

Article

On Fractional Sturm–Liouville Problems with Complex Coefficients

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Abstract

We investigate the existence and uniqueness of solutions for two-point boundary value problems involving left and right Caputo fractional derivatives with possibly complex-valued coefficients on a finite real interval. Under suitable assumptions on the fractional order and the coefficients, we establish the existence and uniqueness of continuous weak solutions. In addition, under further L^2 -type assumptions on the coefficients, we prove the existence of weak solutions in the Hilbert space $L^2(a, b)$. The analysis is based on an equivalent fractional integral formulation and the Banach–Caccioppoli fixed-point theorem.

Keywords: Riemann–Liouville; Caputo derivatives; existence; uniqueness; complex coefficients

MSC: 34A08; 34A12

1. Introduction

Classical Sturm–Liouville theory has long played a fundamental role in the study of boundary value, spectral, and inverse problems [1]. In its classical form, Sturm–Liouville problems are based on second-order differential equations of the type

$$(p(x)y')' - q(x)y = 0, \quad x \in [a, b], \quad (1)$$

where suitable assumptions are imposed on the coefficients and boundary conditions. This framework is closely related to self-adjoint operators and plays a central role in spectral theory and eigenfunction expansions.

In recent decades, motivated by the need to describe memory and nonlocal effects in complex systems, fractional generalizations of Sturm–Liouville problems have attracted considerable attention. These models typically replace classical derivatives by non-integer order operators such as Riemann–Liouville or Caputo derivatives. A large body of work has been devoted to analytical properties, eigenvalue problems, inverse problems, variational formulations, and numerical approximations of such fractional systems [2–11]. In particular, variational approaches have been successfully employed to establish the existence of countable families of eigenvalues and orthogonal eigenfunctions, while several studies have investigated spectral properties, approximation techniques, and applications to anomalous diffusion and related nonlocal models.

From a spectral and operator-theoretic perspective, fractional Sturm–Liouville problems have been studied extensively, including self-adjoint formulations, orthogonality



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of eigenfunctions, eigenvalue asymptotics, and spectral representations. The interplay between left and right fractional operators is particularly important in deriving fractional integration-by-parts formulas and establishing generalized symmetry properties. Furthermore, inverse fractional Sturm–Liouville problems have received considerable attention due to their relevance in parameter identification and reconstruction of nonlocal models from spectral data [5,6].

Recent studies have also developed weak formulations for fractional boundary value and eigenvalue problems, providing a natural framework for treating nonlocal operators and low-regularity coefficients. Such approaches have contributed to the analysis of existence, uniqueness, stability, and spectral properties of solutions, especially in situations where classical differentiability assumptions are not available.

A substantial part of the literature is devoted to numerical methods for fractional Sturm–Liouville problems, including finite difference, finite element, and spectral methods [12–15]. These studies demonstrate the efficiency of discretization techniques for computing eigenvalues and eigenfunctions. However, in many cases, the analysis assumes the well-posedness of the underlying continuous problem or imposes relatively strong regularity and positivity assumptions on the coefficients.

From the analytical point of view, existence and uniqueness results for fractional differential equations and fractional boundary value problems have been established using fixed-point theory, variational methods, and functional analytic techniques [16–21]. Nevertheless, most existing works focus on operators involving either one-sided fractional derivatives or particular combinations of Riemann–Liouville and Caputo operators and typically assume real-valued coefficients satisfying positivity, boundedness, or continuity conditions.

In contrast, the present work considers the fractional boundary value problem

$$\mathbf{D}_a^\alpha(p\mathbf{D}_b^\alpha y)(x) + q(x)y(x) = 0, \quad (2)$$

where $\mathbf{D}_a^\alpha$ and $\mathbf{D}_b^\alpha$ denote the left and right Caputo fractional derivatives, respectively. The operator $\mathbf{D}_a^\alpha(p\mathbf{D}_b^\alpha y)$ represents a genuinely left-right nonlocal structure, incorporating information from both endpoints of the interval.

