datum mismatch. If one dataset stores RO Const/NH normal heights and another stores orthometric heights approximated from a global geoid, the discrepancy is not a random offset that can be removed by local fitting unless the difference field is itself modeled. This reinforces the need for explicit height-system metadata in cartographic and BIM/GIS integration.

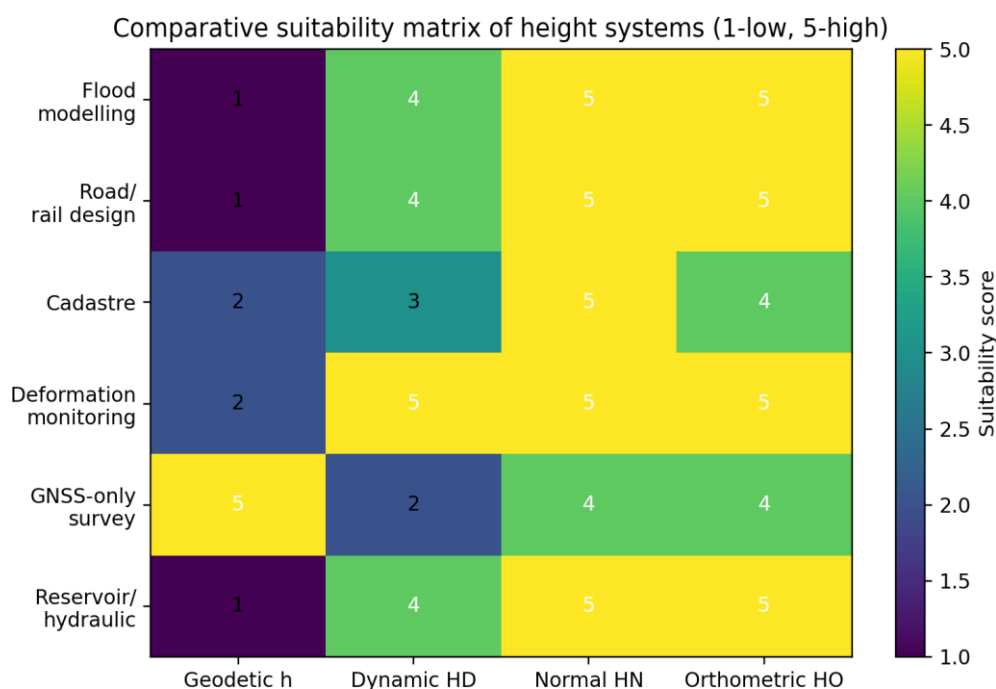
The consequence for other regions is direct. Any country or project area that combines satellite-derived ellipsoidal heights with national normal or orthometric heights faces the same mathematical structure: a precise geometric height does not by itself guarantee a precise physical height if the transformation surface, datum realization, and covariance model remain weak. The Romanian example is therefore generalizable at the level of method, even though the numerical magnitudes are region-specific. Low-relief regions may show small normal-orthometric separations, whereas mountainous regions, cross-border corridors, or areas with sparse gravity data may exhibit larger deterministic and stochastic discrepancies [18,37,38].

A unified stochastic interpretation can be stated by collecting the basic input quantities in a vector  $z = [h, \text{zeta}, N, \bar{g}, \bar{\gamma}, W_0]^T$  and the desired height products in a vector  $y = f(z)$ . First-order propagation then gives  $\text{Sigma}_y = J \text{Sigma}_z J^T$ , where  $J$  is the Jacobian of the selected transformation. For direct normal-height conversion, the relevant sub-Jacobian is  $[1, -1]$  with respect to  $[h, \text{zeta}]$ ; for orthometric conversion, it is  $[1, -1]$  with respect to  $[h, N]$ ; and for geopotential-number adjustment, the covariance of the adjusted numerator is subsequently mapped by the inverse denominator operator. This organization makes explicit why model-surface covariance and spatial correlation may dominate the final vertical error budget, even when the field observations are precise.

The discussion also explains the limits of the Monte Carlo experiment. The simulated standard deviations are nearly location-independent because the same stochastic assumptions are imposed at all points. This is useful for isolating the role of quasi-geoid uncertainty, but it does not reproduce the full spatial correlation, benchmark instability, or terrain-dependent gravity errors expected in a national adjustment. A field-validated implementation should therefore replace the independent synthetic perturbations used here with a covariance model estimated from real benchmark residuals, transformation-grid residuals, and gravity-data quality indicators. The transformation workflow is summarized in Figure 8. The comparative suitability matrix is presented in Figure 9.



**Figure 8.** Algorithmic pipeline for transforming geometric heights into gravity-related heights.



**Figure 9.** Comparative suitability matrix of height systems for engineering, geodetic, and cartographic applications.

## 11. Comparative Discussion for Engineering, Geodetic, and Cartographic Applications

For roadway, railway, and tunnel design, the most robust recommendation is to work in the officially adopted gravity-related system and to ensure that all design surfaces, alignment points, and monitoring marks are expressed in the same height realization. In Romania, that typically means normal heights in the RO Const/NH context unless a project-specific orthometric convention has been formally introduced. The mathematics of consistency is more important than the label alone.

For flood-risk modeling, drainage studies, and reservoir design, both normal and orthometric heights can be physically meaningful if implemented coherently. The decisive issue is whether the adopted surface behaves correctly with respect to equipotential flow and whether the model chain is documented. Dynamic heights deserve special consideration in long hydrological studies because of their clean relation to geopotential, but they are rarely the most convenient public-facing product for engineering drawings.

For cadastral and cartographic production, normal heights often provide the best operational compromise in the Romanian context because they align with national practice and avoid the inaccessible-topographic-density problem of exact orthometric heights. However, GNSS-first workflows must still respect the uncertainty carried by  $\zeta$ . Treating the transformation as a black box can lead to substandard quality assurance, particularly when integrating UAV, LiDAR, and terrestrial total-station surveys [14–17].

For geodetic monitoring and deformation analysis, the correct answer is more nuanced. If the scientific target is pure geometric displacement in the terrestrial frame, ellipsoidal heights may be entirely appropriate. If the target is physical-height change relevant to water or engineering stability, conversion to a gravity-related system becomes necessary. Advanced monitoring networks may therefore need to retain both  $h$  and  $H^N/H^O$  simultaneously, each with its own covariance structure.

Seen from a mathematical angle, the main contribution of the present study is that it separates three levels of decision: the invariant numerator  $C$ , the denominator model  $d$ ,

and the application-specific interpretation of the resulting height. This layered view clarifies many practical misunderstandings. Users do not merely choose among alternative labels; they choose among alternative denominators, each of which implies a different physical abstraction and a different error budget. The application-specific recommendations are summarized in Table 5.

**Table 5.** Recommended height products by application class in a Romania-centered operational perspective.

| Application                   | Preferred Height Output     | Why                                          | Mathematical Concern            |
|-------------------------------|-----------------------------|----------------------------------------------|---------------------------------|
| GNSS network densification    | $h + H^N$                   | Need both geometric and physical coordinates | Frame covariance                |
| Geometric leveling adjustment | $H^N$ or $C$                | Compatible with national control             | Loop corrections in $C$         |
| Hydraulic continuity          | $H^D$ or $H^N$              | Potential continuity matters                 | Reference $\gamma_0$ selection  |
| Road/rail design              | $H^N$ or $H^O$              | Gravity-related elevations required          | Model consistency               |
| Cadastral and mapping         | $H^N$                       | Official national practice                   | Datum metadata                  |
| UAV/LiDAR integration         | $H^N$                       | Fast GNSS-to-physical conversion             | $\sigma_C/\sigma_N$ propagation |
| BIM/GIS interoperability      | $H^N$ with metadata         | Avoid hidden vertical mismatches             | Cross-system transformation     |
| Reservoir and dam monitoring  | $H^N$ and $H^O$ cross-check | High consequence of bias                     | Systematic denominator bias     |

## 12. Algorithmic Recommendations for Software Implementation

A surveying or geospatial software system intended to support advanced Romanian workflows should represent height information through at least four explicitly typed objects: ellipsoidal coordinates, geopotential numbers, gravity-related heights, and vertical metadata. Internally, the adjustment engine should privilege geopotential increments whenever leveling and gravity are available. Only after adjustment should the platform derive  $H^D$ ,  $H^N$ , or  $H^O$  according to user-selected settings and documented models.

A minimal computational sequence is as follows: First, acquire GNSS coordinates and, where available, spirit leveling and gravity data. Second, interpolate  $\zeta$  or  $N$  from the selected national, county, or project quasi-geoid/geoid model. Third, compute provisional physical heights from Equation (5). Fourth, if network data exist, estimate adjusted geopotential numbers by least squares. Fifth, propagate covariance through the selected denominator model and store the result together with the height-system label. Sixth, export all products with explicit metadata so that CAD, GIS, and BIM consumers cannot silently reinterpret the vertical reference.

This layered architecture is mathematically elegant because it separates observable quantities from model-dependent transformations. It is also operationally realistic in Romania, where the same project may involve GNSS-RTK, leveling, total stations, UAV photogrammetry, LiDAR-derived surfaces, and official cadastral deliverables. A well-designed platform should therefore expose the height system as a first-class computational object rather than burying it in import/export settings.

### 12.1. Integrated Software Architecture for Geopotential-Centered Computation

To translate the theoretical framework of the manuscript into an operational research platform, the current software architecture is organized around the same invariant emphasized throughout the paper: the geopotential number  $C$ . In this design, ellipsoidal

heights, geopotential quantities, denominator models, and vertical metadata are represented as typed computational objects rather than as loosely coupled fields. This preserves the mathematical logic of the article, in which  $H^D$ ,  $H^N$ , and  $H^O$  are denominator-dependent realizations of a common physical numerator. The current compile-oriented research prototype is designated GeoHeight Ultimate v1.0.

The architecture is intentionally layered and corresponds to the latest compile-oriented software variant. A core geodetic engine performs the mathematical operations associated with normal gravity, geopotential increments, least-squares adjustment, covariance propagation, Monte Carlo simulation, and cross-system transformation. Above this engine, a WPF interface orchestrates project setup, data import, diagnostics, and reporting. Additional interoperability services handle CSV/XLSX import, GeoJSON and DXF export, and automated DOCX/PDF reporting.

This separation is essential for reproducibility. A single adjusted vector of geopotential quantities can subsequently be mapped into dynamic, normal, or orthometric heights by applying the denominator model chosen by the user, while the same project record preserves datum metadata, model versions, stochastic assumptions, and export provenance. In methodological terms, the software becomes a computational counterpart of the article rather than a detached engineering add-on.

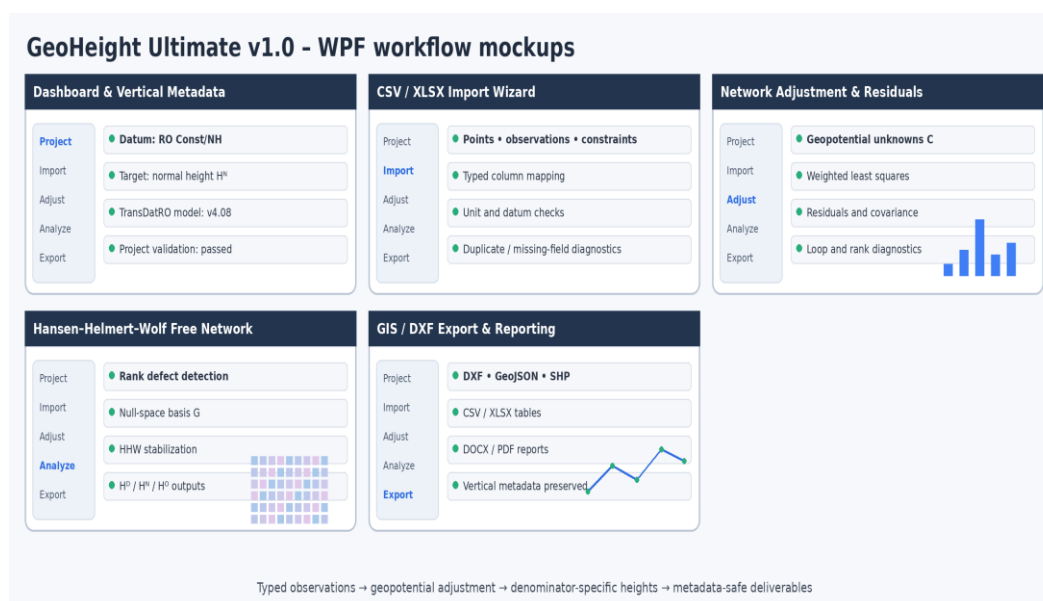
### 12.2. WPF Graphical Interface and User Workflow

The graphical implementation is based on Windows Presentation Foundation (WPF) and targets mixed Romanian GNSS-leveling-gravity projects that require both mathematical transparency and operational usability. The interface is structured around several coordinated workspaces: Project Manager, CSV/XLSX Import Wizard, Height Systems Lab, Network Adjustment, Hansen–Helmert–Wolf analysis, Monte Carlo and covariance diagnostics, GIS/DXF Export, and automated reporting. The prototype uses Microsoft .NET 8.0 and WPF (Microsoft Corporation, Redmond, WA, USA) and was developed and tested at the University of Craiova, Craiova, Romania.

