

Article

Analytical Formulation and Equilibrium Structure of a 26-State Nonlinear Dynamical System for DFIG

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Abstract

We formulate and analyze a 26-dimensional nonlinear dynamical system governing a doubly-fed induction generator (DFIG) wind energy conversion system coupled to an infinite bus through a dynamic transmission line. Seven interacting subsystems— aerodynamics, a two-mass drivetrain, a fourth-order machine, rotor- and grid-side converter controllers, a phase-locked loop, and a pitch regulator—are assembled into a single vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ on $\mathbb{R}^{26}$, derived in dimensionless coordinates. Strict positivity of the determinant $\Delta = L_s L_r - L_m^2 = \sigma L_s L_r$ for every physically admissible machine renders the flux-current map invertible, so the right-hand side is well defined; the nodal Kirchhoff constraint forms a semi-explicit differential-algebraic relation that we eliminate to obtain an explicit ordinary differential equation. The central contribution is a constructive scheme for the equilibria: the 26 stationarity conditions $\mathbf{f}(\mathbf{x}^*, \mathbf{u}) = \mathbf{0}$ are solved by an iterative voltage-matching procedure converging to a residual below 10^{-11} per unit—essentially machine precision—which removes the spurious start-up transients common in reported simulations. Analytically chosen feedback gains induce a hierarchy of well-separated time scales, placing the closed loop in the multiple-time-scale class; the separation is made quantitative through explicit small parameters ε_i formed from the ratios of subsystem time constants. Numerical integration of a GE 3.6 MW configuration confirms the construction: under stationary forcing, the rotor speed stays within 1.32×10^{-5} pu of the equilibrium, and under a large-amplitude wind program ($11 \rightarrow 14 \rightarrow 9$ m/s) spanning the full operating envelope, it is regulated to within 0.065%, while the DC-link voltage deviation remains below 2.4×10^{-5} pu and the power balance closes with residual below 10^{-3} pu, the $\approx 2\%$ mechanical–electrical gap being the modeled losses. Linearization about the computed equilibrium yields a Jacobian whose spectrum lies entirely in the open left half-plane, establishing local asymptotic stability and exposing the individual electromagnetic, torsional, and control modes. The model furnishes a rigorously initialized, analytically transparent basis for linearization, spectral stability analysis, and bifurcation study. Its practical value is that a consistent equilibrium and a certified spectrum remove the start-up transients and undocumented tuning that otherwise let initialization artifacts masquerade as genuine dynamics, so that the model can serve as a trustworthy building block for weak-grid and wind-farm stability studies.



Academic Editors: Manuela Minetti,
Mahmood Hosseini Imani and
Andrea Mazza

Received: 18 June 2026

Revised: 9 July 2026

Accepted: 15 July 2026

Published: 17 July 2026

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Keywords: nonlinear dynamical systems; differential-algebraic equations; equilibrium analysis and consistent initialization; multiple-time-scale (singularly perturbed) systems; nonlinear state-space modeling; numerical integration; per-unit normalization; doubly-fed induction generator (DFIG); wind energy conversion; cascaded and pitch control; phase-locked loop (PLL); weak grid

MSC: 93C05; 93C10; 93C95; 93D15

1. Introduction

Many engineered systems are most naturally described as the interconnection of several physical and control subsystems, each contributing its own state variables, constitutive relations, and characteristic time scale. Coupling these subsystems produces a single high-dimensional nonlinear vector field whose study raises three recurring mathematical questions. First, the heterogeneous constitutive laws—mechanical, electromagnetic, and feedback—must be combined into a well-defined right-hand side, which requires that the algebraic maps relating the internal variables be invertible throughout the operating domain. Second, any meaningful transient analysis presupposes a consistent equilibrium, namely a state at which every component is simultaneously stationary, and equivalently a root of the full coupled balance map. Locating such a root for a tightly coupled, high-dimensional system is a nontrivial nonlinear-algebraic problem in its own right and is the precise mathematical content of what is loosely termed initialization. Third, feedback regulation typically introduces widely separated time scales, placing the closed-loop model within the class of multiple-time-scale (singularly perturbed) systems and shaping both its transient evolution and its spectral structure.

The doubly-fed induction generator (DFIG) wind energy conversion system is a paradigmatic setting in which all three questions arise together and admit a largely analytical treatment. Its dynamics combine a polynomial aerodynamic nonlinearity, a bilinear electromagnetic torque, a strongly cross-coupled four-state machine, several cascaded feedback loops, a phase-locked loop, and a transmission network whose nodal constraint contributes a differential-algebraic component—so that the assembled model is at once high-dimensional, nonlinear, and constrained. The need to analyze this system with mathematical care, rather than merely to simulate it, has sharpened as power grids evolve.

The global installed wind power capacity surpassed 1000 GW in 2023 and continues to accelerate, with projections indicating that wind and solar generation combined will supply over 40% of global electricity by 2030 [1]. Meeting this growth requires accurate energy forecasting and operational models across all renewable generation technologies. As penetration levels rise, wind power plants are increasingly connected to electrically remote locations through long transmission corridors, resulting in weak grid conditions that fundamentally alter the dynamic interaction between generators and the network [2]. Among the variable-speed wind turbine topologies deployed at utility scale, the doubly-fed induction generator (DFIG) remains the dominant technology, commanding roughly half of the global installed base, owing to its partially rated converter, independent active and reactive power controllability, and wide speed range spanning $\pm 30\%$ of synchronous speed [3]. Reliability assessments of deployed fleets have identified the back-to-back converter as the leading contributor to turbine forced-outage hours, underscoring the operational importance of accurate converter-level dynamic models.

Accurate dynamic models are the indispensable foundation for stability assessment, controller design, and grid code compliance verification of DFIG-based wind power plants. The fidelity requirements of such models have sharpened considerably in recent years, driven by two converging trends. First, the displacement of synchronous generators by inverter-based resources reduces system inertia and short-circuit strength, exposing oscillatory modes that were previously well damped [2,4]. Second, documented field incidents of sub-synchronous oscillations in DFIG wind farms connected to series-compensated or weak ac networks—most notably in the ERCOT system—have demonstrated that converter control dynamics, particularly the phase-locked loop (PLL), can interact with grid impedance to produce sustained oscillations at frequencies that simplified models fail to reproduce [5–8].

Despite the practical significance of high-fidelity DFIG modeling, the published literature reveals a persistent gap between the models used for analytical insight and those needed for faithful replication of system-level dynamics. The existing body of work can be broadly categorized into three modeling tiers.

At the first tier, reduced-order models employing two or three differential equations per machine—typically retaining only the rotor flux or voltage-behind-reactance dynamics while neglecting stator electromagnetic transients—have been widely adopted for large-scale power system stability studies [9–11]. These models are computationally efficient and well suited for multi-machine network simulations, but their validity degrades under weak grid conditions and during fast electromagnetic transients where the neglected stator dynamics contribute meaningfully to the system response [12].

At the second tier, medium-fidelity models retaining the fourth-order generator equations (both stator and rotor flux dynamics) together with simplified converter representations have been employed for detailed single-turbine studies and small-signal stability analysis [13–15]. Several recent contributions have advanced analytical modeling within this tier. Fan and Miao developed a 26-state dq -frame model of a Type-3 wind farm incorporating PLL dynamics for sub-synchronous resonance analysis, demonstrating that PLL modeling is essential for replicating field-observed oscillation phenomena [14]. Du et al. showed that strong dynamic interactions between grid-connected DFIGs and the power system can arise from modal coupling, a mechanism invisible to models that omit higher-order converter control states [16]. Model predictive control applied to transient flux linkage and reactive power compensation has further demonstrated that complete fourth-order state representations are essential for enabling advanced DFIG control strategies. Other control methodologies explored at this tier include second-order sliding mode direct power control [17], among a broader family of nonlinear and optimal converter-control strategies. While these contributions advance the state of the art in specific analytical directions, they typically treat individual subsystems in isolation or employ simplified representations for components outside their immediate focus—particularly the mechanical drivetrain, pitch controller, and transmission network.