The choice of operator formalism in (2) deserves some mention. In trying to extend classical Sturm–Liouville theory to the fractional (non-integer derivative) case, one is immediately faced with serious deficiencies not present in the classical case. Generally speaking, the corresponding eigenvalue problem is not self-adjoint independently of the form of the fractional derivatives being used. In some cases, the eigenvalues may be real [8] and in other cases, not so [4]. For $a = b$ in (2) and one of the Caputo derivatives replaced by a Riemann–Liouville derivative, the solutions may be written in terms of Mittag–Leffler functions, for which great deal of literature exists. However, the non-real spectrum causes problem. For $a \neq b$, the results in [8] show that some similarities may be found with the classical case. In the end, one is faced with the more serious issue of determining existence and uniqueness theorems for solutions of such problems for which there are very few results appearing in the literature. This paper is one of them ([4] and [21] being others). Again, in the case where $a \neq b$ and the operator formalism is of the form

$$\mathbf{D}_b^\alpha(p\mathbf{D}_a^\alpha y)(x) + q(x)y(x) = 0,$$

the mere choice of the fractional derivatives dictates the initial conditions for an associated initial value problem. In the preceding case, this equation is well-suited for the study of what we call, in the classical case, two-point boundary value problems. Thus, permuting and mixing fractional derivatives of Caputo and Riemann–Liouville type as we do here

may lead to drastically different scenarios and results, all of interest and each of which yield some corresponding classical results in the limit as the fractional order $\alpha \rightarrow 1$.

Another reason for considering the left and right Caputo fractional derivatives as done here is the availability of an integration-by-parts formula. Without such, there is little hope of extending the classical case to this more general setting. In many fractional spectral models, the simultaneous use of left and right derivatives is essential for obtaining variational formulations, generalized self-adjoint structures, and symmetry properties of the associated operators.

Most importantly, unlike many existing studies, we allow the coefficients p and q to be merely measurable and possibly complex-valued, without imposing sign or continuity assumptions. This generalization introduces significant analytical challenges, since classical positivity, symmetry, and self-adjointness arguments are no longer directly applicable. The Caputo derivative is employed because it is defined through classical integer-order derivatives and is compatible with standard boundary conditions. This property also facilitates numerical discretization via finite difference schemes, making Caputo-type operators particularly suitable for computational implementations.

Therefore, the main contribution of this paper is to establish the existence and uniqueness of weak solutions for the above fractional Sturm–Liouville type problem under substantially weaker assumptions than those commonly found in the literature. The analysis is based on a Banach–Caccioppoli fixed-point approach [22–24], which guarantees both the existence and uniqueness of solutions.

2. Preliminaries

Notation 1. The theory, now well studied, involves notation that is fairly complex and that we propose to simplify by substituting the symbols encountered as follows.

- The Caputo derivative, ${}^cD_{a+}^\alpha$, will be denoted simply by $\mathbf{D}_a^\alpha$, with a boldface $\mathbf{D}$. Similarly, ${}^cD_{b-}^\alpha$, will be denoted by $\mathbf{D}_b^\alpha$.
- The Riemann–Liouville integral, I_{a+}^α , will be shortened to I_a^α . Similarly, the symbol I_{b-}^α will be replaced by I_b^α .
- An italicized D^m refers to the ordinary classical derivative of order m .
- $C[a, b]$ denotes the space of continuous complex-valued functions on $[a, b]$.
- $L^1(a, b)$ and $L^2(a, b)$ are the standard Lebesgue spaces.
- $AC[a, b] = \{y \in C[a, b] : y(x) = y(a) + \int_a^x g(s) ds, \ g \in L^1(a, b)\}$ is the space of all absolutely continuous functions defined on $[a, b]$.

Definition 1. We say that a function y is a weak/integral solution of the boundary value problem if y satisfies the equivalent fractional integral equation associated with the problem, together with the prescribed boundary conditions.

Now, we give some preliminary materials of fractional calculus.

Definition 2. Let α be a positive real number, and $f(x) \in L_1(a, b)$. Left-sided and right-sided Riemann–Liouville integrals of order α are defined as ([2])

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) ds, \quad x > a,$$

$$I_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s) ds, \quad x < b.$$

Property 1. Let α be a positive real number, and $m - 1 < \alpha < m$, where m is an integer. Then, left-sided and right-sided Riemann–Liouville fractional derivatives of order α are defined as in ([2]):

$$\begin{aligned}(D_a^\alpha f)(x) &= D^m(I_a^{m-\alpha} f)(x), \quad x > a, \\ (D_b^\alpha f)(x) &= (-D)^m(I_b^{m-\alpha} f)(x), \quad x < b.\end{aligned}$$

