

## Article

# A Lattice Path Approach to the Multiplicative Powers of Right-Aligned Binomial Matrices

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## Abstract

Pascal's triangle serves as a fundamental source for constructing a diverse array of unique structural matrices in mathematical literature. This study presents a novel combinatorial methodology for computing the multiplicative powers of "right-aligned binomial matrices", where the ones from the right boundary of Pascal's triangle are aligned to the rightmost column of the matrix. Such matrix structures have inherent connections to a broad range of theoretical applications, potentially extending to discrete-time dynamic systems, probability models, and network routing protocols. Currently, while the powers of these matrices are typically evaluated using numerical computation systems, these systems operate via "black-box" algorithms that obscure the underlying structural mechanics and generally preclude symbolic analysis. Furthermore, existing theoretical approaches in the literature rely heavily on complex, indirect recurrence relations. In this work, the computation of matrix powers is approached through lattice path enumeration, specifically utilizing weighted Delannoy paths and Fibonacci numbers. The main theorem yields closed-form symbolic expressions that directly evaluate any arbitrary entry of the matrix power, independent of the exponent's magnitude and order of matrix, without requiring matrix multiplication. Furthermore, a computational implementation in the Wolfram Language is provided to demonstrate the algorithmic efficiency and practical validity of the proposed explicit formulas.

**Keywords:** Fibonacci numbers; Pascal's triangle; binomial matrices; matrix powers; combinatorial matrix theory; combinatorial paths; Delannoy paths; lattice paths

**MSC:** 11B39; 15A15; 05A10



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## 1. Introduction

The study of matrices constructed from binomial coefficients has attracted considerable attention since the pioneering investigations of Carlitz [1,2]. These matrices exhibit remarkable algebraic and combinatorial properties that connect various branches of mathematics. Carlitz's earlier work [2], which established binomial identities arising from sorting problems, laid the structural groundwork for later developments involving generalized Pascal-type matrices [2–5]. A wide range of mathematical tools has contributed to the analysis of binomial matrices. Classical results involving hypergeometric series, as presented by Bailey [6], provided early analytic foundations that later informed the study of structured combinatorial matrices. Further advances include diagonalization and spectral analysis of binomial matrices [7–9], which revealed deep links with algebraic number theory and modular arithmetic.

Beyond combinatorial and spectral approaches, the multiplicative aspects of structured matrices have been extensively studied in numerical linear algebra, such as in multiplicative perturbation analysis for the block Cholesky downdating problem [10]. More recent studies strengthened the connection between binomial matrices and discrete mathematics [11]. Graham, Knuth, and Patashnik [12] highlighted the combinatorial significance of binomial coefficients within discrete structures, while Petkovšek, Wilf, and Zeilberger [13] developed algorithmic methods applicable to recurrence relations inherent in matrix powers. Notably, the work of Peele and Stănică [14–16] investigated eigenvalues, characteristic polynomials, and recurrences for column-justified and right-justified Pascal matrices, establishing important relationships with Fibonacci and Lucas sequences [14–17]. Despite these advances, the explicit computation of matrix powers for right-aligned binomial matrices remains more intricate than that of their left-aligned counterparts. In particular, existing formulations rely heavily on recursive relations, making direct computation challenging. In this study, each entry of the matrix  $R_h^n$  is expressed explicitly in terms of specific Fibonacci numbers and combinatorial lattice paths, expanding upon the growing literature that connects matrix structures with path enumeration [18]. This approach uncovers elegant connections between binomial matrices, Fibonacci sequences, and generalized Delannoy path structures, offering a unified and computationally efficient methodology. To address this computational challenge, this work develops a combinatorial framework that provides direct closed-form expressions for entries of powers of right-aligned binomial matrices. Our primary contributions include the following: a lattice path-based combinatorial interpretation of matrix entries through directional movements mapped to the Delannoy configuration, the formalization of unrestricted and boundary-restricted path enumerations using multinomial coefficients, explicit formulas expressing matrix entries directly in terms of Fibonacci numbers and weighted Delannoy paths, and a comprehensive proof establishing the validity of our combinatorial approach through last-step combinatorial analysis, bypassing complex algebraic inductions. The resulting framework not only simplifies computational procedures but also reveals profound connections between matrix algebra, Fibonacci sequences, and combinatorial enumeration.

## 2. Preliminaries and Definitions

**Definition 1.** The extended Fibonacci sequence  $(f_n)_{n=-\infty}^{\infty}$  is defined by the linear recurrence relation  $f_n = f_{n-1} + f_{n-2}$  for all integers  $n \in \mathbb{Z}$ , with the initial conditions  $f_0 = 0$  and  $f_1 = 1$ . This definition naturally extends to negative indices by the identity  $f_{-n} = (-1)^{n+1}f_n$ .

### 2.1. Right-Aligned Binomial Matrices

**Definition 2 ([1]).** Let  $R_h = [a_{i,j}]$  be an  $h \times h$  matrix defined by  $a_{i,j} = \binom{i-1}{h-j}$  for  $1 \leq i, j \leq h$ , where we adopt the standard convention that  $\binom{m_1}{m_2} = 0$  for any integers  $m_1$  and  $m_2$  satisfying  $m_1 < m_2$  or  $m_2 < 0$ . Then  $R_h$  is called the right-aligned binomial matrix of order  $h$ .

This matrix structure arranges binomial coefficients such that each row contains a segment of a row from Pascal's triangle, right-aligned within the matrix dimensions.

### 2.2. Combinatorial Lattice Paths in Matrix Powers

Let  $R_h^n = [{}_n^h a_{i,j}]$  denote the  $n$ -th power of the matrix  $R_h$ ; where  $n \in \mathbb{Z}$ . To compute the entry  ${}_n^h a_{i,j}$ , we employ a combinatorial approach based on lattice paths through the matrix structure.