At the third tier, detailed electromagnetic transient (EMT) models implemented in commercial software (PSCAD, MATLAB/Simulink, PLECS) provide the highest fidelity by representing switching-level converter dynamics and three-phase electromagnetic transients [18]. However, their complexity renders them opaque to analytical insight, their pre-built architectures resist systematic parameter studies, and the proprietary nature of many implementations hinders reproducibility and independent verification.

The stability challenges arising from PLL-grid interactions under weak grid conditions have attracted particular attention. A common thread runs through this body of work: the phase-locked loop is simultaneously the mechanism that enables grid synchronization and the principal channel through which a converter destabilizes a weak grid, so that the useful PLL bandwidth is bounded above by a stability limit that tightens as the grid weakens. Jeong and Jang demonstrated through impedance-based analysis that high PLL bandwidth introduces negative resistance into the converter output impedance, deteriorating stability under weak grid conditions [19]. The remedies proposed since then can be read as different ways of relaxing this bound: a novel PLL structure proposed by Guo et al. eliminates frequency-coupling effects to improve synchronization stability, though the analysis is limited to generic grid-following converters without DFIG-specific drivetrain and pitch dynamics [20], addressing the coupling at its source; and for DFIG systems specifically, the inter-sequence coupling between positive and negative sequence dynamics introduced by PLL during asymmetric weak grid conditions has been shown to affect system stability, requiring frequency-compensated PLL designs [21], addressing it in the

negative sequence. Comprehensive reviews of sub-synchronous oscillation phenomena in renewable-integrated power systems [4] confirm that modeling efforts are moving toward unified multi-machine frameworks and nonlinear multi-scale dynamics, yet observe that a validated single-turbine model of sufficient fidelity to serve as the building block for such frameworks remains a prerequisite that is not consistently met in the literature. The present work supplies exactly such a building block, and—because every one of these PLL-bandwidth mechanisms ultimately manifests as the migration of a Jacobian eigenvalue toward the imaginary axis—couples it to the spectral apparatus (Section 6) needed to locate those limits.

A further deficiency in the existing literature concerns initialization methodology. Correct steady-state initialization—computing initial conditions such that all state derivatives are simultaneously zero—is a prerequisite for meaningful transient simulation, yet is rarely discussed in detail. Hansen et al. developed initialization procedures for grid-connected wind turbine models within power-system simulation frameworks [22]. However, the initialization of high-order models incorporating controller integrator states, PLL angle, DC-link voltage, filtered power signals, pitch controller states, and transmission line dynamics requires a simultaneous solution of tightly coupled algebraic constraints that is seldom addressed in publications presenting DFIG simulation results. The consequence is that many published simulations exhibit initial transients lasting one to several seconds, during which the system is settling from an inconsistent initial condition rather than responding to the intended disturbance, thereby corrupting the quantitative metrics reported.

This paper addresses the gaps identified above by constructing and analyzing a fully documented 26-dimensional nonlinear dynamical system for a DFIG-based wind energy conversion system that integrates seven subsystems—turbine aerodynamics, a two-mass torsional drivetrain, a fourth-order DFIG machine, a rotor-side converter (RSC) controller, a grid-side converter (GSC) controller, a phase-locked loop, and a pitch regulator—all coupled through a dynamic transmission line to an infinite bus. The principal contributions are the following.

1. *A well-posed vector field.* We derive the coupled 26-dimensional system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ in full in dimensionless coordinates, including every reference-frame transformation and normalization map, and we identify the regularity condition $\Delta = L_s L_r - L_m^2 = \sigma L_s L_r > 0$ under which the flux-current map is invertible and the right-hand side is well defined. The nodal Kirchhoff constraint at the point of common coupling is exhibited as a semi-explicit differential-algebraic relation that is eliminated to yield an explicit state-space form.
2. *Constructive computation of equilibria.* We give an analytical procedure that solves the complete set of 26 simultaneous stationarity conditions $\mathbf{f}(\mathbf{x}^*, \mathbf{u}) = \mathbf{0}$ by an iterative voltage-matching scheme converging to a residual below 10^{-11} per unit. This furnishes a consistent initial state and removes initialization-induced transients, so that computed trajectories reflect genuine forcing rather than relaxation from an inadmissible point.
3. *Time-scale-separated feedback synthesis.* We select the feedback gains analytically from the subsystem time constants so as to realize a prescribed separation of time scales across the control hierarchy, in place of empirical trial-and-error tuning; this makes explicit the multiple-time-scale structure of the closed-loop vector field.
4. *Numerical corroboration.* We validate the model by integration on a GE 3.6 MW configuration under two regimes—stationary forcing and a large-amplitude wind program spanning the full operating envelope—obtaining trajectory–equilibrium agreement to 10^{-5} pu in rotor speed, DC-link regulation within 2.4×10^{-5} pu under large-signal transients, and a verified power balance closing with residual below 10^{-3} pu, with

the $\approx 2\%$ excess of mechanical input over electrical output being exactly the modeled losses.

The mathematical content of these contributions does not lie in the size of the state vector. Raising the dimension of an existing model is, by itself, a bookkeeping operation; what is offered here is structural. Concretely, the paper contributes (i) a *well-posedness result* (Proposition 1) that identifies the exact algebraic condition—strict positivity of a single determinant—under which the constitutive map is a diffeomorphism and the vector field is therefore globally defined on the physical domain; (ii) an *index and reduction analysis* (Proposition 2) showing that the network coupling is an index-one differential-algebraic constraint and giving the explicit elimination that produces the ordinary differential equation; (iii) a *constructive equilibrium map with a convergence guarantee* (Algorithm 1 and Proposition 3), which turns the informal notion of “initialization” into a contraction whose fixed point is the equilibrium; (iv) an *explicit multiple-time-scale decomposition* in which the small parameters are computed rather than asserted (Section 3.6); and (v) a *spectral stability certificate* obtained from the Jacobian at the computed equilibrium (Section 6). Each of these is an analytical statement about the assembled field that no individual subsystem model in the literature provides, and each becomes available only once the subsystems are combined and the coupling is treated with care.

Algorithm 1. Voltage-matching equilibrium computation of the consistent equilibrium.

Input: operating input $\mathbf{u} = (v_w, \omega_{\text{ref}}, Q_s^{\text{ref}}, Q_g^{\text{ref}})$; tolerance $\text{tol} = 10^{-11}$. **Output:** the equilibrium $\mathbf{x}^*$ of (61).

Step 1 (Initial guess). Set $\mathbf{y}^{(0)} = (V_{\text{nom}}, 0)$ (nominal PCC voltage), $\omega_r = \omega_{\text{ref}}$, and $\beta = 0$.

Step 2 (Subsystem back-substitution). At iterate $\mathbf{y}^{(k)}$ compute in order: (i) C_p, P_m, T_m from the aerodynamic map; (ii) $T_e = T_m$ and the rotor currents from torque and reactive-power balance; (iii) the stator currents from (41); (iv) the DC-link and grid-side currents from converter power balance; (v) if the power limit is exceeded, activate pitch and solve the scalar equation $|P_e(\beta)| = 1$ for β (else $\beta = 0$).

Step 3 (Voltage update). Evaluate the PCC mismatch $\mathbf{g}(\mathbf{x}^{(k)}, \mathbf{y}^{(k)})$ of (60) and update $\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} - \mathbf{Y}_{\text{PCC}}^{-1} \mathbf{g}$ (Newton step on the algebraic constraint).

Step 4 (Integrator states). Set every controller integrator so that its output equals the steady control signal (zero steady error), giving $\dot{\mathbf{x}} = \mathbf{0}$ exactly.