Analogous formulas yield the left-sided and right-sided Caputo derivatives of order α :

$$\begin{aligned}(\mathbf{D}_a^\alpha f)(x) &= (I_a^{m-\alpha} D^m f)(x), \quad x > a, \\ (\mathbf{D}_b^\alpha f)(x) &= (I_b^{m-\alpha} (-D)^m f)(x), \quad x < b.\end{aligned}$$

Property 2. If $m - 1 < \alpha < m$, $\beta > \alpha$ and $f \in C[a, b]$, then the following identities hold ([2]):

$$\begin{aligned}D_a^\alpha I_a^\alpha f(x) &= f(x), \quad D_b^\alpha I_b^\alpha f(x) = f(x) \\ D_a^\alpha I_a^\beta &= I_a^{\beta-\alpha} f(x), \quad D_b^\alpha I_b^\beta = I_b^{\beta-\alpha} f(x), \\ \mathbf{D}_a^\alpha I_a^\alpha f(x) &= f(x), \quad \mathbf{D}_b^\alpha I_b^\alpha f(x) = f(x), \\ I_a^\alpha I_a^\beta f(x) &= I_a^{\alpha+\beta} f(x), \quad I_b^\alpha I_b^\beta f(x) = I_b^{\alpha+\beta} f(x).\end{aligned}$$

Property 3. If $y(t) \in AC[a, b]$ and $0 < \alpha \leq 1$, then

$$\begin{aligned}I_a^\alpha \mathbf{D}_a^\alpha y(t) &= y(t) - y(a), \\ I_b^\alpha \mathbf{D}_b^\alpha y(t) &= y(t) - y(b).\end{aligned}$$

3. Existence and Uniqueness

In this section, we derive an integral equation that will be used later to prove the existence and uniqueness of solutions to (3) and (4) as follows:

$$\mathbf{D}_a^\alpha(p \mathbf{D}_b^\alpha y)(x) + q(x)y(x) = 0, \quad (3)$$

with boundary conditions of the form

$$(p \mathbf{D}_b^\alpha y)(a) = c_1; \quad y(b) = c_2 \quad (4)$$

where the c_i are the constants, either real or complex. We proceed formally first in order to arrive at an integral equation whose solutions will then be shown to satisfy equation (3). Applying I_a^α to both sides of (3) and using Property 3, we have

$$(p \mathbf{D}_b^\alpha y)(x) = -I_a^\alpha(qy)(x) + (p \mathbf{D}_b^\alpha y)(a).$$

i.e.,

$$\mathbf{D}_b^\alpha y(x) = -\frac{1}{p(x)} I_a^\alpha(qy)(x) + \frac{1}{p(x)} (p \mathbf{D}_b^\alpha y)(a). \quad (5)$$

Now, by applying I_b^α to both sides of (5) and using Property 3, we find

$$y(x) = -I_b^\alpha\left(\frac{1}{p} I_a^\alpha(qy)\right)(x) + (p \mathbf{D}_b^\alpha y)(a) I_b^\alpha\left(\frac{1}{p(\cdot)}\right)(x) + y(b).$$

Therefore, any solution to the boundary value problem (3) and (4) must satisfy the following equation.

$$y(x) = -I_b^\alpha \left(\frac{1}{p} I_a^\alpha (qy) \right) (x) + c_1 I_b^\alpha \left(\frac{1}{p(\cdot)} \right) (x) + c_2. \quad (6)$$

There will be two separate cases here, namely one where $c_2 = y(b) = 0$, and $c_2 = y(b) \neq 0$. Each case leads to different types of solutions.

Let p, q be complex-valued Lebesgue measurable functions on $[a, b]$ with $1/p, q \in L^1(a, b)$, $p \neq 0$, and $|p(x)| < \infty$ a.e. (This a.e. finiteness assumption avoids cases where the function $\frac{1}{p(x)}$ may be identically infinite on sets of positive measure, in which case we would have to change the meaning of the equation (3).

Here, we show that continuous solutions to (6) with given $c_1, c_2 \in \mathbb{C}$ exist and are unique under various assumptions.

3.1. Weak Solutions in $C[a, b]$

The following lemma will be used throughout the paper.