In practical use, the operator begins by defining the project datum environment, the target height system, and the transformation surface to be used, such as a quasi-geoid or geoid model. The import module maps spreadsheet fields to point objects, observation records, and metadata entities. The computational layer then executes deterministic transformations, stochastic adjustment, or free-network stabilization, after which the export and reporting modules generate engineering tables, GIS/CAD products, and publishable summaries.

Figure 10 presents compile-oriented interface mockups of the principal WPF workspaces. They are included not merely for presentation value but because they encode the workflow implied by the mathematics of the paper: data are ingested as typed observations, transformed into geopotentially coherent unknowns, processed under explicit datum and uncertainty assumptions, and exported together with vertical metadata suitable for downstream CAD/GIS/BIM environments.

The enlarged panels should be read from left to right as a computational workflow rather than as independent screenshots. The dashboard defines the datum environment and target height system; the import panel checks semantic consistency of point, observation, and metadata fields; the adjustment panel estimates geopotential quantities and residuals; the Hansen–Helmert–Wolf panel documents rank defect and datum stabilization; and the export panel writes height products together with vertical-reference metadata. In this way, the interface makes the theoretical separation between numerator, denominator, and metadata visible to the operator.



**Figure 10.** Compile-oriented WPF mockups of the GeoHeight Ultimate v1.0 software platform, including dashboard, CSV/XLSX import wizard, network adjustment, Hansen–Helmert–Wolf processing, and GIS/DXF export views.

### 12.3. Structured Data, Interoperability, and Vertical Metadata

A central requirement of the software is robust handling of structured observational and metadata tables. Instead of emphasizing file formats as a research contribution, the revised architecture emphasizes semantic validation: point identifiers, coordinates, ellipsoidal heights, height anomalies, geoid undulations, leveled height differences, gravity descriptors, stochastic information, datum labels, and model versions are checked before they enter the computational state. This prevents the software from treating physically different vertical quantities as interchangeable numerical columns.

Interoperability is treated as a first-class design constraint, but only as a consequence of the mathematical model. Exported engineering, cartographic, and reporting products must carry the target height system, denominator model, transformation surface, and processing version. The important issue is therefore not the number of available export extensions but whether downstream users can reconstruct the vertical meaning and uncertainty of every height value.

This metadata-aware strategy is particularly important in Romanian workflows combining GNSS, total stations, digital levels, UAV-derived terrain models, and official cadastral products. In such contexts, the main risk is often not the absence of numerical data but the unmarked coexistence of incompatible vertical realizations. The software therefore treats vertical provenance as part of the result itself. The implemented software layers are summarized in Table 6.

**Table 6.** Implemented layered architecture of the geopotential-centered WPF software platform.

| Layer                  | Main Responsibility  | Typical Functions                                                                 |
|------------------------|----------------------|-----------------------------------------------------------------------------------|
| Geodetic core          | Mathematical engine  | $C$ , $H^D$ , $H^N$ , $H^O$ , $\gamma$ , $\bar{g}$ , LSQ, covariance, Monte Carlo |
| Data layer             | Input and validation | CSV/Excel import, schema mapping, metadata checks, project serialization          |
| Interoperability layer | Output packaging     | DXF, GeoJSON/SHP, CSV/Excel, report tables, layer tags                            |

|                        |                  |                                                       |
|------------------------|------------------|-------------------------------------------------------|
| WPF presentation layer | User interaction | Project manager, editors, diagnostics, plots, reports |
|------------------------|------------------|-------------------------------------------------------|

#### 12.4. Hansen–Helmert–Wolf Module for Datum-Defective Networks

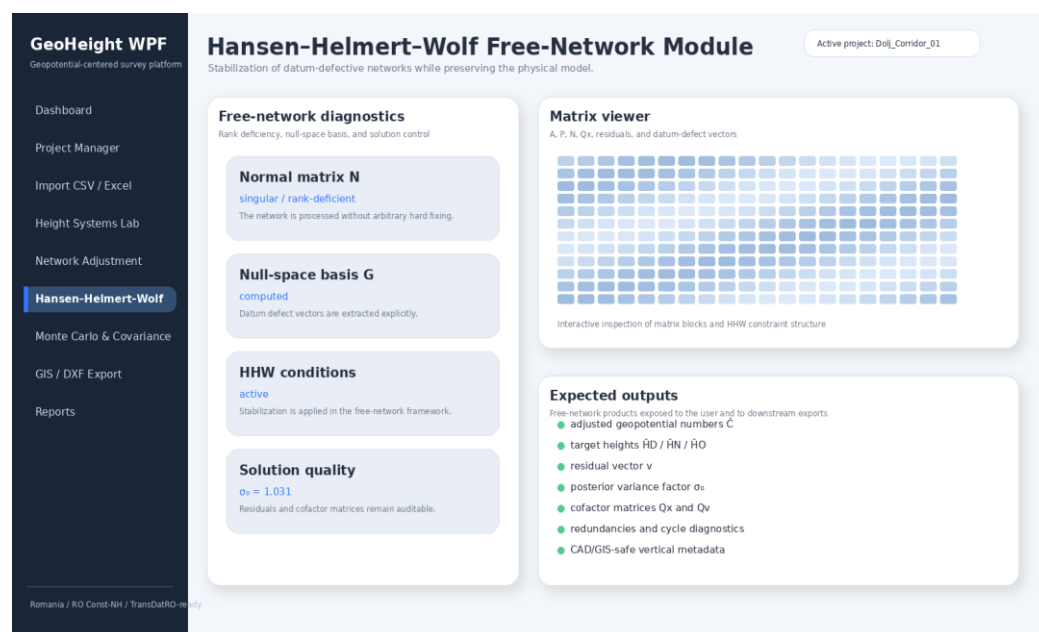
A major extension of the platform is the inclusion of a Hansen–Helmert–Wolf module for free-network adjustment in datum-defective configurations. In mixed vertical-control or combined geodetic networks, the normal matrix may be singular or rank-defective, particularly when only differences or weakly constrained pseudo-observations are available. The HHW module addresses precisely this case by preserving the physical model of the network while avoiding arbitrary hard fixing.

Within the current software distribution, the HHW workflow is implemented for 1D geopotential free networks and is organized around four internal steps: detection of the rank defect, construction of the null-space basis, formation of the stabilized system, and computation of adjusted geopotential numbers together with residuals, posterior variance factors, and cofactor matrices. This makes the free-network solution inspectable rather than opaque and aligns the implementation with the mathematical interpretation developed in Sections 15 and 17.

This module is especially relevant for research-grade Romanian applications involving sparse vertical control, long engineering corridors, monitoring networks, and educational demonstrations of datum defects. Its inclusion raises the platform beyond a simple conversion utility and turns it into a serious computational environment for adjustment, diagnosis, and methodological comparison. The supported import, validation, export, and reporting functions are listed in Table 7. The HHW workspace is detailed in Figure 11.

**Table 7.** Implemented import, validation, export, and reporting capabilities of the current platform.

| Data Block/Export | Examples                                                     | Purpose                                        |
|-------------------|--------------------------------------------------------------|------------------------------------------------|
| Input point table | PointId, X, Y, Lat, Lon, h, $\zeta$ ,<br>N, $\sigma_h$       | Populate point catalog and provisional heights |
| Observation table | From, To, $\Delta n$ , g, $\sigma_{\Delta n}$ , type         | Build mixed observation model                  |
| Constraint table  | Known H, datum condition, sigma                              | Define fixed or stochastic control             |
| GIS/CAD export    | Points, profiles, residuals, $H^N$ – $H^O$ themes            | Engineering and geospatial interoperability    |
| Metadata bundle   | Datum, denominator model, surface version, uncertainty notes | Prevent silent reinterpretation                |



**Figure 11.** Detailed mockup of the Hansen–Helmert–Wolf free-network workspace, highlighting rank-defect diagnostics, matrix inspection, and the expected output set used in geopotential-centered adjustment.

### 12.5. Integration of the Current Software Platform with the Present Article

The software platform developed here is not an external add-on to the manuscript, but a direct computational embodiment of its central argument. The article shows that dynamic, normal, and orthometric heights should be understood as denominator-dependent realizations of a common geopotential content. The software enforces exactly this logic by maintaining a geopotential-centered state model and by delaying system-specific height realization until the mathematically appropriate stage of the workflow.

Likewise, the paper emphasizes that vertical metadata, covariance propagation, Monte Carlo sensitivity testing, robust estimation, free-network stabilization, and cross-system diagnostics must be treated explicitly if height products are to remain reliable in engineering, cartographic, and geospatial integration tasks. For that reason, the present manuscript includes not only software architecture but also implementation-level pseudocode and interface mockups, thereby moving from conceptual recommendation toward a Q1-oriented computational specification.

### Mathematical Traceability of the Software Modules

To avoid interpreting the software section as a parallel engineering add-on, each computational module is tied to a specific mathematical object. The import module constructs typed observations and metadata; the transformation module evaluates h-zeta, h-N, and C/d relations; the adjustment module estimates geopotential-number unknowns and residuals; the Hansen–Helmert–Wolf module stabilizes datum-defective geopotential networks without imposing arbitrary hard constraints; the Monte Carlo and covariance modules propagate uncertainty through the denominator-based transformations; and the export module preserves the selected vertical realization in downstream products.

Within this interpretation, free-network adjustment does not automatically guarantee physical truth; it preserves internal consistency of the estimated geopotential state under the adopted observation model and datum constraints. Geopotential consistency is achieved only when gravity-weighted leveling increments, satellite-derived pseudo-observations, benchmark constraints, and stochastic weights are all expressed in a common

observation equation. The software architecture is therefore conditional on the same assumptions as the mathematical model, which makes the implementation reproducible but not independent of data quality.

#### 12.6. Implementation-Level Pseudocode for the Current Software Variant

To move the manuscript from architectural description toward executable research design, Algorithms 1–3 summarize the internal logic of the current GeoHeight Ultimate workflow. The first pseudocode block formalizes import and semantic validation; the second describes the Hansen–Helmert–Wolf solution path for datum-defective 1D geopotential networks; and the third outlines the automated generation of metadata-safe reports and GIS/CAD deliverables. These blocks are intentionally implementation-level rather than purely conceptual and are aligned with the compile-oriented WPF distribution discussed above. The computational formulae and assumptions are documented in Appendix A and Table A1, and the complete reproducible transformation sequence is summarized in Algorithm A1.

---

#### **Algorithm 1.** CSV/XLSX import, semantic validation, and project-state construction

---

INPUT: workbook\_or\_csv, datum\_settings, target\_height\_system

- 1: open data source and identify sheets/tables {Points, Observations, Constraints, Metadata}
  - 2: map input columns to typed fields {PointId, X, Y, Lat, Lon, h, zeta, N, dH, g, sigma, type}
  - 3: validate mandatory identifiers and numeric domains
  - 4: validate vertical semantics (system labels, datum tags, units, model versions)
  - 5: build typed project objects: PointCollection, ObservationSet, ConstraintSet, MetadataRecord
  - 6: flag duplicate points, missing constraints, inconsistent units, and invalid observation types
  - 7: if validation errors exist, stop processing and emit user-visible diagnostics
  - 8: compute provisional derived quantities where possible (HN, HO, HD, C placeholders)
  - 9: persist clean project state to the computational session
- OUTPUT: validated geopotential-centered project model
- 

---

#### **Algorithm 2.** Hansen–Helmert–Wolf free-network adjustment for a 1D geopotential network

---

INPUT: design matrix  $A$ , weight matrix  $P$ , observation vector  $l$

- 1: form normal matrix  $N = A^T P A$  and right-hand side  $n = A^T P l$
- 2: compute  $\text{rank}(N)$  and determine datum defect  $d$
- 3: if  $d = 0$ , solve classical LSQ solution  $\hat{x} = N^{-1} n$
- 4: else compute null-space basis  $G$  associated with the defect vectors
- 5: form the HHW-stabilized free-network system using the null-space constraints
- 6: solve for adjusted geopotential unknowns  $\hat{C}$  without arbitrary hard fixing
- 7: compute residual vector  $v$ , posterior variance factor  $\sigma_0$ , and cofactor matrices  $Q_x, Q_v$
- 8: map  $\hat{C}$  to target realizations {HD, HN, HO} through the selected denominator model
- 9: expose diagnostics for UI and report generation (rank defect, redundancies, cycles, metadata)

OUTPUT: free-network solution and denominator-dependent height products

---

**Algorithm 3.** Automated technical reporting and geospatial export

---

 INPUT: processed project state, selected figures/tables, export settings

- 1: collect adjusted unknowns, residual statistics, covariance summaries, and metadata provenance
- 2: generate comparative figures (profiles, residual charts, system-difference diagnostics)
- 3: assemble narrative sections for methods, results, and software traceability
- 4: write structured engineering tables to DOCX and PDF report templates
- 5: serialize point and corridor outputs to GeoJSON with vertical metadata attributes
- 6: serialize CAD-oriented layers to DXF using project-safe layer naming conventions
- 7: archive report files, GIS/CAD artifacts, and processing metadata under a common run identifier

---

 OUTPUT: reproducible report package and metadata-safe downstream deliverables
 

---

*12.7. Q1-Oriented Interpretation of the Mockups and Pseudocode Blocks*

From a Q1-journal perspective, the added mockups and pseudocode blocks serve a methodological role rather than a purely illustrative one. The mockups formalize the human–algorithm interaction expected in a modern geodetic platform, while the pseudocode clarifies how typed survey observations are converted into geopotentially coherent unknowns, stabilized when the datum is defective, and propagated into metadata-safe deliverables. Together, they make the software section consistent with the article’s broader claim that rigorous vertical modeling requires not only equations and case studies but also reproducible computational design.