Step 5 (Stopping test). Stop when $\|\mathbf{F}(\mathbf{x}^{(k)}, \mathbf{u})\|_{\infty} < \text{tol}$; otherwise return to Step 2. Saturation limits are enforced by projection at Step 2, and the converged point is accepted only if it lies in the interior of all limits (otherwise the binding limit is imposed and the reduced system re-solved).

The remainder of this paper is organized as follows. Section 2 derives the 26-dimensional vector field, assembling the aerodynamic, drivetrain, machine, and network components, establishing the invertibility of the constitutive map, and casting the interconnection as a semi-explicit differential-algebraic system that is reduced to an explicit state-space form. Section 3 synthesizes the feedback laws. Section 4 computes the equilibrium and validates the model numerically under stationary and time-varying forcing. Section 5 then verifies the energy–power balance at each steady operating point. Section 6 linearizes about the computed equilibrium and reports the eigenvalue spectrum, and Section 7 collects the complete parameter set and reproducibility information. Section 8 concludes and indicates directions for spectral and bifurcation analysis.

2. System Description and Modeling Framework

This section presents a comprehensive mathematical model integrating the wind turbine aerodynamics, mechanical drivetrain, electrical generator, power converters, and transmission network, with the overall system architecture depicted in Figure 1. The model assumes balanced three-phase conditions, which enables representation in the synchronously rotating dq framework. Two reference frames are employed: stator-flux orientation for the DFIG (providing natural active/reactive power decoupling) and grid-voltage orientation for grid-side components via phase-locked loop (PLL). A consistent per-unit system based on the generator rated values provides dimensionless equations with improved numerical conditioning. All subsequent equations are expressed in this per-unit system unless explicitly stated otherwise.

Motor convention is applied consistently: positive power indicates absorption, and negative indicates generation. In super-synchronous operation ($\omega_r > \omega_s$, $s < 0$), both stator and rotor powers are negative ($P_s < 0$, $P_r < 0$), indicating generation. The stator delivers power directly to the grid while the rotor delivers through the back-to-back converter. Total electrical power is $P_{\text{total}} = P_s + P_r$, with rotor power related to slip by $P_r = s \cdot P_s$. In sub-synchronous operation ($\omega_r < \omega_s$, $s > 0$), stator power remains negative ($P_s < 0$, delivering to grid) while rotor power becomes positive ($P_r > 0$, absorbing from grid through the converter to maintain excitation). This mode occurs at lower wind speeds, where the turbine operates below synchronous speed while still generating.

Remark 1 (Per-unit power and torque convention). *Because active power and electromagnetic torque are cross-checked against one another throughout Section 4, we fix their per-unit basis explicitly. In the amplitude-invariant dq transformation the three-phase power and torque carry a factor $3/2$; in the per-unit system, this factor is absorbed into the power base S_{base} and the torque base $T_{\text{base}} = S_{\text{base}}/\omega_{\text{base}}$, so that in per unit*

$$p = v_d i_d + v_q i_q, \quad T_e = \psi_{ds} i_{qs} - \psi_{qs} i_{ds}, \quad (1)$$

with no residual numerical factor. Consequently the electrical, mechanical and aerodynamic powers all live on one basis, and the air-gap power, the mechanical power and the terminal power satisfy $p_{\text{ag}} = \omega_s T_e$, $p_{\text{mech}} = \omega_r T_e$, and $p_{\text{mech}} = |p_e| + (\text{losses})$ without any scale correction. The factor $3/2$ written in the dimensional expressions (51), (53) and (54) is exactly this transformation factor and is removed by the normalization; all per-unit values reported in the case studies are computed from (1). This convention is what makes the power balance of Section 5 close on a single basis.

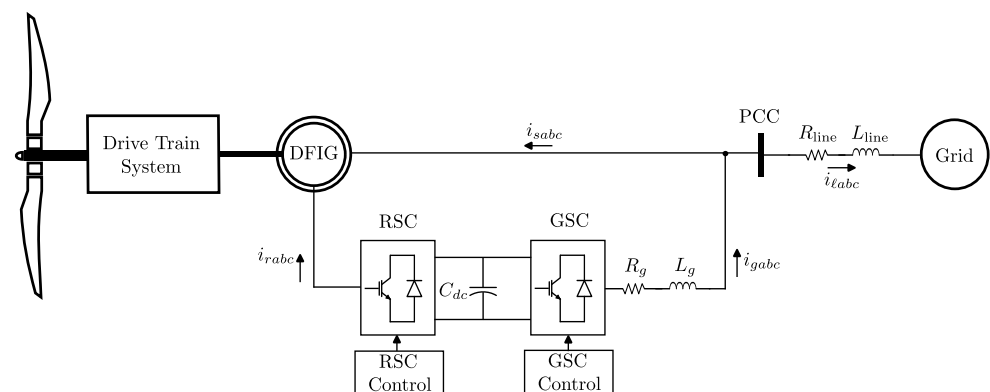


Figure 1. Complete DFIG wind energy conversion system architecture with integrated subsystems and control structure.

2.1. Wind Turbine Aerodynamics

2.1.1. Power Extraction Fundamentals

The wind turbine extracts kinetic energy from the air stream flowing through the rotor swept area. According to actuator disk theory, the maximum theoretical power available in the wind is:

$$P_{\text{wind}} = \frac{1}{2} \rho A v_w^3 \quad (2)$$

where ρ is the air density (standard value 1.225 kg/m^3 at sea level), $A = \pi R^2$ is the rotor swept area, and v_w is the undisturbed wind speed upstream of the rotor. The Betz limit establishes that a maximum of $16/27 \approx 59.3\%$ of this power can theoretically be extracted by an ideal turbine. The actual power extracted depends on the turbine's aerodynamic efficiency, characterized by the power coefficient:

$$P_m = \frac{1}{2} \rho A v_w^3 C_p(\lambda, \beta) \quad (3)$$

where C_p is a dimensionless function of the tip-speed ratio λ and blade pitch angle β .

2.1.2. Tip-Speed Ratio

The tip-speed ratio quantifies the relationship between the rotational speed of the blade tips and the wind speed:

$$\lambda = \frac{\omega_t R}{v_w} \quad (4)$$

where ω_t is the turbine angular velocity and R is the blade radius. In the per-unit system, this becomes:

$$\lambda = \frac{\omega_{t,\text{pu}} \cdot \omega_{m,\text{base}} \cdot R}{v_w} = K_b \frac{\omega_{t,\text{pu}}}{v_w} \quad (5)$$

where K_b is a scaling constant. For numerical convenience, this is normalized to yield tip-speed ratios in the typical operating range of 6–12. The tip-speed ratio is the primary parameter governing turbine aerodynamic efficiency. Each wind speed has an optimal λ_{opt} that maximizes C_p . Below rated wind speed, the turbine operates at variable speed to track this optimum, implementing maximum power point tracking (MPPT). Above rated wind speed, pitch control limits power by increasing β , which reduces C_p .