**Definition 3** (Directional moves). In the matrix  $R_h^n$ , we define three types of directional moves between adjacent entries:

- **Diagonal move**  $({}_n^h a_{s,t}, {}_n^h a_{s+1,t+1})$ : A diagonal move is a move from  ${}_n^h a_{s,t}$  to  ${}_n^h a_{s+1,t+1}$  denoted by  $c$ ,
  - **Downward move**  $({}_n^h a_{s,t}, {}_n^h a_{s+1,t})$ : A downward move is a move from  ${}_n^h a_{s,t}$  to  ${}_n^h a_{s+1,t}$  denoted by  $b$ ,
  - **Rightward move**  $({}_n^h a_{s,t}, {}_n^h a_{s,t+1})$ : A rightward move is a move from  ${}_n^h a_{s,t}$  to  ${}_n^h a_{s,t+1}$  denoted by  $r$ ,
- where  $1 \leq s \leq h-1, 1 \leq t \leq h-1$ .

**Definition 4** (Paths and Routes). For  $1 \leq s \leq i, 1 \leq t \leq j$  with  $(s, t) \neq (i, j)$ , a path from  ${}_n^h a_{s,t}$  to  ${}_n^h a_{i,j}$  is any finite sequence of directional moves connecting the starting entry to the target entry. Each path can be encoded as a word over the alphabet  $\{c, b, r\}$ . A route is the collection of all valid paths sharing the same starting and ending points.

Since these allowable steps correspond exactly to the steps of a Delannoy path in a two-dimensional grid, we can uniquely parameterize any path by the number of diagonal steps it contains. Let  $\Delta i = i - s$  and  $\Delta j = j - t$  denote the row and column span of the route, respectively. A path containing exactly  $d$  diagonal moves ( $c$ ) will necessarily contain exactly  $\Delta i - d$  downward moves ( $b$ ) and  $\Delta j - d$  rightward moves ( $r$ ). The maximum number of diagonal steps permissible is clearly  $\min(\Delta i, \Delta j)$ . The classical Delannoy number  $D(m, n)$  enumerates lattice paths from the southwest corner  $(0, 0)$  to the northeast corner  $(m, n)$  of a rectangular grid, utilizing only single steps directed north, east, or northeast [19]. In the context of the matrix power  $R_h^n$ , we adapt this framework to evaluate single-step paths traversing from the northwest entry  $(1, 1)$  to the southeast target entry  $(i, j)$ . To provide a clear visual representation of these directional moves, the complete family of all valid Delannoy paths on a  $3 \times 3$  subgrid is illustrated in Figure 1. However, certain structural boundary constraints inherent in our matrix architecture dictate that not every path in this ensemble is permissible for the final evaluation.

### 2.3. Weighted Delannoy Paths

We assign multiplicative weights to each directional move using Fibonacci numbers to capture the algebraic properties of the matrix powers.

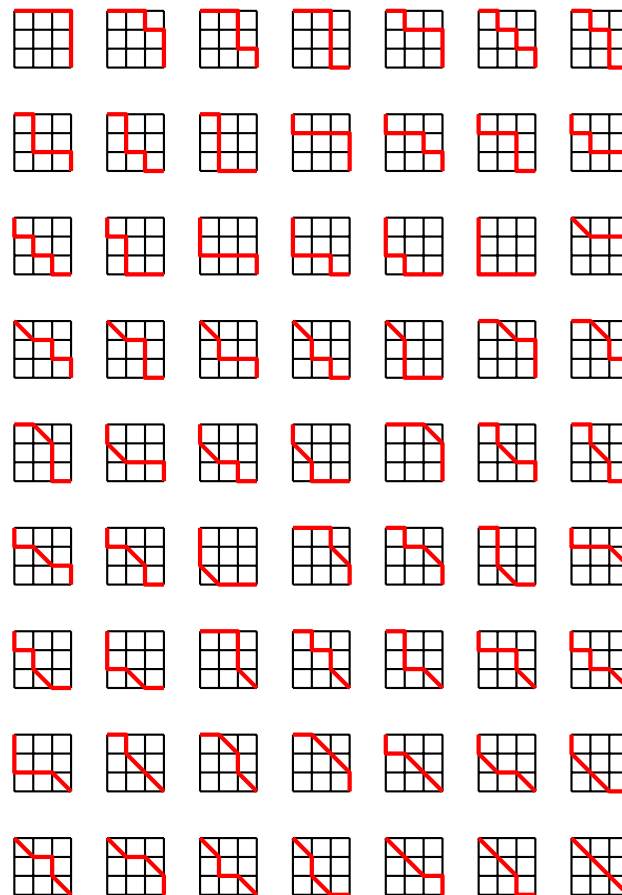
**Definition 5** (Move weights). We define weight functions: for  $n \neq 1$ ,

$$\begin{aligned} c_* &= \frac{f_{n+1}}{f_{n-1}}, \\ b_* &= \frac{f_n}{f_{n-1}}, \\ r_* &= -\frac{f_n}{f_{n-1}}, \end{aligned}$$

for diagonal, downward, and rightward moves, respectively.

Consequently, any path with exactly  $d$  diagonal moves,  $\Delta i - d$  downward moves, and  $\Delta j - d$  rightward moves possesses a uniform weight function given by:

$$\kappa_{i,j}^{s,t}(d) = c_*^d \cdot b_*^{\Delta i - d} \cdot r_*^{\Delta j - d} \quad (1)$$



**Figure 1.** The set of valid Delannoy paths contributing to the entry on the  $3 \times 3$  subgrid.

#### 2.4. Path Enumeration and Boundary Conditions

If there were no structural restrictions, the total number of paths from  $(s, t)$  to  $(i, j)$  using exactly  $d$  diagonal steps would be given by the multinomial coefficient over the total number of steps  $(\Delta i + \Delta j - d)$ , known as the unrestricted Delannoy term:

$$U(s, t, i, j; d) = \binom{\Delta i + \Delta j - d}{d, \Delta i - d, \Delta j - d} = \frac{(\Delta i + \Delta j - d)!}{d! (\Delta i - d)! (\Delta j - d)!}. \quad (2)$$

However, the matrix boundary entries enforce specific initial step constraints to prevent combinatorial overcounting and illegal sequences:

**Remark 1.** In our structure, valid paths originating from the first row or first column are restricted by their initial directional move:

- A path starting from  ${}^h_n a_{1,1}$  **must** begin with a diagonal move (c).
- A path starting from  ${}^h_n a_{s,1}$  ( $s \geq 2$ ) **cannot** begin with a downward move (b).
- A path starting from  ${}^h_n a_{1,t}$  ( $t \geq 2$ ) **cannot** begin with a rightward move (r).