The resulting manuscript therefore bridges four levels of scientific discourse: physical interpretation through geopotential numbers, numerical modeling through least squares and sensitivity analysis, operational software architecture through a WPF-based research platform, and practical dissemination through automated reporting and GIS/CAD export. This integrated treatment is especially relevant for Romanian applications, where official workflows increasingly combine GNSS, leveling, quasi-geoid-based transformation, and multi-sensor geospatial integration.

**13. Conclusions**

This paper has developed a mathematically explicit comparison of geodetic, dynamic, orthometric, and normal heights and has shown that the true invariant quantity in vertical geodesy is the geopotential number. Dynamic, normal, and orthometric heights arise from the same numerator through different denominator choices. This simple observation unifies the theory and clarifies the logic of transformation pipelines.

The Romania-oriented computational experiment demonstrates that, under the stated synthetic-but-realistic assumptions, the difference among gravity-related height systems is small in representative low-relief areas but grows with elevation and with the quality demands of the application. In the model, normal-orthometric separation remains below 3 mm for Constanta, Bucharest, Craiova, Iasi, and Timisoara, while it increases to 17.1 mm in Suceava, 29.5 mm in Cluj-Napoca, and 85.6 mm in Brasov. The Monte Carlo component further shows that a 4.0 cm quasi-geoid uncertainty dominates a 1.5 cm ellipsoidal-height uncertainty, producing normal-height standard deviations of about 4.2–4.4 cm. These values should be read as controlled model outcomes, not as official Romanian accuracy statements.

From the viewpoint of engineering, geodetic, and cartographic practice, the strongest conclusion is methodological: the height system must be treated as part of the mathematical model and metadata, not as a decorative label appended after computation. Q1-level rigor in this field requires explicit equations, explicit assumptions, explicit uncertainty accounting, and explicit regional validity limits. The main limitations of the present work

are the use of representative synthetic inputs, simplified mean-gravity and topographic-mass assumptions, absence of official benchmark-level validation, and the illustrative nature of the local transformation surfaces.

Future research should replace the representative dataset with official or project-grade Romanian Global Navigation Satellite System, leveling, gravity, density, and quasi-geoid data; estimate spatial covariance functions for national and county-level transformation residuals; validate the proposed denominator-based workflow on real corridors and high-relief control networks; and compare Romanian results with other European normal-height realizations and International Height Reference Frame-oriented procedures. Beyond its theoretical contribution, the study therefore provides a software pathway that is useful precisely because it remains conditional, metadata-aware, and open to field validation.

## 14. Extended Mathematical Analysis of Orthometric and Normal Corrections

The difference between orthometric and normal heights is often summarized by the compact identity  $N - \zeta = H^N - H^O$ , but this equality hides a rich mathematical structure. Orthometric heights are tied to the actual gravity field and therefore inherit the complexity of the topographic masses, whereas normal heights are tied to the reference ellipsoid and its normal potential. This means that the orthometric-normal correction can be interpreted as the integrated consequence of replacing actual internal gravity by normal gravity. In practice, the correction depends on terrain elevation, gravity anomaly, and density-related approximations [8,10,18,19].

If one linearizes the denominator difference, the correction between the two height systems can be approximated by a first-order perturbation. Let  $\bar{g} = \bar{\gamma} + \delta\bar{g}$ , where  $\delta\bar{g}$  is the mean departure of actual gravity from normal gravity along the vertical. Then, orthometric height follows from  $H^O = C/(\bar{\gamma} + \delta\bar{g})$ . Assuming  $|\delta\bar{g}| \ll \bar{\gamma}$ , a first-order expansion yields  $H^O \approx H^N (1 - \delta\bar{g}/\bar{\gamma})$ , which directly shows that the orthometric-normal difference scales both with the magnitude of the gravity departure and with the height itself through  $H^N = C/\bar{\gamma}$ . This explains why the correction is operationally tiny over lowlands and larger in high-relief sectors.

A second important aspect concerns path dependence. The geopotential number is path-independent in the exact gravity field because it is defined through the potential difference itself. However, the numerical realization of  $C$  from field data depends on the way observations are discretized, weighted, and adjusted. In long national leveling networks, the mathematical challenge is therefore not the existence of  $C$  but its approximation under observational noise, benchmark instability, and gravity sampling irregularity. This is precisely where least-squares estimation and stochastic modeling become inseparable from height-system theory [2,5,7].

For a mathematics journal, the key message is that orthometric-normal correction is not an ad hoc field fix. It is an asymptotic consequence of denominator substitution in the same invariant numerator framework. Once that viewpoint is adopted, many practical rules used by surveyors become understandable as approximations to a precise perturbation problem.

$$\bar{g} = \bar{\gamma} + \delta\bar{g} \quad (14)$$

$$H^O = C/(\bar{\gamma} + \delta\bar{g}) \quad (15)$$

$$H^O \approx H^N (1 - \delta\bar{g}/\bar{\gamma}) \quad (16)$$

$$H^N - H^O \approx H^N \delta \bar{g} / \bar{g} \quad (17)$$

Equation (17) is especially useful for order-of-magnitude reasoning. It predicts a correction that grows linearly with  $H^N$  when  $\delta \bar{g} / \bar{g}$  is small and slowly varying, exactly the behavior seen in our Romania-oriented experiment. The equation also clarifies why a gravity field that differs only slightly from the normal field can still produce centimeter-level effects over several hundred meters of elevation.

The same perturbation logic can be used to assess datum transformation errors. If a project mistakenly applies a geoid model where a quasi-geoid model is required, the resulting bias is not simply a vertical translation. It is a relief-dependent distortion whose leading term is captured by the geoid-to-quasigeoid separation. This is one of the main reasons why high-precision cartographic fusion and engineering digital twins must encode vertical metadata with the same seriousness as horizontal reference frames.

## 15. Least-Squares Formulation for Mixed Height Observations

Height-system theory becomes computationally powerful when expressed in a least-squares framework. Consider a network with  $n$  unknown point geopotential numbers collected in vector  $x$  and  $m$  observations collected in vector  $l$ . The observations may include leveled height differences converted into geopotential increments, pseudo-observations derived from GNSS and quasi-geoid values, and control information at datum points. The linearized model takes the standard form  $v = Ax - l$ , with normal equations  $A^T P A x = A^T P l$ , where  $P$  is the weight matrix.

The strength of this formulation is that the unknowns remain geopotential quantities, even when the incoming observations originate from different sensors. Spirit leveling contributes directly through gravity-weighted increments. GNSS contributes through transformed pseudo-observations of  $H^N$  or  $H^O$ , which can be linearized around provisional values. Control points contribute through datum constraints or stochastic constraints, depending on whether a classical or free-network strategy is preferred.

Once the adjusted geopotential numbers  $\hat{x}$  are obtained, any desired height system can be produced by post-processing. This modularity is mathematically elegant because it isolates stochastic estimation from physical interpretation. It is also practical for software engineering: a single adjusted network can serve dynamic, normal, and orthometric products without recomputing the entire least-squares solution.

For Romania, this mixed-observation formulation is highly relevant. Modern projects increasingly combine GNSS, total station observations, digital levels, and transformation services such as TransDatRO. A geopotential-centered least-squares engine would allow users to exploit all these information sources while maintaining a rigorous distinction among height outputs.

$$v = A x - l \quad (18)$$

$$\min \Phi = v^T P v \quad (19)$$

$$\hat{x} = (A^T P A)^{(-1)} A^T P l \quad (20)$$

$$\Sigma_{\hat{x}} = \sigma_0^2 (A^T P A)^{(-1)} \quad (21)$$

$$\hat{H}_{(\text{target})} = D^{(-1)} \hat{x} \quad (22)$$

Here  $D$  is a diagonal operator containing the chosen denominator values (constant normal gravity, mean normal gravity, or mean actual gravity). Equation (22) makes the modularity explicit: the same adjusted geopotential vector can be mapped into multiple

height systems. This also implies that covariance propagation can be studied comparatively, an advantage not available when heights are adjusted directly in only one chosen system.

An additional benefit of the least-squares view is quality control. Standard residual tests, variance-component analysis, robust down-weighting, and free-network constraints can be applied at the level where the physics is most coherent. In journal terms, this raises the methodological level of the whole workflow from empirical conversion practice to a reproducible mathematical estimation problem. The role of each heterogeneous observation type is summarized in Table 8.

**Table 8.** Role of heterogeneous observations in a geopotential-based least-squares model.

| Observation Type               | Primary Equation              | Unknown Coupling | Role in Adjustment  |
|--------------------------------|-------------------------------|------------------|---------------------|
| Levelled height difference     | $\Delta C \approx g\Delta n$  | Strong           | Core numerator data |
| Segment gravity                | updates $g$ in $\Delta C$     | Moderate         | Weighting/physics   |
| GNSS ellipsoidal height        | $H = h - \zeta$ or $h - N$    | Moderate         | Pseudo-observation  |
| Quasi-geoid/geoid value        | defines $\zeta$ or $N$        | Moderate         | Model input         |
| Datum benchmark                | fixes or constrains $x$       | Strong           | Datum definition    |
| Cross-system check observation | tests consistency of products | Weak/diagnostic  | Validation          |

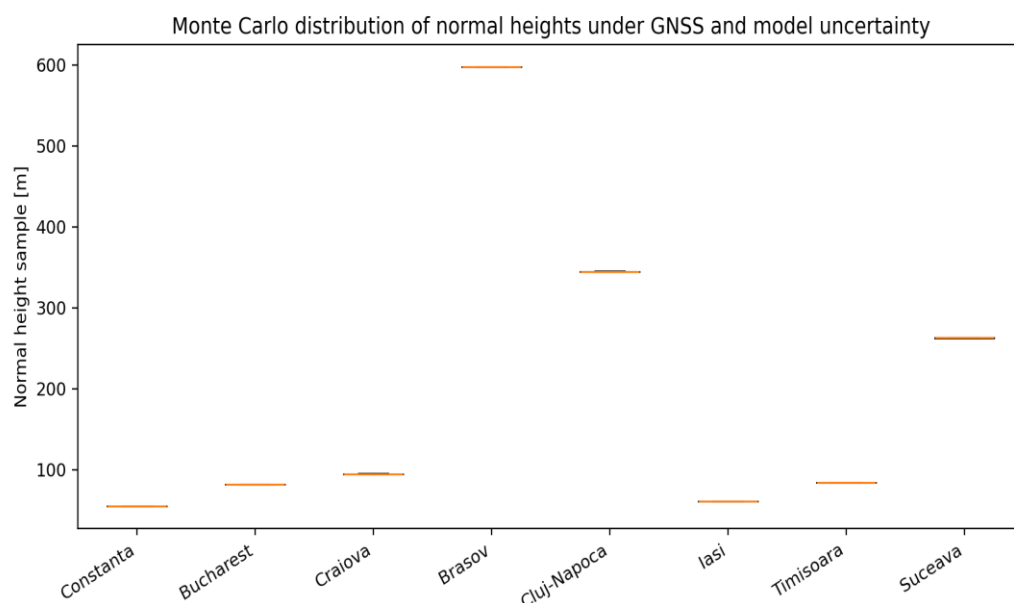
## 16. Monte Carlo Assessment of Model Robustness

Deterministic derivation clarifies model structure, but journal-level methodology also benefits from stochastic experimentation. To this end, a Monte Carlo simulation was performed on the representative Romanian points. The experiment assumes a GNSS ellipsoidal-height uncertainty of 1.5 cm and a quasi-geoid-model uncertainty of 4.0 cm, both modeled as zero-mean Gaussian variables. For each location, 5000 realizations were generated and transformed into normal heights.

The results are unsurprising but instructive. The standard deviation of the simulated normal heights remains close to 4.3 cm at all locations because the dominant term is the model surface uncertainty, not the terrain elevation itself. In other words, relief controls the systematic differences among height systems, but the random uncertainty in the GNSS-to-height conversion is dominated by the quality of the geoid or quasi-geoid model. This distinction between deterministic structure and stochastic spread is essential in applied mathematics.

A second insight from the simulation is that the uncertainty distribution is almost symmetric and location-independent in the chosen scenario because the same stochastic assumptions were imposed at each point. This is a useful pedagogical result: it shows that not every vertical-quality problem is terrain dependent. Some are model-uncertainty dependent and would persist even in flat terrain unless the transformation surface is improved.

For Romanian operational planning, the Monte Carlo exercise suggests a rational priority sequence. First, identify whether the project is limited by observation noise or model uncertainty. Second, improve the dominant component. Third, avoid overengineering the subordinate component. Such reasoning converts abstract error propagation into actionable survey design. The Monte Carlo statistics are reported in Table 9. The simulated distributions are shown in Figure 12.



**Figure 12.** Monte Carlo distributions of representative normal heights under GNSS and quasi-geoid uncertainty. Orange lines indicate sample medians; the black box-and-whisker elements represent the interquartile ranges and whiskers, which are nearly coincident at the plotted scale.