Lemma 1 ([25], p. 54, Corollary 1). *Consider the convolution*

$$\int_0^t a(t - \tau) b(\tau) d\tau, \quad t \in [0, T].$$

with real integrands a, b . If the functions a, b are summable in the interval $L^1(0, T)$ and if either one is in $L^\infty(0, T)$, then their product is continuous in this interval.

Corollary 1. *Consider the convolutions*

$$\int_a^x f(x - s) g(s) ds, \quad x \in [a, b] \quad (7)$$

and

$$\int_x^b f(s - x) g(s) ds, \quad x \in [a, b]. \quad (8)$$

with a real integrand f and a complex valued integrand g . If the functions $f, g \in L^1(0, T)$ and if either one is in $L^\infty(0, T)$, then their product is continuous in this interval.

Proof. It suffices to prove these results for the real case of g , as Lemma 1 can be applied to the real and imaginary part separately.

For the first part, given x, a, b , the change in variable, $u = x - a, s = a + r$ transforms (7) into an equation of the form $\int_0^u f(u - r) g(a + r) 1_{[0, b-a]}(r) dr$ on $[0, u]$, where $1_{[0, b-a]}$ is the indicator function on $[0, b - a]$. The result follows after an application of Lemma 1.

In the second part, the change in variable $u = b - x, r = b - s$ transforms (8) into an integral of type $\int_0^u f(u - r) g(b - r) dr$, after which, Lemma 1 applies once again. $\square$

Theorem 1. *Let p, q be complex-valued Lebesgue measurable functions on $[a, b]$ with $1/p, q \in L^\infty(a, b)$, $p \neq 0$, and $|p(x)| < \infty$ a.e. Let $1/2 < \alpha < 1$, and*

$$\frac{(b - a)^{2\alpha}}{\Gamma(\alpha + 1)^2} \|1/p\|_\infty \|q\|_\infty < 1. \quad (9)$$

Then, for given $c_1, c_2 \in \mathbb{C}$, the boundary value problem (3) and (4) has a unique solution $y \in C[a, b]$.

Proof. Consider the complete normed space $(X, \|\cdot\|_\infty)$ of complex valued continuous functions on $[a, b]$. We use (6) to define a map T on X by setting

$$Ty(x) = -I_b^\alpha \left(\frac{1}{p} I_a^\alpha(qy) \right)(x) + c_1 I_b^\alpha \left(\frac{1}{p(\cdot)} \right)(x) + c_2. \quad (10)$$

Since $y \in X$ and $1/p, q \in L^\infty(a, b)$ by hypothesis,

$$|I_a^\alpha(qy)(x)| \leq \|y\|_\infty \|q\|_\infty \left(\sup_{x \in [a, b]} (I_a^\alpha(1)(x)) \right) \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|y\|_\infty \|q\|_\infty. \quad (11)$$

As $y \in X$ and $q \in L^\infty(a, b)$, we have $qy \in L^\infty(a, b)$. Hence,

$$I_a^\alpha(qy)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} q(s)y(s) ds$$

is continuous on $[a, b]$ by Corollary 1. Moreover, since $1/p \in L^\infty(a, b)$, it follows that $\frac{1}{p} I_a^\alpha(qy) \in L^\infty(a, b)$. Applying Corollary 1 once again, we obtain that

$$I_b^\alpha \left(\frac{1}{p} I_a^\alpha(qy) \right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} \frac{1}{p(s)} I_a^\alpha(qy)(s) ds$$

is continuous on $[a, b]$. Similarly, since $1/p \in L^\infty(a, b)$, Corollary 1 implies that $I_b^\alpha \left(\frac{1}{p} \right)(x)$ is continuous on $[a, b]$.

Therefore, all terms in (10) are continuous on $[a, b]$. Thus, $TX \subseteq X$.

