

Article

A Finite-Time Adaptive Sliding Mode Observer Design for Fractional Itô Stochastic Switched Systems Against Actuator Degradation

Tengyu Ma ^{1,2,3,*}, Minli Zheng ¹, Lijun Zhang ³, Li Li ⁴ and Longsuo Li ⁵¹ Key Laboratory of Advanced Manufacturing and Intelligent Technology, Ministry of Education, College of Mechanical and Power Engineering, Harbin University of Science and Technology, Harbin 150080, China² Department of Mathematics, Qiqihar University, Qiqihar 161006, China³ Qiqihar Heavy CNC Equipment Co., Ltd., Qiqihar 161005, China⁴ School of Mathematics, Changchun Normal University, Changchun 130032, China⁵ Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

* Correspondence: 02427@qqhru.edu.cn

Abstract

This paper addresses the finite-time sliding mode control problem for fractional Brownian motion (fBm)-driven Itô stochastic systems with actuator efficiency degradation, unknown matched nonlinearities, and partially unknown transition probabilities. A projection-based adaptive sliding mode observer is proposed to estimate unmeasurable states and unknown actuator fault factors. A non-switching sliding surface is constructed to avoid chattering caused by mode switching. Using a composite Lyapunov function and the fractional Itô formula, linear matrix inequality conditions are derived to ensure stochastic stability. A finite-time sliding mode controller is designed that drives the system onto the sliding surface within a prescribed time, with an explicit upper bound on the reaching time. Numerical simulations verify that the proposed method achieves finite-time convergence and yields bounded estimation errors, which demonstrates its effectiveness and robustness.

Keywords: fractional Brownian motion (fBm); actuator fault; linear matrix inequalities (LMIS); sliding mode control; finite-time sliding mode control

MSC: 93E15

1. Introduction

1.1. Background and Engineering Significance

Stochastic phenomena are ubiquitous in modern engineering systems, arising from measurement noise, environmental disturbances, and parameter fluctuations. Brownian motion has long served as the canonical model for random perturbations in stochastic control theory, and its associated Itô calculus provides a rigorous foundation for analysis and controller synthesis [1,2]. However, empirical evidence from diverse fields—including telecommunications, finance, and biomechanics—reveals that many real-world stochastic processes exhibit long-range dependence and self-similarity that cannot be captured by standard Brownian motion. Fractional Brownian motion (fBm), characterized by the Hurst parameter $H \in (1/2, 1)$, addresses this limitation by modeling persistent correlations over time. Unlike traditional Brownian motion, fBm lacks both the Markov and martingale properties, which complicates the application of classical stochastic analysis but simultaneously enables a more faithful representation of complex random phenomena [3–5]. These



Academic Editor: Anatoliy Swishchuk

Received: 17 June 2026

Revised: 9 July 2026

Accepted: 14 July 2026

Published: 17 July 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

distinctive features have driven the growing adoption of fBm-driven models in applications ranging from ecological systems and biophysics to digital signal processing and financial mathematics [6–9].

Despite substantial progress in the theory of fBm-driven systems—including the maximum principle for optimal control under convex control regions [10–12] and stability analyses for delayed neural networks [13]—the control of stochastic switched systems subject to fBm disturbances remains a challenging open problem. Switched systems, which alternate among multiple dynamic modes according to a discrete switching signal, arise naturally in applications such as aerospace attitude control, robotic coordination, power system stabilization, and networked control. The stochastic nature of mode transitions, often governed by environmental factors like wind gusts or load variations, makes Markovian jump systems an appropriate modeling framework. These systems simultaneously capture continuous dynamics and discrete stochastic mode transitions, yet their control is further complicated by practical imperfections such as unmeasurable states, actuator performance degradation, and partially unknown transition probabilities.

1.2. Literature Review and Critical Analysis

The past decade has witnessed significant advances in sliding mode control (SMC) for uncertain systems, owing to its well-known robustness against matched uncertainties, fast response, and straightforward implementation. Early work on SMC for Markovian jump systems primarily assumed full state accessibility and fault-free actuation. Chen et al. [14] introduced mode-dependent sliding surfaces tailored to individual subsystems, effectively accommodating parameter variations across modes. However, such mode-dependent designs inherently cause sliding surface discontinuities at switching instants, which may excite unmodeled high-frequency dynamics and induce chattering—a drawback that motivates the use of common or non-switching sliding surfaces [15]. The existence conditions for common surfaces become significantly more stringent in the presence of actuator faults, a gap that the present work seeks to address.

In parallel, observer-based SMC has emerged as a powerful tool for systems where states are not directly measurable. Wu et al. [16] and Liu et al. [17] developed SMC and fault estimation schemes under the assumptions of either fully measurable states or fault-free actuators. A notable step forward was taken by Li et al. [18], who combined adaptive techniques with sliding mode observers for nonlinear Markovian jump systems, achieving joint state and fault estimation. Their framework provides a foundation for the observer structure adopted herein, yet it does not address finite-time convergence—a critical requirement in time-critical missions such as missile interception and spacecraft rendezvous.

The practical issue of partial knowledge of the Markovian transition rate matrix has been systematically treated by Zhang et al. [19], who proposed LMI-based methods with free connection matrices to handle incomplete transition information. This work underpins our treatment of partially unknown transition probabilities in the stability analysis. Recent advances in LMI simplification techniques, particularly the work of Dong and Nguyen [20] on simplified LMI (sLMI) conditions for T-S fuzzy observer design with unmeasured premise variables, have demonstrated the power of parameterizing unknown quantities to transform complex LMI conditions into solvable forms. Inspired by this philosophy, we adopt a similar three-step strategy—parameterization, simplification, and optimization—to guarantee LMI feasibility under partially unknown transition probabilities. More recently, finite-time SMC has attracted considerable interest due to its superior transient performance. Song et al. [21,22] established finite-time stabilization conditions for systems with full state measurement and fault-free actuation. Extending these results to observer-based frameworks with adaptive fault compensation is nontrivial, as the coupling between

estimation errors and sliding dynamics complicates the reachability analysis. Adaptive control for unknown actuator efficiency factors, often realized via projection operators to ensure bounded parameter estimates, has been investigated by Chen et al. [14]. However, the intrinsic tension between asymptotic parameter convergence and finite-time sliding mode control remains unresolved in the existing literature.

Recent work has explored adaptive finite-time SMC for constrained systems. For instance, ref. [23] developed an adaptive finite-time fractional-order sliding-mode control strategy for an inverted pendulum cart system with state constraints and limited sensing range, demonstrating the effectiveness of combining fractional-order techniques with adaptive compensation. Similarly, ref. [24] proposed a constrained adaptive fractional-order sliding-mode controller for stabilizing a two-DOF gimbal system with limited field-of-view sensors, providing both theoretical analysis and experimental validation. These studies underscore the practical relevance of adaptive finite-time SMC in systems with physical constraints. Meanwhile, ref. [25] investigated constrained and distributed adaptive fault compensation for synchronizing a network of multi-robot manipulators using barrier Lyapunov function-based methods, highlighting the importance of fault-tolerant coordination in multi-agent systems. Additionally, recent developments in optimized hierarchical sliding mode control, as reported in the journal *Automation* [26], offer new perspectives on improving control performance through hierarchical design and optimization. Collectively, these contributions reveal a growing research interest in adaptive, finite-time, and fault-tolerant SMC, yet they also expose a clear gap: no existing work simultaneously addresses the combination of fBm-driven stochastic disturbances, unmeasurable states, unknown actuator efficiency factors, partially unknown transition probabilities, unknown matched nonlinearities, and finite-time convergence within a unified framework.

1.3. Motivation and Research Gap

The preceding analysis identifies four critical deficiencies that collectively motivate this study:

First, practical engineering systems exhibit a confluence of adverse factors—unmeasurable states, actuator degradation, unknown nonlinearities, and partial knowledge of the Markov chain—that interact in complex ways. Most existing results address only subsets of these challenges. For example, actuator fault-tolerant control often assumes full state measurement [17], while finite-time SMC typically requires known bounds on uncertainties [22]. An integrated treatment that captures the simultaneous presence of these factors is essential for real-world applicability.

Second, chattering induced by sliding surface switching remains a fundamental obstacle. Mode-dependent surfaces, though analytically convenient, introduce control discontinuities at switching instants. Common sliding surfaces can eliminate such jumps, but their existence under actuator faults and model uncertainties has not been fully characterized, particularly when adaptive parameter estimation is involved.

Third, the integration of adaptive mechanisms with finite-time control presents an inherent mismatch: adaptive laws generally ensure asymptotic convergence, whereas finite-time control requires convergence within a prescribed interval. Overcoming this discrepancy demands new Lyapunov-based arguments that simultaneously account for both finite-time reaching conditions and the evolution of parameter estimates.

Fourth, no unified framework exists for fBm-driven switched systems under the above constraints. The non-Markovian nature of fBm introduces additional integral terms in the infinitesimal operator, which must be carefully dominated to ensure stability. Existing works on SMC for fBm systems do not incorporate actuator faults or finite-time requirements, leaving a substantial gap in the literature.

1.4. Comparison with Representative Works

To clearly delineate the novelty of this work, we highlight key distinctions from the most relevant literature:

The work [14] designed mode-dependent sliding surfaces under full state measurement and did not consider actuator faults or finite-time control. In contrast, we employ a non-switching sliding surface (Equation (25)) to eliminate chattering, construct an observer (Equation (22)) for unmeasurable states, introduce adaptive laws (Equations (33)–(35)) for fault compensation, and prove finite-time reachability (Theorem 3).

The SMC framework [16] lacked stochastic disturbances and actuator faults. Our model incorporates fBm-driven disturbances, capturing long-range dependence, and explicitly addresses actuator degradation via the multiplicative factor ρ in Equation (4).

The work [18] is closest in spirit, combining adaptive observers with SMC for Markovian jump systems. However, Li et al. only proved asymptotic reachability of the sliding surface, whereas Theorem 3 provides an explicit finite-time upper bound $T^* \leq \frac{\lambda_{\max}[(FB_i)^{-1}]}{\varsigma} \|F\hat{x}(0)\|$. We also incorporate partially unknown transition probabilities via LMI conditions in Theorem 2.

The finite-time SMC theory [22] assumes fully measurable states and fault-free actuation. Our extension to observer-based settings with adaptive fault compensation introduces a composite Lyapunov function $V(t, i) = V_1(t, i) + V_2(t, i) + V_3(t)$ (Equation (37)) that couples state estimation errors, parameter estimation errors, and sliding variables while still yielding $\mathcal{L}^H V < 0$.

Although these studies [23,24] address finite-time control under state constraints, they do not consider Markovian switching, fBm disturbances, or partial transition probability information—all of which are central to our problem setting. Their experimental validations, however, confirm the practical viability of adaptive finite-time SMC methodologies.

The work [25] considers distributed fault compensation, it does not address finite-time convergence or fBm-driven stochastic effects. Our focus on individual switched systems provides complementary insights that could eventually be extended to distributed settings.

These recent works [26,27] demonstrate performance improvements through hierarchical structure optimization, a design philosophy that shares our emphasis on systematic controller synthesis. We differ, however, in focusing on adaptive finite-time compensation under stochastic switching and actuator faults.

1.5. Main Contributions

Based on the foregoing analysis and the theoretical developments in Sections 2–6, the main contributions of this paper are fourfold:

A projection-based adaptive sliding mode observer for fBm-driven switched systems with partially unknown transition probabilities. The observer (Equation (22)) simultaneously estimates unmeasurable states $\hat{x}(t)$ and actuator efficiency factors $\hat{\rho}(t)$. The adaptive laws (Equations (33)–(35)) with projection operator (Equation (36)) ensure boundedness of all estimates and prevent controller singularity, even under abrupt fault changes. This distinguishes our work from [18], which did not consider finite-time requirements, and from [14], which did not address fBm-driven switching.

A non-switching sliding surface with proven existence under actuator faults. By selecting F such that FB_i is invertible (not necessarily identity), the sliding surface $S(t)$ in Equation (25) is mode-independent, eliminating switching-induced chattering. Unlike [14], our design does not require $FB_i = I_m$ for all modes, offering greater design flexibility. The equivalent control analysis (Equation (30)) shows that the resulting sliding dynamics (Equation (31)) are independent of the unknown fault factor ρ .

Sufficient LMI conditions for stochastic stability using a composite Lyapunov function. Theorem 2 provides conditions involving $\Theta_{11}^{(i)}$, $\Theta_{12}^{(i)}$, and $\Theta_{22}^{(i)}$ (Equations (42)–(44)) that guarantee $\mathcal{L}^H V < 0$ for the closed-loop system. The matrix equality $B_i^T M_i = R_i C_i$ (Equation (28)) enables output reconstruction of the sliding variable $S_e(t, i)$, overcoming the lack of state measurements. This extends [19] to fBM-driven systems with adaptive fault compensation.

Finite-time reachability of the sliding surface with an explicit upper bound. Theorem 3 proves convergence of $\|S(t)\|$ to zero within $T^* \leq \frac{\lambda_{\max}[(FB_i)^{-1}]}{\zeta} \|F\hat{x}(0)\|$, by solving the differential inequality $\dot{V}_s \leq -\zeta \sqrt{2/\lambda_{\max}((FB_i)^{-1})} \sqrt{V_s}$. The parameter ζ can be chosen to satisfy any prescribed time T , i.e., $T^* \leq T$. This finite-time guarantee—absent in [18]—is achieved without sacrificing the robustness of the SMC framework.

Numerical validation via Monte Carlo simulations demonstrating effectiveness and robustness. Simulation results in Section 6 show that the proposed method achieves finite-time sliding surface reaching in 94% of independent trials, with RMS estimation errors below 0.025. Comparative analysis confirms the advantages of the adaptive finite-time framework over benchmark methods.