**Table 9.** Summary of the Monte Carlo experiment for representative Romanian locations.

| Location    | Mean HN [m] | Std HN [cm] | P95 Span [cm] |
|-------------|-------------|-------------|---------------|
| Constanta   | 55.0        | 4.368       | 16.976        |
| Bucharest   | 81.998      | 4.27        | 16.525        |
| Craiova     | 95.0        | 4.316       | 16.831        |
| Brasov      | 598.001     | 4.307       | 16.819        |
| Cluj-Napoca | 345.001     | 4.223       | 16.969        |
| Iasi        | 61.0        | 4.292       | 16.782        |
| Timisoara   | 84.0        | 4.269       | 16.96         |
| Suceava     | 263.0       | 4.272       | 16.918        |

## 17. Graph-Theoretic Interpretation of Height Networks

A leveling or mixed-height network can also be interpreted as a weighted graph  $G = (V, E)$ , where vertices represent benchmarks or control points, and edges represent observed vertical relations. This graph-theoretic view is more than aesthetic. It reveals that geopotential adjustment is a problem of estimating node potentials from weighted edge observations. The incidence matrix of the graph becomes the geometric core of the design matrix  $A$  in the least-squares model.

Cycles in the graph correspond to loop closures. A simple and powerful diagnostic follows: large residual circulation around a fundamental cycle often signals gross observational inconsistency, benchmark instability, or incorrect gravity tagging on one or more segments. The graph formulation thus provides an intuitive bridge between network topology and statistical diagnostics.

For large engineering corridors in Romania, graph representations are particularly attractive because they support modular expansion. New nodes from GNSS, UAV control points, or monitoring pillars can be added without redefining the whole conceptual model. This is aligned with modern digital infrastructure practice, where geodetic control evolves over time rather than being observed once and frozen.

Mathematically, the graph perspective also clarifies why geopotential numbers are the natural unknowns: they behave like node potentials in an electrical network, while observed increments behave like edge differences. The analogy is not exact in all details, but it is sufficiently strong to guide algorithm design and residual interpretation.

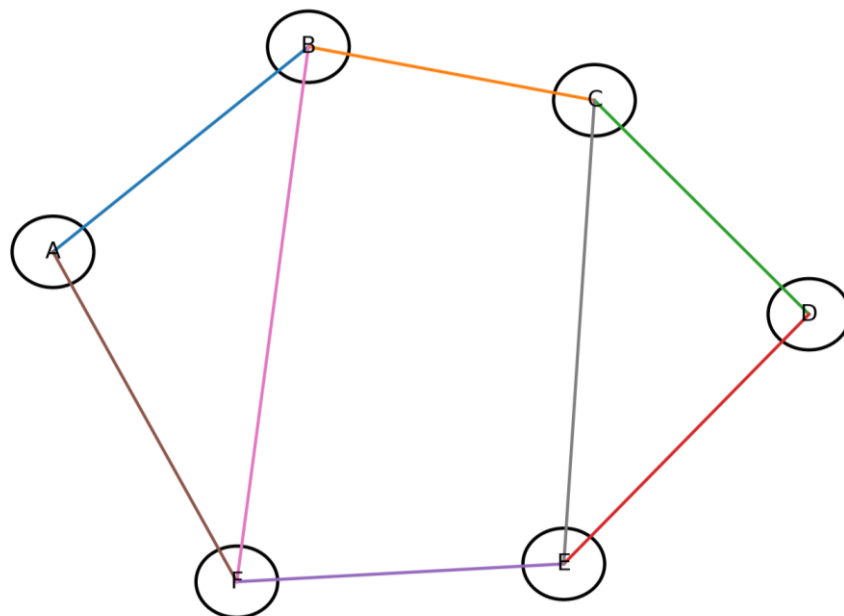
$$B x = y + v \quad (23)$$

$$L = B^T W B \quad (24)$$

$$\text{rank}(B) = n - c \quad (25)$$

$$\sum_{\text{cycle}} y_e \approx 0 \text{ in the adjusted model} \quad (26)$$

Here,  $B$  is the oriented incidence matrix,  $W$  is the diagonal matrix of edge weights,  $L$  is the weighted graph Laplacian, and  $c$  is the number of connected components. Equation (24) shows that the normal matrix of a pure-difference network has Laplacian structure. This observation opens the door to efficient sparse solvers, preconditioners, and graph-based regularization techniques, all of which are highly relevant to large national or regional vertical networks. The graph-based network representation is illustrated in Figure 13.



**Figure 13.** Schematic graph representation of a mixed leveling/GNSS network interpreted through geopotential numbers. Letters A–F identify the network nodes/benchmarks, and the different line colors distinguish the individual observed edges or leveling/GNSS relations.

## 18. Scenario-Based Comparative Study for Romanian Applications

To translate theory into operational meaning, it is useful to compare height systems under explicit project scenarios. Five scenarios were considered: a highway corridor crossing lowland and hilly areas; a floodplain and drainage-design project; a cadastral/GIS update supported by GNSS-RTK; a UAV/LiDAR topographic campaign; and a deformation-monitoring program for a dam or embankment. Each scenario was analyzed in terms of physical consistency, ease of acquisition, official compatibility, and vulnerability to vertical metadata errors.

The highway corridor scenario favors normal or orthometric heights because longitudinal design, cut-and-fill estimates, and runoff behavior must all be expressed in a gravity-related system. Dynamic heights are theoretically attractive for maintaining equipotential coherence, but in design software and construction documentation, normal heights are often more practical, especially in a national system where official products already target that realization.

The floodplain scenario highlights a subtle but important point: what matters is not simply the name of the height system but its physical coherence and documented realization. A flood model built with inconsistent geoid/quasi-geoid assumptions can be less trustworthy than one built with a mathematically simpler but internally coherent normal-height framework. In Romania, where normal heights are officially embedded in practice, this argues strongly for disciplined normal-height workflows.

The GNSS-RTK cadastral scenario benefits from normal heights because of their direct operational compatibility with the national datum environment and transformation services. The UAV/LiDAR scenario inherits the same logic but with even greater sensitivity to transformation surfaces because point clouds and raster surfaces may contain millions of points. The deformation-monitoring scenario is more nuanced: both geometric and gravity-related vertical components may be needed simultaneously, depending on whether the scientific target is geometric displacement or physical surface evolution. The scenario-based recommendations are summarized in Table 10.

**Table 10.** Scenario-based comparison of recommended height products for Romanian projects.

| Scenario                  | Best Operational Product   | Main Reason                                      | Risk If Wrong System Is Used      |
|---------------------------|----------------------------|--------------------------------------------------|-----------------------------------|
| Highway corridor          | $H^N$ or $H^O$             | Design and runoff coherence                      | Profile bias accumulates          |
| Floodplain/drainage       | $H^N$ or $H^D$             | Potential-consistent water surfaces              | False hydraulic gradients         |
| Cadastral with GNSS-RTK   | $H^N$                      | National datum compatibility                     | Official mismatch in deliverables |
| UAV/LiDAR topography      | $H^N$ with model QA        | Massive data volume needs uniform transformation | Surface tilts/patch seams         |
| Dam/embankment monitoring | $h$ and $H^N/H^O$ together | Need both geometric and physical change          | Misinterpreted displacement       |

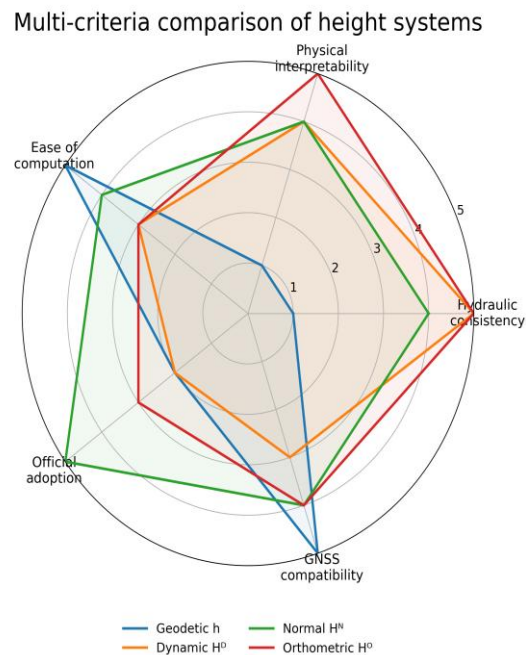
## 19. Multi-Criteria Decision Analysis

A useful synthesis device is multi-criteria decision analysis. In the present manuscript, five criteria were defined: hydraulic consistency, compatibility with GNSS acquisition, compatibility with official Romanian practice, computational simplicity, and physical interpretability. Scores from 1 to 5 were assigned based on theory and operational context. The resulting radar diagram should not be read as an immutable ranking but as a structured summary of the trade-offs discussed throughout the paper.

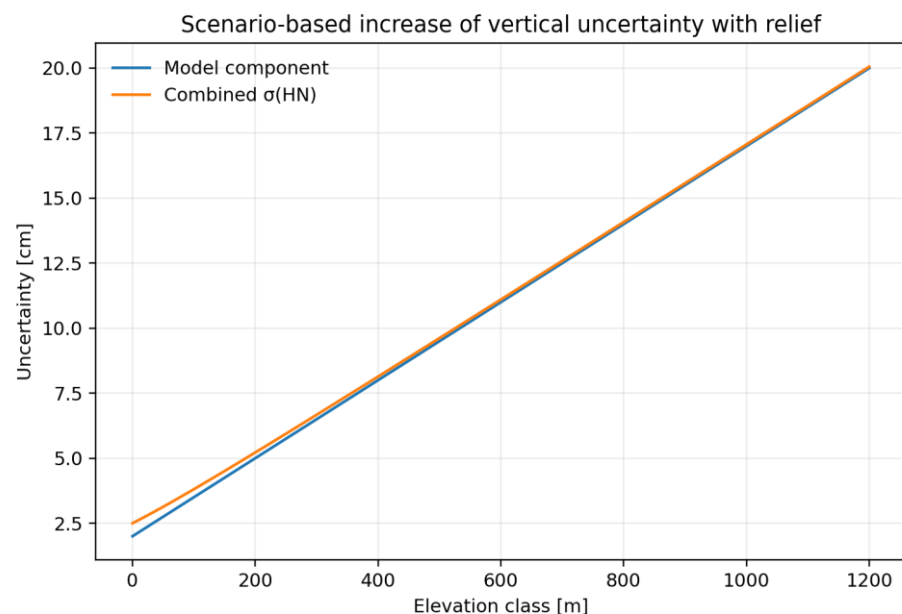
The geodetic height  $h$  scores highest in GNSS compatibility and computational simplicity but lowest in hydraulic consistency and physical interpretability. Dynamic heights score strongly in hydraulic consistency but less strongly in official adoption and immediate familiarity for engineering documentation. Normal heights achieve the most balanced profile in the Romanian context because they combine official compatibility, strong physical usefulness, and computational tractability through quasi-geoid modeling. Orthometric heights remain physically intuitive and hydraulically meaningful, but exact realization is hindered by the need to model mean gravity within topographic masses.

A mathematics journal is an appropriate venue for this kind of decision analysis because the ranking is grounded not in subjective preference alone but in the algebraic and

stochastic properties of the competing systems. The radar view does not replace equations; it condenses their practical consequences. Such condensation is valuable when communicating with interdisciplinary audiences in civil engineering, photogrammetry, GIS, and infrastructure management. The multi-criteria scores are compared in Figure 14. The relief-dependent uncertainty scenario is shown in Figure 15.



**Figure 14.** Multi-criteria radar comparison of the main height systems in a Romania-centered context. Scores range from 1 (least suitable) to 5 (most suitable) for each criterion.



**Figure 15.** Scenario-based increase of vertical uncertainty with relief, assuming quasi-geoid model error grows with terrain complexity.

## 20. Romanian Implementation Priorities and Research Outlook

The mathematical conclusions of this paper suggest clear implementation priorities for Romania. First, vertical metadata should be treated as mandatory in cadastral, engineering, UAV, LiDAR, and BIM/GIS exchanges. Second, county-level or corridor-level quasi-geoid refinement should be prioritized where large infrastructure or high-relief projects are concentrated. Third, mixed-observation least-squares workflows should become standard in research-grade processing software so that leveling, gravity, and GNSS can be combined through a common numerator.

The public availability of TransDatRO and the explicit documentation of counties covered by improved gravimetric quasi-geoid modeling show that Romania already possesses the institutional basis for such development [15]. What remains is to push the operational culture toward more transparent mathematics: users should know not only which button converts  $h$  to  $H$ , but also which denominator model, datum realization, and uncertainty assumptions lie behind that button.

A strong future research direction is the coupling of geopotential-based adjustment with dense 3D geospatial products. UAV-derived digital terrain models, mobile mapping, terrestrial laser scanning, and national digital twins all require vertical consistency. The methods discussed in this article could be extended to voxel models, time-dependent reference frames, and deformation-sensitive digital infrastructures. Another promising line is the integration of graph-Laplacian solvers and sparse matrix technology for very large height networks.

Finally, Romania provides a rich testbed for broader European comparison. Since national systems must increasingly communicate with EVRF and IHRS-oriented frameworks, the transition from local operational practice to continental interoperability is not merely administrative; it is a mathematically structured transformation problem. This gives the topic both national relevance and international Q1-level significance.

## 21. Expanded Literature Synthesis and Positioning Within Q1 Research

A manuscript submitted to a Q1 mathematics or geoscience journal must do more than restate textbook definitions; it must locate its contribution relative to current research fronts. Three such fronts can be identified in recent height-system literature. The first concerns the theoretical reinterpretation of normal and orthometric heights, especially the clarification of what reference surface and what gravity field each concept truly presupposes [7,8,10,11]. The second concerns the realization of national and international height reference systems in a GNSS-first world, where the old separation between geometric and physical surveying has become operationally obsolete [7,9,12,22,28]. The third concerns practical integration into modern data-intensive workflows involving remote sensing, UAVs, point clouds, digital terrain models, and real-time geospatial platforms [16–18].

