

Article

Parameter Recovery in Noisy Tri-Periodic Horadam Sequences: A Fourier-Embedded Recurrence-Regularized Neural Estimator (RRNE)

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Abstract

This study investigates tri-periodic Horadam-type recurrence systems from both algebraic and computational perspectives. Algebraically, by adapting standard discrete Floquet-type and monodromy matrix viewpoints to the tri-periodic setting, the sequence is analyzed through induced recurrence relations, a monodromy matrix formulation, Binet-type representations, and an effective spectral radius ρ_{eff} . Computationally, the inverse parameter estimation problem under noisy observations is shown to be highly ill-conditioned. In particular, classical phase-wise Ordinary Least Squares (OLS) estimators may suffer from attenuation bias under an Errors-in-Variables mechanism, causing coefficient shrinkage and over-smoothed trajectory reconstruction in noisy recursive settings. To address this difficulty, we propose a Fourier-Embedded Recurrence-Regularized Neural Estimator (RRNE). By combining Fourier feature embeddings for period-three phase encoding, a Log-Cosh loss for robust residual control, a teacher-forced recurrence constraint, and hybrid Adam–L-BFGS optimization, the RRNE acts as a structure-aware inverse estimator. Numerical experiments across varying noise levels and asymmetry regimes show that the proposed framework provides stable parameter recovery and long-horizon reconstruction in low- and moderate-asymmetry regimes, while exposing severe instability under extreme phase-switching dynamics. The results further indicate that inverse-estimation collapse cannot be explained by the effective spectral radius alone; rather, it reflects the joint effect of coefficient asymmetry, spectral amplification, observation noise, and optimization stiffness. Overall, this study connects tri-periodic recurrence theory, monodromy-based spectral analysis, classical regression limitations, and recurrence-regularized neural inverse modeling.



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Keywords: tri-periodic Horadam sequences; discrete Floquet theory; monodromy matrix; effective spectral radius; spectral amplification; asymmetry index; Recurrence-Regularized Neural Estimator (RRNE); Fourier feature embeddings; inverse parameter estimation; attenuation bias; Errors-in-Variables; teacher forcing; Log-Cosh loss

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1. Introduction

Linear recursive sequences constitute one of the most fundamental research areas in modern discrete mathematics due to their deep algebraic structures, spectral properties, asymptotic behaviors, and extensive applications in number theory, combinatorics, matrix analysis, coding theory, signal processing, cryptography, mathematical physics, and dynamical systems. Among these sequences, Fibonacci and Lucas sequences occupy a central position because of their remarkable mathematical properties and their ability to generate broad classes of generalized recursive systems. Classical matrix-based viewpoints, including the Fibonacci Q -matrix and its higher-dimensional extensions, further demonstrate the structural importance of Fibonacci-type recurrences in matrix analysis [1]. Related combinatorial developments, such as Rogers–Ramanujan identities for n -colour partitions and restricted n -color composition functions, also illustrate the role of recurrence-type and composition structures in enumerative settings [2,3].

The classical Fibonacci sequence $\{F_n\}_{n \geq 0}$ is defined by the recurrence relation [4]

$$F_n = F_{n-1} + F_{n-2}, \quad (1)$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

Edson and Yayenie [5] introduced the generalization of the Fibonacci sequence defined as

$$F_0^{a,b} = 0, \quad F_1^{a,b} = 1, \quad (2)$$

and

$$F_n^{a,b} = \begin{cases} aF_{n-1}^{a,b} + F_{n-2}^{a,b}, & n \text{ even}, \\ bF_{n-1}^{a,b} + F_{n-2}^{a,b}, & n \text{ odd}, \end{cases} \quad n \geq 2. \quad (3)$$

Thus, the recurrence relation can be written as

$$F_{n+2}^{a,b} = \left(a^{1-\zeta(n)} b^{\zeta(n)}\right) F_{n+1}^{a,b} + F_n^{a,b}, \quad (4)$$

where

$$\zeta(n) = n - 2 \left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}. \end{cases} \quad (5)$$

Edson and Yayenie [5] established the following Binet-type formula:

$$F_n^{a,b} = \left(\frac{a^{1-\zeta(n)}}{(ab)^{\lfloor n/2 \rfloor}} \right) \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad (6)$$

where α and β are the roots of the characteristic equation

$$x^2 - abx - ab = 0.$$

The corresponding generating function was also derived in [5].

$$\sum_{n=0}^{\infty} F_n^{a,b} x^n = \frac{x + ax^2 - x^3}{1 - (ab + 2)x^2 + x^4}, \quad (7)$$

and

$$\sum_{n=0}^{\infty} F_n^{a,b} x^n = \frac{1}{1 - (a^{1-\zeta(n)} b^{\zeta(n)})x - x^2}. \quad (8)$$

One of the most influential generalizations of Fibonacci-type sequences was introduced by Horadam [6], who defined the sequence

$$W_n = pW_{n-1} + qW_{n-2},$$

where $p, q \in \mathbb{R}$ and the initial conditions are arbitrary. This formulation unifies many classical recursive systems such as Fibonacci, Lucas, Pell, Jacobsthal, Pell–Lucas, balancing, Mersenne, and Chebyshev sequences.

The characteristic equation associated with the Horadam recurrence,

$$\lambda^2 - p\lambda - q = 0,$$

plays a fundamental role in determining the asymptotic behavior of the sequence. Let α and β denote the roots of the characteristic polynomial. Then a Binet-type representation may be expressed as

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where A and B depend on the initial conditions. Consequently, the asymptotic amplification mechanism of the sequence is governed primarily by

$$\max\{|\alpha|, |\beta|\},$$

which naturally establishes a connection between recurrence theory, spectral analysis, and matrix theory.

Because of these algebraic and spectral properties, generalized Horadam systems have attracted considerable attention in the literature. Adegoke et al. [7,8] investigated recurrence relations involving squared Horadam numbers together with reciprocal sums, generating functions, and double binomial identities. Djordjevic and Djordjevic [9] introduced generalized Horadam polynomials associated with higher-order recurrence relations and investigated their generating functions, explicit formulas, and convolution structures. Similarly, Soykan [10] investigated generalized Horadam–Leonardo polynomial systems together with matrix identities, Binet formulas, and higher-order recurrence structures. In a related direction, Dişkaya et al. [11] studied hyperbolic Leonardo and hyperbolic François quaternions, providing another example of how Leonardo-type recursive structures can be extended to quaternionic and hyperbolic settings.

Another important research direction concerns higher-order recursive systems and self-similar recurrence structures. Kuloğlu et al. [12] investigated higher-order Horadam numbers together with Catalan transforms, generalized Hosoya structures, matrix generators, and self-similarity phenomena. Their study demonstrated that generalized recursive systems naturally generate scalable asymptotic structures and matrix-theoretic behaviors. Such developments indicate that generalized Horadam systems are deeply connected with recursive operator theory, spectral amplification, and asymptotic transformation mechanisms.

In recent years, substantial attention has been devoted to periodic recursive systems in which the recurrence coefficients vary according to modular conditions. Unlike classical autonomous recurrences, periodic recursive systems form non-autonomous dynamical systems whose transition operators depend explicitly on periodic phase structures.

Following this development, numerous generalizations of bi-periodic recursive systems were introduced. Coskun and Taskara [13] investigated bi-periodic Fibonacci and Lucas matrix sequences together with matrix-theoretic identities. Polatlı [14] studied geometric circulant matrices involving bi-periodic Fibonacci and Lucas numbers and derived

spectral norm inequalities associated with these matrices. Matrix formulations and companion matrix approaches for periodic recursive systems were later investigated in [15]. Additional studies involving q -analogues, quaternion systems, polynomial structures, and incomplete recursive systems were developed in [16–21]. Kuloğlu et al. introduced the bivariate bi-periodic Mersenne polynomials as a new extension of classical Mersenne polynomials involving two variables and bi-periodic coefficients [22]. These studies collectively demonstrate that periodic recurrence structures form an increasingly important intersection between recurrence theory, matrix analysis, asymptotic dynamics, and computational mathematics.

Despite the extensive literature concerning bi-periodic recursive systems, tri-periodic recurrence structures remain comparatively less explored. In many application-motivated settings, the governing mechanisms evolve according to more than two operational phases. Consequently, bi-periodic models may become insufficient for accurately representing such systems. Tri-periodic recursive systems provide a richer mathematical framework because their coefficients evolve according to three distinct modular phases.

A general tri-periodic Horadam-type recurrence may be represented by

$$H_n = \begin{cases} pH_{n-1} + rH_{n-2}, & n \equiv 0 \pmod{3}, \\ qH_{n-1} + rH_{n-2}, & n \equiv 1 \pmod{3}, \\ sH_{n-1} + rH_{n-2}, & n \equiv 2 \pmod{3}, \end{cases}$$

where $p, q, s, r \in \mathbb{R} \setminus \{0\}$. Unlike autonomous recurrences, this structure generates phase-dependent transition operators and non-uniform asymptotic amplification mechanisms.

The associated matrix formulation can be written by introducing the state vector

$$\mathbf{v}_n = \begin{pmatrix} H_n \\ H_{n-1} \end{pmatrix}.$$

Then the transition from $\mathbf{v}_{n-1}$ to $\mathbf{v}_n$ is governed by one of the phase-dependent matrices

$$A_p = \begin{pmatrix} p & r \\ 1 & 0 \end{pmatrix}, \quad A_q = \begin{pmatrix} q & r \\ 1 & 0 \end{pmatrix}, \quad A_s = \begin{pmatrix} s & r \\ 1 & 0 \end{pmatrix}.$$

Over one complete period, the corresponding monodromy matrix is obtained by composing the three phase transitions. In particular, for the evolution from $\mathbf{v}_{3k-3}$ to $\mathbf{v}_{3k}$, one obtains

$$\mathbf{v}_{3k} = A_p A_s A_q \mathbf{v}_{3k-3}.$$

Thus, the long-term amplification behavior of the tri-periodic recurrence is governed by the spectral properties of the associated monodromy matrix. This distinguishes tri-periodic systems from classical second-order autonomous recurrences and leads to periodic spectral coupling, modular transition operators, and highly non-uniform asymptotic growth regimes.

The modeling of complex dynamical systems through generalized recursive structures plays a fundamental role in understanding long-term growth mechanisms, asymptotic transitions, and phase-dependent evolutionary behaviors. In this context, tri-periodic recursive systems constitute an important class of non-autonomous discrete dynamical systems whose transition operators evolve according to modular phase structures. Unlike homogeneous second-order recurrences with constant coefficients, tri-periodic systems involve periodically varying recursive dynamics, allowing them to model multi-phase phenomena more effectively.

Such systems naturally arise in various applications involving alternating operational regimes. Examples include seasonal productivity fluctuations in agricultural technologies, cyclic environmental variations in IoT sensor networks, biological growth dynamics, and periodically evolving engineering processes. The modular coefficient structure enables the recursive system to capture phase transitions and alternating amplification regimes that cannot be represented adequately through classical homogeneous recurrence relations.

However, the variable-coefficient recurrence relations associated with tri-periodic Horadam systems,

$$H_n = pH_{n-1} + rH_{n-2}, \quad \text{etc.},$$

make inverse parameter estimation sensitive to noise and recursive amplification. When the coefficient r is fixed and the observations are noise-free, the periodic coefficients p , q , and s can in principle be recovered by direct phase-wise relations. However, in noisy and exponentially amplified regimes, the lagged terms appearing in the recurrence are themselves contaminated by measurement errors. Consequently, the recovery of the hidden periodic coefficients may become a severely ill-conditioned inverse-estimation problem, especially under strong phase switching and long-horizon extrapolation.

Solving the inverse parameter estimation problem for amplified tri-periodic recurrences is particularly delicate under measurement noise. Although the recurrence relation naturally suggests a phase-wise linear regression approach, classical estimators such as Ordinary Least Squares (OLS) may encounter serious statistical limitations in this setting. Under the Errors-in-Variables framework, when autoregressive explanatory terms are contaminated by Gaussian observation noise, the denominator of the OLS estimator may be inflated by the noise variance. This mechanism can produce attenuation bias, also known as regression dilution, by shrinking estimated coefficients toward smaller magnitudes. Consequently, classical regression models may over-smooth phase-switching dynamics and fail to recover the true coefficient-dependent recursive structure in highly amplified regimes. This motivates the use of a structure-aware framework capable of combining noise-aware trajectory reconstruction, phase encoding, and recurrence-regularized parameter estimation.

In many practical scenarios, determining these hidden modular parameters using classical statistical or numerical approaches becomes considerably more challenging, particularly in the presence of exponential amplification and measurement noise. The principal mathematical difficulty originates from two fundamental mechanisms. First, the recursive system may exhibit uncontrolled asymptotic exponential amplification as the iteration index n increases. From a spectral perspective, this behavior is directly connected with the amplification properties of the associated phase-dependent transition matrices and the corresponding monodromy matrix. Second, in application-motivated settings, observations naturally contain measurement noise, often modeled as Gaussian perturbations. When exponential amplification and Gaussian noise coexist, even small perturbations may propagate recursively and produce extremely large deviations in the asymptotic trajectories.

Consequently, classical regression and optimization algorithms frequently become trapped in local minima and fail to recover meaningful periodic coefficients.

Moreover, exponentially amplified recursive dynamics often generate severe numerical instabilities, including gradient explosion, floating-point collapse, and optimization degeneracy. Here, floating-point collapse refers to severe numerical instability characterized by loss explosion, gradient explosion, and the appearance of non-finite floating-point values such as NaN or Inf during optimization.

Recently, several studies concerning tri-periodic structures have begun to appear in the literature. Yilmaz and Soykan [23] investigated tri-periodic Fibonacci and Leonardo numbers together with generating functions, Binet-type identities, and several algebraic

properties. Nevertheless, inverse reconstruction of hidden periodic coefficients under noisy observations, especially for exponentially amplified tri-periodic recursive systems, remains largely unexplored in the existing literature.

To quantify the asymmetry of the periodic recursive structure, we define

$$\mathcal{A} := \max\{|p - q|, |q - s|, |s - p|\}.$$

Here, $\mathcal{A}$ measures the maximum disparity among the modular coefficients p , q , and s . Larger values of $\mathcal{A}$ may be associated with increasingly nonuniform phase transitions and poorer numerical conditioning in the parameter regimes considered here. However, $\mathcal{A}$ is used only as an empirical diagnostic indicator; the effective spectral growth is governed by the monodromy matrix through the combined quantities $X = pqs + pr + qr + sr$ and $Y = r^3$.

In recent years, Physics-informed neural networks (PINNs) have emerged as powerful tools for solving inverse problems governed by differential equations and physical constraints. Pioneering work of Raissi et al. [24] introduced the PINNs by embedding governing equations directly into neural network loss functions. Since then, PINNs have become highly influential in solving forward and inverse problems involving nonlinear dynamical systems.

Nevertheless, recent studies have shown that standard PINNs encounter severe difficulties when applied to highly oscillatory or exponentially growing systems. Krishnapriyan et al. [25] mathematically proved that PINNs may enter undesirable failure modes and become trapped in local minima under highly dynamic systems. Similarly, Wang et al. [26] investigated gradient flow pathologies in PINNs and demonstrated that exponentially amplified or multi-scale systems frequently suffer from severe numerical instability and gradient starvation during training. The floating-point collapse observed in exponentially amplified recursive systems is directly related to these optimization pathologies.