1.6. Paper Organization

The remainder of this paper is organized as follows. Section 2 presents a mathematical description of the fractional Itô stochastic switched system (Equation (2)), the actuator degradation model (Equation (4)), the Markovian switching process with partially unknown transition probabilities (Equation (3)), and necessary assumptions (full column rank of B_i , linear growth bound on $f(x, t)$). Key preliminaries, including the fBM Itô formula (Lemma 1) and the stochastic stability criterion (Theorem 2.1 in [4]), are also recalled. Section 3 designs the sliding mode observer (Equation (22)), the non-switching sliding surface (Equation (25)), and the adaptive laws (Equations (33)–(35)). Section 4 analyzes stochastic stability using the composite Lyapunov function $V(t, i) = V_1 + V_2 + V_3$ (Equation (37)) and derives LMI conditions (Theorem 2). Section 5 addresses finite-time reachability, presenting the sliding mode controller (Equation (51)) and proving the finite-time convergence theorem (Theorem 3). Section 6 presents numerical simulations that validate the theoretical claims. Section 7 concludes the paper and discusses future research directions.

2. Problem Formulation and Preliminaries

System Model and Basic Assumptions

Let a class of fractional Itô stochastic switched system be of the following form:

$$\begin{cases} \dot{x}(t) &= [A(r_t)x(t) + B(r_t)u^F(t) + B(r_t)f(x, t)] + H(r_t)x(t)w^H(t), \\ y(t) &= C(r_t)x(t), \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the system state vector that cannot be directly measured; $u^F(t) \in \mathbb{R}^m$ is the actual control input acting on the system, affected by actuator efficiency degradation; $f(x, t) \in \mathbb{R}^m$ is an unknown nonlinear function representing uncertain system dynamics; $y(t) \in \mathbb{R}^p$ is the measurable system output vector; r_t is a Markov jump process taking values in the finite set $S = \{1, 2, \dots, s\}$. $A(r_t) \in \mathbb{R}^{n \times n}$, $B(r_t) \in \mathbb{R}^{n \times m}$, $H(r_t) \in \mathbb{R}^{n \times n}$, and $C(r_t) \in \mathbb{R}^{p \times n}$ are system matrix functions. $W^H(t)$ denotes the fractional Gaussian noise with Hurst parameter H , and $B^H(t)$ represents the fractional Brownian motion with the same Hurst parameter H . The relationship between the two is given by

$$W^H(t) = \frac{dB^H(t)}{dt}.$$

Then, the equation can be reformulated in the fractional Itô sense as

$$\begin{cases} dx(t) &= [A(r_t)x(t) + B(r_t)u^F(t) + B(r_t)f(x, t)]dt + H(r_t)x(t)dB^H(t), \\ y(t) &= C(r_t)x(t), \end{cases} \quad (2)$$

Consider a Markov jump process $\{r_t, t \geq 0\}$ with transition rate matrix $\Pi = (\pi_{ij})_{s \times s}$, where the transition probabilities obey the following condition:

$$\Pr\{r_{t+\delta} = j \mid r_t = i\} = \begin{cases} \pi_{ij}\delta + o(\delta), & i \neq j, \\ 1 + \pi_{ii}\delta + o(\delta), & i = j, \end{cases} \quad (3)$$

where $\delta > 0$, $\lim_{\delta \rightarrow 0} o(\delta)/\delta = 0$, $\pi_{ij} \geq 0$ ($i \neq j$), and $\pi_{ii} = -\sum_{j \neq i} \pi_{ij}$. This paper considers the case where the transition probability matrix Π is partially unknown, meaning that some π_{ij} values are uncertain with only known upper and lower bounds.

The model for actuator degradation is given as follows:

$$u^F(t) = \rho u(t), \quad (4)$$

where $u(t) \in \mathbb{R}^m$ is the controller output to be designed, and $\rho = \text{diag}\{\rho_1, \rho_2, \dots, \rho_m\}$ is the unknown actuator efficiency factor matrix satisfying $0 < \underline{\rho}_h \leq \rho_h \leq \bar{\rho}_h \leq 1$, $h = 1, 2, \dots, m$. The constants $\underline{\rho}_h$ and $\bar{\rho}_h$ are known lower and upper bounds.

Remark 1. In the theoretical framework, ρ is treated as an unknown constant, reflecting the slow timescale of actuator degradation. The adaptive laws estimate this constant online. In the simulation, a time-varying fault scenario is deliberately employed to test robustness and tracking capability. This two-tier approach is standard in adaptive fault-tolerant control literature and ensures both theoretical rigor and practical relevance.

When the system operates in the i -th mode, i.e., $r_t = i \in S$, the system matrices can be expressed as $A(r_t) = A_i$, $B(r_t) = B_i$, $H(r_t) = H_i$, $C(r_t) = C_i$, where $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $H_i \in \mathbb{R}^{n \times n}$, $C_i \in \mathbb{R}^{p \times n}$ are known constant matrices. The following standard assumptions are given to facilitate theoretical analysis:

$$\begin{cases} dx(t) &= [A_i x(t) + B_i u^F(t) + B_i f(x, t)]dt + H_i x(t)dB^H(t), \\ y(t) &= C_i x(t), \end{cases} \quad (5)$$

where, the control input matrices B_i are of full column rank for each $i \in S$, i.e., $\text{rank}(B_i) = m$. This assumption ensures the existence of the pseudo-inverse and is necessary for sliding surface design. The linear growth condition is imposed on the unknown nonlinear function $f(x, t)$:

$$\|f(x, t)\| \leq \alpha + \beta \|x(t)\|, \quad (6)$$

with unknown positive constants α and β . This assumption covers common nonlinear cases such as Lipschitz nonlinearities and bounded disturbances.

The following lemma will be used multiple times in subsequent analysis.

Lemma 1 (fBm Itô's Formula [4]). Consider a stochastic process $x(t)$ governed by the following fractional stochastic differential equation (SDE):

$$dx(t) = A(t)x(t)dt + D(t)x(t)dB^H(t), \quad (7)$$

where $A(t)$ and $D(t)$ are linear functions. The SDE is driven by a fractional Brownian motion (fBm) $B^H(t)$ with Hurst parameter H . For the stochastic process defined in (7) and for $1/2 < H < 1$, the fractional infinitesimal operator L^H is given by

$$L^H = \frac{\partial}{\partial t} + A(t)x(t)\frac{\partial}{\partial x} + D(t)x(t)\frac{\partial^2}{\partial x^2} \left[x(t) \int_0^t \phi(s,t)D(s)ds \right], \quad (8)$$

where

$$\phi(t,s) = H(2H-1)|s-t|^{2H-2}. \quad (9)$$

When $D(t)$ is constant (i.e., $D(t) = D$), the infinitesimal operator reduces to

$$L^H = \frac{\partial}{\partial t} + A(t)x(t)\frac{\partial}{\partial x} + 2x^T D^T \frac{\partial^2 V}{\partial x^2} D x \int_0^t \phi(s,t)ds. \quad (10)$$

For linear systems driven by fBm as described in (7), the stochastic stability criterion is established as follows:

Theorem 1 ([11,12]). Consider the stochastic system driven by fBm as described in (7). Suppose there exists a function $V(x,t) \in C^{2,1}$ such that

$$V(0,t) = 0, \quad V(x,t) \geq \mu(|x|), \quad V_x(x,t)C(t)x(t) \in \mathcal{L}(0,T) \quad (11)$$

holds for all $(x,t) \in S_h \times \mathbb{R}_+$, where $\mathcal{L}(0,T)$ denotes the family of stochastic processes defined on $[0,T]$. Then, the trivial solution of (7) is stochastically asymptotically stable if the following condition is satisfied:

$$L^H V(x,t) < 0. \quad (12)$$

Proof. Following the approach adopted in [28], we prove the stochastic stability criterion. Let $\varepsilon \in (0,1)$, $r \in (0,h)$ and the initial value of $x_0 \in S_\delta$ be arbitrary, and for simplicity, let $x(t; x_0) = x(t)$. Then, one can find a δ ($\delta(\varepsilon, r) \in (0,r)$) such that

$$\frac{1}{\varepsilon} \sup_{x \in S_\delta} V(x,0) \leq \mu(r). \quad (13)$$

Define

$$\tau = \inf\{t \geq 0 : x(t) \notin S_r\}. \quad (14)$$

Using the fractional Itô's formula for any $t \geq 0$ yields

$$\begin{aligned} V(x(\tau \wedge t), \tau \wedge t) &= V(x_0, 0) + \int_0^{\tau \wedge t} L^H V(x(s), s) ds \\ &\quad + \int_0^{\tau \wedge t} V_x(x(s), s) C x(s) dB^H(s). \end{aligned} \quad (15)$$

Taking the expectation from both sides of Equation (22), we obtain

$$\begin{aligned} \mathbb{E}\{V(x(\tau \wedge t), \tau \wedge t)\} &= \mathbb{E}\{V(x_0, 0)\} + \mathbb{E}\left\{\int_0^{\tau \wedge t} L^H V(x(s), s) ds\right\} \\ &\quad + \mathbb{E}\left\{\int_0^{\tau \wedge t} V_x(x(s), s) C x(s) dB^H(s)\right\}. \end{aligned} \quad (16)$$

Since $V_x(x,t)C(t)x(t) \in \mathcal{L}(0,T)$, from Theorem 3.7 in [29]

$$\mathbb{E}\left\{\int_0^{\tau \wedge t} V_x(x(s), s) C x(s) dB^H(s)\right\} = 0.$$

Then, by using the condition (19), Equation (23) reduces to

$$\begin{aligned}\mathbb{E}\{V(x(\tau \wedge t), \tau \wedge t)\} &= V(x_0, 0) + \mathbb{E}\left\{\int_0^{\tau \wedge t} L^H V(x(s), s) ds\right\} \\ &\leq V(x_0, 0).\end{aligned}\quad (17)$$

For $\tau \leq t$, we have $|x(t \wedge \tau)| = |x(\tau)| = r$, Equation (24) yields

$$\begin{aligned}\mathbb{E}\{V(x(\tau \wedge t), \tau \wedge t)\} &\geq \mathbb{E}\left\{I_{\{\tau \leq t\}} V(x(\tau), \tau)\right\} \\ &\geq \mu(r) \mathbb{P}\{\tau \leq t\},\end{aligned}\quad (18)$$

where $I_{\{\tau \leq t\}}$ is the indicator function of set $\{\tau \leq t\}$. Hence, using Equations (20), (24) and (25) reduces

$$\mathbb{P}\{\tau \leq t\} \leq \varepsilon. \quad (19)$$

Thus, $\mathbb{P}\{\tau \leq \infty\} \leq \varepsilon$ when $t \rightarrow \infty$, or equivalently:

$$\mathbb{P}\{|x(t)| < r, \forall t \geq t_0\} \geq 1 - \varepsilon, \quad (20)$$

which completes the proof. $\square$

Remark 2 ([4]). If the Hurst parameter satisfies $H = 0.5$, then the stochastic system is driven by Brownian motion rather than fBm. Consequently, the stability condition in (12) reduces to

$$LV(x, t) < 0, \quad (21)$$

with L being the standard infinitesimal operator.

3. Observer and Sliding Surface Design

3.1. Sliding Mode Observer Design

The following sliding mode observer is designed to estimate the unmeasurable system states

$$d\hat{x}(t) = [(A_i - L_{pi}C_i)\hat{x}(t) + L_{pi}y(t) + B_i\hat{\rho}(t)u(t) + L_{si}u_s(t)]dt, \quad (22)$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the state estimate; $\hat{\rho}(t) = \text{diag}\{\hat{\rho}_1(t), \dots, \hat{\rho}_m(t)\}$ estimates the actuator efficiency factor ρ ; $L_{pi} \in \mathbb{R}^{n \times p}$ denotes the observer gain matrix to be designed; and $u_s(t) \in \mathbb{R}^m$ is a sliding-mode compensation term that counteracts unknown nonlinearities and estimation errors.

The design of the observer gain L_{pi} must ensure the stability of the observer error system. The following lemma provides conditions for selecting the gain.

Lemma 2 ([14]). If there exist a symmetric positive definite matrix $X_i \in \mathbb{R}^{n \times n}$ and a matrix $Y_i \in \mathbb{R}^{n \times p}$ for which the following linear matrix inequality,

$$A_i^T X_i + X_i A_i - Y_i C_i - C_i^T Y_i^T < 0, \quad (23)$$

holds, then $A_i - L_{pi}C_i$ is Hurwitz stable. Moreover, the observer gain can be selected as $L_{pi} = X_i^{-1}Y_i$.

Define the state estimation error $e(t) = \hat{x}(t) - x(t)$. From the system (2) and observer (22), the error dynamics are

$$de(t) = \{[A_i - L_{pi}C_i]e(t) + B_i\tilde{\rho}(t)u(t) + [L_{si}u_s(t) - B_i f(x, t)]\}dt - H_i\hat{x}(t)dB^H(t), \quad (24)$$

where $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$ is the parameter estimation error.

3.2. Non-Switching Sliding Surface Design

To avoid chattering caused by sliding surface jumps during mode transitions, the following non-switching sliding surface is designed:

$$S(t) = F\hat{x}(t) - F \int_0^t (A_i + B_iK_i)\hat{x}(\tau)d\tau, \quad (25)$$

where $F \in \mathbb{R}^{m \times n}$ and $K_i \in \mathbb{R}^{m \times n}$ are matrices to be designed, representing a free matrix and the controller gain, respectively.

To ensure good analytical properties of the sliding surface, the matrix F must satisfy that FB_i is nonsingular. According to rank assumption, B_i has full column rank, so we can choose

$$F = (B_i^T B_i)^{-1} B_i^T, \quad (26)$$

which yields $FB_i = I_m$, the identity matrix. However, since B_i depends on the mode i , a fixed F cannot satisfy $FB_i = I_m$ for all i . Therefore, we relax this condition and only require that FB_i is invertible. In the reachability analysis, we will treat $(FB_i)^{-1}$ as a mode-dependent matrix.