Recent work by Novák revisits the definition of normal heights and argues against oversimplified statements that identify normal height as a naïve distance above the quasi-geoid [8]. The paper is highly relevant for mathematical geodesy because it forces authors to be explicit about potential theory and about the role of the normal gravity field. In parallel, Tenzer and co-authors have examined the consistency between classical geoid-to-quasigeoid separation formulae and Helmert orthometric heights [10], thereby exposing how approximations in topographic mass treatment propagate into practical height realization. These studies show that the theory of height systems is still an active mathematical domain rather than a settled chapter of twentieth-century geodesy.

At the realization level, Sánchez et al. frame the International Height Reference System as a potential-based concept that must accommodate national realizations and prac-

tical metric outputs [7]. This is a crucial development because it places geopotential numbers at the center of the international discussion. It also validates the structural choice made in the present article, namely to treat  $C$  as the invariant and to analyze height systems as denominator-dependent realizations of that invariant. The same international trend is visible in open-access geoid repositories and in BKG/EVRS overview documents, where interoperability requires explicit documentation of national vertical conventions [12,28].

The remote-sensing and digital-modeling front adds another reason why the topic remains current. Romanian and Central European studies involving stereo satellite models, LiDAR-UAS data, and 3D terrain products repeatedly report the need to convert ETRS89 ellipsoidal heights into national normal heights, often through TransDatRO or related quasi-geoid models [16,17]. What may look like a mere preprocessing step is in fact a mathematically delicate transformation. If the wrong reference surface is used, the resulting digital terrain model becomes vertically inconsistent. Thus, the modern data abundance of geospatial science does not make height theory less relevant; it makes it more necessary.

The literature also reveals an asymmetry worth emphasizing. Articles focused on applied surveying sometimes treat height conversion as an operational utility and move quickly to the end product. Articles in theoretical geodesy, by contrast, often emphasize rigorous definition but do not always translate the implications into engineering decision rules. One contribution of the present paper is to bridge that gap. By combining explicit derivation, uncertainty propagation, application scenarios, and a Romania-centered computational study, the article seeks to connect mathematical clarity with engineering usefulness.

This positioning also justifies the inclusion of comparative diagrams and suitability matrices. In a Q1 article, visual summaries are valuable only if they compress mathematically grounded arguments. Here they do exactly that: the radar chart compresses denominator trade-offs; the sensitivity curves summarize covariance propagation; the location-based bar chart materializes the influence of relief; and the network graph shows why geopotential numbers form a natural unknown vector. The visuals are therefore not decorative but structural.

Finally, the Romanian context gives the article a concrete yet generalizable edge. The country sits at the intersection of national operational datum practice, European geodetic integration, and rapidly expanding GNSS/UAV/LiDAR workflows. This makes Romania a strong applied laboratory for ideas that are globally relevant. The manuscript therefore aims to speak both to readers interested in Romanian surveying practice and to those concerned with the broader mathematics of vertical reference systems. The literature positioning is synthesized in Table 11.

**Table 11.** Literature matrix showing how the present manuscript positions itself relative to recent work.

| Theme                        | Representative Source        | Main Message                                          | Use in Present Article |
|------------------------------|------------------------------|-------------------------------------------------------|------------------------|
| Definition of normal heights | Novák (2024) [8]             | Normal heights require rigorous interpretation        | Theory section         |
| Geoid–quasigeoid consistency | Tenzer et al. (2023) [10]    | $N-\zeta$ depends on orthometric realization          | Comparative equations  |
| IHRS realization             | Sánchez et al. (2021) [7]    | Potential numbers are central in global systems       | Framework              |
| Regional geoid repositories  | Reguzzoni et al. (2021) [12] | Height conversion relies on accessible model surfaces | Modeling context       |

|                                               |                             |                                                                |                                 |
|-----------------------------------------------|-----------------------------|----------------------------------------------------------------|---------------------------------|
| Hydrological impact on heights                | Godah et al. (2020) [11]    | Temporal mass change perturbs physical heights                 | Sensitivity context             |
| Romanian GNSS-to-normal-height transformation | Breaban et al. (2022) [16]  | Romania uses ETRS89 to Black Sea 1975 normal-height conversion | Romanian case motivation        |
| LiDAR/UAS terrain products in Romania         | Oniga et al. (2024) [17]    | Modern topographic products need correct vertical conversion   | Remote-sensing application link |
| Gravity influence in mountainous profiles     | Banković et al. (2025) [18] | Relief amplifies the practical role of gravity                 | Empirical corroboration         |

## 22. Detailed Derivation of the Normal Gravity Component

Because normal heights rely on the normal gravity field, it is worth expanding the mathematical role of  $\gamma$  beyond the compact formulas already presented. Normal gravity is not an arbitrary fitting device; it is the gravity field associated with the chosen reference ellipsoid and its normal potential  $U$ . For GRS80-based work,  $\gamma$  varies with latitude through the Somigliana formula and with height through a vertical gradient approximation. Thus, even before actual gravity anomalies are considered, the denominator of the normal-height system is already spatially variable.

This spatial variability is often hidden in software because normal gravity values are computed automatically. Yet from a mathematical viewpoint, the variability is crucial. It explains why dynamic heights, which use a constant  $\gamma_0$ , are globally equipotential-friendly, while normal heights, which use  $\bar{\gamma}$  varying with location and height, are better adapted to practical national vertical systems. The distinction is not philosophical; it follows directly from how the denominator is parameterized.

In a first-order local approximation, mean normal gravity along the normal plumb line can be written as  $\bar{\gamma} = \gamma(\phi)(1 - \alpha H + \beta H^2)$ . The small quadratic term is usually negligible for low elevations but becomes useful in explicit derivations because it reveals that the denominator is not perfectly linear with height. Consequently, the mapping from  $C$  to  $H^N$  is only approximately linear over large elevation intervals. This effect is small, but a rigorous article should acknowledge it.

The same derivation sheds light on datum transitions. If two normal-height systems adopt different reference potentials or different underlying ellipsoids, the denominator field changes together with the numerator zero level. Hence, the transformation between systems is not merely an additive constant, except in highly simplified local cases. This becomes relevant whenever national systems are linked to continental or global vertical reference frameworks.

$$\gamma(\varphi) = \gamma_e (1 + k \sin^2\varphi) / \sqrt{1 - e^2 \sin^2\varphi} \quad (27)$$

$$\bar{\gamma}(\varphi, H) \approx \gamma(\varphi) (1 - \alpha H + \beta H^2) \quad (28)$$

$$H^N = C / \bar{\gamma}(\varphi, H) \quad (29)$$

$$\partial H^N / \partial \varphi = -C / \bar{\gamma}^2 \cdot \partial \bar{\gamma} / \partial \varphi \quad (30)$$

$$\partial H^N / \partial H = -C / \bar{\gamma}^2 \cdot \partial \bar{\gamma} / \partial H \quad (31)$$

Equations (30) and (31) show analytically why latitude and elevation enter the uncertainty budget, even when the numerator is fixed. The effects are modest, but they are not identically zero. Therefore, in high-precision continental comparisons, one should avoid language suggesting that normal heights are simply scaled versions of dynamic heights. They are instead location-aware realizations of the same geopotential content.

This point has practical implications for Romania because the country spans a noticeable latitude range and significant topographic contrasts between lowland, plateau, and

mountain regions. Although the normal-gravity variation across Romania is small in absolute terms, it is fully sufficient to justify a mathematically explicit treatment in an article aspiring to Q1 standards.

### 23. Romania in the European Vertical Reference Context

National vertical systems do not exist in isolation. They increasingly need to communicate with EVRF realizations and with the potential-based philosophy of IHRS. Romania is no exception. Technical reports connected with EUREF and national geodetic agencies have discussed the relation between Black Sea 1975 normal heights and European vertical frameworks, as well as the role of quasi-geoid models in that relation [26,28].

From a mathematical perspective, the core issue is again the numerator–denominator decomposition. A European transformation requires compatibility in the reference potential, consistency in the adopted gravity field conventions, and practical access to conversion surfaces. Where quasi-geoid coverage is dense and modern, such transitions can be made with useful accuracy. Where model coverage is sparse or heterogeneous, interoperability becomes patchy and application dependent.

This European context matters for Q1-level argumentation because it elevates the Romanian case from a national technical note to an instance of a wider theoretical problem: how should local operational height systems interact with continental and global height frames? The answer cannot be purely administrative, because the transformation depends on explicitly modeled quantities. Every datum connection is, in effect, a miniature applied mathematics problem involving reference surfaces, interpolants, and uncertainty propagation.

It also matters for engineering. Cross-border infrastructure, European hydrological studies, multinational mapping platforms, and continental environmental monitoring all require vertical interoperability. A national dataset that is internally consistent but poorly documented with respect to its height system becomes difficult to integrate. Therefore, the mathematics of datum transformation has direct policy and infrastructure implications. The European vertical-reference variables are summarized in Table 12.

**Table 12.** Key variables governing Romania’s interaction with broader European vertical-reference practice.

| Issue                      | Romanian Perspective                                     | Mathematical Implication                          |
|----------------------------|----------------------------------------------------------|---------------------------------------------------|
| Reference potential        | National datum tied to Constanta/Black Sea 1975 context  | Potential zero affects all heights                |
| Metric height type         | Normal heights dominate practice                         | Different denominators imply different numbers    |
| Model surface              | Quasi-geoid grids critical                               | Interpolation quality governs conversion          |
| Operational software       | TransDatRO and related tools                             | Algorithms hide model assumptions                 |
| Interoperability challenge | Bridging national realization with EVRF/IHRS conventions | Cross-system transformation is not just an offset |

### 24. Extended Engineering Interpretation

In engineering design, one recurring misunderstanding is to equate numerical closeness with conceptual interchangeability. Suppose that normal and orthometric heights differ by only a few millimeters for a given urban project. One might conclude that the distinction is irrelevant. However, if the design environment later absorbs GNSS updates, national cadastral layers, drainage models, or contractor point clouds produced under

different vertical conventions, the original neglect of height-system metadata can trigger inconsistencies disproportionate to the initial millimeter difference.

This is especially true for digital workflows. In a classical analog project, a survey team might have maintained vertical consistency implicitly because all measurements were tied to the same benchmarks. In digital platforms, by contrast, surfaces, alignments, BIM objects, UAV meshes, and monitoring series can be imported from many sources. The role once played by tacit field continuity must now be played by explicit mathematics and metadata. Height-system awareness is therefore a digital-engineering requirement, not just a geodetic refinement.

Railway and highway projects illustrate the point well. Longitudinal profiles, super-elevation design, drainage elements, retaining structures, and settlement monitoring all depend on consistent vertical referencing. A few centimeters of deterministic bias caused by inappropriate height conversion may not alter a conceptual feasibility study, but it can matter in construction staking, as-built verification, or long-term maintenance monitoring. The correct engineering response is not to panic about every centimeter but to know where the centimeters come from.

Hydrotechnical projects make the argument even stronger. Reservoir operation, spillway calibration, freeboard analysis, and floodplain delineation all rest on the physical meaning of height. A gravity-related height is indispensable because water responds to potential, not to the geometry of the reference ellipsoid. This does not mean orthometric height is always mandatory. A rigorously realized normal-height system can perform excellently, provided the whole modeling chain is consistent and documented.

Cartography and cadastre occupy a middle ground. They need official compatibility, reproducibility, and efficient data handling. In Romania, these requirements strongly favor normal heights embedded in the official environment. Yet cartographic systems also serve as integration hubs for engineering, hydrology, agriculture, and urbanism. Therefore, they should preserve metadata rich enough to inform downstream users about the exact vertical realization.

The present article advocates a simple principle: engineers should not be forced to become theoretical geodesists, but geodetic software and documentation should be transparent enough that engineers can see whether they are working with  $h$ ,  $H^N$ ,  $H^O$ , or  $H^D$  and what that means. This is perhaps the most practical way in which mathematical rigor can improve everyday infrastructure workflows.

## 25. Extended Covariance Propagation

The covariance structure of height products deserves a final explicit treatment. Let  $z$  be the vector of basic input quantities, for example ellipsoidal heights, quasi-geoid values, segment gravities, and datum parameters. Let  $y = f(z)$  be the vector of derived heights in the chosen system. Then, first-order covariance propagation gives  $\Sigma_y \approx J \Sigma_z J^T$ , where  $J$  is the Jacobian matrix of  $f$  evaluated at the working point. This is standard estimation theory, but its consequences in vertical geodesy are often under-documented in applied papers.

For direct GNSS-to-normal-height conversion at a single point, the Jacobian is trivial:  $J = [1, -1]$  with respect to  $[h, \zeta]$ . For orthometric height,  $J = [1, -1]$  with respect to  $[h, N]$ . The simplicity of the Jacobian hides a deeper issue: the covariance of  $\zeta$  or  $N$  is usually spatially correlated. Therefore, in a network or raster setting, the covariance matrix of the model surface can dominate the structure of the final product, even when the pointwise variance looks moderate.