2.1.3. Power Coefficient Representation

For the GE 3.6 MW turbine, the power coefficient surface $C_p(\lambda, \beta)$ is represented using a two-dimensional polynomial expansion:

$$C_p(\lambda, \beta) = \sum_{i=0}^4 \sum_{j=0}^4 a_{ij} \beta^i \lambda^j \quad (6)$$

This 25-coefficient representation (5×5 matrix) captures the complex aerodynamic behavior over the operating envelope. The coefficients a_{ij} are determined through least-squares fitting to manufacturer-provided performance data or computational fluid dynamics (CFD) simulations. The polynomial is valid for $0^\circ \leq \beta \leq 30^\circ$ and $3 \leq \lambda \leq 15$, covering the turbine's operational range. Physical constraints are enforced on C_p :

$$C_p(\lambda, \beta) = \max\left(0, \min\left(C_p^{\text{poly}}(\lambda, \beta), 0.5\right)\right) \quad (7)$$

preventing unphysical negative values and limiting the maximum to a practical achievable value of 0.5 (below the theoretical Betz limit of 0.593). The optimal power coefficient occurs at $\beta = 0^\circ$ (feathered blades) and $\lambda = \lambda_{\text{opt}} \approx 7.55$, yielding $C_{p,\text{max}} \approx 0.50$ for this turbine

design (the fitted polynomial peaks marginally above the extraction cap of (7) near the optimum, so C_p saturates at 0.50 over a narrow plateau). Figure 2 illustrates the wind turbine aerodynamic performance and MPPT tracking quality. Figure 2a shows mechanical power versus wind speed for rotor speeds from 0.7 to 1.2 pu, delineating Region 2 (MPPT operation, 3–11 m/s) where power follows the cubic relationship $P_m \propto v_w^3$ at constant C_p , and Region 3 (power limiting, >11 m/s) where pitch control activates to regulate power. The rated operating point (red star) at 11 m/s and 1.2 pu produces 3.46 MW, representing 50% extraction efficiency from available wind power. Figure 2b quantifies MPPT precision across the operating envelope. At rated conditions, the achieved tip-speed ratio $\lambda = 7.582$ deviates only 0.49% from optimal ($\lambda_{\text{opt}} = 7.545$), enabling $C_p = 0.500$ (at the achievable maximum). This close tracking is a favourable operating point rather than a claim of superiority over other controllers; its significance is best expressed through the local sensitivity of C_p to tip-speed ratio, which we quantify next.

Sensitivity of the power coefficient. Because MPPT holds the turbine near the crest of $C_p(\lambda, 0)$, the relevant quantity is how sharply C_p falls as λ departs from λ_{opt} . At the optimum the first derivative vanishes by definition, $\partial C_p / \partial \lambda|_{\lambda_{\text{opt}}} = 0$, so the leading behavior is quadratic,

$$C_p(\lambda, 0) \approx C_{p,\text{max}} \left[1 - \left(\frac{\lambda - \lambda_{\text{opt}}}{w} \right)^2 \right], \quad E_\lambda := \frac{\lambda}{C_p} \frac{\partial C_p}{\partial \lambda} \approx - \frac{2\lambda_{\text{opt}}(\lambda - \lambda_{\text{opt}})}{w^2}, \quad (8)$$

where w is the half-width at which C_p has fallen appreciably from its peak. For the present surface $w \approx \lambda_{\text{opt}}$, so the dimensionless elasticity E_λ is zero at the optimum and grows only linearly with the fractional detuning $\delta = (\lambda - \lambda_{\text{opt}})/\lambda_{\text{opt}}$: a 10% tip-speed error ($\delta = 0.1$) costs on the order of $\delta^2 \approx 1\%$ of $C_{p,\text{max}}$, while the 0.49% error at the rated point costs of order 10^{-4} and is therefore negligible. The flatness of the optimum—a curvature $\partial^2 C_p / \partial \lambda^2 < 0$ of modest magnitude—is precisely what makes variable-speed MPPT robust to tip-speed-ratio errors and to the smoothing effect of drivetrain inertia, and it is the aerodynamic origin of the small speed-tracking errors reported in Section 4.

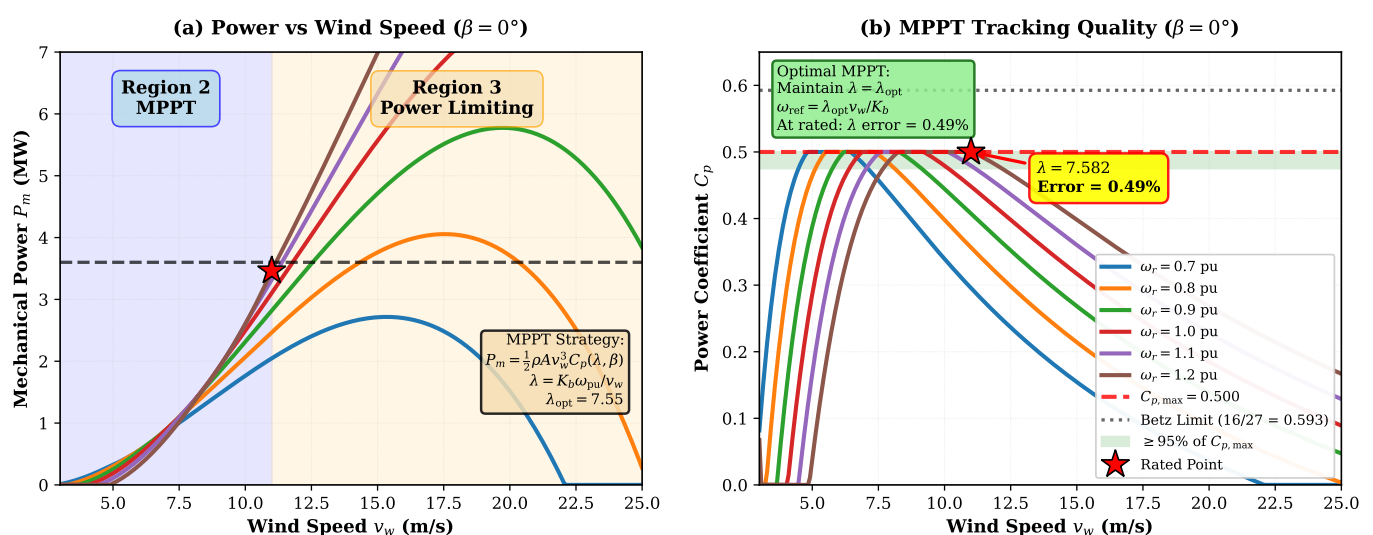


Figure 2. Wind turbine aerodynamic performance and MPPT tracking quality. (a) Mechanical power versus wind speed for different rotor speeds, showing Region 2 (MPPT) and Region 3 (power limiting) operation. (b) Power coefficient C_p tracking quality demonstrating 0.49% tip-speed ratio error at rated conditions.

2.1.4. Mechanical Torque

The mechanical torque applied to the drivetrain is derived from the power-speed relationship:

$$T_m = \frac{P_m}{\omega_t} \quad (9)$$

Substituting (3):

$$T_m = \frac{1}{2\omega_t} \rho A v_w^3 C_p(\lambda, \beta) \quad (10)$$

In per-unit form, with appropriate normalization by base power and speed:

$$T_{m,pu} = \frac{P_{m,pu}}{\omega_{t,pu}} = \frac{1}{\omega_{t,pu} \cdot S_{base}} \cdot \frac{1}{2} \rho A v_w^3 C_p(\lambda, \beta) \quad (11)$$

A singularity avoidance mechanism is implemented for very low speeds ($\omega_{t,pu} < 0.01$) by setting $T_m = 0$, representing a parked or nearly stopped turbine.

2.1.5. Operating Regions

Wind turbine operation is classified into three regions based on wind speed, as illustrated in Figure 3. Region I ($v_w < 3$ m/s) represents parked operation where insufficient wind energy is available for power generation [23]. Region II ($3 \leq v_w \leq 11$ m/s) employs variable-speed MPPT control with pitch angle maintained at $\beta = 0^\circ$ to maximize aerodynamic efficiency [24]. As shown in Figure 3b, the DFIG rotor speed varies from 0.7 pu to 1.3 pu of synchronous speed (slip range ± 0.3) to track the optimal tip-speed ratio, with electrical power output increasing approximately as the cube of wind speed (Figure 3a) [3,10]. Region III ($v_w > 11$ m/s) maintains constant rated power through active pitch control, with rotor speed held at its rated value of 1.2 pu (slip -0.2 , within the ± 0.3 capability) and pitch angle progressively increased to shed excess aerodynamic power (Figure 3c) [24]. The pitch controller activates when rotor speed or electrical power approaches rated values, coordinating the transition between operating regions [25,26].