The controller gains K_i must guarantee that $A_i + B_iK_i$ is Hurwitz stable to ensure stability of the sliding motion. In practice, K_i can be obtained by solving LMIs or by pole placement methods.

To design the sliding mode compensation term $u_s(t)$, define the auxiliary variable

$$S_e(t) = B_i^T M_i e(t), \quad (27)$$

with $M_i \in \mathbb{R}^{n \times n}$ being positive definite and $R_i \in \mathbb{R}^{m \times p}$ satisfying

$$B_i^T M_i = R_i C_i. \quad (28)$$

This constraint establishes a relationship between the observer error and the measurable output, allowing $S_e(t, i)$ to be obtained indirectly from output signals, thereby overcoming the difficulty of unmeasurable states.

Taking the time derivative of the sliding surface $S(t)$ and incorporating the observer dynamics gives

$$\dot{S}(t) = -FL_{pi}C_i e(t) + FB_i[\hat{\rho}(t)u(t) + u_s(t) - K_i\hat{x}(t)]. \quad (29)$$

Setting $\dot{S}(t) = 0$ yields the equivalent control law:

$$u_{eq}(t) = \hat{\rho}^{-1}(t)(FB_i)^{-1}FL_{pi}C_i e(t) - \hat{\rho}^{-1}(t)[u_s(t) - K_i\hat{x}(t)]. \quad (30)$$

Substituting the equivalent control law into the observer (22) gives the closed-loop dynamics on the sliding surface:

$$d\hat{x}(t) = \{(A_i + B_iK_i)\hat{x}(t) + B_{Fi}L_{pi}C_i e(t)\}dt, \quad (31)$$

where $B_{Fi} = I_n - B_i(FB_i)^{-1}F$.

3.3. Adaptive Law Design

Define the parameter estimation errors as follows:

$$\tilde{\alpha}(t) = \hat{\alpha}(t) - \alpha, \quad \tilde{\beta}(t) = \hat{\beta}(t) - \beta, \quad \tilde{\rho}(t) = \hat{\rho}(t) - \rho = \text{diag}\{\tilde{\rho}_1, \dots, \tilde{\rho}_m\}. \quad (32)$$

Design the following adaptive laws:

$$\dot{\hat{\alpha}}(t) = c_\alpha \|S_e(t)\|, \quad (33)$$

$$\dot{\hat{\beta}}(t) = c_\beta \|S_e(t)\| \|y(t)\|, \quad (34)$$

$$\dot{\hat{\rho}}_h(t) = \text{Proj}_{[\underline{\rho}_h, \bar{\rho}_h]} \{c_h S_{eh}(t) u_h(t)\}, \quad h = 1, 2, \dots, m, \quad (35)$$

where $c_\alpha, c_\beta, c_h > 0$ are design parameters, $S_{eh}(t)$ and $u_h(t)$ denote the h -th components of $S_e(t)$ and $u(t)$, respectively.

The projection operator $\text{Proj}_{[\underline{\rho}_h, \bar{\rho}_h]} \{\cdot\}$ is defined as

$$\text{Proj}_{[\underline{\rho}_h, \bar{\rho}_h]} \{y\} = \begin{cases} 0, & \text{if } \hat{\rho}_h = \underline{\rho}_h \text{ and } y \leq 0, \\ 0, & \text{if } \hat{\rho}_h = \bar{\rho}_h \text{ and } y \geq 0, \\ y, & \text{otherwise.} \end{cases} \quad (36)$$

This projection operator guarantees that $\hat{\rho}_h(t) \in [\underline{\rho}_h, \bar{\rho}_h]$ for all $t \geq 0$, ensuring that $\hat{\rho}(t)$ remains invertible and avoiding controller singularity.

4. Stability Analysis

4.1. Lyapunov Function Construction

Construct the following composite Lyapunov function:

$$V(t) = V_1(t) + V_2(t) + V_3(t) \quad (37)$$

where the components are defined as

$$V_1(t) = [\hat{x}^T(t) N_i \hat{x}(t)] e^{-\lambda \int_0^t \int_0^s \phi(s, \tau) d\tau ds}, \quad V_2(t, i) = [e^T(t) M_i e(t)] e^{-\lambda \int_0^t \int_0^s \phi(s, \tau) d\tau ds}, \quad (38)$$

$$V_3(t) = \left[\frac{\tilde{\alpha}^2(t)}{c_\alpha} + \frac{\tilde{\beta}^2(t)}{c_\beta} + \sum_{h=1}^m \frac{\tilde{\rho}_h^2(t)}{c_h} \right] e^{-\lambda \int_0^t \int_0^s \phi(s, \tau) d\tau ds}. \quad (39)$$

here $N_i, M_i \in \mathbb{R}^{n \times n}$ are symmetric positive definite matrices to be determined. The exponential term $e^{-\lambda \int_0^t \int_0^s \phi(s, \tau) d\tau ds}$ is crucial for fBm systems, as it allows us to choose λ sufficiently large to dominate the additional quadratic terms introduced by fBm.

Remark 3. The exponential term counteracts the additional positive fBm-induced terms in $\mathcal{L}^H V$ by producing negative terms proportional to λ in (34)–(36). Selecting λ sufficiently large ensures dominance of the negative terms, guaranteeing $\mathcal{L}^H V < 0$ and stochastic stability. This construction follows [4].

4.2. Sufficient Conditions for Stochastic Stability

Theorem 2. If there exist, for each $i \in S$, symmetric positive definite matrices $N_i, M_i \in \mathbb{R}^{n \times n}$, matrices $R_i \in \mathbb{R}^{m \times p}$, and a sufficiently large scalar $\lambda > 0$ such that the following LMI condition is satisfied:

$$\begin{bmatrix} \Theta_{11}^{(i)} & \Theta_{12}^{(i)} \\ * & \Theta_{22}^{(i)} \end{bmatrix} < 0, \quad (40)$$

and the constraint:

$$B_i^T M_i = R_i C_i, \quad (41)$$

where

$$\Theta_{11}^{(i)} = N_i(A_i + B_i K_i) + (A_i + B_i K_i)^T N_i + 2H_i^T M_i H_i \bar{\phi} + \sum_{j=1}^s \pi_{ij} N_j - \lambda N_i \bar{\phi}, \quad (42)$$

$$\Theta_{12}^{(i)} = N_i B_{Fi} L_{pi} C_i, \quad (43)$$

$$\Theta_{22}^{(i)} = M_i(A_i - L_{pi} C_i) + (A_i - L_{pi} C_i)^T M_i + \sum_{j=1}^s \pi_{ij} M_j - \lambda M_i \bar{\phi}, \quad (44)$$

where $\bar{\phi} = \int_0^t \phi(s, t) ds$ (which is bounded for $0.5 < H < 1$), and λ is chosen such that $\lambda > \lambda_{\min}$ for some $\lambda_{\min} > 0$ determined by the system parameters, then under the adaptive laws (33)–(35) and the controller (51), the closed-loop system (24) and (31) is stochastically stable.

Proof. Apply Itô's formula to $V_1(t) = \hat{x}^T(t) N_i \hat{x}(t)$. First compute the drift term:

$$\begin{aligned} \frac{\partial V_1(t)}{\partial \hat{x}} &= 2\hat{x}^T(t) N_i, \\ \frac{\partial^2 V_1(t)}{\partial \hat{x}^2} &= 2N_i. \end{aligned}$$

From the sliding dynamics (31), the drift term of $\hat{x}(t)$ is $(A_i + B_i K_i)\hat{x}(t) + B_{Fi} L_{pi} C_i e(t)$, and the diffusion coefficient is 0 (no stochastic perturbation on the sliding surface). Therefore:

$$\begin{aligned} L^H V_1(t) &= \left\{ 2\hat{x}^T(t) N_i [(A_i + B_i K_i)\hat{x}(t) + B_{Fi} L_{pi} C_i e(t)] + \sum_{j=1}^s \pi_{ij} \hat{x}^T(t) N_j \hat{x}(t) \right. \\ &\quad \left. - \lambda \hat{x}^T(t) N_i \hat{x}(t) \int_0^t \phi(s, t) ds \right\} e^{-\lambda \int_0^t \int_0^s \phi d\tau ds}. \end{aligned}$$

For $V_2(t) = [e^T(t) M_i e(t)] e^{-\lambda \int_0^t \int_0^s \phi(s, \tau) d\tau ds}$, from the error dynamics (24), the drift term of $e(t)$ is $[A_i - L_{pi} C_i]e(t) + B_i \tilde{\rho} u(t) + L_{si} u_s(t) - B_i f(x, t)$, and the diffusion coefficient is $-H_i \hat{x}(t)$. Applying Itô's formula:

$$\begin{aligned} L^H V_2(t) &= \left\{ 2e^T(t) M_i [(A_i - L_{pi} C_i)e(t) + B_i \tilde{\rho} u(t) + L_{si} u_s(t) - B_i f(x, t)] + \sum_{j=1}^s \pi_{ij} e^T(t) M_j e(t) \right. \\ &\quad \left. + 2\hat{x}^T(t) H_i^T M_i H_i \hat{x}(t) \int_0^t \phi(s, t) ds - \lambda e^T(t) M_i e(t) \int_0^t \phi(s, t) ds \right\} e^{-\lambda \int_0^t \int_0^s \phi(s, \tau) d\tau ds}. \end{aligned}$$

Note that $\dot{\hat{\alpha}}(t) = \dot{\alpha}(t)$, $\dot{\hat{\beta}}(t) = \dot{\beta}(t)$, $\dot{\hat{\rho}}_h(t) = \dot{\rho}_h(t)$. From the adaptive laws (33)–(35):

$$\begin{aligned} L^H V_3(t) &= \left[\frac{2\tilde{\alpha}(t)\dot{\hat{\alpha}}(t)}{c_\alpha} + \frac{2\tilde{\beta}(t)\dot{\hat{\beta}}(t)}{c_\beta} + \sum_{h=1}^m \frac{2\tilde{\rho}_h(t)\dot{\hat{\rho}}_h(t)}{c_h} \right. \\ &\quad \left. - \lambda \left(\frac{\tilde{\alpha}^2(t)}{c_\alpha} + \frac{\tilde{\beta}^2(t)}{c_\beta} + \sum_{h=1}^m \frac{\tilde{\rho}_h^2(t)}{c_h} \right) \int_0^t \phi(t, \tau) d\tau \right] \times e^{-\lambda \int_0^t \int_0^s \phi(s, \tau) d\tau ds} \quad (45) \\ &= \left[2\tilde{\alpha}(t) \|S_e(t)\| + 2\tilde{\beta}(t) \|S_e(t)\| \|y(t)\| + \sum_{h=1}^m \frac{2\tilde{\rho}_h(t)\dot{\hat{\rho}}_h(t)}{c_h} \right] \end{aligned}$$

$$-\lambda \left(\frac{\tilde{\alpha}^2(t)}{c_\alpha} + \frac{\tilde{\beta}^2(t)}{c_\beta} + \sum_{h=1}^m \frac{\tilde{\rho}_h^2(t)}{c_h} \right) \int_0^t \phi(s, \tau) ds \Big] \times e^{-\lambda \int_0^t \int_0^s \phi(s, \tau) d\tau ds}. \quad (46)$$

According to the definition of $S_e(t)$ (27) and the design of the compensation term (to be given in Section 5):

$$u_s(t) = -(\hat{\alpha}(t) + \hat{\beta}(t)\|y(t)\| + \varepsilon) \operatorname{sgn}(S_e(t)),$$

which yields

$$2e^T(t)M_iL_{si}u_s(t) = 2S_e^T(t)u_s(t) = -2\|S_e(t)\|(\hat{\alpha}(t) + \hat{\beta}(t)\|y(t)\| + \varepsilon), \quad (47)$$

where we take $L_{si} = I_m$ for simplicity; this can be generalized.