Using (2) and accounting for these boundary constraints via elementary combinatorial subtraction, we formalize the path counting function  $P(s, t, i, j; d)$ :

**Lemma 1** (Restricted Path Enumeration). Let  $\Delta i = i - s$  and  $\Delta j = j - t$ . The number of valid paths  $P(s, t, i, j; d)$  from  ${}^h_n a_{s,t}$  to  ${}^h_n a_{i,j}$  containing exactly  $d$  diagonal moves is calculated as follows: For paths starting from  ${}^h_n a_{1,1}$  (must start with c):

$$P(1, 1, i, j; d) = \binom{\Delta i + \Delta j - d - 1}{d - 1, \Delta i - d, \Delta j - d},$$

For paths starting from  ${}_n^h a_{s,1}$  where  $s \geq 2$  (cannot start with  $b$ ):

$$P(s, 1, i, j; d) = U(s, 1, i, j; d) - \binom{\Delta i + \Delta j - d - 1}{d, \Delta i - d - 1, \Delta j - d},$$

For paths starting from  ${}_n^h a_{1,t}$  where  $t \geq 2$  (cannot start with  $r$ ):

$$P(1, t, i, j; d) = U(1, t, i, j; d) - \binom{\Delta i + \Delta j - d - 1}{d, \Delta i - d, \Delta j - d - 1}.$$

**Proof.** These formulas are direct consequences of standard multinomial selection and the principle of combinatorial exclusion. For paths starting from  $(1, 1)$ , the structural boundary conditions dictate that the path must begin with a diagonal move ( $c$ ). By fixing this first step as  $c$ , the combinatorial problem reduces to distributing the remaining  $d - 1$  diagonal steps over the remaining  $(\Delta i - 1) + (\Delta j - 1) - (d - 1) = \Delta i + \Delta j - d - 1$  total steps. This directly yields the restricted multinomial coefficient  $P(1, 1, i, j; d)$ .

For paths starting from the boundaries  $(s, 1)$  and  $(1, t)$ , we employ a subtraction methodology by explicitly enumerating the number of “forbidden paths”—those that illegally begin with a prohibited directional move—and subtracting them from the total unrestricted Delannoy paths  $U(s, t, i, j; d)$ . Specifically, for a path originating at  $(s, 1)$  where  $s \geq 2$ , a downward move ( $b$ ) is strictly forbidden as the first step. To count how many paths violate this rule, we deliberately fix the first step as  $b$ . This forced initial move consumes exactly one downward step, leaving  $\Delta i - d - 1$  downward steps,  $d$  diagonal steps, and  $\Delta j - d$  rightward steps to be freely arranged across the remaining  $\Delta i + \Delta j - d - 1$  total steps. The number of these invalid paths is precisely given by the multinomial coefficient:

$$\binom{\Delta i + \Delta j - d - 1}{d, \Delta i - d - 1, \Delta j - d}.$$

Subtracting this exact quantity of forbidden sequences from the total unrestricted path count  $U(s, 1, i, j; d)$  yields the number of valid paths that do not start with  $b$ . Analogously, for paths originating at  $(1, t)$  where  $t \geq 2$ , the path is legally barred from starting with a rightward move ( $r$ ). If we fix the first step as the forbidden move  $r$ , we consume one rightward step. This leaves exactly  $\Delta j - d - 1$  rightward steps to be distributed alongside  $d$  diagonal steps and  $\Delta i - d$  downward steps, across a remaining total of  $\Delta i + \Delta j - d - 1$  steps. This yields exactly:

$$\binom{\Delta i + \Delta j - d - 1}{d, \Delta i - d, \Delta j - d - 1}$$

forbidden paths. Subtracting this from  $U(1, t, i, j; d)$  isolates the valid path count, which completes the proof.  $\square$

### 3. Formulation of Matrix Entries

In this section, we formulate the explicit entries of the matrix powers by first resolving the boundary elements and subsequently applying the restricted Delannoy path enumerations. Our core finding, Theorem 1, synthesizes these components into a unified piecewise explicit formula that bypasses traditional recursive evaluation.

**Lemma 2** ([16]). Let  $n \in \{-1, 0, 1\}$ ;  $h \in \mathbb{Z}^+$  (where  $\mathbb{Z}^+$  denotes the set of positive integers) and let  $R_h^n$  be the  $n$ -th power of the right-aligned binomial matrix of order  $h$ . We have:

$${}_1^h a_{i,j} = \binom{i-1}{h-j},$$

$$R_h^0 = I_h,$$

$${}_{-1}^h a_{i,j} = (-1)^{h-i-j+1} \binom{h-i}{j-1}.$$

**Lemma 3** ([16]). Let  $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$ ;  $h \in \mathbb{Z}^+$  and let  $R_h^n$  be the  $n$ -th power of the right-aligned binomial matrix of order  $h$ . Then the entry  ${}_n^h a_{1,j}$  of  $R_h^n$  is given by:

$${}_n^h a_{1,j} = \binom{h-1}{j-1} f_{n-1}^{h-j} f_n^{j-1}.$$

**Lemma 4** ([16]). Let  $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$ ;  $h \in \mathbb{Z}^+$  and let  $R_h^n$  be the  $n$ -th power of the right-aligned binomial matrix of order  $h$ . Then the entry  ${}_n^h a_{i,1}$  of  $R_h^n$  is given by:

$${}_n^h a_{i,1} = f_{n-1}^{h-i} f_n^{i-1}.$$

### Combinatorial Interpretation via Weighted Delannoy Paths

Before presenting the main theorem, we establish a formal combinatorial link. The directional moves defined in Definition 3 corresponding to diagonal ( $c$ ), downward ( $b$ ), and rightward ( $r$ ) steps are perfectly modeled using a lattice path structure known as Delannoy paths. Let  $\mathcal{P}(s, t \rightarrow i, j)$  denote the set of all valid paths from an initial boundary entry  ${}_n^h a_{s,t}$  to a target entry  ${}_n^h a_{i,j}$ . We assign a multiplicative weight to each path  $p \in \mathcal{P}$ , defined as the product of its individual step weights given in Definition 5. Thus, the summations over  $\mathcal{P}(s, t, i, j; d)$  multiplied by the corresponding  $\kappa_{i,j}^{s,t}(d)$  fundamentally represent the total aggregated sum of the weights of all Delannoy paths terminating at the coordinate  $(i, j)$ . Prior to presenting the generalized combinatorial framework, we note that the matrix powers for the trivial and baseline exponents  $n \in \{-1, 0, 1\}$  are either structurally self-evident or already established in the literature. Explicitly, for  $n = 0$ ,  $R_h^0 = I_h$  represents the identity matrix; for  $n = 1$ , the matrix power reduces to the base right-aligned binomial matrix  $R_h$  itself as defined in Definition 2. Furthermore, for the inverse case  $n = -1$ , Stănică and Peele [16] explicitly determined the entries to be  ${}_{-1}^h a_{i,j} = (-1)^{h-i-j+1} \binom{h-i}{j-1}$ . By excluding these known boundary cases, the true structural and combinatorial mechanics under the weighted Delannoy path model emerge for all remaining non-trivial powers, as formalized in the following theorem.

**Theorem 1.** Let  $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$ ;  $h \in \mathbb{Z}^+$  and let  $R_h^n$  be the  $n$ -th power of the right-aligned binomial matrix of order  $h$ . Then the entry  ${}_n^h a_{i,j}$  of  $R_h^n$  is given by the following simplified piecewise function:

$${}_n^h a_{i,j} = \begin{cases} \binom{h-1}{j-1} f_{n-1}^{h-j} f_n^{j-1}, & \text{if } i = 1 \\ f_{n-1}^{h-i} f_n^{i-1}, & \text{if } i > 1 \text{ and } j = 1 \\ {}_n^h a_{1,1} \sum_{d=1}^{\min(i-1, j-1)} \kappa_{i,j}^{1,1}(d) P(1, 1, i, j; d) \\ + \sum_{t=2}^j {}_n^h a_{1,t} \sum_{d=0}^{\min(i-1, j-t)} \kappa_{i,j}^{1,t}(d) P(1, t, i, j; d), & \text{if } i > 1 \text{ and } j > 1 \\ + \sum_{s=2}^i {}_n^h a_{s,1} \sum_{d=0}^{\min(i-s, j-1)} \kappa_{i,j}^{s,1}(d) P(s, 1, i, j; d) \end{cases}$$

**Proof.** Let  $S_{i,j}$  denote the explicit algebraic expression provided on the right-hand side of the equation in Theorem 1. We must prove that  ${}^h_n a_{i,j} = S_{i,j}$  for all  $i, j \geq 1$ . We proceed by mathematical induction. The fundamental recurrence relation for the entries of the matrix powers, established by Peele and Stănică [14], is given by:

$${}^h_n a_{i,j} = {}^h_n a_{i-1,j-1} \frac{f_{n+1}}{f_{n-1}} + {}^h_n a_{i-1,j} \frac{f_n}{f_{n-1}} - {}^h_n a_{i,j-1} \frac{f_n}{f_{n-1}}. \quad (3)$$

**Base Cases:** By Lemma 3 and Lemma 4, the theorem trivially holds for the boundary conditions. Thus,  ${}^h_n a_{1,j} = S_{1,j}$  and  ${}^h_n a_{i,1} = S_{i,1}$  for all  $i, j$ .

**Inductive Step via Combinatorial Analysis:** Assume that  ${}^h_n a_{x,y} = S_{x,y}$  for all  $x \leq i$  and  $y \leq j$  such that  $(x, y) \neq (i, j)$ . By our construction in Section 2, the expression  $S_{i,j}$  computes the total weighted sum of all valid Delannoy paths from the boundaries to the entry  $(i, j)$ . Let  $\mathcal{P}(i, j)$  be the set of all such paths. We can partition  $\mathcal{P}(i, j)$  into three disjoint subsets based on the strictly penultimate node of each path:

- Paths arriving from  $(i-1, j-1)$  via a diagonal move (c). The sum of the weights of these paths is exactly  $c_* \cdot S_{i-1,j-1}$ .
- Paths arriving from  $(i-1, j)$  via a downward move (b). The sum of the weights of these paths is exactly  $b_* \cdot S_{i-1,j}$ .
- Paths arriving from  $(i, j-1)$  via a rightward move (r). The sum of the weights of these paths is exactly  $r_* \cdot S_{i,j-1}$ .

Since this partition is exhaustive and mutually exclusive, the total weight  $S_{i,j}$  must satisfy the linear combination of its predecessors:

$$S_{i,j} = c_* \cdot S_{i-1,j-1} + b_* \cdot S_{i-1,j} + r_* \cdot S_{i,j-1}. \quad (4)$$

Substituting the weight functions  $c_*, b_*, r_*$  defined in Definition 5 into (4) yields:

$$S_{i,j} = S_{i-1,j-1} \frac{f_{n+1}}{f_{n-1}} + S_{i-1,j} \frac{f_n}{f_{n-1}} - S_{i,j-1} \frac{f_n}{f_{n-1}}. \quad (5)$$

Note that (5) is algebraically identical to (3). Since both the explicit formula sum  $S_{i,j}$  and the matrix entries  ${}^h_n a_{i,j}$  satisfy the exact same linear recurrence relation and share the exact same initial boundary conditions (Base Cases), it follows by the uniqueness of the solution to the recurrence that  ${}^h_n a_{i,j} = S_{i,j}$  for all  $i, j$ .  $\square$

While Theorem 1 yields an explicit formulation to compute all entries of the matrix power  $R_h^n$  for an arbitrary positive integer  $h$  and an arbitrary integer  $n$ , it concurrently enables the derivation of a general closed-form expression for the sequence  ${}^h_n a_{i,j}$  under the condition  $h \geq \max\{i, j\}$ . Furthermore, to investigate the structural behavior of these matrix powers more deeply, the subsequent theorem establishes the symmetry property exhibited by the entries lying on the anti-diagonal of  $R_h^n$ .