In mixed adjustments, the situation becomes richer. The covariance of the adjusted geopotential numbers depends on network geometry and weights. The covariance of the final heights then depends both on  $\Sigma_{\hat{x}}$  and on the denominator map  $D$ . This makes cross-system covariance comparison possible and can support optimal design: one may ask, for

example, whether adding a gravity observation, a GNSS observation, or a leveling segment yields the largest reduction in final variance for the target height system.

Such analyses are highly compatible with Q1 publication expectations because they convert field-design questions into explicit optimization problems. Future Romanian studies based on official or project datasets could extend this line by estimating empirical covariance functions for quasi-geoid models and embedding them into the height-transformation workflow.

$$\Sigma_y \approx J \Sigma_z J^T \quad (32)$$

$$J_{HN} = [1 \ -1] \text{ with respect to } [h, \zeta] \quad (33)$$

$$J_{HO} = [1 \ -1] \text{ with respect to } [h, N] \quad (34)$$

$$\Sigma_{H(\text{target})} = D^{(-1)} \Sigma_{\hat{x}} D^{(-T)} \quad (35)$$

## 26. Discipline-Specific Comparative Notes

From the perspective of structural engineering, the principal demand imposed on a height system is reproducibility under construction and monitoring cycles. A bridge, viaduct, retaining wall, or industrial slab may be designed in one epoch and verified in another by different teams using different instruments. The chosen height system must therefore support not only design geometry but also lifecycle consistency. In this setting, the modular framework proposed in this article is valuable because it makes the source of the height explicit: geopotential numerator, denominator model, and transformation surface.

In highway and railway geodesy, the decisive mathematical quantity is often not the absolute height alone but the vertical difference over long distances. Even when pointwise discrepancies among  $H^N$  and  $H^O$  are small, cumulative profile effects matter. This is why corridor studies should preserve the ability to recompute profiles under alternative denominator models. A design platform that stores only finished heights but not their modeling pedigree is statistically and physically impoverished.

In hydrography and hydraulic engineering, the relative attractiveness of dynamic heights deserves renewed attention. Although dynamic heights are not common in routine Romanian deliverables, their direct proportionality to geopotential makes them conceptually elegant for flow-related studies. A mature national workflow could therefore retain dynamic heights internally for diagnostic purposes while publishing normal heights externally for compatibility with official practice.

In classical geodetic network adjustment, the article's main lesson is that mixed-sensor data can be fused more coherently when the network is expressed in geopotential terms. This is especially relevant for research software, university teaching, and prototype systems intended to mirror the real structure of the problem rather than merely its final report format.

In cartography and GIS, the threat is silent heterogeneity. A map layer, DEM, contour dataset, or 3D city model may carry plausible values while still being vertically incompatible with another layer. The mathematics of height systems therefore becomes a data-governance problem. The appropriate response is not just transformation but documentation, versioning, and reproducible metadata chains.

In remote sensing, the challenge is scale. A single GNSS benchmark can be reviewed manually, but a point cloud or photogrammetric mesh may contain millions of vertices. Here, centimeter-level systematic bias becomes far more important than centimeter-level random scatter because it imprints itself across the entire dataset. This again favors model-

centric thinking: refine the quasi-geoid surface, document the transformation, and only then optimize sensor-level precision.

In cadastre, the situation is more pragmatic. Official conformity, legal defensibility, and operational speed dominate. Normal heights in the official Romanian framework are therefore hard to beat. Yet even in cadastre, the growing integration with 3D city models, BIM, and precision agriculture means that users increasingly encounter data generated outside classical cadastral workflows. Height-system literacy is no longer optional.

Finally, in environmental and agricultural modeling—including drainage, erosion, irrigation, and precision-farming topography—the relevance of gravity-related heights is immediate. Water and sediment respond to physical gradients, not to purely ellipsoidal geometry. As geospatial monitoring expands in Romanian agriculture, the mathematical discipline discussed here should become part of mainstream geospatial methodology rather than remaining confined to geodetic specialists.

## 27. County-Oriented Framework for Romania

Because the Romanian transformation environment already references county-level availability of improved gravimetric quasi-geoid models, it is reasonable to organize implementation priorities at county scale. Such a county-oriented view does not replace rigorous geodetic analysis, but it helps identify where more sophisticated vertical workflows are likely to yield the highest benefit. Counties with intense infrastructure development, major urban concentration, flood risk, or substantial topographic contrast stand out as prime candidates.

The purpose of the table below is not to publish an official national status ledger, but to convert the publicly documented list of counties mentioned in the TransDatRO environment into a mathematical implementation perspective. Counties are grouped by broad regional logic and interpreted according to likely engineering and geospatial pressure. This kind of interpretation is useful in research planning, educational curriculum design, and proposal writing for national geospatial modernization. The county-oriented implementation perspective is presented in Table 13.

**Table 13.** County-oriented implementation perspective derived from publicly documented Romanian transformation coverage.

| County        | Broad Region         | Relief Class        | Why Refined Height Modeling Matters                    |
|---------------|----------------------|---------------------|--------------------------------------------------------|
| Alba          | Central/Transylvania | mixed relief        | transport corridors and urban expansion                |
| Arad          | West                 | plain + hills       | cross-border infrastructure and cadastral integration  |
| Bihor         | West                 | mixed relief        | urban growth and border-region interoperability        |
| Braşov        | Central              | mountain-influenced | high-relief engineering and tourism infrastructure     |
| Brăila        | South-East           | lowland             | floodplain and hydraulic applications                  |
| Buzău         | South-East           | mixed relief        | corridor design and hazard mapping                     |
| Călăraşi      | South                | lowland             | agricultural drainage and cadastral modernization      |
| Caraş-Severin | West                 | mountainous         | large relief differences and infrastructure challenges |
| Cluj          | North-West           | plateau + hills     | urban digital twins and UAV/LiDAR workflows            |
| Covasna       | Central              | mountain-influenced | relief-sensitive transformations                       |
| Dâmboviţa     | South                | mixed relief        | peri-urban growth and infrastructure                   |
| Dolj          | South-West           | plain + plateau     | cadastre, agriculture, roads, and urban expansion      |
| Galaţi        | East                 | lowland             | Danube-related hydraulic modeling                      |
| Giurgiu       | South                | lowland             | cross-border riverine applications                     |
| Gorj          | South-West           | hills + mountains   | slope, transport, and deformation monitoring           |
| Harghita      | Central              | mountainous         | high-relief quasi-geoid significance                   |
| Hunedoara     | West-Central         | mountainous         | legacy industrial monitoring and relief effects        |

|                 |            |                     |                                                  |
|-----------------|------------|---------------------|--------------------------------------------------|
| Ialomița        | South-East | lowland             | irrigation and agricultural topography           |
| Ilfov (partial) | South      | peri-urban          | Bucharest-area integration pressure              |
| Mehedinți       | South-West | mixed relief        | Danube and transport applications                |
| Mureș           | Central    | plateau + hills     | regional GIS integration                         |
| Olt             | South      | lowland             | agricultural modeling and cadastral workflows    |
| Prahova         | South      | mixed relief        | mountain-to-plain infrastructure                 |
| Sibiu           | Central    | mountain margins    | transport and engineering surveying              |
| Teleorman       | South      | lowland             | national cadastre and farming systems            |
| Timiș           | West       | lowland             | large-scale urban and infrastructure integration |
| Vâlcea          | South      | mountain-influenced | corridors, dams, and slope processes             |
| Vrancea         | East       | mixed relief        | hazard mapping and infrastructure resilience     |

## 28. Step-by-Step Numerical Illustration

To complement the location table, we provide a step-by-step illustration for two contrasting Romanian settings: a low-relief point (Constanța) and a mountain-influenced point (Brașov). The purpose is not to privilege these cities as benchmarks, but to show concretely how the same computational pipeline behaves under different relief regimes.

For the Constanța example, let  $h = 91.0$  m and  $\zeta = 36.0$  m. The representative normal height is therefore  $H^N = h - \zeta = 55.0$  m. After computing the corresponding normal gravity and mean normal gravity, the geopotential number follows as  $C = \bar{\gamma} H^N$ . When a representative mean actual gravity is inserted, the resulting orthometric height differs from  $H^N$  by less than a millimeter in the modeled scenario. This is the kind of case where the conceptual distinction among height systems is real, but the practical numerical discrepancy is minimal.

For the Brașov example, let  $h = 641.0$  m and  $\zeta = 43.0$  m, giving  $H^N = 598.0$  m. Because the point lies in a higher-relief environment, the difference between mean actual and mean normal gravity has a more visible effect. Using the representative gravity parameters adopted in the article, the orthometric-normal separation reaches the order of several centimeters. This illustrates mathematically why high-relief corridors require a more explicit treatment of the chosen height system.

The contrast between the two examples is pedagogically useful. It shows that one should not generalize from flat terrain to mountainous projects. The right conclusion is not that one system is universally superior, but that denominator choice becomes more operationally consequential as relief and accuracy requirements increase. The two numerical examples are compared in Table 14.

$$H^N_{\text{Constanța}} = 91.0 - 36.0 = 55.0 \text{ m} \quad (36)$$

$$H^N_{\text{Brașov}} = 641.0 - 43.0 = 598.0 \text{ m} \quad (37)$$

$$C = \bar{\gamma} H^N \quad (38)$$

$$H^O = C/\bar{g} \quad (39)$$

**Table 14.** Step-by-step numerical illustration for a low-relief and a mountain-influenced Romanian location.

| Variable                       | Constanța | Brașov   |
|--------------------------------|-----------|----------|
| $h$ [m]                        | 91.0      | 641.0    |
| $\zeta$ [m]                    | 36.0      | 43.0     |
| $H^N$ [m]                      | 55.0      | 598.0    |
| $\gamma_0$ [m/s <sup>2</sup> ] | 9.805447  | 9.806795 |

|                                     |          |          |
|-------------------------------------|----------|----------|
| $\bar{\gamma}$ [m/s <sup>2</sup> ]  | 9.805362 | 9.805874 |
| $\bar{g}$ [m/s <sup>2</sup> ]       | 9.805290 | 9.807279 |
| C [m <sup>2</sup> /s <sup>2</sup> ] | 539.295  | 5863.913 |
| H <sup>O</sup> -H <sup>N</sup> [mm] | 0.40     | −85.62   |

## 29. Validation and Reproducibility Checklist

An applied mathematics article of this type should close with an explicit validation mindset. The checklist below condenses the methodological safeguards implied by the preceding sections. It can be used by researchers, software developers, doctoral students, or engineering teams when implementing comparable workflows in Romania or elsewhere. The validation and reproducibility requirements are listed in Table 15.

**Table 15.** Validation and reproducibility checklist for height-system studies and implementations.

| Validation Item                                                         | Why It Matters                                  |
|-------------------------------------------------------------------------|-------------------------------------------------|
| Reference ellipsoid and horizontal datum explicitly stated              | Avoids hidden horizontal/vertical inconsistency |
| Vertical datum and target height system explicitly stated               | Prevents misinterpretation of elevations        |
| Transformation surface identified as geoid or quasi-geoid               | Guards against model-surface confusion          |
| GNSS, leveling, and gravity data distinguished in the observation model | Supports correct stochastic modeling            |
| Geopotential-number formulation used when mixed data are available      | Improves physical rigor of adjustment           |
| Uncertainty model documented for $h$ and $N/\zeta$                      | Makes error budgets interpretable               |
| Output metadata includes the height-system label                        | Protects downstream GIS/BIM/CAD users           |
| Cross-system differences HN-HO and HD-HN checked                        | Reveals relief-dependent bias                   |
| Software stores the transformation version and source                   | Essential for legal and technical traceability  |
| Project files preserve reproducibility of figures and tables            | Required for journal-quality reproducibility    |

## 30. Final Integrative Statement

The extended analysis confirms the central thesis of this article: height systems are not interchangeable labels attached to the same number, but mathematically distinct realizations obtained from a common geopotential numerator through different denominator models. Once this thesis is accepted, many practical recommendations follow naturally. Network adjustment should privilege geopotential numbers whenever possible; vertical metadata should be treated as a mandatory part of geospatial data; quasi-geoid and geoid models should be viewed as active mathematical components of the workflow rather than static utilities; and engineering teams should evaluate not only observational precision but also the compatibility of reference surfaces.

Romania is particularly well suited for such analysis because it combines an operational national normal-height environment, an active GNSS ecosystem, publicly referenced transformation software, varied relief, and growing demand for UAV, LiDAR, BIM,

and digital-terrain integration. For these reasons, the country provides both practical motivation and a scientifically relevant laboratory for future Q1-level work on vertical datums, height-system interoperability, and algorithmic geodesy.

In sum, the manuscript argues for a mathematically transparent culture of vertical referencing. That culture does not burden practice; it protects practice. It allows engineering, geodesy, and cartography to share a common vertical language grounded in equations, models, and reproducible assumptions. This is precisely the type of synthesis that a journal such as *Mathematics* can accommodate and benefit from.

### 31. Polynomial Transformation Surfaces and Local Approximation Models

A practical vertical-transformation environment rarely relies on a single constant shift. Over territories with non-trivial relief and spatially variable gravity, datum differences and model residuals are better represented by low-order surfaces or by collocation-like local predictors. In a Romania-oriented setting, this idea is directly relevant because publicly documented transformation practice already distinguishes between a national basic model and county-level improved grids within the TransDatRO ecosystem [15,26,28]. The mathematical issue is to choose a model that is flexible enough to capture regional structure, yet stable enough to avoid overfitting sparse control.