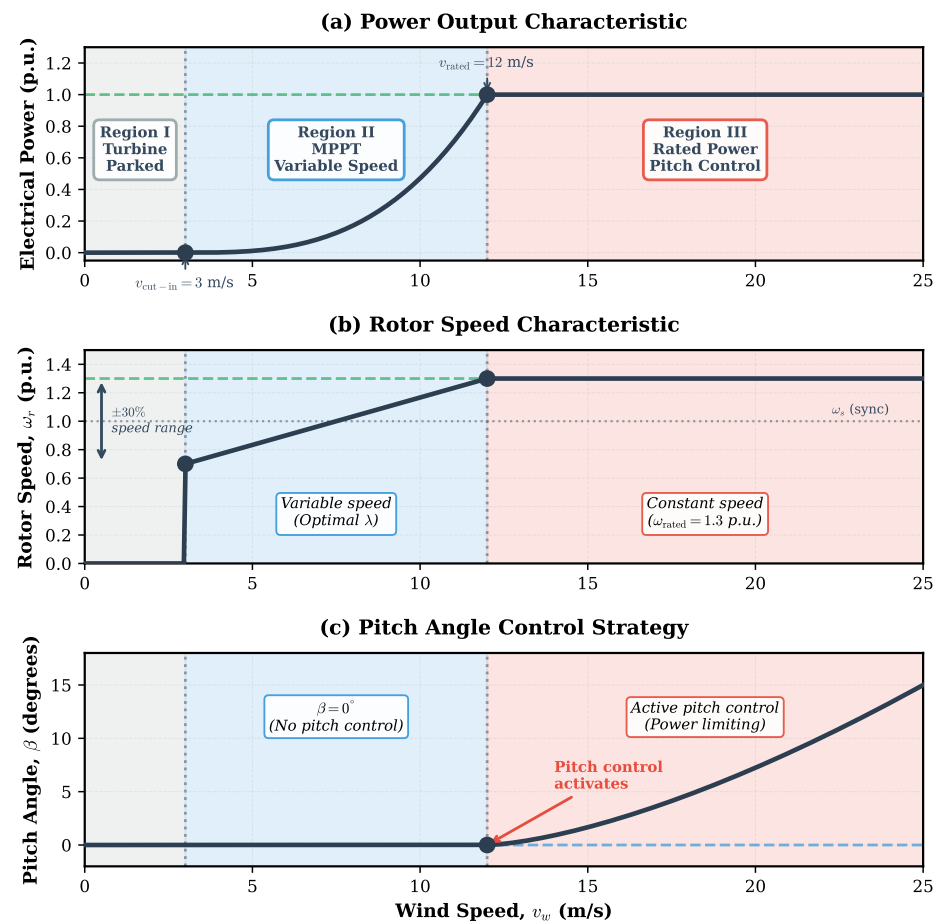


Figure 3. Wind turbine operating regions: (a) power output, (b) rotor speed, and (c) pitch angle characteristics versus wind speed. Region I ($v_w < 3$ m/s): turbine parked. Region II ($3 \leq v_w \leq 11$ m/s): MPPT operation with $\beta = 0^\circ$ and variable speed (0.7–1.3 pu, slip range ± 0.3). Region III ($v_w > 11$ m/s): constant rated power with active pitch control.

2.2. Mechanical Drivetrain Dynamics

2.2.1. Two-Mass Model

The mechanical coupling between the wind turbine rotor and the electrical generator is represented by a two-mass system connected through a compliant shaft. This model captures the dominant torsional oscillation mode while maintaining computational simplicity compared to multi-mass representations. The system consists of the turbine mass (blades, hub, and low-speed shaft with inertia constant H_t), the generator mass (generator rotor and high-speed shaft with inertia constant H_g), and the coupling shaft (torsional stiffness K_{tg} and damping D_{tg}). The inertia constants are defined in the standard power system convention [27]:

$$H = \frac{J\omega_{base}^2}{2S_{base}} \quad (12)$$

where J is the moment of inertia in $\text{kg}\cdot\text{m}^2$. Larger H values indicate slower response to torque imbalances.

2.2.2. Governing Equations

The dynamic equations for the two rotating masses are derived from Newton's second law applied to rotational motion:

$$J_t \frac{d\omega_t}{dt} = T_m - T_{tg} - D_t \omega_t \quad (13)$$

$$J_g \frac{d\omega_r}{dt} = T_{tg} - T_e - D_g \omega_r \quad (14)$$

where T_{tg} is the internal shaft torque transmitted between the masses, D_t and D_g represent damping (friction, windage), and T_e is the electromagnetic torque from the generator. Converting to the per-unit system and utilizing the relationship between inertia and the inertia constant:

$$\frac{d\omega_t}{dt} = \frac{1}{2H_t} (T_m - T_{tg} - D_t \omega_t) \quad (15)$$

$$\frac{d\omega_r}{dt} = \frac{1}{2H_g} (T_{tg} - T_e - D_g \omega_r) \quad (16)$$

2.2.3. Shaft Compliance

The shaft torque evolves according to a spring-damper model relating the angular displacement between the two masses:

$$T_{tg} = K_{tg}(\theta_t - \theta_r) + D_{tg}(\omega_t - \omega_r) \quad (17)$$

Taking the time derivative and noting that $d\theta/dt = \omega$:

$$\frac{dT_{tg}}{dt} = K_{tg}(\omega_t - \omega_r) + D_{tg} \left(\frac{d\omega_t}{dt} - \frac{d\omega_r}{dt} \right) \quad (18)$$

Substituting (15) and (16) into the damping term creates an implicit relationship. To obtain an explicit state-space form, an algebraic manipulation yields:

$$\frac{dT_{tg}}{dt} = \frac{1}{D'} \left[K_{tg}(\omega_t - \omega_r) - D_{tg} \left(\frac{T_m}{2H_t} + \frac{T_e}{2H_g} \right) \right] \quad (19)$$

where the effective damping coefficient is defined as

$$D' = 1 + D_{tg} \left(\frac{1}{2H_t} + \frac{1}{2H_g} \right) \quad (20)$$

For typical parameters where the damping term in the denominator is small ($D_{tg} \ll 1$), this simplifies to:

$$\frac{dT_{tg}}{dt} \approx K_{tg}(\omega_t - \omega_r) \quad (21)$$

2.2.4. Natural Oscillation Frequency

The undamped natural frequency of the two-mass system is:

$$\omega_n = \sqrt{K_{tg} \left(\frac{1}{2H_t} + \frac{1}{2H_g} \right)} \quad (22)$$

For systems where the turbine inertia dominates ($H_t \gg H_g$), this reduces to:

$$\omega_n \approx \sqrt{\frac{K_{tg}}{2H_t}} \quad (23)$$

This frequency, typically in the range 0.5–3 Hz for multi-MW wind turbines, represents a critical resonance that control systems must avoid exciting. The shaft damping coefficient D_{tg} provides damping ratio:

$$\zeta = \frac{D_{tg}}{2\sqrt{K_{tg}(H_t + H_g)}} \quad (24)$$

Values of $D_{tg} = 0.5$ pu typically yield light damping ($\zeta \approx 0.05$), sufficient to prevent sustained oscillations while preserving dynamic response fidelity.

2.3. DFIG Electrical Model

The doubly-fed induction generator is modeled using the classical fourth-order wound-rotor induction machine equations [27]. Unlike reduced-order models that assume stiff grid voltage and neglect stator transients [25], the full fourth-order model retains both stator and rotor flux dynamics, which is essential for accurate representation of transient electromagnetic torque during disturbances and capturing electromechanical oscillations in weak grid conditions [11,12]. The four state variables are the stator and rotor currents in the dq frame: i_{ds} , i_{qs} , i_{dr} , and i_{qr} .