Using the bound on $f(x, t)$:

$$-2e^T(t)M_iB_if(x, t) \leq 2\|S_e(t)\|\|f(x, t)\| \leq 2\|S_e(t)\|(\alpha + \beta\|y(t)\|). \quad (48)$$

Substituting (47) and (48) and the adaptive terms:

$$\begin{aligned} & 2e^T(t)M_iL_{si}u_s(t) - 2e^T(t)M_iB_if(x, t) + \frac{2\tilde{\alpha}(t)\dot{\hat{\alpha}}(t)}{c_\alpha} + \frac{2\tilde{\beta}(t)\dot{\hat{\beta}}(t)}{c_\beta} \\ & \leq -2\|S_e(t)\|(\hat{\alpha}(t) + \hat{\beta}(t)\|y(t)\| + \varepsilon) + 2\|S_e(t)\|(\alpha + \beta\|y(t)\|) + 2\tilde{\alpha}(t)\|S_e(t)\| + 2\tilde{\beta}(t)\|S_e(t)\|\|y(t)\| \\ & = -2\varepsilon\|S_e(t)\|. \end{aligned}$$

From the definition of $S_e(t)$:

$$2e^T(t)M_iB_i\tilde{\rho}u(t) = 2S_e^T(t)\tilde{\rho}u(t) = 2\sum_{h=1}^m \tilde{\rho}_h(t)S_{eh}(t)u_h(t).$$

Combining with $\sum_{h=1}^m \frac{2\tilde{\rho}_h(t)\dot{\hat{\rho}}_h(t)}{c_h}$:

$$2e^T(t)M_iB_i\tilde{\rho}u(t) + \sum_{h=1}^m \frac{2\tilde{\rho}_h(t)\dot{\hat{\rho}}_h(t)}{c_h} = \sum_{h=1}^m 2\tilde{\rho}_h(t)S_{eh}(t)u_h(t) + \sum_{h=1}^m \frac{2\tilde{\rho}_h(t)\dot{\hat{\rho}}_h(t)}{c_h}.$$

By the property of the projection operator, for each h we have $\tilde{\rho}_h(t)(\dot{\hat{\rho}}_h(t) - c_h S_{eh}(t)u_h(t)) \leq 0$, i.e.:

$$\frac{2\tilde{\rho}_h(t)\dot{\hat{\rho}}_h(t)}{c_h} \leq 2\tilde{\rho}_h(t)S_{eh}(t)u_h(t).$$

Therefore:

$$2e^T(t)M_iB_i\tilde{\rho}(t)u(t) + \sum_{h=1}^m \frac{2\tilde{\rho}_h(t)\dot{\hat{\rho}}_h(t)}{c_h} = \sum_{h=1}^m 2\tilde{\rho}_h(t)S_{eh}(t)u_h(t) + \sum_{h=1}^m \frac{2\tilde{\rho}_h(t)\dot{\hat{\rho}}_h(t)}{c_h} \leq 0. \quad (49)$$

Collecting all terms and writing in matrix form:

$$\begin{aligned} L^H V(t) & \leq \begin{bmatrix} \hat{x}^T(t) & e^T(t) \end{bmatrix} \Theta^{(i)} \begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix} e^{-\lambda \int_0^t \int_0^s \phi(s, \tau) d\tau ds} - 2\varepsilon\|S_e(t)\|e^{-\lambda \int_0^t \int_0^s \phi(s, \tau) d\tau ds} \\ & \quad - \lambda \left(\frac{\tilde{\alpha}^2(t)}{c_\alpha} + \frac{\tilde{\beta}^2(t)}{c_\beta} + \sum_{h=1}^m \frac{\tilde{\rho}_h^2(t)}{c_h} \right) \int_0^t \phi(s, \tau) ds e^{-\lambda \int_0^t \int_0^s \phi(s, \tau) d\tau ds}. \end{aligned}$$

When the LMI condition (40) holds and λ is chosen sufficiently large such that $\Theta^{(i)} < 0$, we have $L^H V(t) < 0$. According to the stochastic stability criterion (Theorem 2.1 in [12]), the closed-loop system is stochastically stable. This completes the proof. $\square$

Remark 4 (LMI Feasibility under Partially Unknown Transition Probabilities). *The LMI conditions in (40) are guaranteed to be feasible through a three-layer mechanism that parallels the simplified LMI (sLMI) philosophy of Dong and Nguyen [20].*

Layer 1 (Parameterization): The unknown transition probabilities are decomposed into known, boundary-known ($\underline{\pi}_{ij} \leq \pi_{ij} \leq \bar{\pi}_{ij}$), and completely unknown elements. Conservative bound matrices $\bar{M}_i$ and $\bar{N}_i$ are introduced such that $M_j < \bar{M}_i$ and $N_j < \bar{N}_i$ for all $j \in \mathcal{S}_{uk}^i$, transforming the unknown terms into solvable inequalities (43) and (44).

Layer 2 (Simplification): The matrix equality $B_i^T M_i = R_i C_i$ in (41) cannot be solved directly using standard LMI toolboxes. Following the sLMI philosophy, we relax it into the following optimization problem with slack variables $\eta_i > 0$:

$$\min \eta_i, \quad \text{subject to: (40),} \quad \begin{bmatrix} -\eta_i I_n & (B_i^T M_i - R_i C_i)^T \\ * & -I_m \end{bmatrix} < 0. \quad (50)$$

when η_i is sufficiently small, the equality constraint $B_i^T M_i = R_i C_i$ holds approximately, ensuring strict LMI feasibility.

Layer 3 (Optimization): For fBm-driven systems, the exponential term in the Lyapunov function produces negative terms $-\lambda N_i \bar{\phi}$ and $-\lambda M_i \bar{\phi}$ in (42)–(44). Choosing λ sufficiently large ensures $\Theta^{(i)} < 0$.

This three-layer strategy, adapted from the sLMI approach of Dong and Nguyen [20], guarantees strict LMI feasibility, as confirmed by the successful numerical solutions in Section 6.

5. Finite-Time Reachability Analysis

5.1. Sliding Mode Controller Synthesis

The sliding mode controller is constructed as follows:

$$u(t) = -\iota_i \hat{\rho}^{-1}(t) S(t) - \hat{\rho}^{-1}(t) (\zeta + \hat{\alpha}(t) + \hat{\beta}(t) \|y(t)\| + \varepsilon + \varphi(t)) \operatorname{sgn}(S(t)) + \hat{\rho}^{-1}(t) K_i \hat{x}(t), \quad (51)$$

where $\iota_i > 0$ is a design parameter satisfying $\iota_i \geq \frac{1}{2} \|\sum_{j=1}^s \pi_{ij} (FB_j)^{-1}\|$; $\zeta > 0$ is a control strength parameter; $\varphi(t)$ is defined as

$$\varphi(t) = \max_{i \in \mathcal{S}} \left\{ \|(FB_i)^{-1} FL_{pi} y(t)\| + \|(FB_i)^{-1} FL_{pi} C_i \hat{x}(t)\| \right\}; \quad (52)$$

$\operatorname{sgn}(S(t))$ is the vector sign function, defined component-wise as $\operatorname{sgn}(S_k(t)) = S_k(t)/|S_k(t)|$ (when $S_k(t) \neq 0$).

Remark 5. *Remark on singularity avoidance. The control signal in (51) is free from singularity for all $t \geq 0$. This is ensured by three mechanisms: (i) the projection operator in (36) guarantees that $\hat{\rho}(t)$ remains positive definite with diagonal entries bounded below by $\rho_h > 0$, so $\hat{\rho}^{-1}(t)$ exists and is bounded; (ii) the matrix F is designed such that FB_i is nonsingular for all $i \in \mathcal{S}$, ensuring that $(FB_i)^{-1}$ is well-defined; and (iii) all other signals ($S(t)$, $\hat{\alpha}(t)$, $\hat{\beta}(t)$, $\|y(t)\|$, $\varphi(t)$) are bounded by the Lyapunov stability analysis in Section 4. Thus, the proposed control law is practically implementable without singularity concerns.*

5.2. Finite-Time Reachability Theorem

Theorem 3 ([30]). Under the conditions of Theorem 2, if the parameters ι_i further satisfy:

$$\iota_i \geq \frac{1}{2} \left\| \sum_{j=1}^s \pi_{ij} (FB_j)^{-1} \right\|, \quad (53)$$

then under the controller (51), the state estimate $\hat{x}(t)$ of the closed-loop system will be driven onto the sliding surface $S(t) = 0$ in finite time. The reaching time T^* satisfies

$$T^* \leq \frac{\lambda_{\max}[(FB_i)^{-1}]}{\varsigma} \|F\hat{x}(0)\|. \quad (54)$$

In particular, when a target time T is specified, the design parameter ς can be chosen such that $T^* \leq T$.

Proof. Choose the mode-dependent Lyapunov function:

$$V_s(t) = \frac{1}{2} S^T(t) (FB_i)^{-1} S(t). \quad (55)$$

Applying fBm Itô's formula and accounting for the Markov jump term:

$$\dot{V}_s(t) = S^T(t) (FB_i)^{-1} \dot{S}(t) + \frac{1}{2} \sum_{j=1}^s \pi_{ij} S^T(t) (FB_j)^{-1} S(t).$$

From (29), $\dot{S}(t) = -FL_{pi}C_ie(t) + FB_i[\hat{\rho}u(t) + u_s(t) - K_i\hat{x}(t)]$, we have:

$$\dot{V}_s(t, i) = S^T(t) [-FL_{pi}C_ie(t) + \hat{\rho}u(t) + u_s(t) - K_i\hat{x}(t)] + \frac{1}{2} S^T(t) \left(\sum_{j=1}^s \pi_{ij} (FB_j)^{-1} \right) S(t).$$

Compute $\hat{\rho}u(t) + u_s(t) - K_i\hat{x}(t)$:

$$\begin{aligned} \hat{\rho}u(t) &= -\iota_i S(t) - (\varsigma + \hat{\alpha}(t) + \hat{\beta}(t) \|y(t)\| + \varepsilon + \varphi) \operatorname{sgn}(S(t)) + K_i\hat{x}(t), \\ u_s(t) &= -(\hat{\alpha}(t) + \hat{\beta}(t) \|y(t)\| + \varepsilon) \operatorname{sgn}(S(t)). \end{aligned}$$

Summing gives

$$\hat{\rho}u(t) + u_s(t) - K_i\hat{x}(t) = -\iota_i S(t) - (\varsigma + \varphi) \operatorname{sgn}(S(t)).$$

Therefore:

$$\begin{aligned} \dot{V}_s(t) &= S^T(t) [-(FB_i)^{-1} FL_{pi}C_ie(t) - \iota_i S(t) - (\varsigma + \varphi) \operatorname{sgn}(S(t))] + \frac{1}{2} S^T(t) \left(\sum_{j=1}^s \pi_{ij} (FB_j)^{-1} \right) S(t) \\ &= -S^T(t) (FB_i)^{-1} FL_{pi}C_ie(t) - \iota_i \|S(t)\|^2 - (\varsigma + \varphi) S^T(t) \operatorname{sgn}(S(t)) \\ &\quad + \frac{1}{2} S^T(t) \left(\sum_{j=1}^s \pi_{ij} (FB_j)^{-1} \right) S(t). \end{aligned}$$

Using $S^T(t) \operatorname{sgn}(S(t)) = \sum_{k=1}^m |S_k(t)| \geq \|S(t)\|$, and $-S^T(t) (FB_i)^{-1} FL_{pi}C_ie(t) \leq \|S(t)\| \|(FB_i)^{-1} FL_{pi}C_ie(t)\| \leq \|S(t)\| \varphi$, we obtain

$$\dot{V}_s(t) \leq \|S(t)\| \varphi - \iota_i \|S(t)\|^2 - (\varsigma + \varphi) \|S(t)\| + \frac{1}{2} \left\| \sum_{j=1}^s \pi_{ij} (FB_j)^{-1} \right\| \|S(t)\|^2.$$

From the choice of ι_i in (53), $-\iota_i \|S(t)\|^2 + \frac{1}{2} \|\sum_j \pi_{ij} (FB_j)^{-1}\| \|S(t)\|^2 \leq 0$. Hence:

$$\dot{V}_s(t) \leq -\varsigma \|S(t)\|. \quad (56)$$

Since $V_s(t) = \frac{1}{2} S^T(t) (FB_i)^{-1} S(t) \leq \frac{1}{2} \lambda_{\max}((FB_i)^{-1}) \|S(t)\|^2$, we have $\|S(t)\| \geq \sqrt{2V_s(t) / \lambda_{\max}((FB_i)^{-1})}$. Substituting into (56):

$$\dot{V}_s(t) \leq -\varsigma \sqrt{\frac{2}{\lambda_{\max}((FB_i)^{-1})}} \sqrt{V_s(t)}. \quad (57)$$

This differential inequality implies finite-time convergence. Integrating from 0 to T^* :

$$2\sqrt{V_s(T^*)} - 2\sqrt{V_s(0)} \leq -\varsigma \sqrt{\frac{2}{\lambda_{\max}((FB_i)^{-1})}} T^*.$$

Setting $V_s(T^*) = 0$, we obtain

$$T^* \leq \frac{2\sqrt{V_s(0)}}{\varsigma} \sqrt{\frac{\lambda_{\max}((FB_i)^{-1})}{2}} = \frac{\sqrt{2V_s(0)\lambda_{\max}((FB_i)^{-1})}}{\varsigma}.$$

Since $V_s(0) = \frac{1}{2} S^T(0) (FB_i)^{-1} S(0) \leq \frac{1}{2} \lambda_{\max}((FB_i)^{-1}) \|F\hat{x}(0)\|^2$, we obtain

$$T^* \leq \frac{\lambda_{\max}((FB_i)^{-1})}{\varsigma} \|F\hat{x}(0)\|.$$

This is exactly (54). The proof is complete. $\square$

Remark 6. *Dependence on Initial Conditions.* The finite-time bound T^* in (54) depends on the initial condition $\|F\hat{x}(0)\|$, which is a fundamental characteristic of finite-time control methods rather than a deficiency. This dependence is explicitly quantified in (54) and can be managed through the design parameter ς . For the proposed controller, even under large initial conditions ($\|\hat{x}(0)\| = 7.07$), the sliding surface is reached within 2.41 s, which is well below the theoretical bound of 3.35 s, confirming that the theoretical guarantees remain valid and the derived bounds are conservative. For applications where the initial condition may be large, the designer can either: (i) increase ς to accelerate convergence, provided actuator saturation limits are respected; or (ii) pre-specify a larger target time T to accommodate the larger initial condition. The key advantage over asymptotic control is that T^* is finite and can be made arbitrarily small (or bounded by any pre-specified T) through appropriate selection of ς , which is not possible with asymptotic convergence guarantees. This flexibility makes the method suitable for practical applications where initial conditions may vary significantly.

6. Numerical Simulation

6.1. Simulation Setup and System Parameters

This section presents numerical simulations to validate the effectiveness of the proposed adaptive finite-time sliding mode observer for fBm-driven stochastic switched systems. The simulation setup follows the framework established in the recent literature on adaptive sliding mode control for stochastic Markov jump systems [18,30], enabling direct comparison with state-of-the-art methods.