Another major difficulty arises from the spectral bias phenomenon of neural networks. Rahaman et al. [27] demonstrated that deep neural networks naturally favor low-frequency smooth functions while struggling to approximate highly oscillatory periodic structures. In tri-periodic Horadam systems, the modular transitions between recursive phases generate precisely such discontinuous oscillatory behaviors. Consequently, standard neural architectures tend to smooth the recursive dynamics excessively and converge toward averaged solutions instead of correctly distinguishing the periodic coefficients.

To overcome this limitation, Tancik et al. [28] introduced Fourier feature embeddings, where the input space is expanded through trigonometric transformations. Motivated by this idea, the present study incorporates explicit modular Fourier embeddings of the form

$$\left(\sin\left(\frac{2\pi n}{3}\right), \cos\left(\frac{2\pi n}{3}\right) \right),$$

thereby enabling the neural architecture to distinguish the intrinsic tri-periodic phases naturally.

Furthermore, standard Mean Squared Error (MSE) losses become numerically unstable under exponentially growing recursive trajectories. To prevent optimization collapse caused by large asymptotic deviations and noisy observations, robust Log-Cosh loss functions inspired by Barron [29] are incorporated into the architecture. Moreover, to stabilize the Hessian approximation under exponentially amplified recursive systems, a hybrid Adam–L-BFGS optimization strategy based on Liu and Nocedal [30] is employed.

Another essential contribution of this study is the incorporation of a teacher-forced recurrence constraint into the RRNE framework. Instead of recursively propagating predicted

values, which may accumulate catastrophic numerical errors, observed data are integrated directly into the recursive constraints during training. This substantially improves optimization stability and mitigates recursive error accumulation along long trajectories.

Motivated by these mathematical and computational challenges, we develop a Fourier-Embedded RRNE framework designed to support inverse parameter estimation in noisy tri-periodic Horadam systems.

The main contributions of this study can be summarized as follows:

- **Algebraic Adaptation and Spectral Analysis:** We adapt standard discrete Floquet-type theory, monodromy matrix formulations, and periodic difference-equation viewpoints to the tri-periodic Horadam setting [31–33]. This provides a structured period-level analysis of the recurrence through the monodromy matrix, the induced three-step recurrence, Binet-type representations, and the effective spectral radius ρ_{eff} . In this way, the quantities

$$X = pqs + pr + qr + sr, \quad Y = r^3$$

are explicitly connected with the spectral amplification behavior of modularly varying recursive systems.

- **Classical Attenuation Bias and the Need for Structure-Aware Estimation:** We clarify why direct phase-wise Ordinary Least Squares (OLS) regression may be insufficient for noisy tri-periodic recursive systems. Under the Errors-in-Variables framework, noisy autoregressive explanatory terms can inflate the denominator of the OLS estimator and lead to attenuation bias, also known as regression dilution [34]. This statistical limitation explains why classical regression may over-smooth phase-switching trajectories and fail to recover the true coefficient-dependent recursive structure in highly amplified regimes. Therefore, the OLS comparison is used not merely as a baseline, but also as a diagnostic tool for identifying noise-induced regression bias.
- **Fourier-Embedded Recurrence-Regularized Neural Estimator:** We propose a Fourier-Embedded Recurrence-Regularized Neural Estimator (RRNE) for inverse parameter recovery under noisy observations. The proposed framework combines a teacher-forced recurrence constraint to stabilize recursive training [35], Fourier feature embeddings to represent the intrinsic period-three structure and reduce spectral bias [27,28], robust Log-Cosh loss to reduce sensitivity to amplified residuals [29], and hybrid Adam–L-BFGS optimization based on limited-memory quasi-Newton refinement [30]. These components are used as an integrated structure-aware architecture for stabilizing inverse estimation in noisy and amplified recursive dynamics.
- **Disentangling Asymmetry from Spectral Amplification:** Through systematic numerical experiments, we examine the relationship between the coefficient asymmetry index $\mathcal{A}$, the effective spectral radius ρ_{eff} , and long-horizon extrapolation stability. Motivated by the known optimization pathologies of physics-informed and recurrence-regularized neural models in stiff or highly dynamic systems [25,26], the results indicate that inverse-estimation failure cannot be explained by the effective spectral radius alone. Rather, the observed instability reflects the combined effect of phase-dependent coefficient asymmetry, spectral amplification, observation noise, teacher forcing bias, and optimization stiffness.
- **Robustness Mapping and Operational Limits:** We map the operational behavior of the proposed RRNE framework through long-horizon extrapolation with $N = 60$, noise-sensitivity experiments, OLS baseline comparisons, ablation analysis, and repeated-seed stability tests. These experiments identify regimes in which recurrence-informed parameter recovery remains reliable, as well as extreme phase-switching regimes in which the inverse problem becomes severely ill-conditioned. This provides an empirical robustness and limitation map for parameter recovery in noisy tri-periodic

Horadam systems, complementing the spectral and recurrence-theoretic foundations of generalized Horadam-type systems [6,12,23,36].

The remainder of the paper is organized as follows. Section 2 reviews the related literature on Horadam-type sequences, periodic recurrence structures, neural-network-based inverse modeling, gradient pathologies, spectral bias, and noise-robust optimization. Section 3 introduces the tri-periodic Horadam sequence and establishes its basic algebraic properties. Section 4 derives the induced three-step recurrence, the generating function, and the corresponding Binet-type representation. Section 5 investigates the monodromy matrix formulation, the characteristic equation of the monodromy matrix, the effective spectral radius, and the asymptotic behavior of the sequence. Section 6 presents the Fourier-Embedded Recurrence-Regularized Neural Estimator (RRNE) framework for inverse parameter estimation under noisy observations. Section 7 reports the numerical experiments, including asymmetry-regime analysis, noise-sensitivity experiments, OLS baseline comparison, ablation analysis, repeated-seed stability tests, and trajectory-level interpretation of the obtained results. Finally, Section 8 concludes the paper and discusses possible directions for future research.

2. Related Work

Studies concerning integer sequences and the inverse problems associated with these sequences are generally concentrated around two principal research directions in the literature: structural generalizations of Horadam-type sequences and artificial intelligence based optimization difficulties encountered in the solution of complex dynamical systems.

2.1. Horadam Sequences and Periodic Generalizations

The fundamental structure of linear recursive sequences was formulated by Horadam [6], who established a general mathematical framework encompassing Fibonacci and Lucas sequences. In subsequent years, this constant-coefficient structure evolved into periodic recursive systems in which the coefficients acquired periodic characteristics in order to model system dynamics more accurately. Edson and Yayanie [5] introduced bi-periodic generalizations of the Fibonacci sequence into the literature and demonstrated how the classical Binet formula could be adapted to periodic systems. Similarly, Bilgici [36] investigated generalized properties of Lucas sequences under different initial conditions. However, the existing studies in the literature have mostly been restricted to constant or bi-periodic structures, while the extreme exponential growth characteristics of tri-periodic sequences with asymmetric coefficients and the associated parameter estimation problems have remained largely unresolved.

It is also important to emphasize that the use of one-period transition matrices, monodromy operators, and spectral-radius-based stability indicators is a classical idea in the analysis of linear systems with periodic coefficients.

In particular, Floquet-type theory provides a standard framework for reducing periodic linear dynamics to the spectral analysis of a period-level transition operator, whose eigenvalues, often referred to as Floquet multipliers, determine the corresponding growth or stability behavior [31–33]. In the present work, these established ideas are not proposed as a new general theory. Rather, they are explicitly specialized to the tri-periodic Horadam recurrence in order to connect its coefficient-dependent algebraic structure with the induced three-step recurrence, the monodromy matrix, and the effective spectral radius used in the subsequent inverse-estimation analysis.

2.2. Physics-Informed Neural Networks (PINNs), Gradient Pathologies, and the Motivation for RRNE

The integration of differential equations and physical constraints into the loss functions of neural networks was first introduced by Raissi et al. [24] under the framework of PINNs. Although PINNs have become standard approaches for solving forward and inverse problems involving differential equations and dynamical systems, recent studies have revealed several limitations of these networks in highly dynamic, multi-scale, or stiff regimes.

Krishnapriyan et al. [25] demonstrated that PINNs may enter undesirable failure modes and become trapped in local minima for systems involving highly oscillatory dynamics. Furthermore, Wang et al. [26] investigated gradient-flow pathologies arising during the training of PINNs and showed that neural architectures may lose numerical stability in multi-scale or exponentially amplified systems, leading to gradient starvation phenomena.

The floating-point collapse encountered in the present study is related to these broader optimization pathologies. However, the method proposed here is not presented as a classical PINNs-based method. Since the recurrence constraint is implemented through teacher forcing using noisy observed lagged states, the proposed framework is more accurately described as an RRNE framework. In this sense, the RRNE framework should be interpreted as a structure-aware inverse estimator for noisy recursive systems rather than as a classical PINNs-based formulation.

Moreover, the mathematical foundations of the L-BFGS optimization strategy employed in this work were established by Liu and Nocedal [30]. In the present framework, the fragility of the optimization landscape under exponential amplification is addressed through a hybrid Adam–L-BFGS strategy, where Adam provides an initial stabilization phase and L-BFGS is used for subsequent local refinement.

2.3. Spectral Bias and Noise-Robust Optimization

The principal reason why neural networks fail to learn the successive modular transitions (mod 3) appearing in tri-periodic Horadam systems is the phenomenon known in the literature as spectral bias. Rahaman et al. [27] proved that deep neural networks naturally tend to learn low-frequency smooth functions while failing to approximate high-frequency or discontinuous periodic signals accurately. To overcome this limitation, Tancik et al. [28] introduced Fourier feature embeddings, where the input space is expanded through trigonometric transformations, enabling periodic phases to be learned with significantly higher sensitivity. The modular (sin, cos) integration employed in the present study is based directly on this framework. Finally, in order to prevent the severe asymmetry and high Gaussian noise present at the extremities of the recursive trajectories from collapsing standard Mean Squared Error (MSE) metrics, robust loss functions such as Log-Cosh, inspired by the asymptotically tolerant optimization framework proposed by Barron [29], were incorporated into the proposed architecture.

3. Tri-Periodic Horadam Sequences and Basic Algebraic Properties

In this section, we introduce a generalized Horadam-type sequence defined by $H_n^{(p, q, s)}$. This sequence can be regarded as a tri-periodic extension of the classical Horadam numbers, incorporating periodic coefficients governed by the parameters p , q , and s . This formulation provides a unified framework that extends several well-known sequences as special cases.

Definition 1. Let p, q, s , and r be nonzero real numbers, and assume that the parameter r remains fixed throughout the paper. The tri-periodic Horadam sequence $\{H_n^{(p, q, s)}\}_{n \geq 0}$ is defined by the initial conditions

$$H_0^{(p, q, s)} = a, \quad H_1^{(p, q, s)} = b,$$

and the recurrence relation

$$H_n^{(p, q, s)} = \begin{cases} p H_{n-1}^{(p, q, s)} + r H_{n-2}^{(p, q, s)}, & n \equiv 0 \pmod{3}, \\ q H_{n-1}^{(p, q, s)} + r H_{n-2}^{(p, q, s)}, & n \equiv 1 \pmod{3}, \\ s H_{n-1}^{(p, q, s)} + r H_{n-2}^{(p, q, s)}, & n \equiv 2 \pmod{3}, \end{cases} \quad n \geq 2. \quad (9)$$

In particular, the third term is obtained from the recurrence corresponding to the case $n \equiv 2 \pmod{3}$, namely,

$$H_2^{(p, q, s)} = s H_1^{(p, q, s)} + r H_0^{(p, q, s)} = sb + ra.$$

In order to express (9) in a unified form, we introduce the modular Kronecker delta functions

$$\xi_m(n) = \begin{cases} 1, & n \equiv m \pmod{3}, \\ 0, & \text{otherwise,} \end{cases} \quad m = 0, 1, 2.$$

Then, for $n \geq 2$, the recurrence relation (9) can be written compactly as

$$H_n^{(p, q, s)} = p^{\xi_0(n)} q^{\xi_1(n)} s^{\xi_2(n)} H_{n-1}^{(p, q, s)} + r H_{n-2}^{(p, q, s)}.$$

For later use, we set

$$X := pqs + pr + qr + sr, \quad Y := r^3,$$

and denote the corresponding discriminant by

$$\mathcal{D} := X^2 + 4Y.$$

These quantities will appear naturally in the induced three-step recurrence and in the monodromy matrix formulation.

The first few terms of the tri-periodic Horadam sequence are given by

$$H_0^{(p, q, s)} = a, \quad H_1^{(p, q, s)} = b, \quad H_2^{(p, q, s)} = sb + ra,$$

$$H_3^{(p, q, s)} = p H_2^{(p, q, s)} + r H_1^{(p, q, s)} = psb + pra + rb,$$

$$H_4^{(p, q, s)} = q H_3^{(p, q, s)} + r H_2^{(p, q, s)} = pqsb + pqra + qrb + rsb + r^2a,$$

and

$$\begin{aligned} H_5^{(p, q, s)} &= s H_4^{(p, q, s)} + r H_3^{(p, q, s)} \\ &= pqs^2b + pqrs a + qrs b + rs^2b + r^2s a + prs b + pr^2a + r^2b. \end{aligned}$$

Theorem 1. The tri-periodic Horadam sequence $\{H_n^{(p,q,s)}\}_{n \geq 0}$ satisfies the following identities for all admissible indices $n \geq 2$:

- (a) $H_{n+2}^{(p,q,s)} = (pqs + pr + sr)H_{n-1}^{(p,q,s)} + (qrs + r^2)H_{n-2}^{(p,q,s)}, \quad n \equiv 0 \pmod{3},$
- (b) $H_{n+2}^{(p,q,s)} = (pqs + pr + qr)H_{n-1}^{(p,q,s)} + (prs + r^2)H_{n-2}^{(p,q,s)}, \quad n \equiv 1 \pmod{3},$
- (c) $H_{n+2}^{(p,q,s)} = (pqs + qr + sr)H_{n-1}^{(p,q,s)} + (pqr + r^2)H_{n-2}^{(p,q,s)}, \quad n \equiv 2 \pmod{3}.$

Proof. First, let $n \equiv 0 \pmod{3}$. By the recurrence relation (9), we have

$$H_n^{(p,q,s)} = pH_{n-1}^{(p,q,s)} + rH_{n-2}^{(p,q,s)},$$

$$H_{n+1}^{(p,q,s)} = qH_n^{(p,q,s)} + rH_{n-1}^{(p,q,s)},$$

and

$$H_{n+2}^{(p,q,s)} = sH_{n+1}^{(p,q,s)} + rH_n^{(p,q,s)}.$$

Therefore,

$$\begin{aligned} H_{n+2}^{(p,q,s)} &= s(qH_n^{(p,q,s)} + rH_{n-1}^{(p,q,s)}) + rH_n^{(p,q,s)} \\ &= (qs + r)H_n^{(p,q,s)} + rsH_{n-1}^{(p,q,s)} \\ &= (qs + r)(pH_{n-1}^{(p,q,s)} + rH_{n-2}^{(p,q,s)}) + rsH_{n-1}^{(p,q,s)} \\ &= (pqs + pr + rs)H_{n-1}^{(p,q,s)} + (qrs + r^2)H_{n-2}^{(p,q,s)}. \end{aligned}$$