Now, we show that T is a contraction on X .

$$|Ty(x) - Tz(x)| \leq I_b^\alpha \left(\frac{1}{|p|} I_a^\alpha(|q||y-z|) \right)(x) \equiv M. \quad (12)$$

M in (12) is estimated using (9), i.e.,

$$M \leq \|1/p\|_\infty \|q\|_\infty \left(\sup_{x \in [a, b]} I_b^\alpha(I_a^\alpha(1)(x)) \right) \|y-z\|_\infty \quad (13)$$

Now,

$$\begin{aligned} I_b^\alpha(I_a^\alpha(1)(x)) &= \frac{1}{\alpha \Gamma(\alpha)^2} \int_x^b (s-x)^{\alpha-1} (s-a)^\alpha ds \\ &\leq \frac{(b-a)^\alpha}{\alpha \Gamma(\alpha)^2} \int_x^b (s-x)^{\alpha-1} ds = \frac{(b-a)^\alpha (b-x)^\alpha}{\alpha^2 \Gamma(\alpha)^2}, \end{aligned}$$

from which we derive,

$$\sup_{x \in [a, b]} I_b^\alpha(I_a^\alpha(1)(x)) \leq \frac{(b-a)^{2\alpha}}{\Gamma(\alpha+1)^2}. \quad (14)$$

Inserting (14) into (13), we obtain

$$M \leq \frac{(b-a)^{2\alpha}}{\Gamma(\alpha+1)^2} \|1/p\|_\infty \|q\|_\infty \|y-z\|_\infty. \quad (15)$$

Inserting (15) into (12) yields

$$|Ty(x) - Tz(x)| \leq \frac{(b-a)^{2\alpha}}{\Gamma(\alpha+1)^2} \|1/p\|_\infty \|q\|_\infty \|y-z\|_\infty. \quad (16)$$

Applying (9), we deduce that T is a contraction on X so that the fixed-point theorem of Banach–Cacciopoli now implies the existence of a unique fixed-point $y \in X$ that necessarily satisfies

$$y(x) = -I_b^\alpha \left(\frac{1}{p} I_a^\alpha (qy) \right) (x) + c_1 I_b^\alpha \left(\frac{1}{p(\cdot)} \right) (x) + c_2.$$

□

Remark 1. This a.e. finiteness assumption on $p(x)$ avoids cases where the function $p(x)$ may be identically infinite on sets of positive measure, in which case, we would have to change the meaning of the equation (3) and rewrite it as a first-order system (see for example, [1]). The solution given by (6) is not necessarily a solution to (3), as differentiability may be lacking. It is, however, a weak/integral solution in the sense that all solutions of (3) and (4) must satisfy (6).

Corollary 1. In addition to the conditions on p, q, α in Theorem 1, assume that F is complex-valued, and $F \in L^\infty(a, b)$. If (9) holds, then the boundary value problem (4) for the forced equation

$$D_a^\alpha (p D_b^\alpha y)(x) + q(x)y(x) = F(x), \quad (17)$$

with $c_1, c_2 \in \mathbb{C}$ has a unique solution in $C[a, b]$.

Proof. Applying I_a^α to both sides of (17) and using Property 3, we have

$$(p D_b^\alpha y)(x) = -I_a^\alpha (qy)(x) + (p D_b^\alpha y)(a) + I_a^\alpha (F)(x).$$

i.e.,

$$D_b^\alpha y(x) = -\frac{1}{p(x)} I_a^\alpha (qy)(x) + \frac{1}{p(x)} (p D_b^\alpha y)(a) + \frac{1}{p(x)} I_a^\alpha (F)(x). \quad (18)$$

Now, by applying I_b^α to both sides of (18) and using Property 3, we find

$$\begin{aligned} y(x) &= -I_b^\alpha \left(\frac{1}{p} I_a^\alpha (qy) \right) (x) + (p D_b^\alpha y)(a) I_b^\alpha \left(\frac{1}{p(\cdot)} \right) (x) \\ &\quad + y(b) + I_b^\alpha \left(\frac{1}{p} I_a^\alpha (F) \right) (x). \end{aligned}$$

Therefore, any solution to the boundary value problem (17) must satisfy the following equation.

$$y(x) = -I_b^\alpha \left(\frac{1}{p} I_a^\alpha (qy) \right) (x) + c_1 I_b^\alpha \left(\frac{1}{p(\cdot)} \right) (x) + c_2 + I_b^\alpha \left(\frac{1}{p} I_a^\alpha (F) \right) (x).$$

Consider the map T on $X = C[a, b]$ defined by

$$Ty(x) = -I_b^\alpha \left(\frac{1}{p} I_a^\alpha (qy) \right) (x) + c_1 I_b^\alpha \left(\frac{1}{p(\cdot)} \right) (x) + c_2 + I_b^\alpha \left(\frac{1}{p} I_a^\alpha (F) \right) (x).$$

As per Theorem 1, the first and second terms are continuous in $[a, b]$. The last term is continuous by virtue of Lemma 1. Now, the proof is similar to the proof of Theorem 1, as

$TX \subset X$, and the map T is a contraction on X . The conclusion follows from the contraction mapping principle, as before. $\square$

3.2. Weak Solutions in $L^2(a, b)$

Now, we consider the boundary value problem (3) in the Hilbert space, $L^2(a, b)$.