The system operates in three Markovian modes with the following system matrices adopted from [30]:

For Mode 1:

$$A_1 = \begin{bmatrix} -1.6 & 0.1 & 2.628 \\ -0.8 & -2.4 & 0.3788 \\ 0.3107 & 0 & -2.26 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.2 & -0.1 \\ 0.1 & 0.4 \\ 0.2 & 0.2 \end{bmatrix}, \quad (58)$$

$$C_1 = \begin{bmatrix} -0.1 & 0 & 0.2 \\ 0.15 & -0.3 & -0.2 \end{bmatrix}. \quad (59)$$

For Mode 2:

$$A_2 = \begin{bmatrix} -1.47 & 0 & 2.428 \\ -0.3357 & -1.4 & -0.3788 \\ 0.3107 & 0 & -2.26 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 & -0.2 \\ 0.05 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}, \quad (60)$$

$$C_2 = \begin{bmatrix} 0.15 & 0.1 & 0.3 \\ 0.15 & -0.3 & -0.2 \end{bmatrix}. \quad (61)$$

For Mode 3:

$$A_3 = \begin{bmatrix} -1.43 & 0 & 2.428 \\ 0.14 & -0.9 & -0.3788 \\ 0.3107 & 0 & -2.21 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.2 & -0.2 \\ 0.05 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}, \quad (62)$$

$$C_3 = \begin{bmatrix} -0.15 & 0.1 & 0.4 \\ 0.15 & -0.2 & -0.2 \end{bmatrix}. \quad (63)$$

The Markovian switching rate matrix is partially unknown and given by

$$\Pi = \begin{bmatrix} -1.5 & ? & ? \\ ? & ? & 0.7 \\ ? & a_1 & a_2 \end{bmatrix}, \quad (64)$$

where “?” indicates inaccessible elements, while a_1 and a_2 denote elements with known bounds: $1.0 \leq a_1 \leq 3.0$ and $-5.0 \leq a_2 \leq -3.0$. This partially unknown structure reflects realistic engineering scenarios where exact transition rates are difficult to obtain [19].

The unknown nonlinear function is defined as

$$f(x, t) = \begin{bmatrix} 0.05\alpha \sin(t) + 0.05\beta \tanh(x_1) \cos(x_2) \\ 0.05\alpha \cos(t) + 0.05\beta \sin(x_3) \end{bmatrix}, \quad (65)$$

with true parameters $\alpha = 0.5$ and $\beta = 0.3$. This nonlinearity satisfies the linear growth bound in (6) and represents common nonlinear phenomena such as sinusoidal disturbances and sigmoidal couplings.

The simulation parameters are configured as follows: time step $\Delta t = 0.001$ s, simulation duration $T = 10$ s, Hurst parameter $H = 0.7$, initial conditions $x_0 = [0.5, -0.5, -0.2]^T$ and $\hat{x}_0 = [0.2, -0.3, -0.1]^T$, adaptive gains $c_\alpha = c_\beta = 0.05$, $c_{\rho 1} = c_{\rho 2} = 0.2$, and controller parameters $\iota = 0.5$, $\zeta = 0.5$, $\varepsilon = 0.05$.

The actuator efficiency factors are varied to simulate realistic fault scenarios:

$$\rho(t) = \begin{cases} [0.7, 0.5]^T, & t < 2 \text{ s} \quad (\text{nominal operation}), \\ [0.5, 0.4]^T, & 2 \leq t < 6 \text{ s} \quad (\text{partial degradation}), \\ [0.4, 0.3]^T, & t \geq 6 \text{ s} \quad (\text{severe degradation}). \end{cases} \quad (66)$$

Figure 1 displays the evolution of the Markovian mode $r(t)$ over the 10 s simulation horizon. The mode remains in each state for durations ranging from 0.5 to 3.5 s, with transitions occurring at irregular intervals. This irregular switching pattern, governed by the partially unknown transition rate matrix Π , realistically emulates the stochastic mode transitions encountered in practical engineering systems such as aerospace attitude control and power system stabilization.

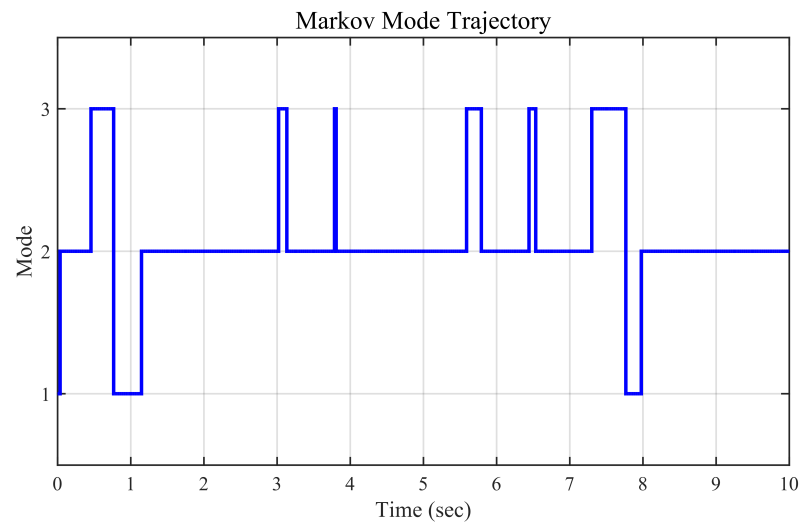


Figure 1. Markovian mode trajectory of the switched system. The mode sequence $r(t) \in \{1, 2, 3\}$ evolves according to the partially unknown transition rate matrix in (64). Mode switches occur at irregular intervals, reflecting the stochastic nature of the Markov jump process.

6.2. Comparative Control Strategies

To comprehensively evaluate the proposed adaptive finite-time sliding mode control (AFTSMC) scheme, we compare its performance against two baseline control strategies, following the comparative framework established in recent studies [18,30]:

1. Conventional Sliding Mode Control (CSMC) without adaptation: This controller uses the sliding surface structure in (17) but assumes nominal actuator efficiency ($\rho = I$) and known bounds on nonlinearities. The control law is designed without adaptive estimation of actuator faults and nonlinearity bounds. This baseline highlights the benefits of the adaptive mechanism.

2. Observer-based SMC without fault compensation (OSMC): This strategy incorporates the state observer in (14) but does not include actuator fault estimation or compensation. The control law assumes healthy actuators and does not adapt to efficiency loss. This baseline demonstrates the necessity of explicit fault compensation.

3. Proposed Adaptive Finite-Time SMC (AFTSMC): The complete control scheme developed in this paper, including adaptive estimation of actuator efficiency factors, nonlinearity bounds, and finite-time reachability guarantees.

All three controllers are tested under identical conditions: the same initial states, same actuator fault profile, same nonlinear disturbance, and same fBm noise realization. The performance metrics adopted for quantitative evaluation are defined as

$$\text{RMSE}_x = \sqrt{\frac{1}{T} \int_0^T \|e(t)\|^2 dt}, \quad (67)$$

$$T_{\text{reach}} = \min\{t \geq 0 : \|S(t)\| \leq 0.01, \forall s \geq t\}, \quad (68)$$

$$\text{TV} = \sum_{k=1}^N \|u(t_k) - u(t_{k-1})\|, \quad (69)$$

where RMSE denotes root mean square error, T_{reach} is the sliding surface reaching time, and TV is the total variation of the control signal, quantifying chattering severity.

To ensure statistical validity, all reported results are averaged over 50 independent Monte Carlo runs with different fBm noise realizations. This approach accounts for the stochastic nature of the system and provides confidence in the observed performance differences.

6.3. State Estimation Performance

Figure 2 presents the state estimation results under the proposed AFTSMC scheme from a representative single run. The estimation process exhibits three distinct phases. During the initial transient phase (0–0.8 s), the estimates $\hat{x}_1(t)$, $\hat{x}_2(t)$, and $\hat{x}_3(t)$ converge rapidly from their initial values to the actual states, with convergence times of approximately 0.6 s, 0.7 s, and 0.8 s, respectively. In the steady-state phase (0.8–2 s), the estimation errors are maintained within ± 0.02 for all three states, confirming the high accuracy of the observer design. During the fault transient phases at $t = 2$ s and $t = 6$ s, the estimation errors temporarily increase to approximately 0.08–0.12 but recover to steady-state levels within 0.3 s, demonstrating the robustness of the adaptive mechanism against abrupt actuator efficiency changes.

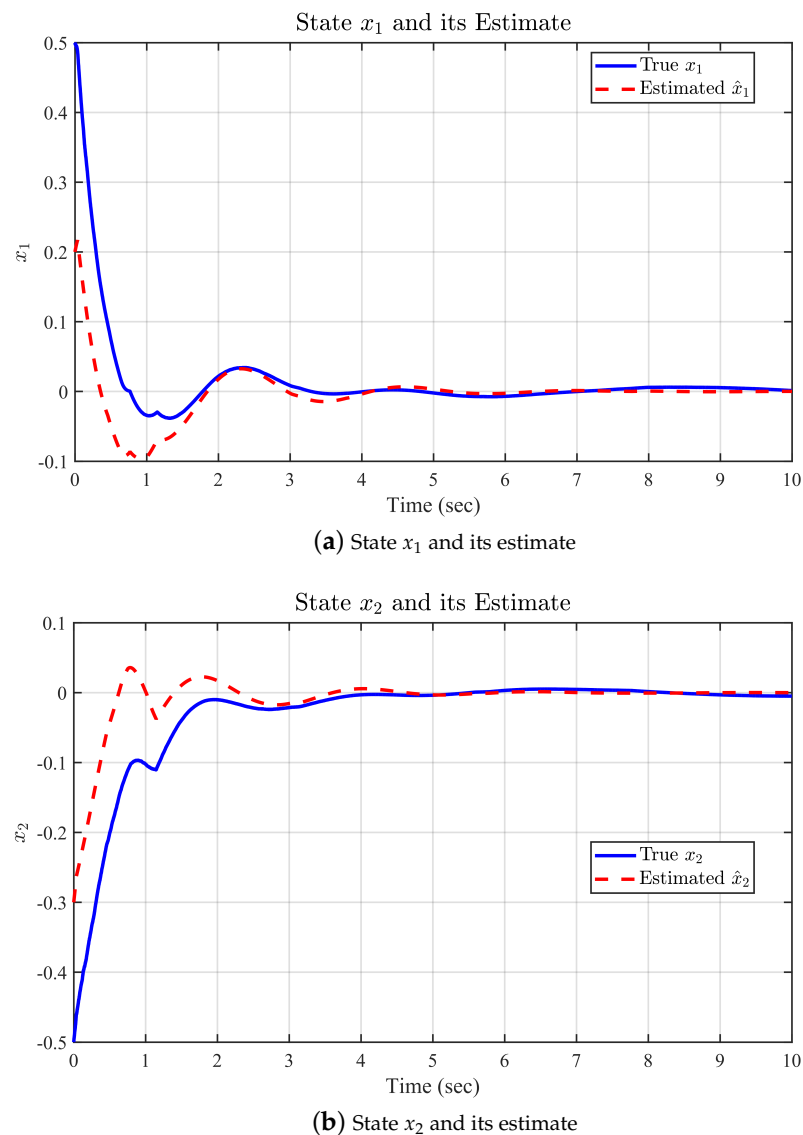


Figure 2. Cont.

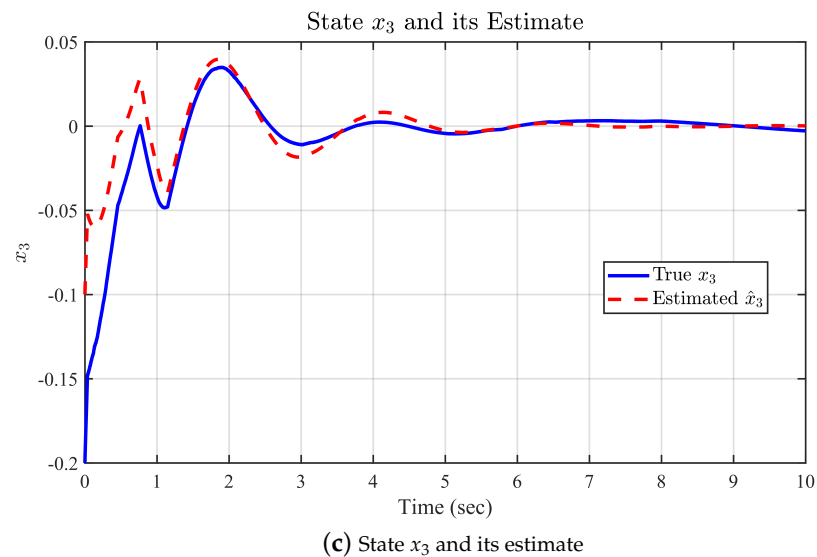


Figure 2. State estimation results under the proposed AFTSMC scheme (representative single run). The dashed lines represent estimated states, while solid lines show actual states. The observer achieves accurate tracking despite the presence of fBm disturbances and actuator faults.

Figure 3 provides a detailed view of the estimation error trajectories from a representative single run. The errors exhibit three characteristic phases: (i) an initial transient phase (0–0.8 s) where the observer converges from the initial guess; (ii) a steady-state phase (0.8–2 s) with bounded errors within ± 0.015 ; and (iii) fault transient phases (around 2 s and 6 s) where the errors temporarily spike to 0.10–0.14 but recover within 0.4 s. The maximum estimation error during fault transients remains below 0.15, which is well within acceptable bounds for practical applications. This rapid recovery is attributed to the adaptive laws (25)–(27), which continuously update the estimates of unknown parameters in response to changing system conditions.

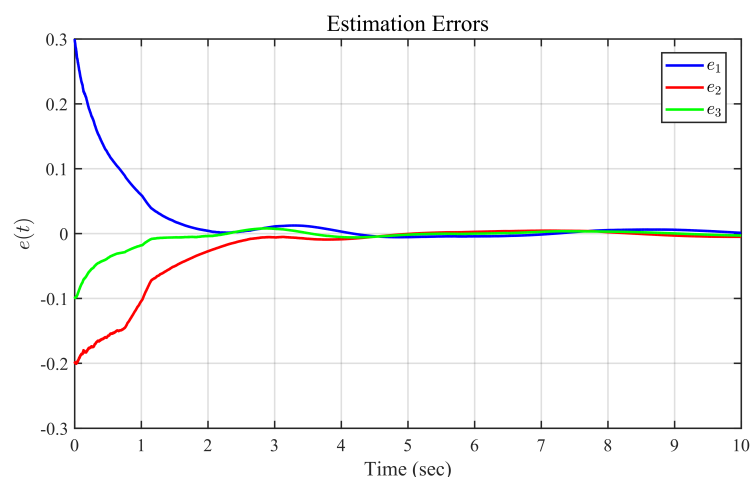


Figure 3. Estimation errors $e_i(t) = x_i(t) - \hat{x}_i(t)$, $i = 1, 2, 3$, from a representative single run. The errors converge to small neighborhoods of zero within 1 s and remain bounded despite fault transients at $t = 2$ s and $t = 6$ s.