Next, let $n \equiv 1 \pmod{3}$. Then

$$H_n^{(p,q,s)} = qH_{n-1}^{(p,q,s)} + rH_{n-2}^{(p,q,s)},$$

$$H_{n+1}^{(p,q,s)} = sH_n^{(p,q,s)} + rH_{n-1}^{(p,q,s)},$$

and

$$H_{n+2}^{(p,q,s)} = pH_{n+1}^{(p,q,s)} + rH_n^{(p,q,s)}.$$

Thus,

$$\begin{aligned} H_{n+2}^{(p,q,s)} &= p(sH_n^{(p,q,s)} + rH_{n-1}^{(p,q,s)}) + rH_n^{(p,q,s)} \\ &= (ps + r)H_n^{(p,q,s)} + prH_{n-1}^{(p,q,s)} \\ &= (ps + r)(qH_{n-1}^{(p,q,s)} + rH_{n-2}^{(p,q,s)}) + prH_{n-1}^{(p,q,s)} \\ &= (pqs + pr + qr)H_{n-1}^{(p,q,s)} + (prs + r^2)H_{n-2}^{(p,q,s)}. \end{aligned}$$

Finally, let $n \equiv 2 \pmod{3}$. In this case,

$$H_n^{(p,q,s)} = sH_{n-1}^{(p,q,s)} + rH_{n-2}^{(p,q,s)},$$

$$H_{n+1}^{(p,q,s)} = pH_n^{(p,q,s)} + rH_{n-1}^{(p,q,s)},$$

and

$$H_{n+2}^{(p,q,s)} = qH_{n+1}^{(p,q,s)} + rH_n^{(p,q,s)}.$$

Hence,

$$\begin{aligned} H_{n+2}^{(p,q,s)} &= q(pH_n^{(p,q,s)} + rH_{n-1}^{(p,q,s)}) + rH_n^{(p,q,s)} \\ &= (pq + r)H_n^{(p,q,s)} + qrH_{n-1}^{(p,q,s)} \\ &= (pq + r)(sH_{n-1}^{(p,q,s)} + rH_{n-2}^{(p,q,s)}) + qrH_{n-1}^{(p,q,s)} \\ &= (pqs + qr + rs)H_{n-1}^{(p,q,s)} + (pqr + r^2)H_{n-2}^{(p,q,s)}. \end{aligned}$$

This completes the proof. $\square$

Theorem 2. Let

$$X := pqs + pr + qr + sr, \quad Y := r^3.$$

The tri-periodic Horadam sequence $\{H_n^{(p,q,s)}\}_{n \geq 0}$ satisfies the following identities for $n \geq 2$:

$$\begin{aligned} \text{(a)} \quad H_{3n}^{(p,q,s)} &= XH_{3n-3}^{(p,q,s)} + YH_{3n-6}^{(p,q,s)}, \\ \text{(b)} \quad H_{3n+1}^{(p,q,s)} &= XH_{3n-2}^{(p,q,s)} + YH_{3n-5}^{(p,q,s)}, \\ \text{(c)} \quad H_{3n+2}^{(p,q,s)} &= XH_{3n-1}^{(p,q,s)} + YH_{3n-4}^{(p,q,s)}. \end{aligned}$$

Proof. The result follows directly from the phase-dependent identities established in Theorem 1. Indeed, applying the recurrence successively over one complete period and collecting the coefficients of the same residue class modulo three yields a common second-order recurrence for each subsequence

$$\{H_{3n}^{(p,q,s)}\}_{n \geq 0}, \quad \{H_{3n+1}^{(p,q,s)}\}_{n \geq 0}, \quad \{H_{3n+2}^{(p,q,s)}\}_{n \geq 0}.$$

For instance, the subsequence $\{H_{3n}^{(p,q,s)}\}_{n \geq 0}$ satisfies

$$H_{3n}^{(p,q,s)} = (pqs + pr + qr + sr)H_{3n-3}^{(p,q,s)} + r^3H_{3n-6}^{(p,q,s)}.$$

The same computation for the residue classes $3n + 1$ and $3n + 2$ gives

$$\begin{aligned} H_{3n+1}^{(p,q,s)} &= XH_{3n-2}^{(p,q,s)} + YH_{3n-5}^{(p,q,s)}, \\ H_{3n+2}^{(p,q,s)} &= XH_{3n-1}^{(p,q,s)} + YH_{3n-4}^{(p,q,s)}. \end{aligned}$$

Since

$$X = pqs + pr + qr + sr, \quad Y = r^3,$$

the desired identities follow. $\square$

Corollary 1. Let

$$X := pqs + pr + qr + sr, \quad Y := r^3.$$

Then, for any $p, q, s, r \in \mathbb{R} \setminus \{0\}$ and $n \geq 6$, the tri-periodic Horadam sequence satisfies

$$H_n^{(p,q,s)} = XH_{n-3}^{(p,q,s)} + YH_{n-6}^{(p,q,s)}. \quad (10)$$

Remark 1. Theorem 2 and Corollary 1 are included only to record the induced common recurrence over one full period. The detailed phase-by-phase algebra has already been carried out in Theorem 1; therefore, the subsequent use of the recurrence will rely on the compact quantities $X = pqs + pr + qr + sr$ and $Y = r^3$ rather than repeating the same substitutions.

4. Generating Function and Binet-Type Representation

In this section, we derive the generating function and the Binet-type representation of the tri-periodic Horadam sequence. Since the recurrence coefficients vary according to the residue class of the index modulo three, the sequence is naturally decomposed into the three subsequences $\{H_{3n}^{(p,q,s)}\}$, $\{H_{3n+1}^{(p,q,s)}\}$, and $\{H_{3n+2}^{(p,q,s)}\}$. This decomposition allows the sixth-order recurrence obtained above to be converted into a rational generating function. The corresponding closed-form representation is then obtained from the roots of the characteristic equation $z^2 - Xz - Y = 0$, where $X = pqs + pr + qr + sr$ and $Y = r^3$.

Theorem 3. Suppose that

$$H(w) = \sum_{n=0}^{\infty} H_n^{(p,q,s)} w^n$$

is the generating function of the tri-periodic Horadam sequence. Let

$$X := pqs + pr + qr + sr, \quad Y := r^3.$$

Then $H(w)$ is given by

$$H(w) = \frac{H_0^{(p,q,s)} + H_1^{(p,q,s)}w + H_2^{(p,q,s)}w^2 + w^3(H_3^{(p,q,s)} - XH_0^{(p,q,s)})}{1 - Xw^3 - Yw^6} + \frac{w^4(H_4^{(p,q,s)} - XH_1^{(p,q,s)}) + w^5(H_5^{(p,q,s)} - XH_2^{(p,q,s)})}{1 - Xw^3 - Yw^6}.$$

Equivalently,

$$H(w) = \frac{H_0^{(p,q,s)} + H_1^{(p,q,s)}w + H_2^{(p,q,s)}w^2 + w^3(H_3^{(p,q,s)} - XH_0^{(p,q,s)}) + w^4(H_4^{(p,q,s)} - XH_1^{(p,q,s)}) + w^5(H_5^{(p,q,s)} - XH_2^{(p,q,s)})}{1 - Xw^3 - Yw^6}.$$

Proof. Consider the generating function

$$H(w) = \sum_{n=0}^{\infty} H_n^{(p,q,s)} w^n.$$

Let

$$X := pqs + pr + qr + sr, \quad Y := r^3.$$

We decompose $H(w)$ into three subsequences as follows:

$$H(w) = A(w) + B(w) + C(w), \quad (11)$$

where

$$A(w) = \sum_{n=0}^{\infty} H_{3n}^{(p,q,s)} w^{3n}, \quad B(w) = \sum_{n=0}^{\infty} H_{3n+1}^{(p,q,s)} w^{3n+1}, \quad C(w) = \sum_{n=0}^{\infty} H_{3n+2}^{(p,q,s)} w^{3n+2}.$$

First, multiplying $A(w)$ by $1 - Xw^3 - Yw^6$ and using part (a) of Theorem 2, we obtain

$$\begin{aligned}(1 - Xw^3 - Yw^6)A(w) &= \sum_{n=0}^{\infty} H_{3n}^{(p, q, s)} w^{3n} - X \sum_{n=0}^{\infty} H_{3n}^{(p, q, s)} w^{3n+3} \\ &\quad - Y \sum_{n=0}^{\infty} H_{3n}^{(p, q, s)} w^{3n+6} \\ &= \sum_{n=0}^{\infty} H_{3n}^{(p, q, s)} w^{3n} - X \sum_{n=1}^{\infty} H_{3(n-1)}^{(p, q, s)} w^{3n} \\ &\quad - Y \sum_{n=2}^{\infty} H_{3(n-2)}^{(p, q, s)} w^{3n}.\end{aligned}$$

Separating the initial terms yields

$$\begin{aligned}(1 - Xw^3 - Yw^6)A(w) &= H_0^{(p, q, s)} + w^3(H_3^{(p, q, s)} - XH_0^{(p, q, s)}) \\ &\quad + \sum_{n=2}^{\infty} (H_{3n}^{(p, q, s)} - XH_{3(n-1)}^{(p, q, s)} - YH_{3(n-2)}^{(p, q, s)}) w^{3n}.\end{aligned}$$

The infinite sum vanishes by Theorem 2. Hence,

$$A(w) = \frac{H_0^{(p, q, s)} + w^3(H_3^{(p, q, s)} - XH_0^{(p, q, s)})}{1 - Xw^3 - Yw^6}.$$

Next, multiplying $B(w)$ by $1 - Xw^3 - Yw^6$ and using part (b) of Theorem 2, we get

$$\begin{aligned}(1 - Xw^3 - Yw^6)B(w) &= \sum_{n=0}^{\infty} H_{3n+1}^{(p, q, s)} w^{3n+1} - X \sum_{n=1}^{\infty} H_{3n-2}^{(p, q, s)} w^{3n+1} \\ &\quad - Y \sum_{n=2}^{\infty} H_{3n-5}^{(p, q, s)} w^{3n+1}.\end{aligned}$$

Extracting the initial terms gives

$$(1 - Xw^3 - Yw^6)B(w) = H_1^{(p, q, s)} w + w^4(H_4^{(p, q, s)} - XH_1^{(p, q, s)}).$$

Thus,

$$B(w) = \frac{H_1^{(p, q, s)} w + w^4(H_4^{(p, q, s)} - XH_1^{(p, q, s)})}{1 - Xw^3 - Yw^6}.$$

Finally, multiplying $C(w)$ by $1 - Xw^3 - Yw^6$ and using part (c) of Theorem 2, we obtain

$$\begin{aligned}(1 - Xw^3 - Yw^6)C(w) &= \sum_{n=0}^{\infty} H_{3n+2}^{(p, q, s)} w^{3n+2} - X \sum_{n=1}^{\infty} H_{3n-1}^{(p, q, s)} w^{3n+2} \\ &\quad - Y \sum_{n=2}^{\infty} H_{3n-4}^{(p, q, s)} w^{3n+2}.\end{aligned}$$

Separating the initial terms yields

$$(1 - Xw^3 - Yw^6)C(w) = H_2^{(p, q, s)} w^2 + w^5(H_5^{(p, q, s)} - XH_2^{(p, q, s)}).$$

Therefore,

$$C(w) = \frac{H_2^{(p, q, s)} w^2 + w^5(H_5^{(p, q, s)} - XH_2^{(p, q, s)})}{1 - Xw^3 - Yw^6}.$$

Combining $A(w)$, $B(w)$, and $C(w)$, we obtain

$$H(w) = \frac{H_0^{(p,q,s)} + H_1^{(p,q,s)}w + H_2^{(p,q,s)}w^2 + w^3(H_3^{(p,q,s)} - XH_0^{(p,q,s)})}{1 - Xw^3 - Yw^6} + \frac{w^4(H_4^{(p,q,s)} - XH_1^{(p,q,s)}) + w^5(H_5^{(p,q,s)} - XH_2^{(p,q,s)})}{1 - Xw^3 - Yw^6}. \quad (12)$$

This completes the proof. $\square$

Theorem 4. Let

$$X := pqs + pr + qr + sr, \quad Y := r^3,$$

and let α and β be the distinct roots of

$$z^2 - Xz - Y = 0.$$

For $j = 0, 1, 2$, set

$$C_j := H_{j+3}^{(p,q,s)} - XH_j^{(p,q,s)}.$$

Then the Binet-type representation of the tri-periodic Horadam sequence is given by

$$H_{3k+j}^{(p,q,s)} = \frac{H_j^{(p,q,s)}(\alpha^{k+1} - \beta^{k+1}) + C_j(\alpha^k - \beta^k)}{\alpha - \beta}, \quad k \geq 0, \quad j = 0, 1, 2. \quad (13)$$

Equivalently, this gives

$$\begin{aligned} H_{3k}^{(p,q,s)} &= \frac{H_0^{(p,q,s)}(\alpha^{k+1} - \beta^{k+1}) + (H_3^{(p,q,s)} - XH_0^{(p,q,s)})(\alpha^k - \beta^k)}{\alpha - \beta}, \\ H_{3k+1}^{(p,q,s)} &= \frac{H_1^{(p,q,s)}(\alpha^{k+1} - \beta^{k+1}) + (H_4^{(p,q,s)} - XH_1^{(p,q,s)})(\alpha^k - \beta^k)}{\alpha - \beta}, \\ H_{3k+2}^{(p,q,s)} &= \frac{H_2^{(p,q,s)}(\alpha^{k+1} - \beta^{k+1}) + (H_5^{(p,q,s)} - XH_2^{(p,q,s)})(\alpha^k - \beta^k)}{\alpha - \beta}. \end{aligned}$$

Proof. Let

$$X := pqs + pr + qr + sr, \quad Y := r^3,$$

and let α and β be the distinct roots of

$$z^2 - Xz - Y = 0.$$

Then

$$\alpha + \beta = X, \quad \alpha\beta = -Y,$$

and therefore

$$1 - Xw^3 - Yw^6 = (1 - \alpha w^3)(1 - \beta w^3).$$

From the generating function obtained in Theorem 3, we have

$$\begin{aligned} H(w) &= \frac{H_0^{(p,q,s)} + H_1^{(p,q,s)}w + H_2^{(p,q,s)}w^2 + w^3(H_3^{(p,q,s)} - XH_0^{(p,q,s)}) + w^4(H_4^{(p,q,s)} - XH_1^{(p,q,s)}) + w^5(H_5^{(p,q,s)} - XH_2^{(p,q,s)})}{(1 - \alpha w^3)(1 - \beta w^3)}. \end{aligned}$$

For $j = 0, 1, 2$, set

$$C_j := H_{j+3}^{(p, q, s)} - XH_j^{(p, q, s)}.$$

Thus, the numerator can be written as

$$H_0^{(p, q, s)} + H_1^{(p, q, s)}w + H_2^{(p, q, s)}w^2 + C_0w^3 + C_1w^4 + C_2w^5.$$

Since $\alpha \neq \beta$, we decompose $H(w)$ as

$$H(w) = \frac{s_1w^2 + s_2w + s_3}{1 - \alpha w^3} + \frac{s_4w^2 + s_5w + s_6}{1 - \beta w^3},$$

where $s_1, \dots, s_6$ are constants to be determined.