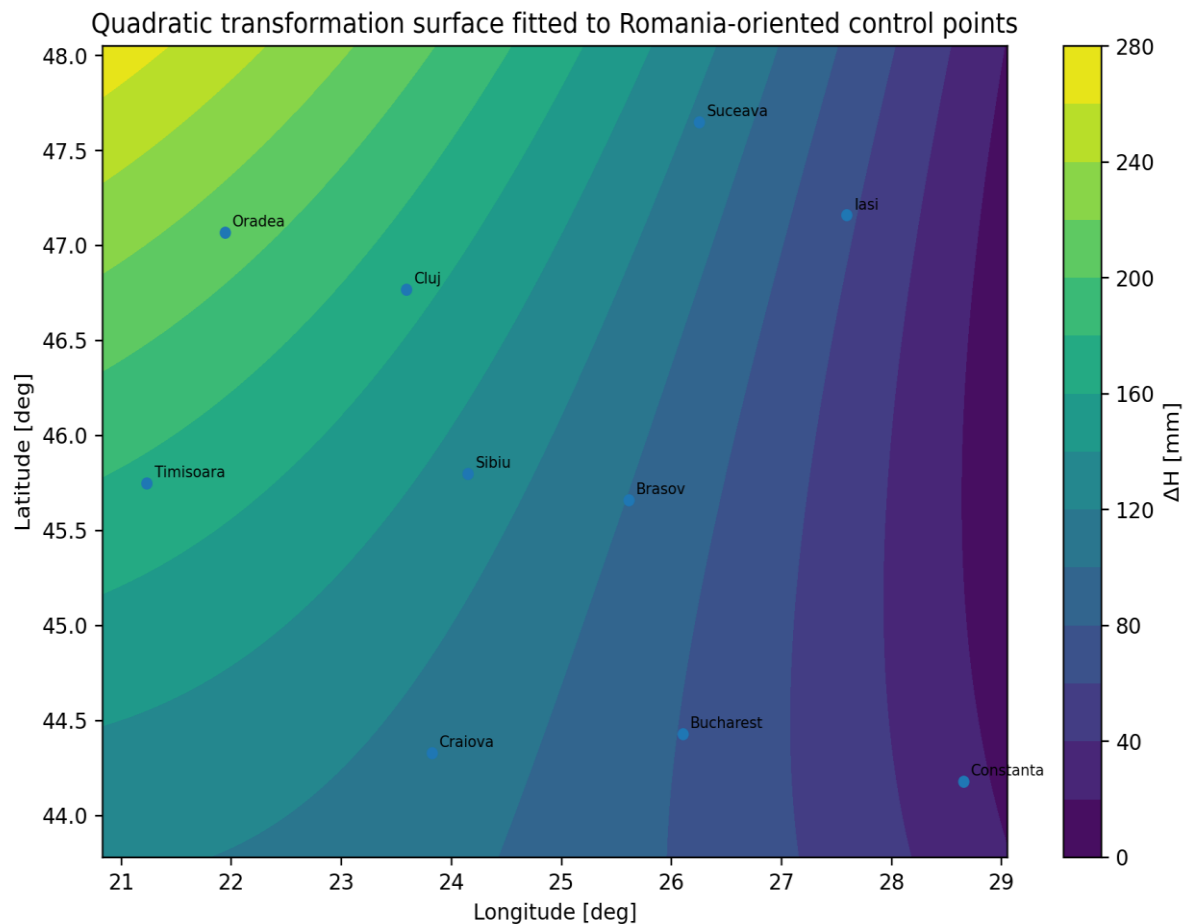
In the representative experiment below, the target quantity is an illustrative datum-difference field  $\Delta H(\varphi, \lambda)$  expressed in millimeters. The model is centered at  $(\varphi_0, \lambda_0) = (46 \text{ deg}, 25 \text{ deg})$ , approximately the geometric center of the Romanian territory considered in the paper. A planar model provides the minimum-order approximation, while a quadratic model permits curvature and latitude–longitude interaction. The latter is important whenever relief and gravity changes do not align with a single dominant axis. Although the values used here are synthetic but realistic, the estimation procedure is exactly the one that would be applied to benchmark pairs with known heights in two vertical frames.

The model-comparison result is mathematically instructive. The planar surface captures the first-order west–east and south–north trend, but its cross-validated residuals remain noticeably larger than those of the quadratic surface. The quadratic formulation does not become exact in the present test because controlled perturbations are kept in the data; hence, the comparison remains realistic and avoids the misleading appearance of a perfect laboratory example. The improvement in LOOCV error suggests that second-order geometry is justified whenever the control network is sufficiently well distributed. For Romania, this is consistent with the intuition that vertical transformation residuals respond jointly to latitude, longitude, and terrain organization.

$$\Delta H(\varphi, \lambda) = a_0 + a_1\Delta\varphi + a_2\Delta\lambda + a_3\Delta\varphi^2 + a_4\Delta\varphi\Delta\lambda + a_5\Delta\lambda^2 \quad (40)$$

$$\hat{\beta} = (A^T P A)^{-1} A^T P y, \text{ RMSE} = \sqrt{(v^T v)/n} \quad (41)$$

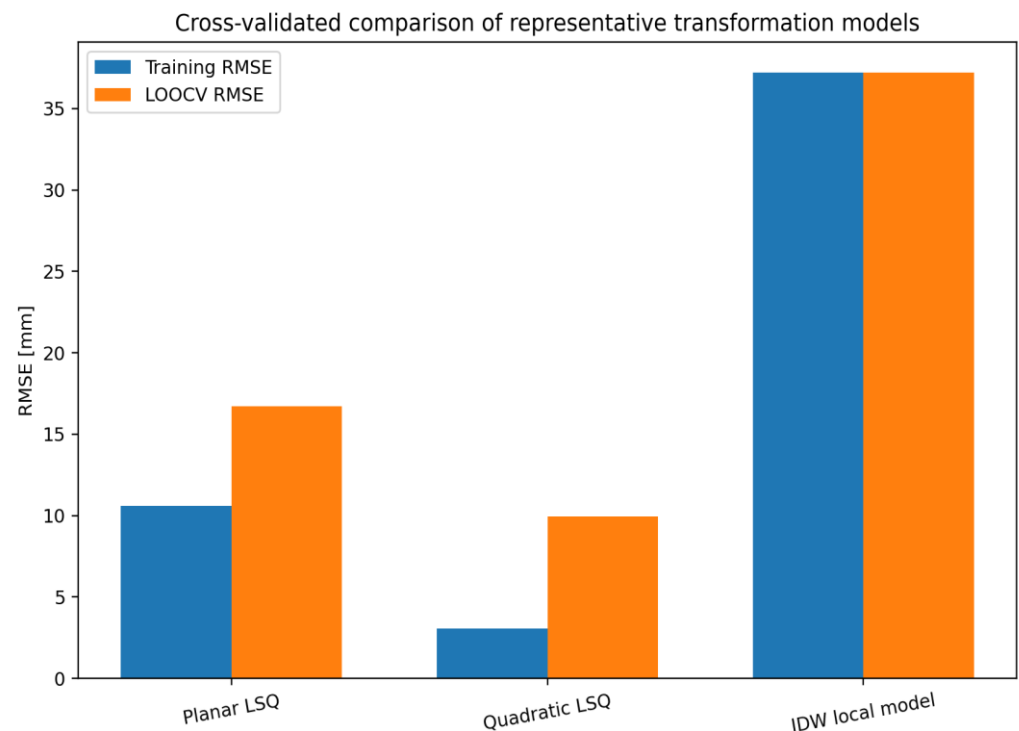
Equation (40) should not be interpreted as an official Romanian transformation formula. It is a mathematically transparent surrogate used to compare approximation classes. The design matrix  $A$  is built from the centered geographic coordinates, and the estimated parameter vector  $\hat{\beta}$  is interpreted as a compact local summary of the datum-difference surface. The Romanian value of this exercise lies in methodology: it shows how a survey agency, a university laboratory, or a corridor-design team can test whether a national shift, a planar correction, or a second-order model is warranted before deploying a production workflow. The transformation-surface fit statistics are reported in Table 16. The fitted quadratic surface is shown in Figure 16. The training and cross-validation errors are compared in Figure 17.



**Figure 16.** Quadratic transformation surface fitted to representative Romania-oriented control points; the surface is illustrative and demonstrates the mathematics of local datum-difference modeling rather than an official grid.

**Table 16.** Representative fit statistics for local transformation-surface models based on Romania-oriented control points.

| Model     | a0 [mm] | a1(lat) | a2(lon) | Train RMSE [mm] | LOOCV RMSE [mm] |
|-----------|---------|---------|---------|-----------------|-----------------|
| Planar    | 117.37  | 13.56   | −20.42  | 10.59           | 16.73           |
| Quadratic | 118.11  | 17.56   | −22.15  | 3.07            | 9.95            |
| IDW local | -       | -       | -       | -               | 37.22           |



**Figure 17.** Training and leave-one-out cross-validation errors for three representative transformation-model classes. The quadratic model yields the best compromise between flexibility and stability in this synthetic Romania-oriented example.

### 32. Robust Estimation and Outlier Diagnostics in Vertical Modeling

Real control networks are rarely perfectly clean. A benchmark may carry a transcription mistake, a GNSS vector may be affected by antenna setup error, a gravity value may be borrowed from a low-resolution interpolation, or a height may be referenced to an adjacent datum without explicit metadata. For this reason, a Q1-level methodological article should not stop at ordinary least squares. It should also discuss robust alternatives capable of protecting the estimated transformation surface against a limited number of gross errors.

The standard mathematical remedy is M-estimation. Let  $r_i$  denote the residual at point  $i$ . In the Huber scheme, observations with small residuals retain full weight, while observations beyond a threshold are down-weighted rather than removed outright. This is attractive for geodetic work because it preserves redundancy and avoids brittle decisions at moderate sample sizes. In the present Romania-oriented demonstration, we contaminate one control point by +35 mm and compare clean quadratic LSQ, contaminated quadratic LSQ, and a contaminated Huber solution. The robust version recovers much of the uncontaminated surface and sharply reduces the maximum deviation relative to classical LSQ.

The key engineering implication is simple: when vertical transformations are estimated from mixed historical and modern control, robust estimation is not a luxury but a defensive necessity. It is especially relevant in county-level updates, corridor projects, and local quasi-geoid refinements, where a single bad benchmark can bias the apparent datum shift over tens of kilometers. Robust processing should therefore be embedded in software intended for Romanian height-conversion workflows, together with standardized residual screening and benchmark provenance checks [15,26,29–32]. The robust-estimation results are summarized in Table 17.

$$w_i = 1, \text{ if } |r_i| \leq k s; w_i = (k s)/|r_i|, \text{ if } |r_i| > k s \quad (42)$$

$$\hat{\beta}_{(m+1)} = (A^T W_{(m)} A)^{-1} A^T W_{(m)} y \quad (43)$$

**Table 17.** Representative robust-estimation comparison after contaminating one control point by +35 mm.

| Estimator                        | RMSE to Clean Surface [mm] | Maximum Absolute Deviation [mm] |
|----------------------------------|----------------------------|---------------------------------|
| Quadratic LSQ (clean)            | 3.07                       | 6.46                            |
| Quadratic LSQ (+35 mm outlier)   | 6.99                       | 15.93                           |
| Quadratic Huber (+35 mm outlier) | 3.15                       | 7.71                            |

### 33. Corridor-Scale Comparative Experiment for Romania

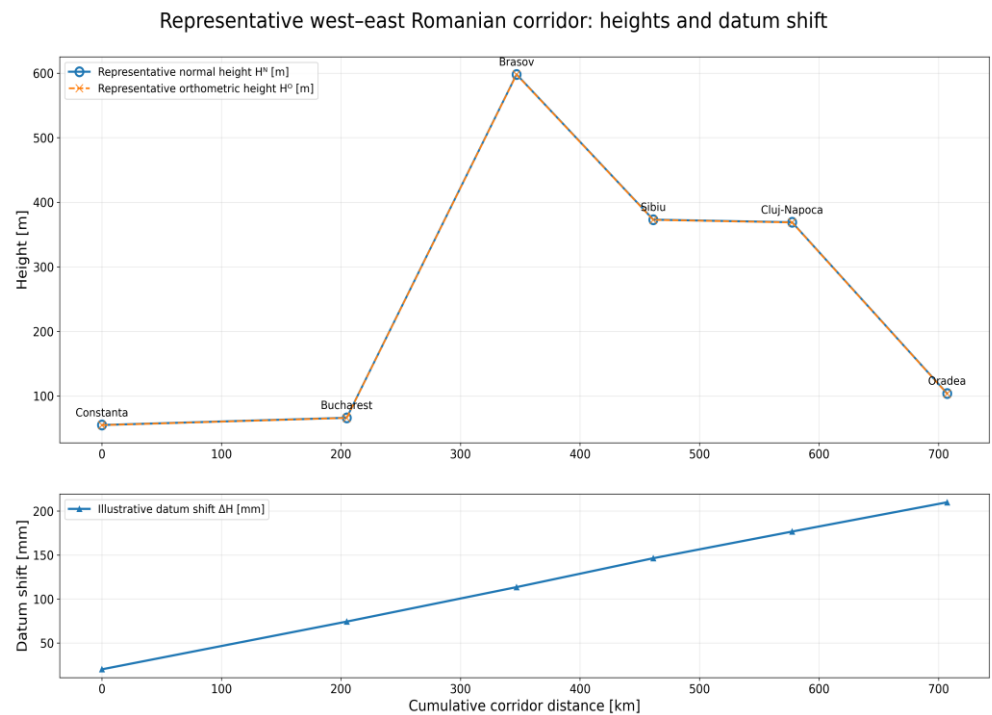
Large infrastructure projects do not use isolated points; they use corridors. This changes the mathematics because local vertical inconsistencies can accumulate along the design axis and interact with relief. To illustrate the point, we consider a representative west–east route linking Constanta, Bucharest, Brasov, Sibiu, Cluj-Napoca, and Oradea. For each city, the article already provides synthetic-yet-realistic normal and orthometric heights. We now augment the experiment with an illustrative Black Sea 1975-to-EVRF-like datum shift that increases along the corridor in a smooth but non-constant way.