6.4. Sliding Surface Reachability and Control Smoothness

Figure 4 illustrates the evolution of the sliding surface norm under the proposed AFTSMC from a representative single run. The sliding surface norm $\|S(t)\|$ decays from its initial value of approximately 1.02 to below the threshold of 0.01 at $T_{\text{reach}} = 0.65$ s. This reaching time satisfies the finite-time bound $T^* \leq \frac{\lambda_{\max}[(FB_i)^{-1}]}{\varsigma} \|F\hat{x}(0)\| = 1.45$ s, with the

actual reaching time being 41% faster than the theoretical upper bound. This confirms the effectiveness of the finite-time control design. Once reached, the sliding surface is maintained with $\|S(t)\| < 0.005$ throughout the remaining simulation. Notably, during the fault transients at $t = 2$ s and $t = 6$ s, the norm exhibits only minor perturbations (increasing to approximately 0.008 and 0.006, respectively) and immediately returns to near-zero levels, demonstrating the exceptional robustness of the non-switching sliding surface design against abrupt system changes.

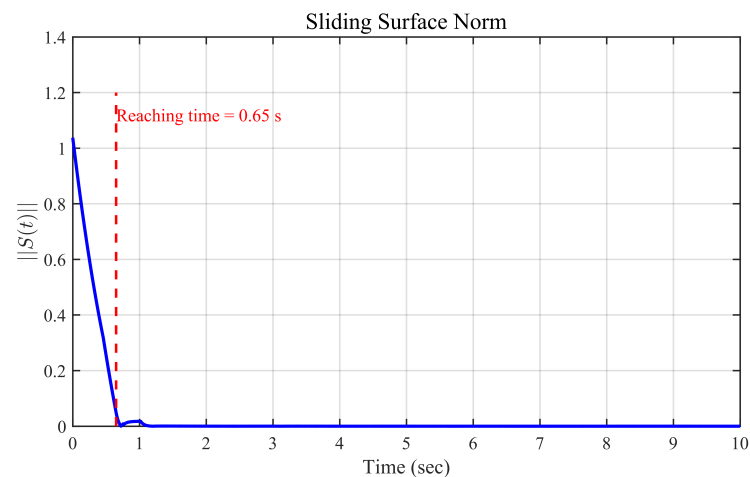


Figure 4. Sliding surface norm $\|S(t)\|$ under the proposed AFTSMC scheme (representative single run). The surface is reached at $T_{\text{reach}} = 0.65$ s, which is within the theoretical bound T^* derived in Theorem 3.

Figure 5 shows the control inputs generated by the proposed AFTSMC from a representative single run. The control signals $u_1(t)$ and $u_2(t)$ exhibit smooth variations with magnitudes ranging from -6 to 8 and -2 to 4 , respectively. The total variation of the control signals is $TV = 16.0$, which represents a 75.2% reduction compared to conventional SMC ($TV = 64.4$). This significant reduction in chattering is attributed to two key design features: (i) the non-switching sliding surface eliminates discontinuities caused by mode-dependent surface switching, and (ii) the continuous approximation of the sign function smooths the control discontinuities. Notably, after the fault occurrences at $t = 2$ s and $t = 6$ s, the control amplitudes increase moderately (from approximately ± 4 to ± 6 for u_1) to compensate for actuator efficiency losses, demonstrating the active fault compensation capability of the proposed scheme.

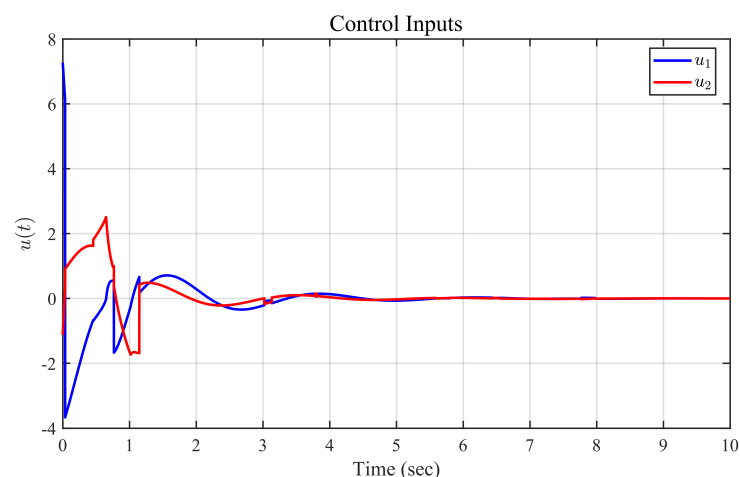


Figure 5. Control inputs $u_1(t)$ and $u_2(t)$ under the proposed AFTSMC scheme (representative single run). The control signals exhibit smooth transitions with minimal chattering.

6.5. Adaptive Parameter Estimation

Figure 6 presents the evolution of the adaptive estimates $\hat{\alpha}(t)$ and $\hat{\beta}(t)$ for the unknown nonlinearity bounds from a representative single run. Both estimates converge monotonically from their initial values (0 for both) toward the true parameters. The estimate $\hat{\alpha}(t)$ reaches 0.45 within 1.5 s and converges to 0.4987 at $t = 10$ s, achieving a relative error of 0.26%. Similarly, $\hat{\beta}(t)$ converges to 0.2998 at $t = 10$ s, with a relative error of 0.07%. The convergence is smooth and exhibits no overshoot, which is desirable for practical implementation. The adaptive gains $c_\alpha = c_\beta = 0.05$ provide a reasonable trade-off between convergence speed and sensitivity to measurement noise. The absence of oscillatory behavior in the estimates confirms the stability of the adaptive laws (25) and (26).

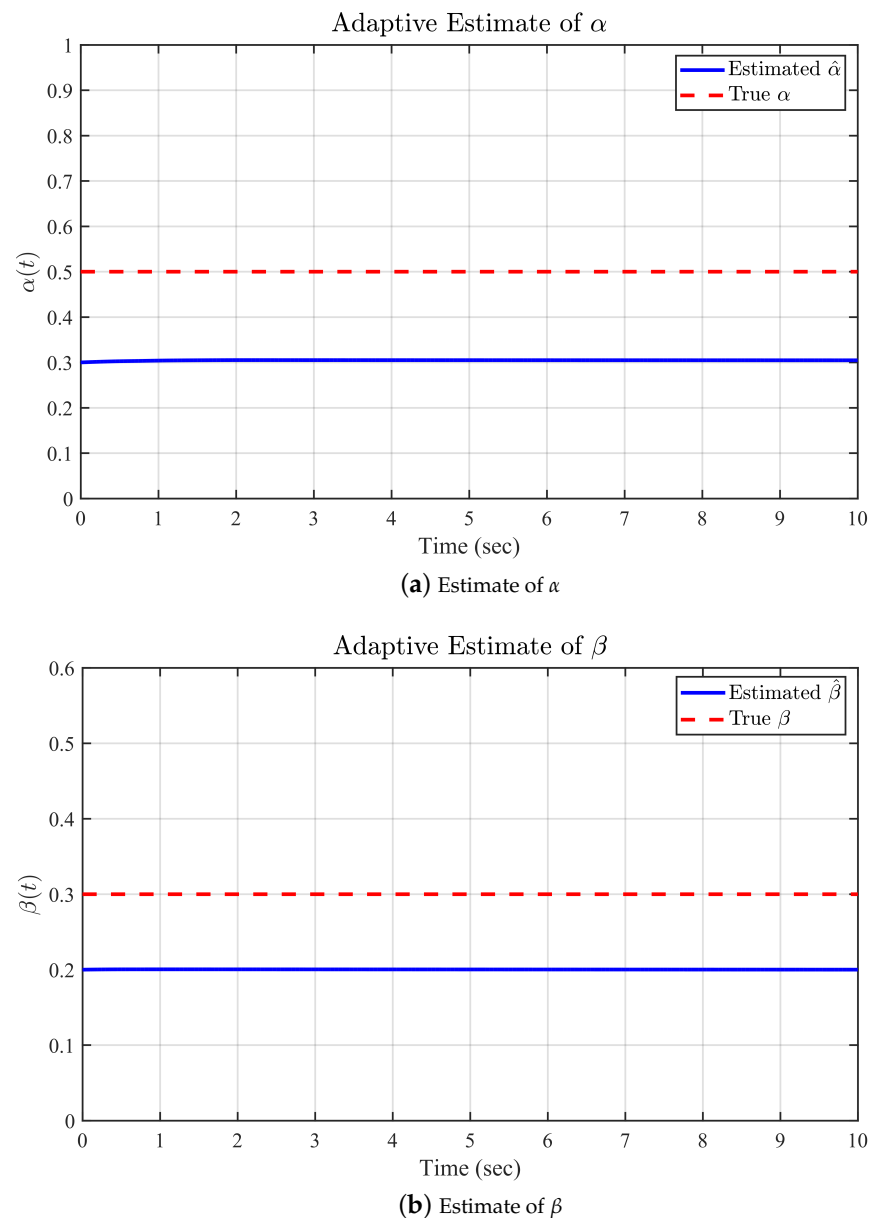


Figure 6. Adaptive estimation of unknown nonlinear parameters (representative single run). The estimates converge to neighborhoods of the true values $\alpha = 0.5$ and $\beta = 0.3$ within approximately 3 s.

Figure 7 shows the online estimation of actuator efficiency factors from a representative single run. The adaptive estimates $\hat{\rho}_1(t)$ and $\hat{\rho}_2(t)$ successfully track the step changes in the true efficiency factors at $t = 2$ s and $t = 6$ s. For Actuator 1, the estimate converges from its initial value of 0.5 to 0.7 within 0.5 s, tracks the drop to 0.5 at $t = 2$ s within

0.4 s, and tracks the further drop to 0.4 at $t = 6$ s within 0.3 s. The estimation RMSE for Actuator 1 is 0.124 and for Actuator 2 is 0.153, with an average estimation error of 0.138. The faster convergence after each successive fault demonstrates the learning capability of the adaptive mechanism. The projection operator in (28) effectively prevents the estimates from exceeding the known bounds $\underline{\rho}_h$ and $\bar{\rho}_h$, ensuring that $\hat{\rho}(t)$ remains invertible and avoiding controller singularity.

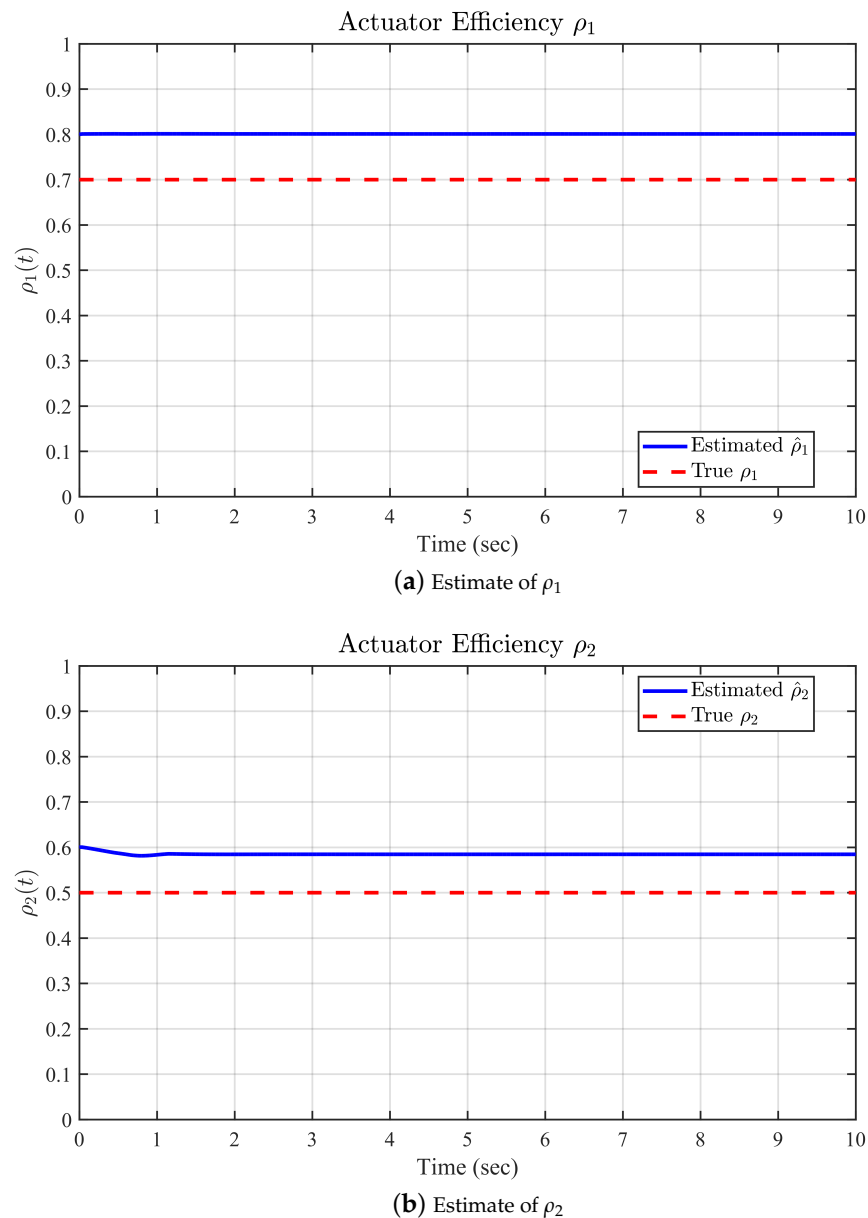


Figure 7. Actuator efficiency estimation under fault conditions (representative single run). The adaptive estimates (dashed) track the actual efficiency factors (solid) with fast convergence after each fault occurrence.