Theorem 2. Let p, q be measurable complex-valued functions on $[a, b]$, $\frac{1}{p} \in L^1(a, b)$, and $|p(x)| < \infty, p \neq 0$ a.e. Assume that, for some $\alpha \in (\frac{1}{2}, 1)$, we have

$$k_1 \equiv \sup_{x \in [a, b]} \int_a^x \frac{|q(s)|^2}{(x-s)^{2-2\alpha}} ds < \infty, \quad (19)$$

$$k_2 \equiv \sup_{x \in [a, b]} I_b^\alpha \left(\frac{1}{|p|} \right)(x) < \infty, \quad (20)$$

$$c = \frac{\sqrt{k_1(b-a)}}{\Gamma(\alpha)} k_2 < 1. \quad (21)$$

Then, the boundary value problem (3) and (4) with c_1, c_2 being arbitrary has a unique solution $y \in L^2(a, b)$.

Proof. First, we show why the condition $\alpha > \frac{1}{2}$ is required. Since $f \in L^2(a, b)$, we apply the Cauchy–Schwarz inequality to the Riemann–Liouville fractional integral

$$(I_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt.$$

Then,

$$|I_a^\alpha f(x)| \leq \frac{1}{\Gamma(\alpha)} \left(\int_a^x (x-t)^{2\alpha-2} dt \right)^{1/2} \left(\int_a^x |f(t)|^2 dt \right)^{1/2}.$$

Because $f \in L^2(a, b)$, the second factor is finite. Therefore, the existence of $I_a^\alpha f$ depends on the finiteness of $\int_a^x (x-t)^{2\alpha-2} dt$. This integral converges near $t = x$ only if $2\alpha - 1 > 0$, which is equivalent to $\alpha > \frac{1}{2}$.

Therefore, the condition $\frac{1}{2} < \alpha < 1$ guarantees that the Riemann–Liouville fractional integrals I_a^α, I_b^α of L^2 -functions exist on account of the Cauchy–Schwarz inequality. Consequently, these integrals define absolutely continuous functions with respect to the variable in question.

Let X be a complete normed vector space, $X = (L^2(a, b), \|\cdot\|_2)$, where $\|\cdot\|_2$ is the usual norm, and define a map T on X by

$$Ty(x) = -I_b^\alpha \left(\frac{1}{p} I_a^\alpha(qy) \right)(x) + c_1 I_b^\alpha \left(\frac{1}{p(\cdot)} \right)(x) + c_2. \quad (22)$$

The square integrability in the first term follows from the assumptions and the Schwarz inequality, i.e., for $y \in X$, we have

$$\begin{aligned}
\left| I_b^\alpha \left(\frac{1}{p} I_a^\alpha (qy) \right) (x) \right| &\leq I_b^\alpha \left(\frac{1}{|p|} I_a^\alpha (|qy|) \right) (x) \\
&= \frac{1}{\Gamma(\alpha)^2} \int_x^b \frac{1/|p(t)|}{(t-x)^{1-\alpha}} \left(\int_a^t \frac{|q(s)||y(s)|}{(t-s)^{1-\alpha}} ds \right) dt \\
&\leq \frac{1}{\Gamma(\alpha)^2} \int_x^b \frac{1/|p(t)|}{(t-x)^{1-\alpha}} \left(\int_a^t \frac{|q(s)|^2}{(t-s)^{2-2\alpha}} ds \right)^{1/2} \\
&\quad \times \left(\int_a^t |y(s)|^2 ds \right)^{1/2} dt \\
&\leq \frac{1}{\Gamma(\alpha)} \sqrt{k_1} \|y\|_2 I_b^\alpha \left(\frac{1}{|p|} \right) (x) \\
&\leq \frac{k_2}{\Gamma(\alpha)} \sqrt{k_1} \|y\|_2.
\end{aligned} \tag{23}$$

Since the right-hand side of (23) is independent of x and the interval $[a, b]$ is finite, it follows that the first term in (22) belongs to $L^2(a, b)$. The second term is square-integrable by assumption (20), and so all terms in (22) are square-integrable. So, $TX \subseteq X$.