**Theorem 2.** Let  $R_h^n$  be the  $n$ -th power of the right-aligned binomial matrix of order  $h$ . The elements lying on the anti-diagonal (secondary diagonal) of  $R_h^n$  exhibit symmetry with respect to the center of the anti-diagonal. Specifically, for any cell  $a_{i,j}$  lying on the anti-diagonal under the condition  $i + j = h + 1$ , the following identity holds:

$${}^h_n a_{i,j} = {}^h_n a_{j,i}.$$

**Proof.** To establish the anti-diagonal symmetry of the matrix, we analyze the structural behavior of the entries by dividing the verification into the boundary elements and the inner elements.

### Case 1: Boundary Elements ( $i = 1$ or $j = 1$ )

Under the condition  $i + j = h + 1$ , setting the row index  $i = 1$  yields the column index  $j = h$  for the first row element. The corresponding symmetric counterpart on the first column is obtained by setting  $j = 1$ , which directly implies  $i = h$ . Thus, we must verify the boundary equality:

$${}_n^h a_{1,h} = {}_n^h a_{h,1}.$$

By utilizing the boundary definitions of the matrix recurrence relation, since there are no preceding paths to branch from, the explicit forms reduce to the boundary binomial coefficients:

$${}_n^h a_{1,h} = \binom{h-1}{h-1} f_n^{h-1} \quad \text{and} \quad {}_n^h a_{h,1} = \binom{h-1}{0} f_n^{h-1}.$$

Since  $\binom{h-1}{h-1} = \binom{h-1}{0} = 1$ , the boundary values are identically equal, confirming that the anti-diagonal symmetry holds perfectly at the outer edges of the matrix.

### Case 2: Inner Elements of the Matrix ( $i, j \geq 2$ )

For the inner entries, we explicitly expand both sides of the target equation  ${}_n^h a_{i,j} = {}_n^h a_{j,i}$  using the general path representation formula. The expansions are given as:

$$\begin{aligned} {}_n^h a_{i,j} = & {}_n^h a_{1,1} \sum_{d=1}^{\min(i-1,j-1)} \kappa_{i,j}^{1,1}(d) P(1, 1, i, j; d) \\ & + \sum_{t=2}^j {}_n^h a_{1,t} \sum_{d=0}^{\min(i-1,j-t)} \kappa_{i,j}^{1,t}(d) P(1, t, i, j; d) \\ & + \sum_{s=2}^i {}_n^h a_{s,1} \sum_{d=0}^{\min(i-s,j-1)} \kappa_{i,j}^{s,1}(d) P(s, 1, i, j; d) \end{aligned}$$

and

$$\begin{aligned} {}_n^h a_{j,i} = & {}_n^h a_{1,1} \sum_{d=1}^{\min(j-1,i-1)} \kappa_{j,i}^{1,1}(d) P(1, 1, j, i; d) \\ & + \sum_{t=2}^i {}_n^h a_{1,t} \sum_{d=0}^{\min(j-1,i-t)} \kappa_{j,i}^{1,t}(d) P(1, t, j, i; d) \\ & + \sum_{s=2}^j {}_n^h a_{s,1} \sum_{d=0}^{\min(j-s,i-1)} \kappa_{j,i}^{s,1}(d) P(s, 1, j, i; d). \end{aligned}$$

Given the condition  $i + j = h + 1$ , the effective upper bounds controlled by the minimum functions match identically via cross-correspondence, where the row-boundary paths of one element map directly to the column-boundary paths of its symmetric counterpart:

1.  $\min(i-1, j-1) = \min(j-1, i-1)$ ,
2.  $\min(i-1, j-t) = \min(j-s, i-1)$  for  $s = t$ ,
3.  $\min(i-s, j-1) = \min(j-1, i-t)$  for  $t = s$ .

Furthermore, the explicit algebraic computation of the weight parameters demonstrates that the coefficients reduce to identical core components under index permutation ( $i \leftrightarrow j$ ):

$$\kappa_{i,j}^{1,1}(d) = \kappa_{j,i}^{1,1}(d) = (-1)^{j-1-d} f_n^{h-2d-1} f_{n-1}^{d-h+1} f_{n+1}^d,$$

$$\kappa_{i,j}^{1,t}(d) = \kappa_{j,i}^{s,1}(d) \quad \text{and} \quad \kappa_{i,j}^{s,1}(d) = \kappa_{j,i}^{1,t}(d),$$

for  $s = t$  and  $t = s$ , respectively. By employing the same index relation, the path counting functions also preserve this structural equivalence:

$$\begin{aligned} P(1, 1, i, j; d) &= P(1, 1, j, i; d), \\ P(1, t, i, j; d) &= P(s, 1, j, i; d) \quad (\text{for } s = t), \\ P(s, 1, i, j; d) &= P(1, t, j, i; d) \quad (\text{for } t = s). \end{aligned}$$

Consequently, due to the one-to-one cross-correspondence of all corresponding terms and the exact equivalence of their effective summation intervals, the identity  ${}^h_n a_{i,j} = {}^h_n a_{j,i}$  is fully satisfied for the inner entries. Combining the inner elements with the boundary conditions analyzed in Case 1, it is established that, under the restriction  $i + j = h + 1$ , the elements lying on the anti-diagonal exhibit perfect symmetry with respect to the center of the matrix.  $\square$

## 4. Discussion

This study develops a concise combinatorial framework for determining the powers of right-aligned binomial matrices. Our main result, Theorem 1, provides explicit closed formulas that go well beyond earlier recurrence-based methods. Notably, Peele and Stănică [16] obtained direct closed-form expressions for the powers of left-justified Pascal matrices, yet explicitly noted that the corresponding problem for the right-justified case proved considerably more resistant to such direct methods, relying instead on recurrences and congruence relations. Theorem 1 addresses precisely this gap by providing explicit closed-form expressions for the right-aligned case as well. Recent work—see, for example, ref. [20]—identifies 36 distinct alignment variants of binomial matrices, showing that the right-aligned case is only one part of a much richer structural family. The explicit restriction to the right-aligned configuration in this study arises from the fact that alternative alignments possess inherently distinct combinatorial topologies and underlying recurrences; hence, attempting a generalized unified formula without treating their discrete structural properties would obscure the precision of our Delannoy-based path enumeration. Thus, our framework serves as a foundational step for developing parallel, alignment-specific combinatorial methodologies across these variants. Regarding computational complexity, traditional recursive “black-box” methods rely on binary matrix exponentiation algorithms, which evaluate the entire matrix and operate with an overall arithmetic complexity bounded by  $\mathcal{O}(h^\omega \log n)$  (where  $\omega \approx 2.37$  for optimized matrix multiplication). These standard algorithms are highly dependent on the magnitude of the exponent  $n$ . In stark contrast, our explicit combinatorial formula (Theorem 1) fundamentally shifts this computational paradigm. It directly targets a single arbitrary entry  $(i, j)$  without requiring the evaluation of the full matrix. By analyzing the summation bounds of our explicit formula, the number of algebraic terms required to compute any single entry is bounded by  $\mathcal{O}(h^2)$ . Most notably, the number of combinatorial summation steps in our method is completely independent of the matrix power  $n$ . This property makes our proposed methodology computationally superior for extremely large exponents, offering a highly efficient and structurally transparent tool for symbolic analysis in linear algebra.