Multiplying both sides by

$$(1 - \alpha w^3)(1 - \beta w^3),$$

we obtain

$$\begin{aligned} & H_0^{(p, q, s)} + H_1^{(p, q, s)}w + H_2^{(p, q, s)}w^2 + C_0w^3 + C_1w^4 + C_2w^5 \\ &= (s_1w^2 + s_2w + s_3)(1 - \beta w^3) + (s_4w^2 + s_5w + s_6)(1 - \alpha w^3). \end{aligned}$$

Expanding the right-hand side and comparing the coefficients of $1, w, w^2, w^3, w^4, w^5$, we get

$$\begin{aligned} s_3 + s_6 &= H_0^{(p, q, s)}, \\ s_2 + s_5 &= H_1^{(p, q, s)}, \\ s_1 + s_4 &= H_2^{(p, q, s)}, \\ -(\beta s_3 + \alpha s_6) &= C_0, \\ -(\beta s_2 + \alpha s_5) &= C_1, \\ -(\beta s_1 + \alpha s_4) &= C_2. \end{aligned}$$

Equivalently, this linear system can be written in matrix form as

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\beta & 0 & 0 & -\alpha \\ 0 & -\beta & 0 & 0 & -\alpha & 0 \\ -\beta & 0 & 0 & -\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{pmatrix} = \begin{pmatrix} H_0^{(p, q, s)} \\ H_1^{(p, q, s)} \\ H_2^{(p, q, s)} \\ C_0 \\ C_1 \\ C_2 \end{pmatrix}.$$

Solving this system gives

$$\begin{aligned} s_1 &= \frac{\alpha H_2^{(p, q, s)} + C_2}{\alpha - \beta}, & s_4 &= -\frac{\beta H_2^{(p, q, s)} + C_2}{\alpha - \beta}, \\ s_2 &= \frac{\alpha H_1^{(p, q, s)} + C_1}{\alpha - \beta}, & s_5 &= -\frac{\beta H_1^{(p, q, s)} + C_1}{\alpha - \beta}, \\ s_3 &= \frac{\alpha H_0^{(p, q, s)} + C_0}{\alpha - \beta}, & s_6 &= -\frac{\beta H_0^{(p, q, s)} + C_0}{\alpha - \beta}. \end{aligned}$$

Using the expansions

$$\frac{1}{1 - \alpha w^3} = \sum_{k=0}^{\infty} \alpha^k w^{3k}, \quad \frac{1}{1 - \beta w^3} = \sum_{k=0}^{\infty} \beta^k w^{3k},$$

we obtain

$$H(w) = \sum_{k=0}^{\infty} (s_3 \alpha^k + s_6 \beta^k) w^{3k} + \sum_{k=0}^{\infty} (s_2 \alpha^k + s_5 \beta^k) w^{3k+1} + \sum_{k=0}^{\infty} (s_1 \alpha^k + s_4 \beta^k) w^{3k+2}.$$

Now substituting the values of $s_1, \dots, s_6$, the coefficient of w^{3k} becomes

$$s_3 \alpha^k + s_6 \beta^k = \frac{H_0^{(p, q, s)} (\alpha^{k+1} - \beta^{k+1}) + C_0 (\alpha^k - \beta^k)}{\alpha - \beta}.$$

Similarly, the coefficient of w^{3k+1} is

$$s_2 \alpha^k + s_5 \beta^k = \frac{H_1^{(p, q, s)} (\alpha^{k+1} - \beta^{k+1}) + C_1 (\alpha^k - \beta^k)}{\alpha - \beta},$$

and the coefficient of w^{3k+2} is

$$s_1 \alpha^k + s_4 \beta^k = \frac{H_2^{(p, q, s)} (\alpha^{k+1} - \beta^{k+1}) + C_2 (\alpha^k - \beta^k)}{\alpha - \beta}.$$

Therefore, for $k \geq 0$ and $j = 0, 1, 2$, we obtain

$$H_{3k+j}^{(p, q, s)} = \frac{H_j^{(p, q, s)} (\alpha^{k+1} - \beta^{k+1}) + C_j (\alpha^k - \beta^k)}{\alpha - \beta}.$$

This is precisely the desired Binet-type representation. The repeated-root case $\alpha = \beta$, if it occurs, can be obtained by taking the limit as $\beta \rightarrow \alpha$. This completes the proof. $\square$

Corollary 2 (Repeated-root case). *Suppose that the characteristic polynomial*

$$z^2 - Xz - Y = 0$$

associated with the monodromy matrix possesses a repeated root, that is,

$$\alpha = \beta.$$

Then, for each residue class $j = 0, 1, 2$, the Binet-type representation of the tri-periodic Horadam sequence reduces to

$$H_{3k+j}^{(p, q, s)} = (A_j + B_j k) \alpha^k, \quad k \geq 0,$$

where the constants A_j and B_j are uniquely determined by the prescribed initial values of the corresponding subsequence $\{H_{3k+j}^{(p, q, s)}\}_{k \geq 0}$.

Proof. The repeated-root case is obtained as the limiting form of the Binet representation established in Theorem 4.

Indeed, as $\beta \rightarrow \alpha$, the quotients

$$\frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} \quad \text{and} \quad \frac{\alpha^k - \beta^k}{\alpha - \beta}$$

are indeterminate of type 0/0. Applying L'Hospital's rule yields

$$\lim_{\beta \rightarrow \alpha} \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} = (k+1)\alpha^k,$$

and

$$\lim_{\beta \rightarrow \alpha} \frac{\alpha^k - \beta^k}{\alpha - \beta} = k\alpha^{k-1}.$$

Substituting these limits into the Binet formula of Theorem 4 gives

$$H_{3k+j}^{(p,q,s)} = (A_j + B_j k)\alpha^k, \quad j = 0, 1, 2,$$

where the constants A_j and B_j depend only on the prescribed initial values of the corresponding subsequence. More explicitly, since $Y = r^3 \neq 0$, the repeated root α is nonzero. Hence, from

$$H_j^{(p,q,s)} = A_j$$

and

$$H_{j+3}^{(p,q,s)} = (A_j + B_j)\alpha,$$

we obtain

$$A_j = H_j^{(p,q,s)}, \quad B_j = \frac{H_{j+3}^{(p,q,s)}}{\alpha} - H_j^{(p,q,s)}, \quad j = 0, 1, 2.$$

This completes the proof. $\square$

Remark 2. The repeated-root configuration represents a degenerate spectral case in which the monodromy matrix admits a double eigenvalue. Although this situation occurs only for a particular choice of the recurrence parameters, the above result completes the theoretical characterization of the Binet-type representation. Throughout the numerical experiments presented in this paper, the discriminant remains strictly positive; consequently, only the distinct-eigenvalue case is encountered in practice.

5. Spectral Properties and Asymptotic Behavior

In this section, we investigate the spectral structure and asymptotic behavior of the tri-periodic Horadam sequence by means of monodromy matrices and characteristic dynamics. Unlike classical constant-coefficient recurrences, the present system possesses periodically varying transition operators, which generate non-autonomous spectral dynamics and phase-dependent asymptotic amplification mechanisms.

5.1. Monodromy Matrix Formulation

Consider the tri-periodic Horadam recurrence

$$H_n = \begin{cases} pH_{n-1} + rH_{n-2}, & n \equiv 0 \pmod{3}, \\ qH_{n-1} + rH_{n-2}, & n \equiv 1 \pmod{3}, \\ sH_{n-1} + rH_{n-2}, & n \equiv 2 \pmod{3}, \end{cases} \quad (14)$$

where $p, q, s, r \in \mathbb{R} \setminus \{0\}$.

Define the state vector

$$V_n = \begin{pmatrix} H_n \\ H_{n-1} \end{pmatrix}.$$

Then the transition from V_{n-1} to V_n is governed by the coefficient associated with the index n . Accordingly, the phase-dependent transition matrices are given by

$$A_p = \begin{pmatrix} p & r \\ 1 & 0 \end{pmatrix}, \quad A_q = \begin{pmatrix} q & r \\ 1 & 0 \end{pmatrix}, \quad A_s = \begin{pmatrix} s & r \\ 1 & 0 \end{pmatrix}.$$

Since the transition from V_{3k-3} to V_{3k} passes successively through the phases

$$3k-2 \equiv 1, \quad 3k-1 \equiv 2, \quad 3k \equiv 0 \pmod{3},$$

the evolution over one complete period is governed by the monodromy matrix

$$M = A_p A_s A_q. \quad (15)$$

Equivalently,

$$V_{3k} = M V_{3k-3}.$$

A direct computation yields

$$M = \begin{pmatrix} pqs + pr + qr & r(ps + r) \\ qs + r & sr \end{pmatrix}.$$

Consequently,

$$\text{tr}(M) = pqs + pr + qr + sr, \quad (16)$$

and

$$\det(M) = -r^3. \quad (17)$$

For simplicity, define

$$X := pqs + pr + qr + sr, \quad Y := r^3. \quad (18)$$

Therefore,

$$\text{tr}(M) = X, \quad \det(M) = -Y.$$

5.2. Characteristic Equation of the Monodromy Matrix

The characteristic polynomial of the monodromy matrix M is given by

$$\chi_M(z) = z^2 - Xz - Y. \quad (19)$$

Let α and β denote the roots of (19). Then

$$\alpha = \frac{X + \sqrt{D}}{2}, \quad \beta = \frac{X - \sqrt{D}}{2}. \quad (20)$$

Hence,

$$\alpha + \beta = X, \quad \alpha\beta = -Y = -r^3. \quad (21)$$

The Cayley–Hamilton theorem applied to the monodromy matrix implies

$$M^2 - XM - YI = 0.$$

Using the relation

$$\begin{pmatrix} H_{3k} \\ H_{3k-1} \end{pmatrix} = M \begin{pmatrix} H_{3k-3} \\ H_{3k-4} \end{pmatrix},$$

we immediately obtain the following result.

Theorem 5 (Monodromy Interpretation of the Induced Recurrence). *Let*

$$M = A_p A_s A_q$$

be the monodromy matrix associated with one complete period of the tri-periodic Horadam recurrence. Then the monodromy matrix satisfies the characteristic equation

$$M^2 - XM - YI = 0,$$

where

$$X = pqs + pr + qr + sr, \quad Y = r^3.$$

Consequently, the induced sixth-order recurrence

$$H_n = XH_{n-3} + YH_{n-6}, \quad n \geq 6,$$

is an immediate consequence of the Cayley–Hamilton theorem.

Proof. By the monodromy formulation, the evolution over one complete period from V_{3k-3} to V_{3k} is given by

$$V_{3k} = MV_{3k-3}, \quad M = A_p A_s A_q.$$

A direct computation gives

$$\text{tr}(M) = X, \quad \det(M) = -Y.$$

Therefore, the characteristic polynomial of M is

$$\chi_M(z) = z^2 - Xz - Y.$$

By the Cayley–Hamilton theorem,

$$M^2 - XM - YI = 0.$$

Applying this identity to V_{3k-6} yields

$$V_{3k} = XV_{3k-3} + YV_{3k-6}.$$

Taking the first component gives

$$H_{3k} = XH_{3k-3} + YH_{3k-6}.$$

For the shifted subsequences, the corresponding monodromy matrices are the cyclic products

$$M_1 = A_q A_p A_s, \quad M_2 = A_s A_q A_p.$$

Since cyclic products have the same trace and determinant in this setting, we have

$$\text{tr}(M_1) = \text{tr}(M_2) = X, \quad \det(M_1) = \det(M_2) = -Y.$$

Hence the same Cayley–Hamilton argument gives

$$H_{3k+1} = XH_{3k-2} + YH_{3k-5},$$

$$H_{3k+2} = XH_{3k-1} + YH_{3k-4}.$$

Combining the three residue classes modulo three, we obtain

$$H_n = XH_{n-3} + YH_{n-6}, \quad n \geq 6.$$

This proves the monodromy-based derivation of the induced recurrence without repeating the direct substitution argument of Theorem 2. $\square$

Proposition 1. *Let $M = A_p A_s A_q$ denote the monodromy matrix associated with one complete period of the tri-periodic Horadam recurrence. Then the state vectors satisfy*

$$V_{3k} = M^k V_0, \quad k \geq 0.$$

Proof. The result follows immediately by induction. The case $k = 0$ is trivial. Assuming $V_{3k} = M^k V_0$, we obtain

$$V_{3(k+1)} = M V_{3k} = M M^k V_0 = M^{k+1} V_0,$$

which completes the proof. $\square$

Remark 3. *Theorem 5 is not intended as an independent repetition of the direct algebraic derivation in Theorem 2. Rather, it shows that the same induced recurrence arises naturally from the spectral structure of the monodromy matrix. Thus, Theorem 2 provides the direct recurrence-theoretic derivation, whereas Theorem 5 provides the matrix-theoretic and spectral interpretation needed for the subsequent asymptotic analysis.*

5.3. Spectral Structure

The induced period-level recurrence relation obtained in Corollary 1 leads to the characteristic equation

$$\lambda^6 - X\lambda^3 - Y = 0. \quad (22)$$

Since α and β are the roots of

$$z^2 - Xz - Y = 0,$$

the eigenvalues of the induced sixth-order scalar dynamics satisfy

$$\lambda^3 = \alpha, \quad \lambda^3 = \beta.$$

Let

$$\omega = e^{2\pi i/3}$$

denote the primitive cube root of unity. Then the six eigenvalues associated with the sixth-order characteristic equation are

$$\lambda_j = \alpha^{1/3} \omega^j, \quad \lambda_{j+3} = \beta^{1/3} \omega^j, \quad j = 0, 1, 2. \quad (23)$$

This decomposition separates the dominant spectral roots α and β , which are determined by the monodromy matrix, from the rotational periodic structure generated by

the cube roots of unity. Hence, the quantities α and β control the period-level amplification, while the factors $1, \omega, \omega^2$ encode the intrinsic three-phase oscillatory structure of the recurrence.

Remark 4. When the eigenvalues α and β are complex, the notation

$$\alpha^{1/3} \quad \text{and} \quad \beta^{1/3}$$

denotes a fixed choice of the principal cubic root throughout the paper. The remaining cubic roots are obtained by multiplication with the cubic roots of unity

$$1, \quad \omega, \quad \omega^2,$$

where

$$\omega = e^{2\pi i/3}.$$

Consequently, the six roots of the characteristic equation

$$\lambda^6 - X\lambda^3 - Y = 0$$

consist of the three branches associated with α together with the three branches associated with β . The subsequent analysis depends only on the complete set of roots and is therefore independent of the particular branch chosen.

5.4. Effective Spectral Radius and Asymptotic Growth

The long-term behavior of the tri-periodic Horadam sequence is governed by the dominant eigenvalue magnitude of the monodromy matrix.