The result is not intended as a national transformation product. Its value is structural. Even when the pointwise difference between normal and orthometric heights remains only a few decimeters or centimeters, the datum-transition term can vary by more than one decimeter between corridor extremities. In route design, hydrological chainage interpretation, or cross-border infrastructure planning, this makes a uniform vertical shift mathematically risky. A corridor should therefore be treated as a sequence of location-dependent transformations governed by control density, relief, and datum metadata.

The same lesson generalizes beyond transportation. Pipelines, energy corridors, long irrigation channels, and river-engineering works all inherit cumulative exposure to vertical-reference inconsistency. The western and eastern ends of a project may each look acceptable when checked locally, yet still be mutually incompatible if they were transformed under oversimplified assumptions. A corridor-scale analysis therefore provides an operational bridge between point geodesy and project engineering, which is one of the central themes of this manuscript. The corridor values used in the experiment are listed in Table 18. The corridor-scale height and datum-shift profiles are shown in Figure 18.

$$H_{\text{target}(s)} = H_{\text{source}(s)} + \Delta H_{\text{datum}(s)}, s \in [0, L] \quad (44)$$

$$\Delta H_{\text{datum}(s)} \neq \text{const. for a spatially varying transformation surface} \quad (45)$$



**Figure 18.** Representative west-east Romanian corridor showing normal and orthometric heights together with an illustrative datum-shift term that varies with cumulative project distance. The upper panel distinguishes normal and orthometric heights by marker and line style, while the lower panel presents the datum-shift series.

**Table 18.** Corridor-oriented representative values for a west-east Romanian route used to study the accumulation of datum effects.

| City        | Cumulative Distance [km] | $H_N$ [m] | $H_O$ [m] | Illustrative $\Delta H$ [mm] |
|-------------|--------------------------|-----------|-----------|------------------------------|
| Constanta   | 0.0                      | 55.0      | 54.8      | 20.0                         |
| Bucharest   | 204.8                    | 66.2      | 66.0      | 74.3                         |
| Brasov      | 346.9                    | 598.0     | 597.6     | 113.5                        |
| Sibiu       | 461.3                    | 373.0     | 372.7     | 146.4                        |
| Cluj-Napoca | 577.4                    | 369.0     | 368.7     | 176.7                        |
| Oradea      | 707.1                    | 103.9     | 103.6     | 210.0                        |

### 34. Explicit Covariance Example for Height Conversion

A final mathematical point concerns covariance interpretation. It is common in practice to quote a normal or orthometric height without recording the covariance structure inherited from GNSS, geoid/quasi-geoid modeling, and any latent correlation between those inputs. Yet the converted height is a composite random variable. Its dispersion depends not only on the individual standard deviations but also on the sign and magnitude of the cross-covariance terms. This matters because positive correlation can either increase or decrease the output uncertainty, depending on the algebraic form of the transformation.

Consider a representative single-point conversion in which  $\sigma_h = 18$  mm,  $\sigma_\zeta = 30$  mm, and  $\sigma_N = 35$  mm. Assume further  $\text{corr}(h, \zeta) = 0.20$  and  $\text{corr}(h, N) = 0.15$ . The covariance-propagation result shows that the representative standard uncertainty of the normal height is approximately 31.7 mm, while that of the orthometric height is approximately 36.9 mm. In other words, the output precision is not merely the worst of the two components; it is the result of the full covariance algebra. Reporting only the input standard deviations would miss this distinction.

The same principle scales to networks and rasters. A county-scale DEM converted from ellipsoidal heights into normal heights inherits spatial correlation from both GNSS control and the quasi-geoid model. A corridor adjustment inherits covariance from the adjusted geopotential vector. A national transformation grid inherits covariance from the fit of its coefficients. For Q1-level reproducibility, a vertical product should therefore be accompanied by uncertainty descriptors that are explicit about what has been propagated and what has been ignored. This is where mathematical geodesy directly improves engineering governance. The single-point covariance results are summarized in Table 19.

$$\Sigma_{[h,\zeta]} = [[\sigma_h^2, \rho_{h\zeta} \sigma_h \sigma_\zeta], [\rho_{h\zeta} \sigma_h \sigma_\zeta, \sigma_\zeta^2]] \quad (46)$$

$$\sigma_{HN}^2 = \sigma_h^2 + \sigma_\zeta^2 - 2 \rho_{h\zeta} \sigma_h \sigma_\zeta \quad (47)$$

$$\sigma_{HO}^2 = \sigma_h^2 + \sigma_N^2 - 2 \rho_{hN} \sigma_h \sigma_N \quad (48)$$

**Table 19.** Explicit covariance-propagation example for single-point GNSS-to-physical-height conversion.

| Case                     | Input Standard Deviations [mm]     | Correlation                    | Output Uncertainty [mm] |
|--------------------------|------------------------------------|--------------------------------|-------------------------|
| Normal height $H^N$      | $\sigma_h = 18, \sigma_\zeta = 30$ | $\text{corr}(h, \zeta) = 0.20$ | 31.7                    |
| Orthometric height $H^O$ | $\sigma_h = 18, \sigma_N = 35$     | $\text{corr}(h, N) = 0.15$     | 36.9                    |

### 35. Final Note on Methodological Scope

The four sections added above deepen the central argument of the manuscript in a way that is especially appropriate for a mathematics-oriented Q1 journal. They move the discussion from definitions toward model selection, robust estimation, corridor-scale interpretation, and covariance transparency. Together, they show that vertical referencing is not just a matter of choosing between named height systems; it is a coupled problem of approximation theory, statistical estimation, uncertainty propagation, and engineering decision support.

For Romania, the message is constructive rather than merely critical. The existence of a documented national reference framework, public coordinate-system information through ROMPOS, transformation software that distinguishes improved local models, and expanding public or project-level elevation data creates a strong technological opportunity for spreading rigorous vertical-reference practice [13,15,26,28]. At the same time, the present manuscript remains limited by its representative data and by the absence of official adjustment residuals. The next research step is therefore clear: replace the synthetic dataset with benchmarked national observations, calibrate local covariance models, and publish a reproducible comparison of local, county, national, and European vertical-conversion strategies using the state-of-the-art procedures now emerging in the 2025–2026 literature [18,23,36–41].

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**Data Availability Statement:** The numerical experiment uses representative Romania-oriented inputs constructed to be mathematically consistent with official framework descriptions and literature-reported ranges. The dataset is synthetic and is intended for model-behavior analysis rather than empirical validation of an official national transformation. The tabulated inputs are embedded in the manuscript; future validated work should use official or project-grade Global Navigation Satellite System, leveling, gravity, benchmark, and quasi-geoid datasets.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## Nomenclature

The following main symbols are used in this manuscript:

| Symbol         | Meaning                                  |
|----------------|------------------------------------------|
| $W$            | Actual gravity potential                 |
| $W_0$          | Reference potential of the adopted datum |
| $C$            | Geopotential number                      |
| $h$            | Ellipsoidal height                       |
| $H^D$          | Dynamic height                           |
| $H^N$          | Normal height                            |
| $H^O$          | Orthometric height                       |
| $N$            | Geoid undulation                         |
| $\zeta$        | Height anomaly                           |
| $\bar{g}$      | Mean actual gravity along plumb line     |
| $\bar{\gamma}$ | Mean normal gravity                      |
| $\gamma_0$     | Reference normal gravity constant        |
| $\sigma^2$     | Posterior variance factor                |

## Appendix A. Key Formulae Used in the Computational Experiment

The representative calculations were based on the GRS80 normal gravity formula and on simplified but internally consistent models for mean normal and mean actual gravity along the plumb line. The purpose of the appendix is not to replace official reduction formulae but to document the mathematical structure used in the reproducible experiment.

$$\gamma(\varphi) = \gamma^e (1 + k \sin^2 \varphi) / \sqrt{1 - e^2 \sin^2 \varphi} \quad (A1)$$

$$\bar{\gamma} \approx \gamma(\varphi) [1 - \alpha H^N + \beta (H^N)^2] \quad (A2)$$

$$\bar{g} \approx g_{\text{surface}} + c_H H^N \quad (A3)$$

$$N = h - H^O, \zeta = h - H^N \quad (A4)$$

Here,  $\alpha$  and  $\beta$  are small coefficients used to represent the decrease of normal gravity with height, and  $c_H$  is a simplified Helmert-type correction coefficient. In the present paper, they are selected to ensure physical plausibility over Romanian topographic ranges rather than to emulate one specific official national model.

## Appendix B. Reproducibility Notes

All tables and figures included in this manuscript can be reproduced from the representative input set by applying Equations (A1)–(A4) together with the definitions introduced in Sections 2–4. The coordinates of the representative Romanian points are geographic placeholders for regional interpretation; the resulting heights are synthetic but mathematically consistent values chosen for comparative research exposition.

This distinction is important. A rigorous mathematics article may legitimately use representative computational data when the purpose is to analyze model behavior rather than to publish a national control dataset. Such practice is acceptable provided that the assumptions, ranges, and interpretive limits are declared explicitly, which has been done throughout this manuscript.

## Appendix C. Pseudocode for the Computational Workflow

### Algorithm A1. Geopotential-centered transformation and comparison workflow

Input: Geographic coordinates ( $\phi$ ,  $\lambda$ ), ellipsoidal heights  $h$ , optional leveling increments  $\Delta n$ , optional segment gravities  $g$ , quasi-geoid/geoid model values  $\zeta$  or  $N$ , datum information, stochastic model.

Step 1. Compute normal gravity  $\gamma(\phi)$  on GRS80 or the adopted ellipsoid.

Step 2. If leveling and gravity are available, convert observed segments into geopotential increments  $\Delta C = g \Delta n$  and adjust the network by least squares.

Step 3. If only GNSS and model surfaces are available, compute provisional  $H^N = h - \zeta$  and/or  $H^O = h - N$ .

Step 4. If an adjusted geopotential vector exists, derive  $H^D$ ,  $H^N$ , and  $H^O$  from the common numerator by applying the selected denominator map.

Step 5. Propagate covariance from input quantities or from adjusted geopotential numbers to the output heights.

Step 6. Store each output together with its height-system label, datum metadata, and uncertainty descriptors.

Step 7. Generate comparative diagnostics:  $H^N - H^O$ ,  $H^D - H^N$ , sensitivity to model uncertainty, loop residuals, and application-specific recommendations.

Output: Consistent vertical products for engineering, geodetic, and cartographic use.

**Table A1.** Explicit assumptions used in the representative computational study.

| Assumption                                                                    | Reason for Acceptance                                  | Implication If Violated                      |
|-------------------------------------------------------------------------------|--------------------------------------------------------|----------------------------------------------|
| Representative points reflect realistic Romanian ranges                       | Study is comparative, not an official adjustment       | Results lose regional relevance              |
| Quasi-geoid uncertainty dominates GNSS uncertainty in the Monte Carlo test    | Consistent with many operational GNSS-height workflows | Uncertainty curves would change shape        |
| First-order covariance propagation is acceptable                              | Sufficient for methodological exposition               | Higher-order terms would be needed           |
| Linearized orthometric-normal correction is valid for small $\delta g/\gamma$ | Perturbation terms remain small                        | Correction may need full nonlinear treatment |

|                                                                  |                                                  |                                           |
|------------------------------------------------------------------|--------------------------------------------------|-------------------------------------------|
| Denominator maps are known or approximated from available models | Necessary for transforming C into metric heights | Height outputs become model-indeterminate |
|------------------------------------------------------------------|--------------------------------------------------|-------------------------------------------|

## Appendix D. Closing Remarks on Mathematical Scope

The mathematical scope of this article is intentionally broader than that of a standard surveying note. It treats height systems as a family of denominator-dependent realizations of geopotential information, links their derivation to least-squares estimation, interprets their networks through weighted graphs, and extends the discussion to uncertainty propagation and multi-criteria decision analysis. This scope is, in our view, appropriate for a submission to a mathematics-oriented Q1 journal because it combines formal structure with operational meaning.

At the same time, the paper remains honest about its computational status. The Romania-oriented numerical experiment is representative and mathematically consistent, but it is not presented as an official national dataset. This distinction protects scientific rigor. It allows the article to demonstrate principles, compare models, and propose workflows without claiming access to benchmark-level control information that has not been reproduced here.

The next natural step is obvious and scientifically valuable: replace the representative dataset with dense Romanian GNSS-leveling-gravity observations, calibrate local covariance models for quasi-geoid interpolation, and assess the effect of height-system choice on actual engineering or cadastral deliverables. Such work would deepen the bridge between geodetic mathematics and national geospatial infrastructure.

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