### 4.1. Parity Independence and Invariance

To explicitly demonstrate why structural parity (even versus odd indices) does not introduce additional combinatorial effects, we compare the representative even-indexed entry  ${}^h_n a_{2,4}$  and the odd-indexed entry  ${}^h_n a_{3,5}$ . The foundational weight function (Definition 5) is governed by  $\kappa_{i,j}^{s,t}(d) = c_*^d b_*^{\Delta i - d} r_*^{\Delta j - d}$ . This function depends strictly on the relative index

spans  $\Delta i = i - s$  and  $\Delta j = j - t$ , rather than the absolute coordinates  $i$  and  $j$ . For instance, evaluating the path contributions originating from the top-left boundary  ${}_n^h a_{1,1}$ :

- For the even case  ${}_n^h a_{2,4}$ : The paths traverse a span of  $\Delta i = 1$  and  $\Delta j = 3$ .
- For the odd case  ${}_n^h a_{3,5}$ : The paths traverse a span of  $\Delta i = 2$  and  $\Delta j = 4$ .

While the absolute distance from the boundary increases the overall path count (as enumerated by Lemma 1), the algebraic structure of the paths remains perfectly identical. Our framework does not contain any alternating sign mechanisms or phase shifts (such as  $(-1)^i$  or  $(-1)^j$ ) that would evaluate differently for odd or even positions. Consequently, the lattice path configurations and the resulting symbolic expansions are structurally invariant with respect to absolute index parity.

#### 4.2. Computational Examples

To illustrate the application of our main result, we present detailed computations:

**Example 1.** Consider the element  ${}_5^8 a_{3,5}$  of  $R_5^n$ . According to Theorem 1, this entry receives contributions from paths originating at seven starting points:  ${}_n^h a_{1,1}, {}_n^h a_{2,1}, {}_n^h a_{3,1}, {}_n^h a_{1,2}, {}_n^h a_{1,3}, {}_n^h a_{1,4}, {}_n^h a_{1,5}$ . The combinatorial analysis reveals two equivalence classes for paths from  $a_{1,1}$ :

- Class 2:  $\{ccrr, crcr, crrc\}$  with weight  $\kappa_{3,5}^{1,1}(2) = c_*^2 r_*^2$  and count  $P(1, 1, 3, 5; 2) = 3$ .
- Class 1:  $\{cbrrr, crbrr, crrbr, crrrb\}$  with weight  $\kappa_{3,5}^{1,1}(1) = c_* r_*^3 b_*$  and count  $P(1, 1, 3, 5; 1) = 4$ .

Similar analyses for other starting points yield the complete expression. For  $n = 5$ , our formula computes  ${}_5^8 a_{3,5} = f_5^2 f_6^2 = 25 \cdot 64 = 1600$ , which matches direct computation.

**Example 2.** Consider the combinatorial calculation and Fibonacci expansion of  ${}_8^8 a_{3,5}$ . The target element  ${}_8^8 a_{3,5}$  of  $R_8^n$  is computed by summing the boundary contributions using path weights  $\kappa_{3,5}^{s,t}(d)$  and valid path counts  $P(s, t, 3, 5; d)$ :

$$\begin{aligned} {}_8^8 a_{3,5} = & {}_8^8 a_{1,1} \left[ \kappa_{3,5}^{1,1}(2) P(1, 1, 3, 5; 2) + \kappa_{3,5}^{1,1}(1) P(1, 1, 3, 5; 1) \right] \\ & + {}_8^8 a_{1,2} \left[ \kappa_{3,5}^{1,2}(2) P(1, 2, 3, 5; 2) + \kappa_{3,5}^{1,2}(1) P(1, 2, 3, 5; 1) + \kappa_{3,5}^{1,2}(0) P(1, 2, 3, 5; 0) \right] \\ & + {}_8^8 a_{1,3} \left[ \kappa_{3,5}^{1,3}(2) P(1, 3, 3, 5; 2) + \kappa_{3,5}^{1,3}(1) P(1, 3, 3, 5; 1) + \kappa_{3,5}^{1,3}(0) P(1, 3, 3, 5; 0) \right] \\ & + {}_8^8 a_{1,4} \left[ \kappa_{3,5}^{1,4}(1) P(1, 4, 3, 5; 1) + \kappa_{3,5}^{1,4}(0) P(1, 4, 3, 5; 0) \right] \\ & + {}_8^8 a_{1,5} \left[ \kappa_{3,5}^{1,5}(0) P(1, 5, 3, 5; 0) \right] \\ & + {}_8^8 a_{2,1} \left[ \kappa_{3,5}^{2,1}(1) P(2, 1, 3, 5; 1) + \kappa_{3,5}^{2,1}(0) P(2, 1, 3, 5; 0) \right] \\ & + {}_8^8 a_{3,1} \left[ \kappa_{3,5}^{3,1}(0) P(3, 1, 3, 5; 0) \right]. \end{aligned} \quad (6)$$