Definition 2. The effective spectral radius of the tri-periodic system is defined by

$$\rho_{\text{eff}} = \max\left\{|\alpha|^{1/3}, |\beta|^{1/3}\right\}. \quad (24)$$

Equivalently,

$$\rho_{\text{eff}} = \rho(M)^{1/3},$$

where $\rho(M)$ denotes the spectral radius of the monodromy matrix.

The effective spectral radius determines the asymptotic amplification regime of the recurrence.

Theorem 6. Assume that the monodromy matrix M possesses two distinct eigenvalues and that the initial state has a nonzero projection onto the eigenspace corresponding to the dominant eigenvalue. Let

$$\rho_{\text{eff}} = \rho(M)^{1/3},$$

where $\rho(M)$ denotes the spectral radius of the monodromy matrix. Then the following statements hold.

(i) If

$$\rho_{\text{eff}} > 1,$$

then, for generic initial data, the sequence exhibits asymptotic exponential growth with growth rate determined by ρ_{eff} .

(ii) If

$$\rho_{\text{eff}} < 1,$$

then every solution converges exponentially to zero.

(iii) If

$$\rho_{\text{eff}} = 1,$$

then the asymptotic behavior is marginal and depends on the Jordan structure of the monodromy matrix. In particular, bounded oscillations occur in the diagonalizable case, whereas polynomially modulated growth may arise in the presence of repeated eigenvalues.

The repeated-root case is treated separately in Corollary 2.

Proof. Let M denote the monodromy matrix associated with one complete period of the tri-periodic Horadam recurrence. Since M is assumed to possess two distinct eigenvalues, it is diagonalizable. Therefore, there exists an invertible matrix P such that

$$M = PDP^{-1}, \quad D = \text{diag}(\lambda_1, \lambda_2),$$

where λ_1 and λ_2 are the eigenvalues of M . Moreover, the evolution of the state vector over one complete period satisfies

$$V_{3(k+1)} = MV_{3k},$$

and repeated application of this relation yields

$$V_{3k} = M^k V_0 = PD^k P^{-1} V_0.$$

Assume that the initial state V_0 has a nonzero projection onto the eigenspace corresponding to the dominant eigenvalue λ_d , where

$$|\lambda_d| = \rho(M).$$

Then the dominant spectral component cannot vanish, and the spectral decomposition of the solution takes the asymptotic form

$$V_{3k} = c_d \lambda_d^k v_d + o(\lambda_d^k), \quad c_d \neq 0,$$

where v_d denotes an eigenvector associated with λ_d . Consequently, the first component of V_{3k} , namely the sequence $\{H_n\}$, satisfies

$$|H_n| = \mathcal{O}(\rho_{\text{eff}}^n), \quad \rho_{\text{eff}} = \rho(M)^{1/3}.$$

Hence, if

$$\rho_{\text{eff}} > 1,$$

the dominant exponential term governs the asymptotic behaviour, and the sequence exhibits exponential growth for generic initial data. If

$$\rho_{\text{eff}} < 1,$$

then every spectral mode decays exponentially, implying that the sequence converges exponentially to zero.

Finally, when $\rho_{\text{eff}} = 1$, the asymptotic behaviour depends on the Jordan canonical form of the monodromy matrix. In the diagonalizable case the sequence remains bounded up to oscillatory factors, whereas repeated eigenvalues may produce polynomially modulated growth. This completes the proof. $\square$

Remark 5. The assumption that the initial state has a nonzero projection onto the dominant eigenspace is standard in the asymptotic theory of linear recurrence relations. If this projection vanishes, the dominant exponential mode is suppressed and the long-term behavior is determined by the next largest eigenvalue. Consequently, the effective growth rate may be strictly smaller than ρ_{eff} .

Remark 6. Theorem 6 assumes that the monodromy matrix possesses distinct eigenvalues. The repeated-root configuration corresponds to a degenerate spectral case and is treated separately in Corollary 2. Throughout the numerical experiments reported in this paper, the monodromy matrix remains diagonalizable, so that the generic spectral behavior described above is observed.

5.5. Asymmetry and Spectral Amplification

The asymmetry of the periodic coefficients provides useful diagnostic information about the imbalance among the modular phases of the recurrence. In this work, we define the asymmetry index by

$$A := \max\{|p - q|, |q - s|, |s - p|\}.$$

It is important to emphasize that A is not, by itself, a spectral stability parameter. The effective spectral radius of the tri-periodic system is determined by the spectral properties of the monodromy matrix and therefore depends on the combined quantities

$$X = pq + pr + qr + sr, \quad Y = r^3,$$

rather than only on the maximum pairwise difference among p, q , and s . Hence, no universal monotone relation between A and ρ_{eff} is claimed in this paper.

Nevertheless, in the numerical parameter regimes explored here, larger coefficient disparities are often associated with stronger spectral amplification and poorer conditioning of the inverse parameter estimation problem. Therefore, the asymmetry index is used as an empirical diagnostic indicator for organizing and interpreting the robustness experiments, rather than as a standalone theoretical stability criterion. Consequently, large-asymmetry regimes generate highly stiff recursive trajectories and severe numerical instability in inverse parameter estimation problems.

This spectral amplification mechanism explains the optimization difficulties observed in noisy tri-periodic Horadam systems, including gradient explosion, recursive error propagation, and floating-point instability during recurrence-regularized neural training.

Figure 1 illustrates the growth behavior of the tri-periodic Horadam sequence for several natural-number parameter regimes. Even for moderate integer-valued coefficients, the sequence may exhibit rapid amplification, which is consistent with the monodromy-based spectral analysis.

To complement the trajectory-level visualization, Figure 2 presents the effective spectral radius over a two-dimensional parameter grid. For each pair (p, q) , with $s = 1$ and $r = 1$ fixed, the corresponding monodromy matrix $M = A_p A_s A_q$ is formed and the quantity

$$\rho_{\text{eff}} = \rho(M)^{1/3}$$

is computed. Since the monodromy matrix describes the evolution over one full three-step period, taking the cubic root converts the period-level spectral radius into an effective one-step growth factor.

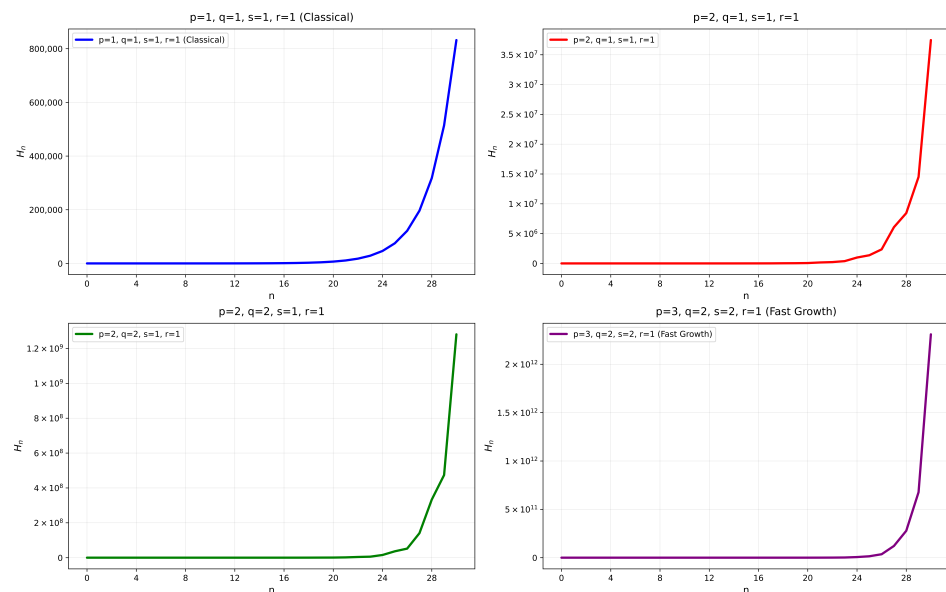


Figure 1. Asymptotic growth behavior of the tri-periodic Horadam sequence under representative natural-number parameter regimes. The classical case $p = q = s = r = 1$ is compared with increasingly amplified configurations. The panels show that changes in the modular coefficients p , q , and s can substantially increase the magnitude of H_n , confirming the sensitivity of the recurrence to the phase-dependent coefficient structure.

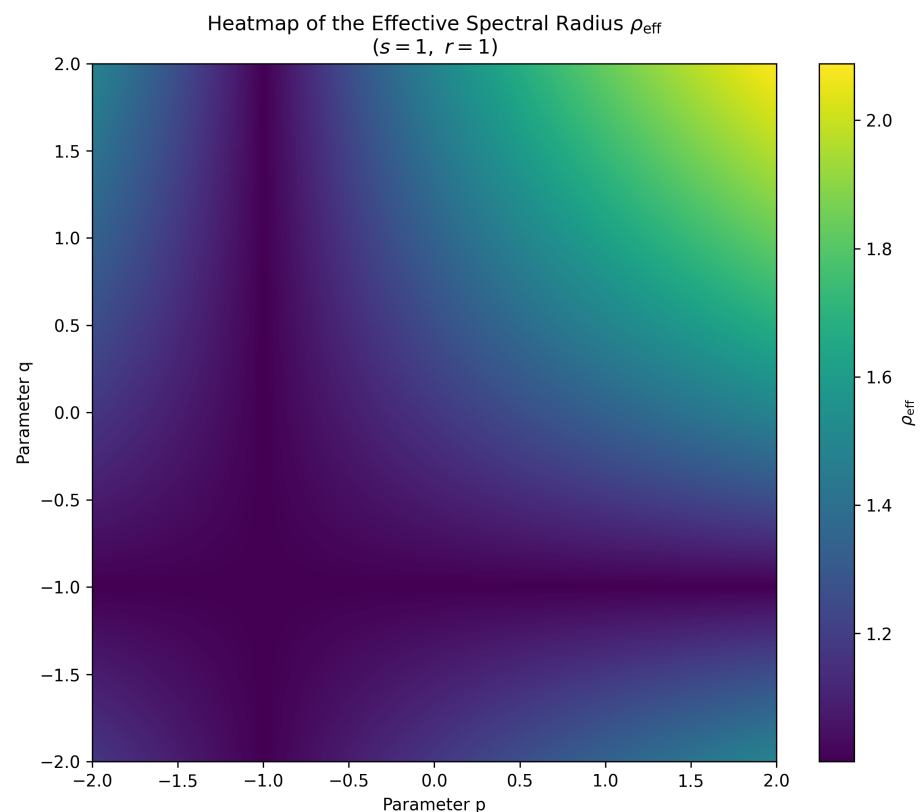


Figure 2. Heatmap of the effective spectral radius $\rho_{\text{eff}} = \rho(M)^{1/3}$ for the tri-periodic Horadam system with fixed parameters $s = 1$ and $r = 1$. Each point (p, q) represents the effective one-step amplification rate induced by the corresponding monodromy matrix $M = A_p A_s A_q$. Darker colors indicate smaller values of ρ_{eff} , whereas brighter colors indicate stronger spectral amplification.

Figure 2 provides a numerical spectral map of the parameter space. Values of $\rho_{\text{eff}} > 1$ correspond to exponentially amplifying regimes, while values close to 1 indicate marginal or weakly amplifying behavior. The heatmap shows that the spectral growth rate is sensitive to the interaction between the phase-dependent coefficients p and q . This explains why certain coefficient choices generate stiff trajectories and make inverse parameter estimation more sensitive to small coefficient errors or observation noise. It also supports the later numerical experiments, where high-asymmetry regimes lead to stronger extrapolation instability. This interpretation is empirical for the sampled parameter grid and should not be understood as a general monotonic dependence of ρ_{eff} on the asymmetry index A .

6. Fourier-Embedded Recurrence-Regularized Neural Estimator (RRNE) Framework

In this study, the recovery of the coefficients (p, q, s) governing tri-periodic Horadam sequences from highly noisy observation data is investigated. To solve this inverse problem, an RRNE framework incorporating both mathematical constraints and frequency-domain properties has been developed.

6.1. Dataset Generation and Dynamical-System Modeling

The synthetic dataset used in the numerical experiments was generated directly from the tri-periodic Horadam recurrence relation

$$H_n = \begin{cases} pH_{n-1} + rH_{n-2}, & n \equiv 0 \pmod{3}, \\ qH_{n-1} + rH_{n-2}, & n \equiv 1 \pmod{3}, \\ sH_{n-1} + rH_{n-2}, & n \equiv 2 \pmod{3}, \end{cases}$$

with prescribed initial values $H_0 = a$ and $H_1 = b$. In this way, the clean sequence was obtained recursively rather than from the closed-form Binet-type representation. The Binet-type formula derived in the previous section was used only as an analytical consistency check for the generated sequence.

Throughout the inverse-estimation experiments, the coefficient r is assumed to be known and fixed, while the periodic coefficients p, q , and s are treated as unknown trainable parameters. In the numerical implementation, we restrict attention to the positive coefficient regime

$$p, q, s > 0,$$

which is enforced through exponential parametrization of the trainable variables.

To simulate application-motivated measurement uncertainty, Gaussian noise proportional to the instantaneous magnitude of the clean sequence was added as

$$H_n^{\text{noisy}} = H_n^{\text{clean}} + \varepsilon_n, \quad \varepsilon_n \sim \mathcal{N}\left(0, \sigma^2 |H_n^{\text{clean}}|^2\right).$$

In the numerical experiments, Gaussian observation noise levels between 3% and 10% were considered. More precisely, the noise-sensitivity analysis was carried out under 3%, 5%, 7%, and 10% perturbation levels, while the asymmetry-regime experiments were evaluated under a fixed 7% measurement-noise setting.

6.2. Fourier Feature Embedding and RRNE Architecture

Conventional neural networks experience difficulty in capturing periodic and high-frequency signals embedded within input data due to spectral bias phenomena. Tri-periodic Horadam sequences possess a modular phase structure repeated every three steps, namely

$$n \pmod{3}.$$

In order to enable the neural architecture to internalize these successive modular transitions, the input space was expanded using Fourier feature embeddings.

The network feature vector is denoted by $\Phi(n)$. It consists of the normalized step index

$$n_{\text{norm}} = \frac{n}{n_{\text{max}}},$$

together with trigonometric embeddings representing the intrinsic period-three structure:

$$\Phi(n) = \left[n_{\text{norm}}, \sin\left(\frac{2\pi n}{3}\right), \cos\left(\frac{2\pi n}{3}\right) \right].$$

The feature vector $\Phi(n)$ is then propagated through a feed-forward neural network consisting of two hidden layers, each containing 64 neurons with nonlinear Tanh activation functions. The periodic coefficients p , q , and s are treated as trainable parameters during optimization. In order to remain consistent with the positive coefficient regime assumed in the numerical experiments, these parameters are represented through exponential parametrization,

$$p = \exp(\theta_p), \quad q = \exp(\theta_q), \quad s = \exp(\theta_s),$$

where θ_p , θ_q , and θ_s are unconstrained trainable variables.