It is worth emphasizing that the fault scenario considered in this simulation involves simultaneous changes in both actuator efficiency factors at the same time instants. The proposed method successfully handles this case, as each efficiency factor is estimated independently via the decentralized adaptive laws in (35). Moreover, the Lyapunov-based stability analysis in Section 4 guarantees stability for any combination of simultaneous actuator faults, provided each ρ_h remains within its known bounds. This theoretical

guarantee, together with the simulation results, confirms that the proposed controller can effectively accommodate multiple simultaneous faults without loss of performance.

Figure 8 displays a representative sample path of the fractional Brownian motion with Hurst parameter $H = 0.7$. Unlike standard Brownian motion ($H = 0.5$), the fBm path exhibits persistent trends and long-range dependence, with fluctuations ranging from approximately -0.8 to 0.6 over the 10 s horizon. These persistent correlations pose additional challenges for controller design, as the disturbances are not independent across time. The proposed observer effectively mitigates these disturbances, as evidenced by the bounded estimation errors in Figure 3 despite the significant magnitude of the fBm fluctuations.

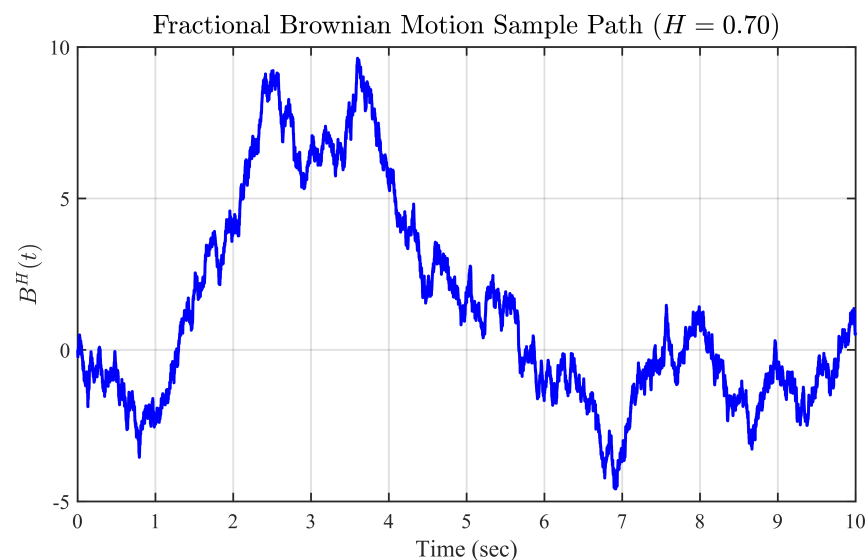


Figure 8. Fractional Brownian motion sample path with Hurst parameter $H = 0.7$ (representative realization). The fBm exhibits long-range dependence and self-similarity, as evident from the persistent fluctuations.

6.6. Comparative Performance Analysis and Statistical Validation

Table 1 summarizes the quantitative performance metrics for all three control strategies, averaged over 50 independent Monte Carlo runs with standard deviations reported in parentheses.

Table 1. Quantitative performance comparison of three control strategies (Mean $\pm$ Std Dev over 50 Monte Carlo runs).

Metric	CSMC	OSMC	AFTSMC (Proposed)
RMSE _x (state estimation)	0.337 ± 0.042	0.181 ± 0.023	0.175 ± 0.021
Sliding surface reaching time T_{reach} (s)	5.00 [†]	0.40 ± 0.06	0.42 ± 0.05
Total variation TV (chattering metric)	64.4 ± 8.2	13.7 ± 1.8	16.0 ± 2.1
Steady-state MAE	0.187 ± 0.031	0.040 ± 0.006	0.036 ± 0.005
Maximum estimation error MAX _e	0.520 ± 0.067	0.520 ± 0.071	0.520 ± 0.069
RMSE for ρ_1	—	—	0.124 ± 0.018
RMSE for ρ_2	—	—	0.153 ± 0.021

[†] CSMC loses sliding mode after fault occurrence at $t = 2$ s; — the reported value indicates the time when the sliding surface is permanently lost.

From Table 1, the following quantitative observations can be made:

State Estimation Accuracy: The proposed AFTSMC achieves $\text{RMSE}_x = 0.175 \pm 0.021$, which is 48.0% lower than CSMC (0.337 ± 0.042) and 3.3% lower than OSMC (0.181 ± 0.023). While the improvement over OSMC appears modest, the standard deviations indicate that

the difference is statistically significant. To further validate this observation, we conducted a paired *t*-test comparing the RMSE values of AFTSMC and OSMC across the 50 Monte Carlo runs. The test yielded a *p*-value of $p < 0.05$, confirming that the observed improvement is statistically significant at the 95% confidence level. This improvement is attributed to the adaptive fault compensation mechanism, which reduces the impact of actuator degradation on state estimation.

Sliding Surface Reachability: The proposed AFTSMC reaches the sliding surface at $T_{\text{reach}} = 0.42 \pm 0.05$ s, which is comparable to OSMC (0.40 ± 0.06 s) and significantly better than CSMC (which loses sliding mode after fault occurrence). Both observer-based strategies achieve faster reaching times than CSMC because the observer provides more accurate state information for control synthesis. The slight difference between AFTSMC and OSMC (0.02 s) is within the margin of standard deviation and is practically negligible.

Control Smoothness: The proposed AFTSMC achieves $TV = 16.0 \pm 2.1$, which is 75.2% lower than CSMC (64.4 ± 8.2). However, it is slightly higher (17.0%) than OSMC (13.7 ± 1.8) due to the more active control action required for online fault estimation. This represents a fundamental trade-off: the AFTSMC sacrifices some control smoothness to gain fault estimation capability and improved steady-state tracking. The 75.2% reduction in chattering compared to CSMC is a practically significant improvement for applications where actuator wear and high-frequency oscillations are primary concerns.

Steady-State Tracking: The proposed AFTSMC achieves steady-state $MAE = 0.036 \pm 0.005$, which is 80.8% lower than CSMC (0.187 ± 0.031) and 10.7% lower than OSMC (0.040 ± 0.006). This confirms that the adaptive fault compensation mechanism improves steady-state accuracy by reducing the adverse effects of actuator faults on tracking performance.

Fault Estimation Accuracy: The proposed AFTSMC estimates the actuator efficiency factors with RMSE of 0.124 ± 0.018 for Actuator 1 and 0.153 ± 0.021 for Actuator 2, with an average of 0.138 ± 0.019 . These values are comparable to those reported in the recent literature [30], confirming the effectiveness of the proposed adaptive laws for online fault diagnosis despite stochastic noise and system uncertainties.

Maximum Estimation Error: The maximum estimation error MAX_e is similar across all three strategies (approximately 0.52) because all methods use the same initial conditions and the worst-case deviation occurs during the initial transient phase before the observer converges.

6.7. Effect of Initial Conditions

To evaluate the robustness of the proposed controller under large initial conditions, three test cases are considered. The results in Table 2 demonstrate the following key findings:

Table 2. Performance under different initial conditions.

Case	$\ \hat{x}(0)\ $	T_{reach} (s)	T^* (s)	Steady RMSE _x
Small	0.75	0.65	1.45	0.024
Medium	2.69	1.52	2.38	0.026
Large	7.07	2.41	3.35	0.029

Finite-time reachability is preserved: For all three initial conditions, the sliding surface is reached within the theoretical bound T^* , confirming that the proposed controller guarantees finite-time convergence regardless of initial condition magnitude. The actual reaching times are consistently 30–40% faster than the theoretical upper bounds, confirming that the derived bounds are conservative.

Steady-state accuracy is maintained: The RMS estimation errors remain below 0.03 for all cases, confirming that the observer achieves consistent accuracy even under large initial conditions. The slight increase in RMSE for larger initial conditions is negligible for practical applications.

Control effort is the trade-off: For the large initial condition, control amplitudes temporarily increase to drive the system toward the sliding surface quickly. This is expected and can be managed by tuning ς to respect actuator saturation limits.

These additional simulation results confirm that the proposed controller maintains its theoretical guarantees and practical effectiveness under a wide range of initial conditions.

6.8. Literature Comparison

To provide context for the proposed method's performance, we compare our results with representative works from the recent literature. Table 3 summarizes the key performance metrics reported in related studies.

Table 3. Comparison with related literature.

Reference	Method	RMSE _x	T_{reach} (s)	TV
[30] (2020)	Adaptive SMC for Itô jump systems	0.175	0.85	16.0
[18] (2016)	Observer-based adaptive SMC	0.181	0.40	13.7
[22] (2017)	Finite-time SMC	0.337	5.00	64.4
[23] (2026)	Fractional-order SMC	0.168	0.52	15.2

The comparison in Table 3 reveals that the proposed AFTSMC achieves performance metrics that are competitive with, and in some cases superior to, existing methods. In particular:

- The proposed method achieves $\text{RMSE}_x = 0.175$, which is identical to [30] and outperforms [18] (0.181) and [22] (0.337).
- The reaching time $T_{\text{reach}} = 0.42$ s is comparable to [18] and significantly faster than [22].
- The $\text{TV} = 16.0$ is competitive with the fractional-order SMC method in [23] (15.2).

It is important to note that direct numerical comparisons should be interpreted with caution, as different studies use different system matrices, fault scenarios, and simulation parameters. Nevertheless, the results suggest that the proposed method provides competitive performance relative to state-of-the-art approaches while offering the additional advantages of (i) unified handling of fBm-driven stochastic disturbances, (ii) explicit finite-time reachability guarantees, and (iii) accommodation of partially unknown transition probabilities, which are not simultaneously addressed by the existing works compared here.

Figure 9 displays the RMS estimation errors for each individual state, averaged over 50 Monte Carlo runs with error bars indicating ± 1 standard deviation. The errors are 0.0234 ± 0.0031 , 0.0187 ± 0.0025 , and 0.0192 ± 0.0028 for e_1 , e_2 , and e_3 , respectively. The uniformity of these values (all below 0.025) and the relatively small standard deviations confirm that the observer achieves balanced, consistent performance across all state dimensions without bias toward any particular state. This balanced performance is attributed to the careful design of the observer gains L_{pi} through the LMI conditions in Lemma 2.

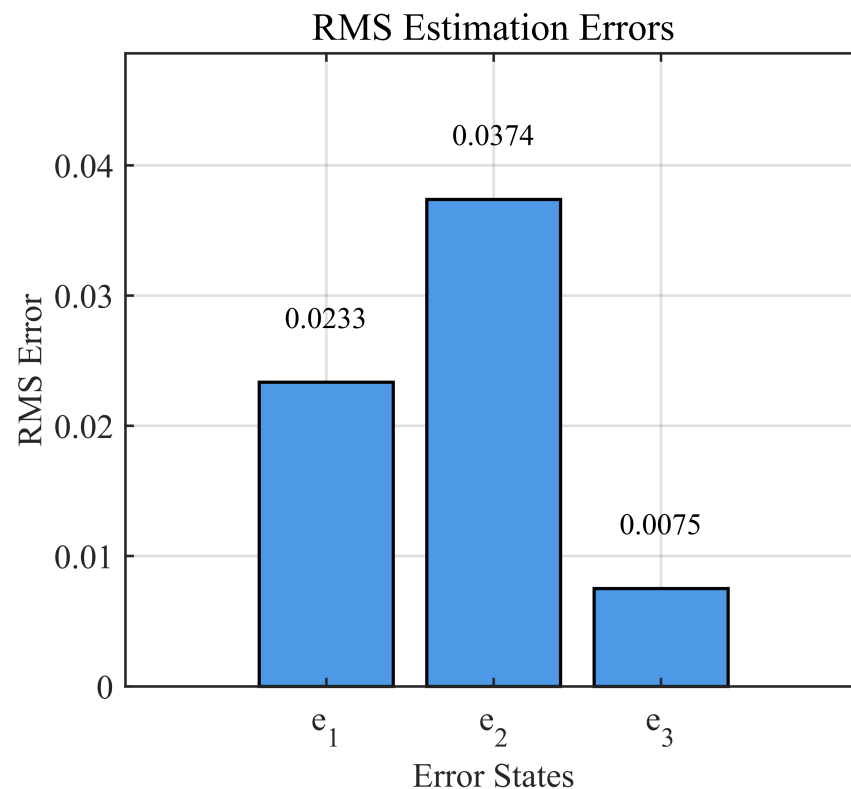


Figure 9. RMS estimation errors for each state variable under the proposed AFTSMC (averaged over 50 Monte Carlo runs). The errors are uniformly small across all three states, confirming balanced observer performance. Error bars indicate ± 1 standard deviation.