Substituting the Fibonacci ratios  $c_* = \frac{f_{n+1}}{f_{n-1}}$ ,  $b_* = \frac{f_n}{f_{n-1}}$ , and  $r_* = \frac{-f_n}{f_{n-1}}$ , the full Fibonacci expansion is:

$$\begin{aligned} {}_8^8 a_{3,5} = & {}_8^8 a_{1,1} \left[ 3 \left( \frac{f_{n+1}}{f_{n-1}} \right)^2 \left( \frac{-f_n}{f_{n-1}} \right)^2 + 4 \left( \frac{f_{n+1}}{f_{n-1}} \right) \left( \frac{f_n}{f_{n-1}} \right) \left( \frac{-f_n}{f_{n-1}} \right)^3 \right] \\ & + {}_8^8 a_{1,2} \left[ 2 \left( \frac{f_{n+1}}{f_{n-1}} \right)^2 \left( \frac{-f_n}{f_{n-1}} \right) + 6 \left( \frac{f_{n+1}}{f_{n-1}} \right) \left( \frac{f_n}{f_{n-1}} \right) \left( \frac{-f_n}{f_{n-1}} \right)^2 + 4 \left( \frac{f_n}{f_{n-1}} \right)^2 \left( \frac{-f_n}{f_{n-1}} \right)^3 \right] \\ & + {}_8^8 a_{1,3} \left[ \left( \frac{f_{n+1}}{f_{n-1}} \right)^2 + 4 \left( \frac{f_{n+1}}{f_{n-1}} \right) \left( \frac{f_n}{f_{n-1}} \right) \left( \frac{-f_n}{f_{n-1}} \right) + 3 \left( \frac{f_n}{f_{n-1}} \right)^2 \left( \frac{-f_n}{f_{n-1}} \right)^2 \right] \\ & + {}_8^8 a_{1,4} \left[ 2 \left( \frac{f_{n+1}}{f_{n-1}} \right) \left( \frac{f_n}{f_{n-1}} \right) + 2 \left( \frac{f_n}{f_{n-1}} \right)^2 \left( \frac{-f_n}{f_{n-1}} \right) \right] \\ & + {}_8^8 a_{1,5} \left[ \left( \frac{f_n}{f_{n-1}} \right)^2 \right] \\ & + {}_8^8 a_{2,1} \left[ 4 \left( \frac{f_{n+1}}{f_{n-1}} \right) \left( \frac{-f_n}{f_{n-1}} \right)^3 + 4 \left( \frac{f_n}{f_{n-1}} \right) \left( \frac{-f_n}{f_{n-1}} \right)^4 \right] \\ & + {}_8^8 a_{3,1} \left[ \left( \frac{-f_n}{f_{n-1}} \right)^4 \right] \\ = & 5f_n^2 f_{n-1} (f_n^4 + 4f_n^2 f_{n-1} f_{n+1} + 2f_{n-1}^2 f_{n+1}^2). \end{aligned} \quad (7)$$

For the specific case  $n = 4$ :

$${}^8_n a_{3,5} \Big|_{n=4} = 57690. \quad (8)$$

This numerical result directly matches the  $(3, 5)$ -entry of the fourth power matrix  $R_8^4$ , which is explicitly given below along with the base matrix  $R_8$  for verification:

$$R_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 \\ 0 & 0 & 1 & 5 & 10 & 10 & 5 & 1 \\ 0 & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \end{pmatrix},$$

$$R_8^4 = \begin{pmatrix} 128 & 1344 & 6048 & 15120 & 22680 & 20412 & 10206 & 2187 \\ 192 & 2048 & 9360 & 23760 & 36180 & 33048 & 16767 & 3645 \\ 288 & 3120 & 14480 & 37320 & 57690 & 53487 & 27540 & 6075 \\ 432 & 4752 & 22392 & 58592 & 91947 & 86535 & 45225 & 10125 \\ 648 & 7236 & 34614 & 91947 & 146480 & 139950 & 74250 & 16875 \\ 972 & 11016 & 53487 & 144225 & 233250 & 226250 & 121875 & 28125 \\ 1458 & 16767 & 82620 & 226125 & 371250 & 365625 & 200000 & 46875 \\ 2187 & 25515 & 127575 & 354375 & 590625 & 590625 & 328125 & 78125 \end{pmatrix}.$$

Furthermore, for any arbitrary integer  $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$  and any arbitrary matrix order  $h \geq 5$ , the entry  ${}^h_n a_{3,5}$  of the matrix power  $R_h^n$  can be obtained by the following explicit formula:

$${}^h_n a_{3,5} = \frac{1}{24}(h-4)(h-3)f_{n-1}^{h-7}f_n^2 \left( (h-6)(h-5)f_n^4 + 8(h-5)f_{n-1}f_n^2f_{n+1} + 12f_{n-1}^2f_{n+1}^2 \right).$$

This equation provides a general term formula to compute the  $(3, 5)$ -entries of the infinitely many matrices of the form  $R_5^1, R_5^2, R_5^3, \dots, R_6^1, R_6^2, R_6^3, \dots$  as shown below:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & \mathbf{1} \\ 0 & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 9 & 7 & 2 \\ 1 & 6 & 13 & 12 & \mathbf{4} \\ 1 & 7 & 18 & 20 & 8 \\ 1 & 8 & 24 & 32 & 16 \end{pmatrix}, \quad \begin{pmatrix} 1 & 8 & 24 & 32 & 16 \\ 2 & 15 & 42 & 52 & 24 \\ 4 & 28 & 73 & 84 & \mathbf{36} \\ 8 & 52 & 126 & 135 & 54 \\ 16 & 96 & 216 & 216 & 81 \end{pmatrix}, \dots$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & \mathbf{2} & 1 \\ 0 & 0 & 1 & 3 & 3 & 1 \\ 0 & 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 5 & 10 & 10 & 5 & 1 \\ 1 & 6 & 14 & 16 & 9 & 2 \\ 1 & 7 & 19 & 25 & \mathbf{16} & 4 \\ 1 & 8 & 25 & 38 & 28 & 8 \\ 1 & 9 & 32 & 56 & 48 & 16 \\ 1 & 10 & 40 & 80 & 80 & 32 \end{pmatrix}, \quad \begin{pmatrix} 1 & 10 & 40 & 80 & 80 & 32 \\ 2 & 19 & 72 & 136 & 128 & 48 \\ 4 & 36 & 129 & 230 & \mathbf{204} & 72 \\ 8 & 68 & 230 & 387 & 324 & 108 \\ 16 & 128 & 408 & 648 & 513 & 162 \\ 32 & 240 & 720 & 1080 & 810 & 243 \end{pmatrix}, \dots$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

## 5. Conclusions

In this paper, we have successfully established a novel combinatorial framework to evaluate the powers of right-aligned binomial matrices using weighted Delannoy paths

and Fibonacci sequences. The explicit, closed-form expressions obtained herein eliminate the need for iterative matrix multiplications and complex recursive relations, thereby significantly improving computational efficiency and symbolic tractability. Future work will investigate the generalizability of these path-based techniques to other structured Pascal-type matrices, multivariate Fibonacci sequences, and alternative alignment variants, further establishing combinatorial links to matrix exponentiation in applications ranging from dynamic systems to discrete combinatorial structures.