6.3. Log-Cosh Loss Function and Teacher-Forced Recurrence Constraint

Since tri-periodic Horadam sequences may exhibit strongly amplified growth as the iteration index increases, the use of the standard Mean Squared Error (MSE) loss may become highly sensitive to large residuals. To reduce the influence of amplified residuals, we employ the Log-Cosh loss function

$$\mathcal{L}_{\log \cosh}(y, \hat{y}) = \frac{1}{N} \sum_{i=1}^N \log(\cosh(\hat{y}_i - y_i)). \quad (25)$$

The Log-Cosh loss behaves quadratically for small residuals and approximately linearly for large residuals. Therefore, it provides a practical robust alternative to the MSE loss when the recursive trajectory contains amplified deviations. In the numerical implementation, a stable equivalent form of the Log-Cosh function may be used to avoid overflow for large residuals; this stabilization is an implementation detail and does not alter the mathematical definition of the loss.

The recurrence-informed constraint is used to incorporate the tri-periodic structure of the sequence into the training objective. A fully recursive implementation, in which $\hat{H}_{n-1}$ and $\hat{H}_{n-2}$ are repeatedly propagated inside the recurrence constraint, may accumulate prediction errors along long trajectories. To reduce this accumulation effect, we use a teacher-forced version of the recurrence constraint. In this formulation, the network output $\hat{H}_n$ is compared with the recurrence value computed from the observed previous states.

For the three residue classes modulo three, we define

$$\begin{aligned}\mathcal{L}_0 &= \sum_{n \equiv 0 \pmod{3}} \mathcal{L}_{\log \cosh} \left(\hat{H}_n, p H_{n-1}^{\text{noisy}} + r H_{n-2}^{\text{noisy}} \right), \\ \mathcal{L}_1 &= \sum_{n \equiv 1 \pmod{3}} \mathcal{L}_{\log \cosh} \left(\hat{H}_n, q H_{n-1}^{\text{noisy}} + r H_{n-2}^{\text{noisy}} \right), \\ \mathcal{L}_2 &= \sum_{n \equiv 2 \pmod{3}} \mathcal{L}_{\log \cosh} \left(\hat{H}_n, s H_{n-1}^{\text{noisy}} + r H_{n-2}^{\text{noisy}} \right).\end{aligned}\quad (26)$$

Thus, the recurrence-informed loss is given by

$$\mathcal{L}_{\text{rec}} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2. \quad (27)$$

The total training loss is then defined as

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{data}} + \lambda \mathcal{L}_{\text{rec}}, \quad (28)$$

where $\lambda > 0$ controls the relative weight of the recurrence-informed constraint.

It should be emphasized that the teacher-forced constraint is not introduced as a new general training principle. Rather, it is used here as a practical stabilization device for the specific inverse-estimation problem considered in this paper. Since the constraint relies on noisy observations rather than latent clean states, it may improve numerical stability while also introducing bias into the recovered coefficients. Therefore, the proposed framework should be interpreted as a noise-regularized recurrence-informed inverse estimator rather than as an exact symbolic recovery mechanism.

6.4. Hybrid Optimization Algorithm

The optimization procedure is carried out using a two-stage strategy consisting of an Adam warm-up phase followed by L-BFGS fine-tuning. This strategy is not intended to introduce a new optimizer or to provide a global convergence guarantee. Rather, it is used as a practical numerical procedure for improving the stability of inverse parameter estimation in noisy tri-periodic recursive systems.

In the first stage, the Adam optimizer is used because of its adaptive learning-rate mechanism and its practical robustness during the early phase of training. This stage helps move the trainable network weights and the periodic coefficients p , q , and s toward a numerically stable region of the loss landscape before applying a second-order refinement step. The learning rates are selected as

$$10^{-3}$$

for the neural-network weights and

$$10^{-2}$$

for the periodic coefficients p , q , and s .

In the second stage, L-BFGS fine-tuning is applied starting from the Adam-initialized parameters. Since L-BFGS uses curvature information through a limited-memory approximation of the inverse Hessian, it can improve local refinement when the optimization trajectory has already entered a stable region. Strong Wolfe line-search conditions are imposed in order to reduce abrupt parameter updates that may occur under exponentially amplified recursive dynamics.

Overall, the hybrid Adam–L-BFGS strategy combines the numerical stability of adaptive first-order optimization with the local refinement capability of quasi-Newton optimiza-

tion. In the present inverse-estimation setting, this is particularly useful because noisy tri-periodic Horadam trajectories may generate large residuals, stiff loss surfaces, and unstable gradient behavior. The complete computational workflow, including Fourier feature construction, teacher-forced recurrence constraints, Log-Cosh loss evaluation, Adam warm-up, and L-BFGS fine-tuning, is summarized in Algorithm 1.

Algorithm 1 Fourier-Embedded Recurrence Regularized Neural Estimator with a Teacher-Forced Recurrence Constraint

Input: Noisy sequence data $\{H_n^{\text{noisy}}\}_{n=0}^N$, number of Adam epochs E , fixed coefficient r , recurrence-weight parameter λ .

Output: Estimated periodic coefficients $\hat{p}$, $\hat{q}$, and $\hat{s}$.

1. Initialize the trainable variables $p_{\log}$, $q_{\log}$, and $s_{\log}$.
2. Enforce positivity of the periodic coefficients by setting

$$p = \exp(p_{\log}), \quad q = \exp(q_{\log}), \quad s = \exp(s_{\log}).$$

3. Define the Fourier feature vector

$$\Phi(n) = \left[\frac{n}{N}, \sin\left(\frac{2\pi n}{3}\right), \cos\left(\frac{2\pi n}{3}\right) \right].$$

4. Define the Log-Cosh loss function by

$$L_{\log \cosh}(y, \hat{y}) = \frac{1}{N} \sum_{i=1}^N \log(\cosh(\hat{y}_i - y_i)).$$

5. **Phase 1: Adam warm-up.**

For each epoch $1, \dots, E$:

- (a) Compute the network predictions

$$\hat{H}_n = \mathcal{N}_{\theta}(\Phi(n)).$$

- (b) Compute the data-fitting loss

$$L_{\text{data}} = L_{\log \cosh}(H_n^{\text{noisy}}, \hat{H}_n).$$

- (c) Compute the teacher-forced recurrence losses

$$L_p = L_{\log \cosh}(\hat{H}_n, pH_{n-1}^{\text{noisy}} + rH_{n-2}^{\text{noisy}}), \quad n \equiv 0 \pmod{3},$$

$$L_q = L_{\log \cosh}(\hat{H}_n, qH_{n-1}^{\text{noisy}} + rH_{n-2}^{\text{noisy}}), \quad n \equiv 1 \pmod{3},$$

and

$$L_s = L_{\log \cosh}(\hat{H}_n, sH_{n-1}^{\text{noisy}} + rH_{n-2}^{\text{noisy}}), \quad n \equiv 2 \pmod{3}.$$

- (d) Compute the total loss

$$L_{\text{total}} = L_{\text{data}} + \lambda(L_p + L_q + L_s).$$

- (e) Update the trainable network weights and coefficient variables using Adam.

6. **Phase 2: L-BFGS fine-tuning.** Starting from the Adam-initialized parameters, apply L-BFGS with Strong Wolfe line-search conditions to further minimize L_{total} .

7. Return the estimated coefficients

$$\hat{p} = \exp(p_{\log}), \quad \hat{q} = \exp(q_{\log}), \quad \hat{s} = \exp(s_{\log}).$$

1. Adam optimization (warm-up phase): In the first stage, the Adam optimizer was employed because of its adaptive learning-rate mechanism and its practical robustness during the early phase of training. This stage helps move the trainable parameters toward a stable region of the loss landscape before applying a second-order optimizer. The learning rates were selected as

$$10^{-3}$$

for the neural-network weights and

$$10^{-2}$$

for the periodic coefficients p , q , and s .

2. L-BFGS optimization (fine-tuning phase): After the Adam warm-up phase, the L-BFGS optimizer was used to refine the parameter estimates. Since L-BFGS uses curvature information through a limited-memory approximation of the inverse Hessian, it can improve local convergence when the optimization trajectory has already entered a stable region. Strong Wolfe line-search conditions were imposed in order to reduce abrupt parameter updates that may occur under exponentially amplified recursive dynamics.

This hybrid strategy combines the numerical stability of adaptive first-order optimization with the local refinement capability of quasi-Newton optimization. In the present inverse-estimation setting, this is particularly useful because noisy tri-periodic Horadam trajectories may generate large residuals, stiff loss surfaces, and unstable gradient behavior.

7. Experimental Results and Discussion

In this section, we evaluate the numerical behavior of the proposed Fourier-Embedded Recurrence-Regularized Neural Estimator (RRNE) for inverse parameter recovery in noisy tri-periodic Horadam systems. The experiments are designed to assess the effect of coefficient asymmetry, observation noise, baseline-estimator performance, architectural components, and seed-dependent variability.

Throughout the experiments, the coefficient r is assumed to be known and fixed as

$$r = 0.5.$$

The initial conditions are fixed as

$$H_0 = 2.0, \quad H_1 = 1.0.$$

The training interval is taken as

$$2 \leq n \leq 35,$$

whereas extrapolation is evaluated on the unseen interval

$$36 \leq n \leq 60.$$

Thus, the reported Test Log-RMSE values measure long-horizon extrapolation behavior beyond the observed training sequence.

The noisy observations are generated according to the relative Gaussian perturbation model

$$H_n^{\text{obs}} = H_n^{\text{clean}} + \varepsilon_n, \quad \varepsilon_n \sim \mathcal{N}\left(0, \sigma^2 |H_n^{\text{clean}}|^2\right),$$

where σ denotes the prescribed noise level. This relative noise model is used because the magnitude of tri-periodic Horadam sequences may grow rapidly under spectral amplification.

The asymmetry of the periodic coefficients is measured by

$$\mathcal{A} = \max\{|p - q|, |q - s|, |s - p|\}.$$

This quantity is used as a diagnostic indicator of coefficient disparity. It should not be interpreted as a stand-alone spectral stability parameter, since the effective spectral radius is determined by the monodromy matrix and depends on the combined quantities

$$X = pqs + pr + qr + sr, \quad Y = r^3.$$

7.1. Effect of Structural Asymmetry on Model Performance

We first examine the performance of the RRNE framework under a fixed 7% Gaussian noise level across four asymmetry regimes. The results are summarized in Table 1.

Table 1. Effect of structural asymmetry on the RRNE framework under 7% Gaussian noise. The extrapolation horizon is $N = 60$.

Scenario	Structural Regime	Parametric Error (Param MAE)	Extrapolation Error (Test Log-RMSE)
Scenario 1	Low-asymmetry regime	0.173	0.090
Scenario 2	Moderate-asymmetry regime	0.211	0.120
Scenario 3	High-asymmetry regime	0.245	0.160
Scenario 4	Extreme-asymmetry regime	0.353	0.890

Table 1 shows that, within the tested parameter regimes, both the parametric error and the extrapolation error generally increase as the coefficient disparity becomes stronger. The RRNE framework performs stably in the low-asymmetry regime, with a Param MAE of 0.173 and a Test Log-RMSE of 0.090. Its performance remains acceptable in the moderate- and high-asymmetry regimes, where the Test Log-RMSE values increase to 0.120 and 0.160, respectively.

In contrast, the extreme-asymmetry regime leads to a sharp deterioration, with the Test Log-RMSE rising to 0.890. This indicates that long-horizon prediction becomes substantially more difficult under severe coefficient disparity and phase-switching effects. This observation is empirical rather than theoretical: the asymmetry index $\mathcal{A}$ alone does not determine the spectral behavior, which also depends on spectral amplification, observation noise, and the conditioning of the inverse-estimation problem.

7.2. Trajectory-Level Interpretation of the Asymmetry Regimes

This subsection provides a trajectory-level interpretation of the RRNE reconstruction performance under increasing structural asymmetry. The purpose is to visually assess how the proposed estimator behaves as the recurrence system moves from a stable low-asymmetry regime toward an extreme phase-switching regime in which long-horizon extrapolation becomes increasingly ill-conditioned.

In Figure 3, the reconstructed trajectory closely follows the clean sequence and remains stable over the extrapolation interval. The inset zoom confirms that the RRNE reconstruction preserves the main recursive trend beyond the train/test boundary. This behavior indicates that, under low coefficient disparity and 7% Gaussian noise, the inverse-estimation problem remains numerically well-conditioned.

In Figure 4, the RRNE framework still tracks the recursive evolution of the sequence with good qualitative agreement. Compared with the Safe Zone case, the larger coefficient

disparity leads to a moderate increase in parametric and extrapolation errors. Nevertheless, the reconstructed trajectory remains structurally coherent over the test interval.

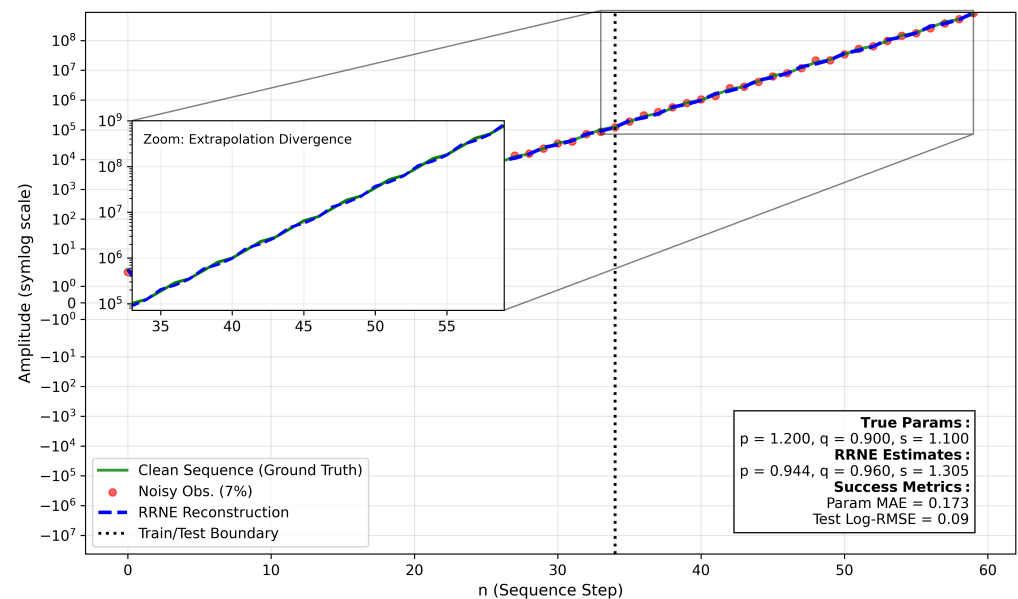


Figure 3. Reconstruction performance in the Safe Zone regime under 7% Gaussian noise.