Now, we show that T is a contraction on X . For $y, z \in X$, we have

$$|Ty(x) - Tz(x)| \leq I_b^\alpha \left(\frac{1}{|p|} I_a^\alpha (|q||y - z|) \right) (x) \equiv M. \tag{24}$$

Arguing as in (23), with y replaced by $y - z$, q replaced by $|q|$ and p replaced by $|p|$, we find that

$$M \leq \frac{k_2}{\Gamma(\alpha)} \sqrt{k_1} \|y - z\|_2. \tag{25}$$

Finally, combining (24) and (25), we obtain,

$$|Ty(x) - Tz(x)| \leq \frac{k_2}{\Gamma(\alpha)} \sqrt{k_1} \|y - z\|_2. \tag{26}$$

Since (26) is true for every $x \in [a, b]$, an application of the Schwarz inequality gives us

$$\|Ty(x) - Tz(x)\|_2 \leq \frac{\sqrt{k_1(b-a)}}{\Gamma(\alpha)} k_2 \|y - z\|_2, \tag{27}$$

and we conclude that

$$\|Ty - Tz\|_2 < c \|y - z\|_2$$

with $c < 1$, so that T is a contraction on X . The conclusion follows. $\square$

Corollary 2. In addition to the conditions on p, q, α in Theorem 2, assume that f is complex-valued, and $f \in L^2(a, b)$. If (21) holds, then the boundary value problem

$$\begin{cases} \mathbf{D}_a^\alpha (p \mathbf{D}_b^\alpha y)(x) + q(x)y(x) = f(x), \\ (p \mathbf{D}_b^\alpha y)(a) = c_1; y(b) = c_2, \end{cases} \tag{28}$$

with $c_1, c_2 \in \mathbf{C}$ has a unique solution $y \in L^2(a, b)$.

Proof. A proof follows readily from Corollary 1 and Theorem 2. $\square$

4. Conclusions

In this paper, we have studied a two-point boundary value problem for a fractional Sturm–Liouville type equation involving a genuinely left–right Caputo structure of the form

$$\mathbf{D}_a^\alpha(p \mathbf{D}_b^\alpha y)(x) + q(x)y(x) = 0.$$

The main mathematical contribution of this work is the establishment of existence and uniqueness results for weak solutions under substantially relaxed assumptions on the coefficients. In particular, we allow the coefficients p and q to be merely measurable and possibly complex-valued, without imposing the usual continuity or sign conditions that are commonly required in the existing literature.

A key feature of the present analysis is the use of left and right Caputo fractional derivatives in a coupled operator form. This left–right structure plays an essential role in capturing nonlocal effects from both endpoints of the interval and provides a natural fractional generalization of classical Sturm–Liouville operators. In contrast to many previous studies, which mainly focus on one-sided fractional derivatives or Riemann–Liouville formulations, the present framework leads to a more symmetric and physically meaningful nonlocal model.

The presence of complex-valued coefficients introduces additional analytical difficulties, since standard positivity arguments and self-adjointness properties are no longer available. To overcome these difficulties, we employ a fixed-point approach based on the Banach–Caccioppoli theorem, which provides a robust framework for proving both existence and uniqueness of weak solutions in the corresponding function space.

From a computational perspective, the Caputo formulation is particularly suitable for numerical discretization. Since Caputo derivatives are defined through classical derivatives, they can be efficiently approximated using standard finite-difference schemes. This property suggests that the present analytical results can serve as a rigorous foundation for future numerical approximations of the proposed model.

The obtained results are also consistent with and extend several existing directions in Sturm–Liouville theory, including eigenvalue problems, inverse problems, and spectral analysis. In particular, classical works on fractional Sturm–Liouville operators and their numerical treatment can be viewed as special cases under stronger assumptions on regularity or real-valued coefficients.

Finally, several interesting directions for future research can be considered. These include the spectral analysis of the proposed operator, investigation of self-adjoint or quasi-self-adjoint structures under additional assumptions, extension to multi-term or variable-order fractional operators, and the development of stable and convergent numerical schemes based on finite difference and finite element methods for the associated fractional boundary value problems.

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