2.3.1. Stator-Flux Oriented Reference Frame

The model is formulated in a reference frame where the d -axis is aligned with the stator flux linkage vector. In this orientation:

$$\lambda_{ds} = \lambda_s, \quad \lambda_{qs} = 0 \quad (25)$$

This orientation provides several advantages: power decoupling ($P_s \propto i_{qs}$, $Q_s \propto i_{ds}$), torque simplification ($T_e \propto i_{qr}$ when $i_{ms} \approx \text{const}$), voltage reduction to $v_{ds} \approx 0$ and $v_{qs} \approx V_{\text{grid}}$ under balanced conditions, and natural coordinates for cascaded control design. The frame rotates at synchronous speed $\omega_s = 1.0$ pu, and is the standard orientation adopted across the DFIG modeling and identification literature [28].

2.3.2. Flux Linkage Relationships

The magnetic coupling between stator and rotor windings is described by:

$$\lambda_{ds} = L_s i_{ds} + L_m i_{dr} \quad (26)$$

$$\lambda_{qs} = L_s i_{qs} + L_m i_{qr} \quad (27)$$

$$\lambda_{dr} = L_m i_{ds} + L_r i_{dr} \quad (28)$$

$$\lambda_{qr} = L_m i_{qs} + L_r i_{qr} \quad (29)$$

where $L_s = L_{ls} + L_m$ is the total stator self-inductance, $L_r = L_{lr} + L_m$ is the total rotor self-inductance, and L_m is the magnetizing inductance representing mutual coupling. The leakage inductances L_{ls} and L_{lr} account for flux that does not link both windings. The leakage coefficient:

$$\sigma = 1 - \frac{L_m^2}{L_s L_r} = \frac{L_s L_r - L_m^2}{L_s L_r} \quad (30)$$

characterizes the magnetic coupling strength. Smaller σ indicates tighter coupling and faster transient response.

2.3.3. Voltage Equations

The stator and rotor voltage equations in the synchronously rotating dq frame are derived from Faraday's law with terms accounting for reference frame rotation. In per-unit form with time normalized by the base frequency $\omega_b = 2\pi f_b$:

$$v_{ds} = r_s i_{ds} - \omega_s \lambda_{qs} + \frac{1}{\omega_b} \frac{d\lambda_{ds}}{dt} \quad (31)$$

$$v_{qs} = r_s i_{qs} + \omega_s \lambda_{ds} + \frac{1}{\omega_b} \frac{d\lambda_{qs}}{dt} \quad (32)$$

$$v_{dr} = r_r i_{dr} - \omega_{sl} \lambda_{qr} + \frac{1}{\omega_b} \frac{d\lambda_{dr}}{dt} \quad (33)$$

$$v_{qr} = r_r i_{qr} + \omega_{sl} \lambda_{dr} + \frac{1}{\omega_b} \frac{d\lambda_{qr}}{dt} \quad (34)$$

where ω_s is the synchronous speed (typically 1.0 pu), $\omega_{sl} = \omega_s - \omega_r$ is the slip frequency, and r_s, r_r represent winding resistances.

2.3.4. Current Dynamic Equations

Solving Equations (31)–(34) for flux derivatives yields:

$$\frac{d\lambda_{ds}}{dt} = \omega_b (v_{ds} - r_s i_{ds} + \omega_s \lambda_{qs}) \quad (35)$$

$$\frac{d\lambda_{qs}}{dt} = \omega_b (v_{qs} - r_s i_{qs} - \omega_s \lambda_{ds}) \quad (36)$$

$$\frac{d\lambda_{dr}}{dt} = \omega_b (v_{dr} - r_r i_{dr} + \omega_{sl} \lambda_{qr}) \quad (37)$$

$$\frac{d\lambda_{qr}}{dt} = \omega_b (v_{qr} - r_r i_{qr} - \omega_{sl} \lambda_{dr}) \quad (38)$$

To obtain current derivatives, we invert the flux–current relationships (26)–(29). The 2×2 block-diagonal structure allows analytical inversion using the determinant:

$$\Delta = L_s L_r - L_m^2 = \sigma L_s L_r \quad (39)$$

which is always positive for physically realizable machines. The intermediate steps are as follows. Collecting the d -axis pair (26) and (28) (the q -axis pair is identical in structure) gives the linear system

$$\begin{bmatrix} \lambda_{ds} \\ \lambda_{dr} \end{bmatrix} = \underbrace{\begin{bmatrix} L_s & L_m \\ L_m & L_r \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} i_{ds} \\ i_{dr} \end{bmatrix}, \quad \det \mathbf{M} = L_s L_r - L_m^2 = \Delta > 0. \quad (40)$$

Since $\det \mathbf{M} = \Delta > 0$ the matrix is nonsingular, and Cramer's rule (or the closed-form inverse of a 2×2 matrix) yields

$$\mathbf{M}^{-1} = \frac{1}{\Delta} \begin{bmatrix} L_r & -L_m \\ -L_m & L_s \end{bmatrix}, \quad \begin{bmatrix} i_{ds} \\ i_{dr} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} L_r \lambda_{ds} - L_m \lambda_{dr} \\ -L_m \lambda_{ds} + L_s \lambda_{dr} \end{bmatrix}. \quad (41)$$

Because $\mathbf{M}$ is constant, differentiating (41) in time commutes with the inverse and replaces each flux by its derivative, giving directly the compact form below. The compact matrix form of the current derivatives is:

$$\frac{di_{ds}}{dt} = \frac{1}{\Delta} \left(L_r \frac{d\lambda_{ds}}{dt} - L_m \frac{d\lambda_{dr}}{dt} \right) \quad (42)$$

$$\frac{di_{qs}}{dt} = \frac{1}{\Delta} \left(L_r \frac{d\lambda_{qs}}{dt} - L_m \frac{d\lambda_{qr}}{dt} \right) \quad (43)$$

$$\frac{di_{dr}}{dt} = \frac{1}{\Delta} \left(L_s \frac{d\lambda_{dr}}{dt} - L_m \frac{d\lambda_{ds}}{dt} \right) \quad (44)$$

$$\frac{di_{qr}}{dt} = \frac{1}{\Delta} \left(L_s \frac{d\lambda_{qr}}{dt} - L_m \frac{d\lambda_{qs}}{dt} \right) \quad (45)$$

Substituting the flux derivatives (35)–(38) and flux linkages (26)–(29) into (42)–(44) yields the fully expanded current derivatives. Define:

$$D = \frac{\omega_b}{L_m^2 - L_s L_r} = -\frac{\omega_b}{\Delta} \quad (46)$$

Then the complete state-space equations are:

$$\begin{aligned} \frac{di_{ds}}{dt} = D \left[r_s L_r i_{ds} + \left(\frac{\omega_b \omega_s}{D} - L_m^2 \omega_r \right) i_{qs} - r_r L_m i_{dr} \right. \\ \left. - \omega_r L_r L_m i_{qr} - L_r v_{ds} + L_m v_{dr} \right] \end{aligned} \quad (47)$$

$$\begin{aligned} \frac{di_{qs}}{dt} = D \left[\left(L_m^2 \omega_r - \frac{\omega_b \omega_s}{D} \right) i_{ds} + r_s L_r i_{qs} + \omega_r L_r L_m i_{dr} \right. \\ \left. - r_r L_m i_{qr} - L_r v_{qs} + L_m v_{qr} \right] \end{aligned} \quad (48)$$

$$\begin{aligned} \frac{di_{dr}}{dt} = D \left[-r_s L_m i_{ds} + \omega_r L_s L_m i_{qs} + r_r L_s i_{dr} \right. \\ \left. + \left(L_s L_r \omega_r + \frac{\omega_b \omega_s}{D} \right) i_{qr} + L_m v_{ds} - L_s v_{dr} \right] \end{aligned} \quad (49)$$

$$\begin{aligned} \frac{di_{qr}}{dt} = D \left[-\omega_r L_s L_m i_{ds} - r_s L_m i_{qs} - \left(\frac{\omega_b \omega_s}{D} + \omega_r L_s L_r \right) i_{dr} \right. \\ \left. + r_r L_s i_{qr} + L_m v_{qs} - L_s v_{qr} \right] \end{aligned} \quad (50)$$

These Equations (47)–(50) explicitly show all electromagnetic couplings: resistance terms, cross-coupling between d and q axes through ω_s and ω_r , stator-rotor coupling through L_m , and voltage forcing terms.