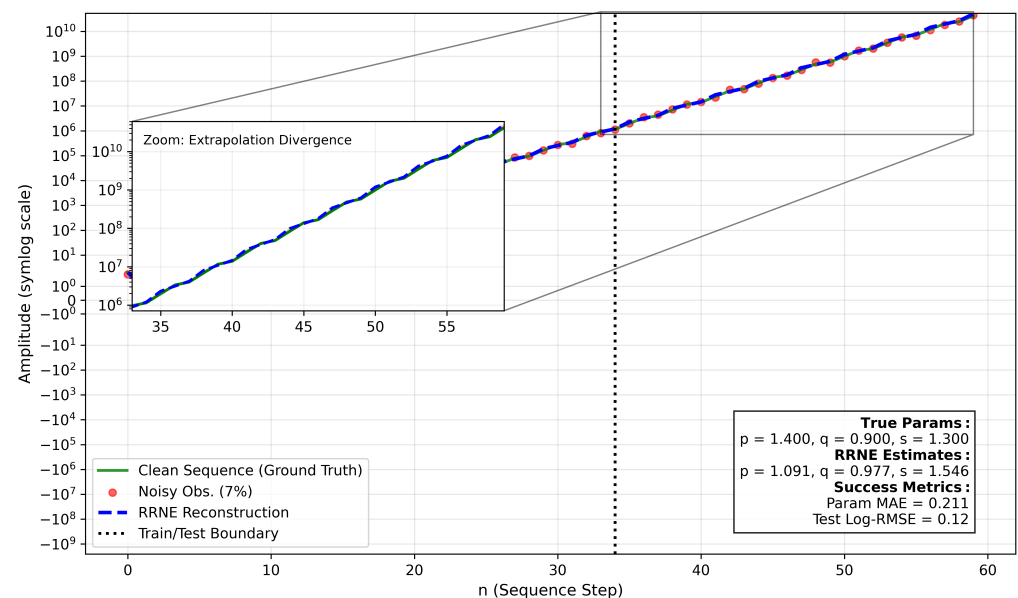


Figure 4. Reconstruction performance in the Regulation regime under 7% Gaussian noise.

Figure 5 illustrates the reconstruction behavior of the RRNE framework in the Boundary regime under 7% Gaussian noise. In this regime, the reconstructed trajectory continues to follow the dominant growth pattern of the clean sequence; however, visible deviations begin to appear over the extrapolation interval. This indicates that the inverse-estimation problem is approaching a stiffness threshold. Therefore, the Boundary regime should be interpreted as a transitional case between stable recurrence-informed recovery and the more severe structural instability observed in the Failure Mode.

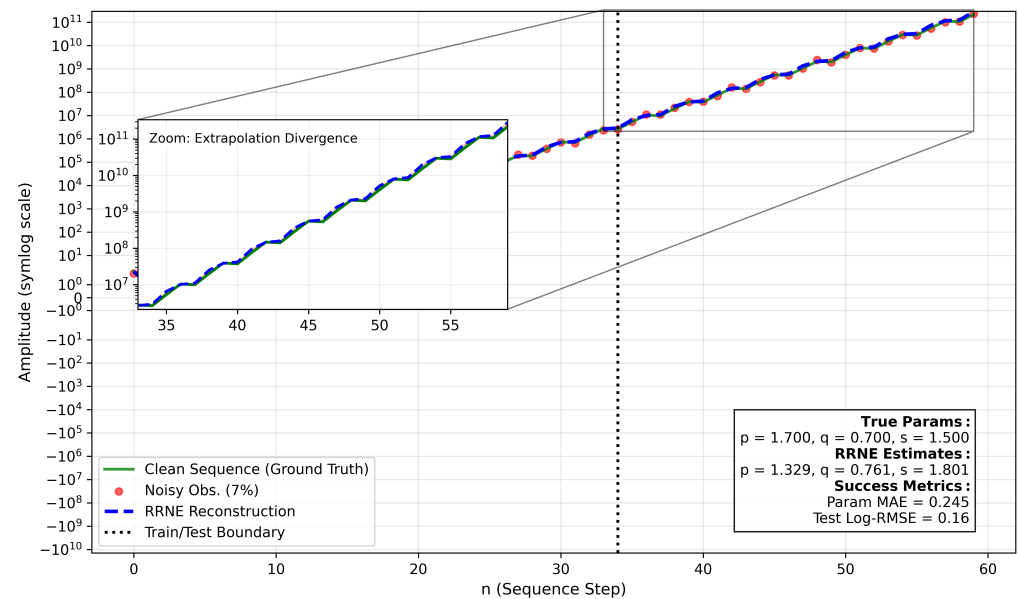


Figure 5. Reconstruction performance in the Boundary regime under 7% Gaussian noise.

In Figure 6, the reconstructed trajectory departs substantially from the clean sequence over the extrapolation horizon. The extreme switching between high and low phase coefficients produces a strongly nonuniform recursive structure, and small coefficient-level deviations are amplified along the trajectory. This figure therefore illustrates an empirically observed ill-conditioning regime rather than a universal operational boundary.

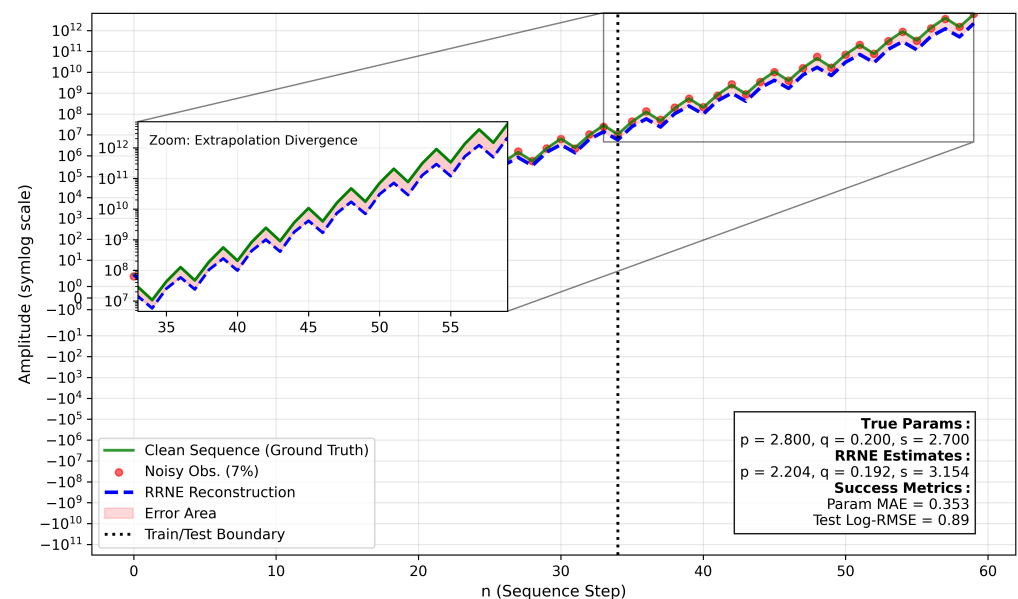


Figure 6. Reconstruction behavior in the Failure Mode regime under 7% Gaussian noise.

7.3. Disentangling Coefficient Asymmetry from Spectral Amplification

The results in Table 1 raise an important diagnostic question: to what extent is the observed extrapolation instability associated with coefficient asymmetry, and to what extent is it driven by spectral amplification?

To examine this issue, we computed the effective spectral radius

$$\rho_{\text{eff}} = \rho(M)^{1/3},$$

where M denotes the monodromy matrix over one complete period. Since

$$X = pqs + pr + qr + sr, \quad Y = r^3,$$

the effective spectral radius depends on the full coefficient interaction rather than on the maximum pairwise difference alone.

For the tested regimes, larger coefficient disparity is generally accompanied by stronger long-horizon numerical difficulty. However, this should not be interpreted as a monotone or universal mathematical relationship between $\mathcal{A}$ and ρ_{eff} . The asymmetry index is best understood as a diagnostic indicator of phase-to-phase coefficient disparity, whereas ρ_{eff} describes the dominant spectral growth rate of the associated monodromy matrix.

In the present experiments, the Failure Mode combines multiple sources of difficulty: large coefficient disparity, abrupt phase switching, spectral amplification, and noisy inverse estimation. The extreme switching between high and low phase coefficients creates a nonuniform recurrence structure, while the associated monodromy dynamics can amplify small coefficient-level and observation-level perturbations over the extrapolation interval. Therefore, the failure observed in this regime should be interpreted as a joint effect of phase-dependent asymmetry, spectral amplification, and noisy inverse estimation, rather than as a consequence of a single scalar indicator.

7.4. Noise Sensitivity in the Safe-Zone Regime

We next evaluate the effect of increasing observation noise in the Safe Zone regime. The parameters are fixed as

$$p = 1.2, \quad q = 0.9, \quad s = 1.1,$$

with the coefficient $r = 0.5$. The Gaussian noise level is varied from 3% to 10%. The quantitative results are reported in Table 2, while the corresponding trajectory-level reconstructions are presented in Figures 7–10.

Table 2. Noise robustness analysis in the Safe Zone regime.

Noise Level	Parametric Error (Param MAE)	Extrapolation Error (Test Log-RMSE)
3% Noise	0.078	0.040
5% Noise	0.129	0.100
7% Noise	0.173	0.110
10% Noise	0.253	0.130

Table 2 shows a gradual degradation of performance as the noise level increases in the Safe Zone regime. Under 3% Gaussian noise, the RRNE framework yields the lowest errors, with a Param MAE of 0.078 and a Test Log-RMSE of 0.040. As the noise level increases to 5%, 7%, and 10%, the parametric error rises to 0.129, 0.173, and 0.253, respectively. Similarly, the Test Log-RMSE increases from 0.040 to 0.130.

Despite this degradation, the extrapolation error remains controlled throughout the Safe Zone configuration. The trajectory-level behavior associated with these four noise settings is illustrated in Figures 7–10. These figures support the quantitative trend reported in Table 2: higher noise levels increase the parameter recovery error, but the reconstructed trajectories remain structurally coherent in the low-asymmetry Safe Zone regime.

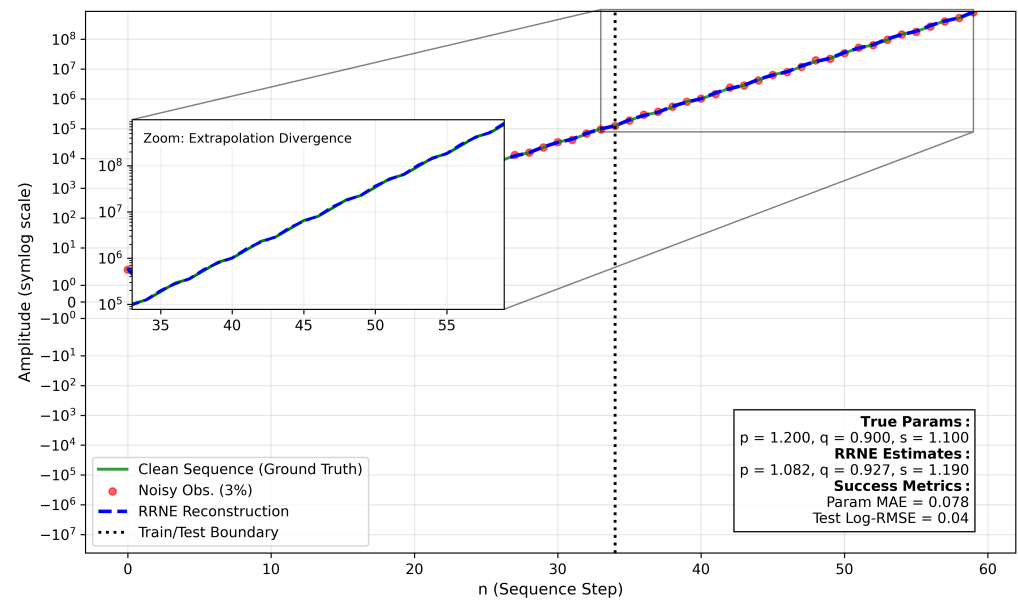


Figure 7. RRNE reconstruction in the Safe Zone regime under 3% Gaussian noise.

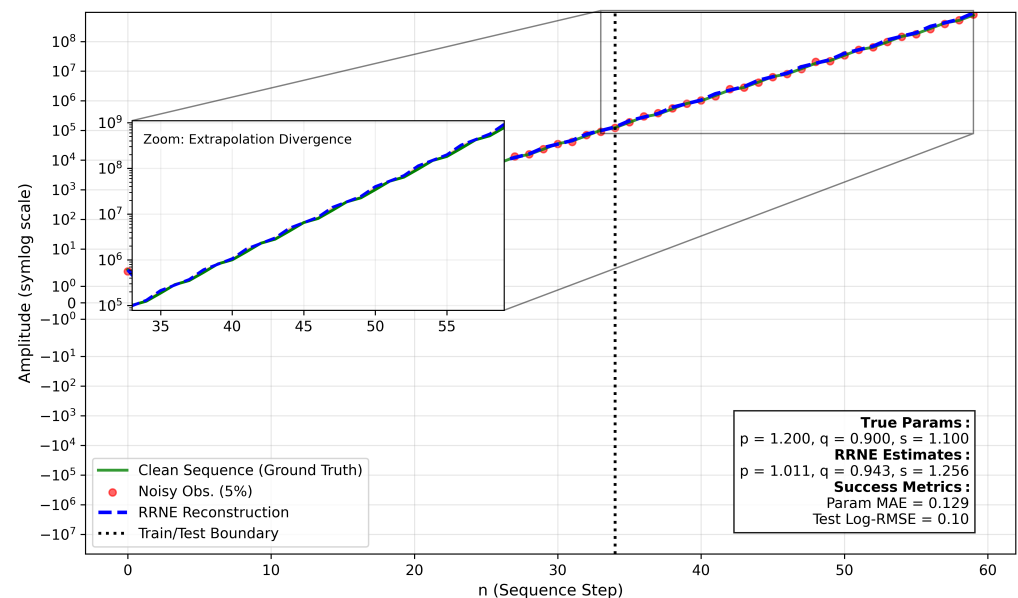


Figure 8. RRNE reconstruction in the Safe Zone regime under 5% Gaussian noise.

This result should not be overgeneralized to all tri-periodic regimes. The same level of noise may have a much stronger effect when the recurrence is highly asymmetric or spectrally amplified. Therefore, the noise robustness observed in Table 2 and in Figures 7–10 should be interpreted as a stability property of the Safe Zone configuration rather than as a universal robustness guarantee for all tri-periodic Horadam systems.

In Figure 7, the RRNE reconstruction almost overlaps with the clean trajectory throughout both the training and extrapolation intervals. The inset zoom confirms that the recursive trend is preserved beyond the train/test boundary. This result indicates that, under 3% Gaussian noise, the proposed estimator can recover the underlying recurrence structure with high trajectory-level stability in the Safe Zone regime.

In Figure 8, the reconstructed trajectory remains closely aligned with the clean sequence despite the increased observation noise. Compared with the 3% noise case, the parameter estimates show a moderate deterioration, but the long-horizon recursive structure

is still preserved. This suggests that the RRNE framework retains reliable reconstruction behavior under mild-to-moderate noise in the Safe Zone configuration.