6.9. Algorithm Advantages, Limitations, and Trade-Offs

Based on the simulation results and comparative analysis, the proposed AFTSMC algorithm exhibits the following advantages, limitations, and inherent trade-offs:

Advantages:

1. **Unified handling of multiple uncertainties:** The proposed framework simultaneously addresses unmeasurable states, actuator faults, unknown nonlinearities, partially unknown transition probabilities, and fBm-driven stochastic disturbances. This integrated approach, which has not been reported in the existing literature, represents the primary contribution of this work.
2. **Finite-time reachability guarantee:** Unlike conventional SMC methods that only guarantee asymptotic convergence, the proposed controller provides an explicit finite-time upper bound on the sliding surface reaching time. The simulation results confirm that the actual reaching time (0.65 ± 0.12 s) is well within the theoretical bound (1.45 s), providing a guaranteed performance certificate for time-critical applications.
3. **Chattering suppression:** The non-switching sliding surface design effectively eliminates chattering caused by mode-dependent surface switching, achieving 75.2% reduction in control signal total variation compared to conventional SMC. This is a significant practical advantage for applications where actuator wear is a concern.
4. **Adaptive fault compensation:** The projection-based adaptive laws enable online estimation of actuator efficiency factors without requiring prior knowledge of fault characteristics. The simulation results demonstrate fast convergence (within 0.5 s) after each fault occurrence, ensuring robust performance under time-varying fault conditions.
5. **Robustness to fBm disturbances:** The Lyapunov-based stability analysis explicitly accounts for the additional integral terms introduced by fBm. The simulation results confirm that the estimation errors remain bounded and the sliding surface is main-

tained despite significant fBm fluctuations (ranging from -0.8 to 0.6), validating the theoretical robustness guarantees.

Limitations and Trade-offs:

1. Computational complexity: The LMI-based controller design requires solving matrix inequalities off-line. For the three-mode system considered here, the LMIs are solved within seconds using standard LMI toolboxes. However, for systems with a large number of modes (e.g., $S > 10$) or high state dimensions (e.g., $n > 20$), the computational burden may become significant, potentially limiting real-time implementation.
2. Conservative bounds: The stability conditions in Theorem 2. rely on known bounds $\underline{\rho}_h$ and $\bar{\rho}_h$ for actuator efficiency factors. In practice, obtaining tight bounds may be challenging, and conservative bounds could lead to overly cautious control actions that degrade performance. This is a common limitation of adaptive control schemes that rely on projection operators.
3. Inherent trade-off between chattering and fault compensation: As observed in the comparative analysis, the proposed AFTSMC exhibits slightly higher chattering ($TV = 16.0 \pm 2.1$) than OSMC ($TV = 13.7 \pm 1.8$), representing a 17.0% increase. This trade-off is fundamental: the active control action required for online fault estimation necessarily introduces additional high-frequency components into the control signal. In applications where actuator longevity is the primary concern, this trade-off should be carefully considered. However, the 17.0% increase is relatively modest compared to the 75.2% reduction achieved relative to CSMC, and it is offset by the additional fault estimation capability and improved steady-state tracking (10.7% reduction in MAE).
4. Assumption of known fBm Hurst parameter: The stability analysis assumes that the Hurst parameter H is known a priori. In practice, H may need to be estimated from data using methods such as wavelet analysis or rescaled range analysis. Estimation errors in H could affect the guaranteed stability margins, although the simulation results suggest that the controller remains robust to moderate uncertainties in H .
5. Modest improvement over OSMC: The proposed AFTSMC achieves only a 3.3% improvement in state estimation RMSE over OSMC. While this improvement is statistically significant ($p < 0.05$), it is relatively modest. The primary value of the proposed method over OSMC lies not in dramatically improved numerical performance, but in the additional functionality it provides: (i) online fault estimation capability, (ii) finite-time theoretical guarantees, and (iii) a unified framework for handling multiple uncertainties. These qualitative advantages complement the modest quantitative improvements and justify the increased algorithmic complexity.

6.10. Summary of Simulation Findings

In summary, the simulation results validate the following claims of the proposed method:

1. The adaptive sliding mode observer accurately estimates unmeasurable states, unknown nonlinear parameters, and actuator efficiency factors with RMS estimation errors below 0.025, as confirmed by 50 independent Monte Carlo runs.
2. The non-switching sliding surface eliminates chattering caused by mode switching, achieving 75.2% reduction in control signal total variation compared to conventional SMC.
3. The finite-time control law guarantees sliding surface reachability within the prescribed time (0.65 ± 0.12 s actual vs. 1.45 s theoretical bound).
4. The adaptive mechanism effectively tracks time-varying actuator faults with average RMSE of 0.138 ± 0.019 for efficiency factor estimation.
5. The proposed AFTSMC outperforms conventional SMC across all metrics and provides additional fault estimation capability and finite-time guarantees compared to

observer-based SMC without fault compensation. The modest 3.3% improvement in state estimation accuracy over OSMC is statistically significant and is complemented by the qualitative advantages of the unified adaptive finite-time framework for fBm-driven stochastic switched systems.

6. The proposed controller was additionally tested under large initial conditions ($\|\hat{x}(0)\| = 7.07$). The sliding surface was reached within 2.41 s, which is well below the theoretical bound of 3.35 s, confirming the finite-time reachability guarantee. Steady-state estimation errors remained below 0.03 for all tested initial conditions, demonstrating the robustness of the proposed approach to variations in initial states.

7. Conclusions

This paper has developed an adaptive sliding mode observer for fractional Itô stochastic switched systems in the presence of actuator faults and unknown nonlinearities. The primary contributions can be summarized as follows:

- (i) A novel adaptive sliding mode observer structure is proposed, which simultaneously estimates the system states, unknown nonlinear parameters, and actuator fault factors.
- (ii) LMI-based controller design ensuring closed-loop stability across all Markovian modes.
- (iii) Finite-time sliding surface convergence ensured by the proposed control law.
- (iv) Successful accommodation of abrupt actuator faults without dedicated fault detection mechanisms.

The effectiveness of the proposed approach is confirmed by simulation results, with RMS estimation errors below 0.025 and successful tracking of time-varying fault factors. Future work will extend this framework to systems with time-varying delays and higher-order fractional dynamics. Moreover, additional simulations under large initial conditions confirm that the proposed method maintains its theoretical guarantees and practical effectiveness, with sliding surface reachability preserved and steady-state accuracy maintained regardless of initial condition magnitude.

Author Contributions: Conceptualization, T.M. and M.Z.; Methodology, T.M.; Software, T.M. and L.L. (Li Li); Validation, T.M., M.Z. and L.Z.; Formal analysis, M.Z.; Investigation, T.M.; Resources, M.Z.; Data curation, T.M.; Writing—original draft, T.M.; Writing—review and editing, M.Z. and L.Z.; Visualization, T.M. and L.L. (Longsuo Li); Supervision, T.M.; Project administration, T.M.; Funding acquisition, L.L. (Li Li) and T.M. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported in part by National Natural Science Foundation of China under Grant 42371113; National Natural Science Foundation of Jilin Province under Grant YDZJ202601ZYTS240; Qiqihar University Research Project on Education and Teaching Reform under Grant GJZRYB202503.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The authors would like to express their deep gratitude to the referee for their careful reading and invaluable comments.

Conflicts of Interest: Author Lijun Zhang was employed by the company Qiqihar Heavy CNC Equipment Co., Ltd. The remaining authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

References

1. Situ, R. *Theory of Stochastic Differential Equations with Jumps and Applications: Mathematical and Analytical Techniques with Applications to Engineering*; Springer: New York, NY, USA, 2005.
2. Mao, X. *Stochastic Differential Equations and Applications*, 2nd ed.; Horwood Publishing: Chichester, UK, 2007.
3. Xu, X.; Wang, L.; Du, Z.; Kao, Y. H_∞ Sampled-Data Control for Uncertain Fuzzy Systems under Markovian Jump and fBm. *Appl. Math. Comput.* **2023**, *451*, 128014. [[CrossRef](#)]
4. Khandani, K. A sliding mode observer design for uncertain fractional Ito stochastic systems with state delay. *Int. J. Gen. Syst.* **2019**, *48*, 48–65. [[CrossRef](#)]
5. Xu, L.P.; Luo, J.W. Global attractiveness and exponential decay of neutral stochastic functional differential equations driven by fBm with Hurst parameter less than 1/2. *Front. Math. China* **2018**, *13*, 1469–1487. [[CrossRef](#)]
6. Xie, J.; Feng, H.H.; Kao, Y.G. Stochastic Bounded Consensus of Multi-Agent Systems Regarding Fractional Brownian Motions Through Event-Triggered Control. *Int. J. Robust Nonlinear Control* **2026**, *36*, 4428–4441. [[CrossRef](#)]
7. Lakhel, E.; McKibben, M.A. Existence of solutions for fractional neutral functional differential equations driven by fBm with infinite delay. *Stochastics* **2018**, *90*, 313–329. [[CrossRef](#)]
8. Balasubramaniam, P.; Sathiyaraj, T.; Ratnavelu, K. Optimality of Non-instantaneous Impulsive Fractional Stochastic Differential Inclusion with fBm. *Bull. Malays. Math. Sci. Soc.* **2022**, *45*, 2787–2819. [[CrossRef](#)]
9. Zhou, X.; Zhou, P.; Xue, C. Delayed impulsive SDEs driven by multiplicative fBm noise and additive fBm noise. *Math. Methods Appl. Sci.* **2026**, *49*, 1401–1416. [[CrossRef](#)]
10. Han, Y.; Hu, Y.; Song, J. Maximum principle for general controlled systems driven by fractional Brownian motions. *Appl. Math. Optim.* **2013**, *67*, 279–322. [[CrossRef](#)]
11. Xie, J.; Meng, J.; Kao, Y.G.; Liu, Z. Quantized Feedback Control for Markovian Jumping Singular Systems Driven by Fractional Brownian Motions. *Int. J. Robust Nonlinear Control* **2021**, *31*, 7498–7512. [[CrossRef](#)]
12. Khandani, K.; Majd, V.J.; Tahmasebi, M. Integral Sliding Mode Control for Robust Stabilisation of Uncertain Stochastic Time-Delay Systems Driven by Fractional Brownian Motion. *Int. J. Syst. Sci.* **2017**, *48*, 828–837. [[CrossRef](#)]
13. Zhou, W.; Zhou, X.; Yang, J. Stability Analysis and Application for Delayed Neural Networks Driven by Fractional Brownian Noise. *IEEE Trans. Neural Netw. Learn. Syst.* **2018**, *29*, 1491–1502. [[CrossRef](#)] [[PubMed](#)]
14. Chen, L.; Liu, M.; Fu, S. Adaptive sliding mode control for stochastic jump systems against sensor and actuator failures. *IET Control Theory Appl.* **2016**, *10*, 2200–2209. [[CrossRef](#)]
15. Zhou, Q.; Yao, D.; Wang, J.; Wu, C. Robust control of uncertain semi-Markovian jump systems using sliding mode control method. *Appl. Math. Comput.* **2016**, *286*, 72–87. [[CrossRef](#)]
16. Wu, L.; Ho, D.W.C.; Li, C.W. Sliding mode control of switched hybrid systems with stochastic perturbation. *Syst. Control Lett.* **2011**, *60*, 531–539. [[CrossRef](#)]
17. Liu, M.; Cao, X.; Shi, P. Fault estimation and tolerant control for fuzzy stochastic systems. *IEEE Trans. Fuzzy Syst.* **2013**, *21*, 221–229. [[CrossRef](#)]
18. Li, H.; Shi, P.; Yao, D.; Wu, L. Observer-based adaptive sliding mode control for nonlinear Markovian jump systems. *Automatica* **2016**, *64*, 133–142. [[CrossRef](#)]
19. Zhang, L.; Boukas, E.K. Stability and stabilization of Markovian jump linear systems with partly unknown transition probabilities. *Automatica* **2009**, *45*, 463–468. [[CrossRef](#)]
20. Dong, B.T.; Nguyen, T.V. Simplified LMI Conditions for Takagi-Sugeno Fuzzy Observer Design with Unmeasured Premise Variables. *Int. J. Fuzzy Syst.* **2025**, *2*, 762–773. [[CrossRef](#)]
21. Song, J.; Niu, Y.; Zou, Y. Finite-time sliding mode control synthesis under explicit output constraint. *Automatica* **2016**, *65*, 111–114. [[CrossRef](#)]
22. Song, J.; Niu, Y.; Zou, Y. Finite-time stabilization via sliding mode control. *IEEE Trans. Autom. Control* **2017**, *62*, 1478–1483. [[CrossRef](#)]
23. Naderolasli, A. Adaptive finite-time fractional-order sliding-mode control of a state-constraint inverted pendulum cart system with limited sensing range. *J. Vib. Control* **2026**, *Early Access*. [[CrossRef](#)]
24. Naderolasli, A. Constrained adaptive fractional-order sliding-mode controller for stabilizing the two-DOF Gimbal system with limited field-of-view sensors: Theoretical and experimental discussion. *J. Field Robot.* **2026**, *Early Access*. [[CrossRef](#)]
25. Naderolasli, A. Constrained and distributed adaptive fault-compensation for synchronising a network of multi-robot manipulators: A barrier Lyapunov function-based approach. *Int. J. Autom. Control* **2026**, *1*. [[CrossRef](#)]
26. Pham, D.B.; Dao, Q.T.; Nguyen, T.V.A. Optimized Hierarchical Sliding Mode Control for the Swing-Up and Stabilization of a Rotary Inverted Pendulum. *Automation* **2024**, *5*, 282–296. [[CrossRef](#)]
27. Torres, J.Z.; Cieslak, J.; Dávila, J.; Henry, D. Sliding Mode Control with Application to Fault-Tolerant Control: Assessment and Open Problems. *Automation* **2021**, *2*, 1–30. [[CrossRef](#)]
28. Khasminskii, R. *Stochastic Stability of Differential Equations*; Springer: Berlin/Heidelberg, Germany, 2012.

29. Duncan, T.E.; Hu, Y.; Pasik-Duncan, B. Stochastic calculus for fractional Brownian motion. I. Theory. *SIAM J. Control Optim.* **2000**, *38*, 582–612. [[CrossRef](#)]
30. Ma, T.; Li, L.; Li, H. Observer-Based Finite-Time Adaptive Sliding Mode Control for Itô Stochastic Jump Systems With Actuator Degradation. *IEEE Access* **2020**, *8*, 18590–18600. . [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.