2.3.5. Electromagnetic Torque

The electromagnetic torque coupling the electrical and mechanical subsystems is:

$$T_e = \frac{3}{2} L_m (i_{qs} i_{dr} - i_{ds} i_{qr}) \quad (51)$$

where the factor $3/2$ accounts for the three-phase to two-axis transformation. In the stator-flux frame with unity power factor operation ($i_{ds} \approx 0$):

$$T_e \approx \frac{3}{2} L_m i_{qs} i_{dr} \quad (52)$$

revealing that torque is directly proportional to the product of stator q -axis current and rotor d -axis current. With the magnetizing current i_{ms} held approximately constant by the reactive power controller, the equivalent expression $T_e \approx \frac{3}{2} \frac{L_m^2}{L_s} i_{ms} i_{qr}$ identifies i_{qr} as the primary torque control variable, consistent with the RSC speed controller design.

2.3.6. Power Flow Equations

The three-phase active and reactive powers at the stator and rotor terminals are:

$$P_s = \frac{3}{2}(v_{ds}i_{ds} + v_{qs}i_{qs}), \quad Q_s = \frac{3}{2}(v_{qs}i_{ds} - v_{ds}i_{qs}) \quad (53)$$

$$P_r = \frac{3}{2}(v_{dr}i_{dr} + v_{qr}i_{qr}), \quad Q_r = \frac{3}{2}(v_{qr}i_{dr} - v_{dr}i_{qr}) \quad (54)$$

In the stator-flux frame ($v_{ds} \approx 0$, $v_{qs} \approx V_{\text{grid}}$), the stator powers simplify to $P_s \approx \frac{3}{2}V_{\text{grid}}i_{qs}$ and $Q_s \approx \frac{3}{2}V_{\text{grid}}i_{ds}$, providing natural power decoupling. The total electrical power is $P_e = P_s + P_r$. Following motor convention, $P_e < 0$ indicates generation. For super-synchronous operation, both $P_s < 0$ (stator delivers to grid) and, typically, $P_r < 0$ (the rotor delivers through converters).

2.4. Transmission Line and Grid Interface

2.4.1. RL Line Model in dq Coordinates

The transmission line connecting the DFIG to the grid is represented as a series RL impedance. While single-phase analysis uses phasors, the time-domain simulation in dq coordinates requires explicit representation of the line dynamics. Applying Kirchhoff's voltage law around the loop from grid to PCC:

$$V_{\text{PCC},d} = V_{\text{grid},d} + R_{\text{line}}i_{\ell,d} - \omega_s L_{\text{line}}i_{\ell,q} + \frac{L_{\text{line}}}{\omega_b} \frac{di_{\ell,d}}{dt} \quad (55)$$

$$V_{\text{PCC},q} = V_{\text{grid},q} + R_{\text{line}}i_{\ell,q} + \omega_s L_{\text{line}}i_{\ell,d} + \frac{L_{\text{line}}}{\omega_b} \frac{di_{\ell,q}}{dt} \quad (56)$$

Solving for the line current dynamics:

$$\frac{di_{\ell,d}}{dt} = \frac{\omega_b}{L_{\text{line}}} \left(V_{\text{PCC},d} - V_{\text{grid},d} - R_{\text{line}}i_{\ell,d} + \omega_s L_{\text{line}}i_{\ell,q} \right) \quad (57)$$

$$\frac{di_{\ell,q}}{dt} = \frac{\omega_b}{L_{\text{line}}} \left(V_{\text{PCC},q} - V_{\text{grid},q} - R_{\text{line}}i_{\ell,q} - \omega_s L_{\text{line}}i_{\ell,d} \right) \quad (58)$$

2.4.2. Current Balance at PCC

The voltage at the PCC is determined by the current balance at this node. Applying Kirchhoff's current law at the PCC node:

$$i_{\ell,dq} = i_{s,dq} + i_{g,dq} + i_{c,dq} \quad (59)$$

where $i_{s,dq}$ represents the stator currents from the DFIG, $i_{g,dq}$ denotes the grid-side converter currents, and $i_{c,dq}$ are the shunt compensation currents when present. This algebraic constraint couples the DFIG electrical states with the line dynamics, creating a differential-algebraic system that requires careful numerical handling.

2.4.3. Well-Posedness and Differential-Algebraic Reduction

Two regularity questions must be settled before the interconnection (64) can be called a dynamical system: whether the constitutive (flux-current) map is invertible so that the right-hand side exists, and whether the algebraic constraint (59) can be eliminated to yield an explicit ordinary differential equation. We record the standing physical assumption and then answer both.

Assumption 1 (Admissible machine). *The inductances satisfy $L_s, L_r, L_m > 0$ with $L_s L_r > L_m^2$ (equivalently the leakage coefficient $\sigma \in (0, 1]$), the winding resistances $r_s, r_r > 0$, and the operating*

domain $\mathcal{D} \subset \mathbb{R}^{26}$ excludes machine standstill, $\omega_t, \omega_r \geq \omega_{\min} > 0$, and keeps the DC-link and PCC voltages bounded away from zero.

Proposition 1 (Well-posed vector field). *Under Assumption 1, the flux–current map $\mathbf{i} \mapsto \lambda$ defined by (26)–(29) is a linear diffeomorphism of $\mathbb{R}^4$, with block inverse (41), and the assembled right-hand side $\mathbf{f}$ in (64) is well defined and C^∞ on $\mathcal{D}$.*

Proof. The map has the block-diagonal matrix $\text{diag}(\mathbf{M}, \mathbf{M})$ of (40), and $\det \mathbf{M} = L_s L_r - L_m^2 = \Delta = \sigma L_s L_r > 0$ by Assumption 1; hence the matrix is invertible with the stated inverse, and being linear it is a global diffeomorphism. Every remaining constitutive relation in $\mathbf{f}$ —the polynomial $C_p(\lambda, \beta)$, the bilinear torque (51), the divisions by ω_t , V_{dc} and v_{sm} in the torque and controller maps, and the reference-frame rotations—is a composition of polynomials, smooth trigonometric maps, and divisions by quantities bounded away from zero on $\mathcal{D}$. A finite composition of C^∞ maps is C^∞ , so $\mathbf{f} \in C^\infty(\mathcal{D})$. $\square$

The nodal constraint (59) makes (64) a semi-explicit differential-algebraic equation (DAE). Writing the differential states as $\mathbf{x}$ and the PCC voltage as the algebraic variable $\mathbf{y} = (V_{\text{PCC},d}, V_{\text{PCC},q})$, the system has the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{u}), \quad \mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{y}) := \mathbf{i}_\ell(\mathbf{x}, \mathbf{y}) - \mathbf{i}_s(\mathbf{x}, \mathbf{y}) - \mathbf{i}_g(\mathbf{x}, \mathbf{y}) - \mathbf{i}_c(\mathbf{x}, \mathbf{y}), \quad (60)$$

where $\mathbf{g} = \mathbf{0}$ is Kirchhoff's current law (59) at the PCC.