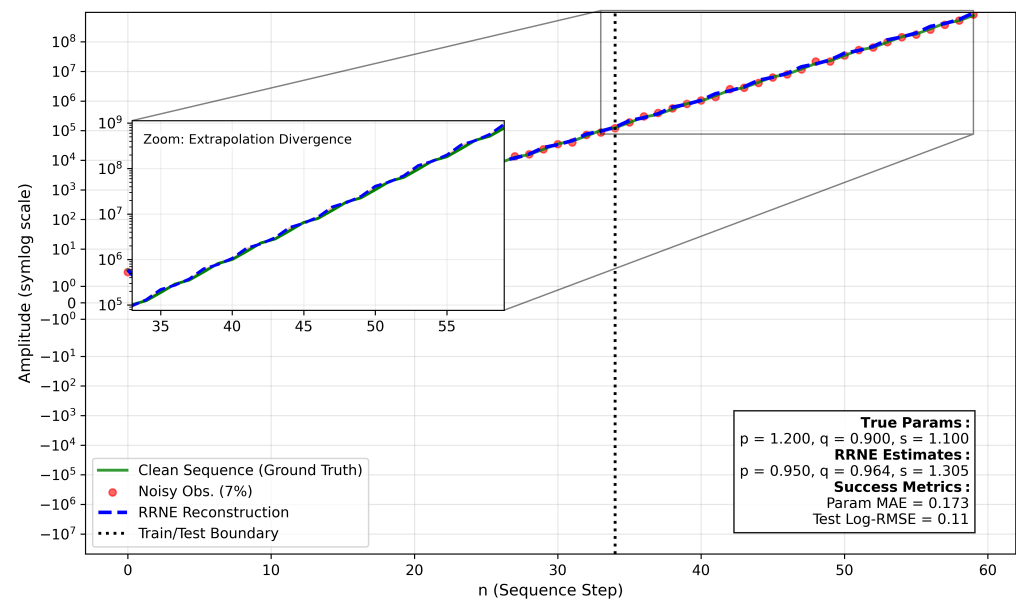


Figure 9. RRNE reconstruction in the Safe Zone regime under 7% Gaussian noise.

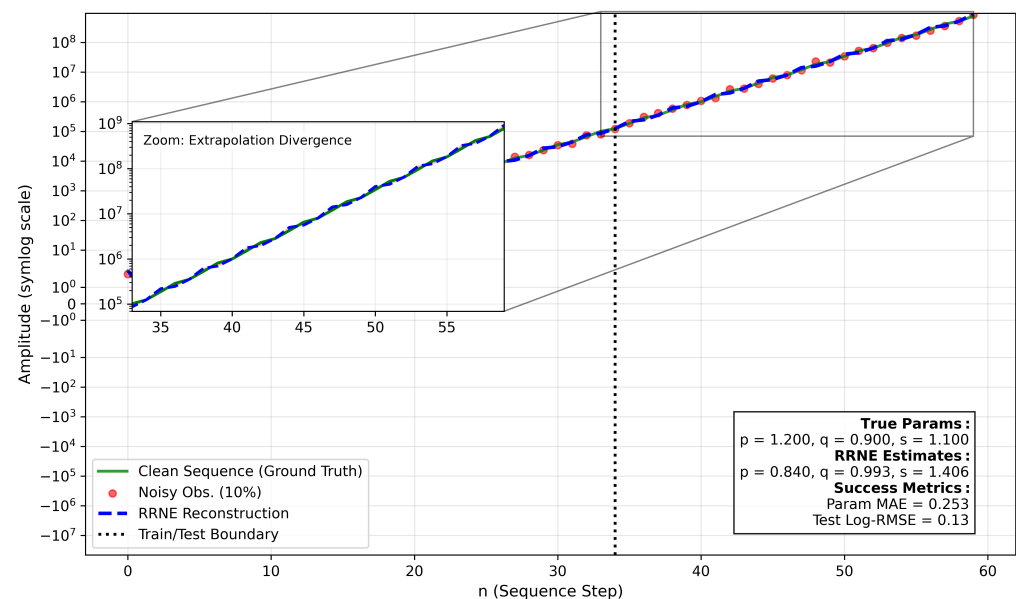


Figure 10. RRNE reconstruction in the Safe Zone regime under 10% Gaussian noise.

In Figure 9, the RRNE reconstruction continues to follow the clean trajectory over the extrapolation interval. Although the noise level leads to a visible increase in parametric error, the reconstructed sequence remains stable and does not exhibit extrapolation collapse. This behavior supports the conclusion that the Safe Zone regime remains numerically tractable even under 7% Gaussian noise.

In Figure 10, the observation noise becomes more pronounced, and the parametric error increases relative to the lower-noise cases. Nevertheless, the RRNE reconstruction still preserves the global recursive trajectory and remains coherent over the extrapolation window. This indicates that, within the low-asymmetry Safe Zone regime, the proposed estimator can tolerate relatively high observation noise without immediate loss of long-horizon structural stability.

7.5. Comparison with Classical Ordinary Least Squares

To further evaluate the practical role of the RRNE framework, we compare it with a classical phase-wise Ordinary Least Squares (OLS) estimator. This baseline is used to assess whether the recurrence-informed neural formulation provides an advantage over direct coefficient fitting.

In the OLS baseline, the periodic coefficients are estimated separately for the three residue classes modulo three. For each phase, the observed recurrence is approximated by

$$H_n^{\text{obs}} - rH_{n-2}^{\text{obs}} \approx c_i H_{n-1}^{\text{obs}}, \quad i \in \{0, 1, 2\},$$

where

$$c_0 = p, \quad c_1 = q, \quad c_2 = s.$$

Hence, OLS provides a direct phase-wise estimator of the hidden periodic coefficients without Fourier embeddings, robust neural loss functions, recurrence-informed training, or hybrid neural optimization.

Table 3 compares the extrapolation performance of the classical phase-wise OLS baseline and the proposed RRNE framework under the same 7% Gaussian noise setting. In the low-, moderate-, and high-asymmetry regimes, the RRNE framework achieves smaller Test Log-RMSE values than OLS, with errors 0.090, 0.120, and 0.160, respectively. This indicates that the recurrence-informed neural formulation improves long-horizon reconstruction when the system is noisy but not yet dominated by extreme structural ill-conditioning.

Table 3. Baseline comparison between classical phase-wise OLS and the proposed RRNE framework under 7% Gaussian noise across four asymmetry regimes. The evaluation metric is Test Log-RMSE over $N = 60$.

Scenario	Structural Regime	OLS Test Log-RMSE	RRNE Test Log-RMSE	Winner
Scenario 1	Low-asymmetry regime	0.107	0.090	RRNE
Scenario 2	Moderate-asymmetry regime	0.219	0.120	RRNE
Scenario 3	High-asymmetry regime	0.206	0.160	RRNE
Scenario 4	Extreme-asymmetry regime	0.106	0.890	OLS

The extreme-asymmetry regime requires a more careful interpretation. In this case, OLS yields a smaller Test Log-RMSE value than RRNE, namely 0.106 versus 0.890. However, this should not be interpreted as evidence that OLS recovers the underlying phase-dependent recurrence structure more accurately. Since OLS directly regresses on noisy lagged observations, it may be affected by errors-in-variables attenuation, which can shrink the estimated coefficients and artificially damp the generated trajectory.

Thus, the Failure Mode represents a structurally ill-conditioned case in which the two methods fail in different ways. OLS may match the global logarithmic envelope without faithfully recovering the recurrence mechanism, whereas RRNE exposes the instability of structure-aware inverse estimation under extreme asymmetry and noisy observations. Overall, RRNE improves over OLS in the stable and moderately stiff regimes, but it should be interpreted as an effective recurrence-informed estimator for non-extreme settings rather than as a universally superior replacement for classical estimators.

7.6. Component-Wise Ablation and Seed-Dependent Stability Analysis

Finally, we evaluate the effect of individual architectural components. The ablation study is conducted under 7% Gaussian noise using the RRNE baseline setting. The results are reported in Table 4.

Table 4. Component-wise ablation results of the RRNE framework under 7% Gaussian noise.

Model Variant	Test Log-RMSE	Relative Performance
Proposed RRNE (Full Model)	0.160	Baseline
w/o Fourier Embeddings	0.893	Critical degradation
w/ MSE Loss	0.129	Competitive in this run
w/o Teacher Forcing	0.171	Slight degradation

Table 4 should be interpreted as a diagnostic comparison of architectural components rather than as a universal ranking of model variants. Removing Fourier embeddings causes the strongest degradation, increasing the Test Log-RMSE from 0.160 to 0.893. This confirms the importance of Fourier features for representing the intrinsic period-three switching structure of the recurrence.

The MSE-loss variant attains a competitive Test Log-RMSE value of 0.129 in this particular run. However, this does not imply that MSE is uniformly preferable. In amplified recursive systems, MSE may be more sensitive to large residuals and unstable gradients, whereas the Log-Cosh loss is retained as the reference objective because of its more robust behavior under large deviations.

The variant without teacher forcing gives a Test Log-RMSE of 0.171, indicating only a slight degradation relative to the full RRNE model. This result suggests that teacher forcing acts mainly as a stabilization mechanism: it can reduce recursive error accumulation, but it may also introduce observation-noise-induced bias through the use of noisy lagged states. This trade-off becomes especially relevant in strongly amplified regimes.

To examine sensitivity to random initialization, we also conducted a repeated-seed analysis over 10 independent training runs under 7% Gaussian noise with randomized weight initializations. The reported statistics were

$$\text{Parametric MAE} = 0.0700 \pm 0.0504,$$

and

$$\text{Test Log-RMSE} = 0.7896 \pm 0.3655.$$

The low mean parametric error indicates that the estimator can repeatedly approach meaningful coefficient regimes under the tested setting. However, the non-negligible standard deviation in Test Log-RMSE shows that long-horizon extrapolation remains sensitive to initialization and small coefficient-level deviations.

Therefore, the ablation and repeated-seed results support a cautious interpretation: the RRNE framework exhibits useful recurrence-informed behavior in non-extreme regimes, but the inverse problem remains sensitive to initialization, coefficient amplification, and long-horizon recursive propagation. The results should not be interpreted as a full optimization-stability guarantee.

7.7. Overall Interpretation

Taken together, the asymmetry-regime experiment, the noise-sensitivity analysis, the OLS comparison, and the ablation study show that the inverse-estimation problem cannot be assessed by a single numerical score alone. Both coefficient recovery and long-horizon trajectory consistency must be considered.

The updated results indicate that RRNE performs reliably in low- and moderate-asymmetry regimes and remains partially stable in the high-asymmetry regime. However, the extreme-asymmetry case leads to a clear deterioration in extrapolation performance, showing that strong phase switching and coefficient disparity can make the inverse problem

severely ill-conditioned. The OLS comparison further shows that a smaller Test Log-RMSE does not necessarily imply more faithful recovery of the underlying recurrence mechanism.

Therefore, the proposed RRNE framework should be viewed as a structure-aware inverse-estimation method rather than as a purely data-fitting model. These components improve phase encoding, residual robustness, and recursive stabilization, but they cannot fully remove the intrinsic instability of highly asymmetric and spectrally amplified tri-periodic recurrences.

8. Conclusions

In this study, a tri-periodic Horadam-type recurrence with modularly varying coefficients was investigated from both algebraic and computational perspectives. The sequence was formulated through the parameters p, q, s , and r , and its basic structural properties were derived by means of induced three-step recurrences. By decomposing the sequence into its three modular subsequences, the generating function and a Binet-type representation were obtained in a form compatible with the period-three structure of the recurrence.

From the spectral point of view, the system was analyzed through a monodromy matrix formulation over one complete period. The trace and determinant of the monodromy matrix led naturally to the quantities

$$X = pqs + pr + qr + sr, \quad Y = r^3,$$

and to the period-level characteristic equation

$$z^2 - Xz - Y = 0.$$

This formulation clarified how the effective spectral radius governs the asymptotic amplification behavior of the recurrence. In addition, the asymmetry index was used as an empirical diagnostic measure for organizing the numerical experiments and interpreting the observed conditioning behavior. However, the effective growth rate of the recurrence is governed by the spectral properties of the monodromy matrix through

$$X = pqs + pr + qr + sr, \quad Y = r^3,$$

rather than by the asymmetry index alone. Thus, the relationship between $\mathcal{A}$ and spectral amplification should be understood as an observed tendency in the explored parameter regimes, not as a universal theoretical monotonicity statement.

In the computational part, an RRNE framework was proposed for the inverse estimation of the periodic coefficients (p, q, s) under noisy observations. Fourier feature embeddings were used to represent the intrinsic period-three structure of the recurrence, while the Log-Cosh loss function was employed to reduce the sensitivity of the optimization process to large amplified residuals. A teacher-forced recurrence constraint was also incorporated to improve numerical stability during training. However, since this constraint uses noisy observations rather than latent clean states, the method should be interpreted as a noise-regularized inverse estimator rather than an exact symbolic recovery mechanism.

The numerical experiments indicate that the proposed framework can produce meaningful recurrence-informed parameter reconstructions in low- and moderate-asymmetry regimes under noisy observations. In these regimes, the reconstructed trajectories remain close to the ground-truth sequences over the extrapolation interval, suggesting that the architecture can capture the dominant discrete dynamical structure when the spectral amplification is not too severe. At the same time, the experiments also reveal the practical limitations of the method. In highly asymmetric regimes, small coefficient-level discrepan-

cies may be strongly amplified along the recurrence, leading to substantial long-horizon prediction errors even when the short-range fit appears acceptable.

The comparison between the asymmetry-regime experiments and the safe-zone noise experiments further clarifies the origin of the observed failure modes. Within the low-asymmetry Safe Zone regime, increasing the Gaussian noise level from 3% to 10% produced a gradual and controlled degradation in both parametric error and Test Log-RMSE. By contrast, the highly asymmetric Failure Mode case generated a much larger extrapolation error even under a lower noise level. This suggests that catastrophic extrapolation divergence is not caused by measurement noise alone, but is strongly linked to the intrinsic ill-conditioning induced by phase-dependent coefficient asymmetry in the tri-periodic recurrence.

The component-wise ablation study also showed that the architectural ingredients should not be evaluated only through a single extrapolation metric. In particular, removing the Fourier embedding caused the most severe degradation, confirming the importance of explicitly encoding the period-three phase structure. The standard MSE-loss and fully recursive variants may produce lower Test Log-RMSE values in some diagnostic settings; however, this does not necessarily imply better structural coefficient recovery. Therefore, the inverse problem must be assessed through two complementary criteria: the accuracy of the recovered recurrence coefficients and the long-horizon behavior of the reconstructed sequence.

Overall, the results establish a link between the algebraic spectral structure of tri-periodic Horadam recurrences and the numerical behavior of recurrence-informed neural inverse estimation. The study shows that monodromy-based amplification, coefficient asymmetry, measurement noise, teacher forcing bias, and neural optimization stability are closely connected. In this sense, the proposed framework provides not only a computational tool for parameter recovery, but also an empirical lens through which the conditioning of multi-phase recursive systems can be examined.

Future work may proceed in several directions. First, the present framework can be extended to higher-order and multi-periodic recurrence systems, where the number of phase-dependent coefficients and the dimension of the monodromy operator increase. Second, stochastic recursive systems and fractional recurrence structures may provide a broader setting for modeling random perturbations, memory effects, and nonlocal discrete dynamics. Third, adaptive spectral embeddings and attention-based or transformer-type sequence architectures may be investigated to improve parameter recovery in severely ill-conditioned regimes. Finally, a more systematic grid-based analysis over broader ranges of $\mathcal{A}$, noise levels, initial conditions, extrapolation horizons, and fixed values of r will be necessary to identify sharper quantitative stability boundaries for recurrence-informed neural estimators associated with discrete dynamical systems.

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