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## Appendix A. Mathematica Implementation

To facilitate the verification of Theorem 1 and demonstrate its practical utility, we provide the following implementation in Wolfram Cloud (Wolfram Research, Inc., Champaign, IL, USA). This code utilizes memoization and explicitly implements the unrestricted and restricted Delannoy path enumerations using built-in optimized combinatorial functions to maximize symbolic evaluation speed and memory efficiency.

```
(* Explicit Formulation for Right-Aligned Binomial Matrix Powers via
   Delannoy Paths *)
ClearAll[a, beta, di, dj, b, c, r, Upath, P, BaseMatrix,
  CustomMatrixPower, AnyOrdr, AnyPwr];
(* Row and Column Span Functions *)
di[s_, i_] := i - s;
dj[t_, j_] := j - t;
(* Weight Functions based on Fibonacci Sequences (with Memoization)
   *)
c[n_] := c[n] = Fibonacci[n + 1] / Fibonacci[n - 1];
b[n_] := b[n] = Fibonacci[n] / Fibonacci[n - 1];
r[n_] := r[n] = -b[n];
(* Uniform Path Weight Function *)
beta[s_, t_, i_, j_, d_, n_] := Power[c[n], d] * Power[b[n], di[s, i]
  ] - d] * Power[r[n], dj[t, j] - d];
(* Unrestricted Delannoy Enumeration (Using Built-in Multinomial) *)
Upath[s_, t_, i_, j_, d_] := Module[{x = di[s, i], y = dj[t, j]},
  If[d < 0 || x - d < 0 || y - d < 0, 0, Multinomial[d, x - d,
    y - d]]
];
(* Restricted Path Counting Function with Boundary Conditions *)
P[s_, t_, i_, j_, d_] := P[s, t, i, j, d] = Module[{x = di[s, i], y
  = dj[t, j]},
```

```

Which[
    s == 1 && t == 1,
    If[d - 1 < 0 || x - d < 0 || y - d < 0, 0,
        Multinomial[d - 1, x - d, y - d]],
    s != 1 && t == 1,
    Upath[s, t, i, j, d] - If[x - d - 1 < 0 || y - d <
        0, 0, Multinomial[d, x - d - 1, y - d]],
    s == 1 && t != 1,
    Upath[s, t, i, j, d] - If[x - d < 0 || y - d - 1 <
        0, 0, Multinomial[d, x - d, y - d - 1]]
]
];
(* Main Matrix Entry Calculation with Memoization *)
a[i_, j_, h_, n_] := a[i, j, h, n] = Which[
    j == 1,
    Power[Fibonacci[n - 1], h - i] * Power[Fibonacci[n], i - 1],
    i == 1,
    Binomial[h - 1, j - 1] * Power[Fibonacci[n - 1], h - j] *
        Power[Fibonacci[n], j - 1],
    True,
    a[1, 1, h, n] * Sum[beta[1, 1, i, j, d, n] * P[1, 1, i, j, d
        ], {d, 1, Min[i - 1, j - 1]}] +
    Sum[a[1, t, h, n] * Sum[beta[1, t, i, j, d, n] * P[1, t, i,
        j, d], {d, 0, Min[i - 1, j - t]}], {t, 2, j}] +
    Sum[a[s, 1, h, n] * Sum[beta[s, 1, i, j, d, n] * P[s, 1, i,
        j, d], {d, 0, Min[i - s, j - 1]}], {s, 2, i}]
];
(* Generate the Base Right-Aligned Binomial Matrix Rh *)
BaseMatrix[h_] := Table[Binomial[i - 1, h - j], {i, 1, h}, {j, 1, h
}];
(* Generate the Full Matrix Power using our Explicit Formula *)
CustomMatrixPower[h_, n_] := Table[a[i, j, h, n], {i, 1, h}, {j, 1,
h}];
(* ===== *)
(* TEST, COMPARISON AND VISUALIZATION SECTION *)
(* ===== *)
orderH = 5; (* Size of the matrix *)
powerN = 4; (* The exponent *)
Print["--- MATRIX INITIALIZATION ---"];
Rh = BaseMatrix[orderH];
Print["Base Matrix R_", orderH, ":"];
Print[MatrixForm[Rh]];
Print["\n--- PERFORMANCE COMPARISON ---"];
(* 1. Built-in Mathematica Function (Black-Box Algorithm) *)
{timeBuiltIn, resultBuiltIn} = AbsoluteTiming[MatrixPower[Rh, powerN
]];
Print["Built-in 'MatrixPower' Computation Time : ", timeBuiltIn, "
seconds. "];
(* 2. Custom Explicit Formula (Combinatorial Delannoy Algorithm) *)
{timeCustom, resultCustom} = AbsoluteTiming[CustomMatrixPower[orderH
, powerN]];
Print["Explicit 'Delannoy Paths' Computation Time : ", timeCustom, "
seconds. "];
Print["\n--- VALIDATION & RESULTS ---"];
If[resultBuiltIn == resultCustom,

```

```

        Print["SUCCESS: Both methods yielded the exact same powered
              matrix!"],
        Print["ERROR: Matrix mismatch!"]
];
Print["\nPowered Matrix (R_", orderH, ")^", powerN, ":"];
Print[MatrixForm[resultCustom]];
Print["All a[3,5]=", FullSimplify[a[3, 5, AnyOrdr, AnyPwr]] //
      TraditionalForm];

```

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