Proposition 2 (Index-one reduction). *Let the aggregate small-signal admittance seen from the PCC, $\mathbf{Y}_{\text{PCC}}(\mathbf{x}) := \partial \mathbf{g} / \partial \mathbf{y}$, be nonsingular on $\mathcal{D}$. Then (60) has differentiation index one, the constraint defines a unique C^∞ solution map $\mathbf{y} = \mathbf{h}(\mathbf{x})$, and substituting it yields the explicit ordinary differential equation*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x}), \mathbf{u}) =: \mathbf{F}(\mathbf{x}, \mathbf{u}) \quad (61)$$

on the constraint manifold $\{\mathbf{g} = \mathbf{0}\}$.

Proof. The stator, converter and line currents are affine in the nodal voltage through the series impedances and the controller feedforward terms, so $\mathbf{g}$ is affine in $\mathbf{y}$ with Jacobian $\mathbf{Y}_{\text{PCC}}(\mathbf{x})$. A single differentiation of $\mathbf{g} = \mathbf{0}$ produces $\mathbf{Y}_{\text{PCC}} \dot{\mathbf{y}} = -(\partial \mathbf{g} / \partial \mathbf{x}) \dot{\mathbf{x}}$, which is solvable for $\dot{\mathbf{y}}$ precisely because $\mathbf{Y}_{\text{PCC}}$ is nonsingular; hence the index is one. Nonsingularity of $\mathbf{Y}_{\text{PCC}}$ is exactly the hypothesis of the implicit function theorem applied to $\mathbf{g}(\mathbf{x}, \cdot) = \mathbf{0}$, which furnishes the unique smooth solution map $\mathbf{y} = \mathbf{h}(\mathbf{x})$; substitution gives (61). $\square$

Physically, $\mathbf{Y}_{\text{PCC}}$ is nonsingular whenever the PCC is not exactly at a series or parallel resonance of the aggregate line-plus-converter network, which holds throughout the operating range considered here. When a physical shunt element (line charging or a filter capacitor of susceptance B_c) is retained at the PCC, the algebraic variable $\mathbf{y}$ becomes a fast differential state and (61) is recovered in the singular limit $B_c \rightarrow 0$ as the slow manifold of the resulting two-time-scale system, connecting the reduction to the multiple-time-scale structure of Section 3.6. In the case studies, the constraint is enforced directly, so the reported trajectories evolve on $\{\mathbf{g} = \mathbf{0}\}$ and the state count is the 26 differential states of (65).

2.4.4. Short-Circuit Ratio and Grid Strength Classification

The grid strength at the PCC is quantified by the short-circuit ratio (SCR), defined as the ratio of the short-circuit capacity at the PCC to the rated power of the connected generator:

$$\text{SCR} = \frac{S_{\text{SC}}}{S_{\text{rated}}} \quad (62)$$

For a predominantly inductive line ($R_{\text{line}} \ll X_{\text{line}}$), this can be expressed as:

$$\text{SCR} \approx \frac{V_{\text{PCC}}^2}{Z_{\text{line}} \cdot S_{\text{rated}}} = \frac{1}{Z_{\text{line,pu}}} \approx \frac{1}{X_{\text{line,pu}}} \quad (63)$$

where all impedances are expressed in per-unit on the generator base. Grid strength is typically classified based on SCR values as follows. A stiff or strong grid ($\text{SCR} > 10$) exhibits minimal voltage variations and stable PLL operation with negligible coupling between the converter and grid dynamics. A moderate grid ($5 < \text{SCR} < 10$) shows noticeable coupling effects and requires careful controller tuning to maintain stability. A weak grid ($3 < \text{SCR} < 5$) exhibits significant coupling between converter and grid dynamics with potential for low-frequency oscillations. Finally, a very weak grid ($\text{SCR} < 3$) presents strong coupling effects and substantial stability challenges, typically requiring specialized control strategies such as modified PLL designs or voltage-following control. Long transmission lines or radial feeders can easily create weak grid conditions.

2.5. Integrated System State-Space Representation

The complete DFIG wind turbine system is represented as a nonlinear state-space model [29]:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \quad (64)$$

The state vector $\mathbf{x}$ comprises 26 elements for the complete system with pitch control (22 without):

$$\mathbf{x} = [\mathbf{x}_{\text{mech}} \quad \mathbf{x}_{\text{DFIG}} \quad \mathbf{x}_{\text{RSC}} \quad \mathbf{x}_{\text{GSC,curr}} \quad \mathbf{x}_{\text{DC}} \quad \mathbf{x}_{\text{PLL}} \quad \mathbf{x}_{\text{GSC,int}} \quad \mathbf{x}_{\text{filt}} \quad \mathbf{x}_{\text{pitch}} \quad \mathbf{x}_{\text{line}}]^T \quad (65)$$

where the subsystem state vectors are defined as:

$\mathbf{x}_{\text{mech}} = [\omega_t, \omega_r, T_{ig}]^T$	(3 states: mechanical dynamics)
$\mathbf{x}_{\text{DFIG}} = [i_{ds}, i_{qs}, i_{dr}, i_{qr}]^T$	(4 states: DFIG electrical)
$\mathbf{x}_{\text{RSC}} = [\xi_d, \xi_q, \xi_\omega, \xi_Q]^T$	(4 states: RSC integrators)
$\mathbf{x}_{\text{GSC,curr}} = [i_{dg}, i_{qg}]^T$	(2 states: GSC currents)
$\mathbf{x}_{\text{DC}} = [V_{dc}]^T$	(1 state: DC-link voltage)
$\mathbf{x}_{\text{PLL}} = [\theta_{\text{PLL}}, \xi_{\text{PLL}}]^T$	(2 states: PLL)
$\mathbf{x}_{\text{GSC,int}} = [\xi_{id}, \xi_{iq}, \xi_{Vdc}]^T$	(3 states: GSC integrators)
$\mathbf{x}_{\text{filt}} = [P_{r,\text{filt}}]^T$	(1 state: power filter)
$\mathbf{x}_{\text{pitch}} = [\beta, \xi_{\beta,\omega}, \xi_{\beta,P}, \xi_{\beta,r}]^T$	(4 states: pitch controller)
$\mathbf{x}_{\text{line}} = [i_{\ell,d}, i_{\ell,q}]^T$	(2 states: line currents)

The pitch controller states include the current blade pitch angle β and three integrator states for the cascaded control structure: $\xi_{\beta,\omega}$ for speed error integration, $\xi_{\beta,P}$ for power error integration, and $\xi_{\beta,r}$ for pitch rate integration. When pitch control is disabled (for

studies at rated or below-rated wind speeds), the pitch angle becomes an external input rather than a state variable, reducing the system to 22 states. The input vector includes:

$$\mathbf{u} = [v_w, \omega_{\text{ref}}, Q_{s,\text{ref}}, Q_{g,\text{ref}}, V_{\text{grid},d}, V_{\text{grid},q}]^T \quad (66)$$

where v_w is the wind speed, ω_{ref} is the rotor speed reference, $Q_{s,\text{ref}}$ and $Q_{g,\text{ref}}$ are the reactive power references for the stator and grid-side converter, respectively, and $V_{\text{grid},d}$, $V_{\text{grid},q}$ are the grid voltage components. Note that when pitch control is enabled, the pitch angle β is determined internally by the pitch controller and is not an external input.

This formulation provides a complete small-signal stability analysis through linearization around operating points, controller design and validation, parametric sensitivity studies, and weak grid stability assessment. The model's modularity allows selective inclusion of subsystems based on study requirements: the pitch controller can be disabled for below-rated wind speed studies (reducing to 22 states), and the transmission line can be replaced with an algebraic model for stiff grid studies.

3. Control System Design

The DFIG wind turbine system employs a sophisticated hierarchical control architecture comprising four major subsystems: the rotor-side converter (RSC) control for torque and reactive power regulation, the grid-side converter (GSC) control for DC-link voltage maintenance and grid current management, the phase-locked loop (PLL) for grid synchronization, and the pitch angle controller for power limiting.

3.1. Control Architecture Overview and Reference Frames

The control system operates in multiple reference frames optimized for different subsystems. The RSC employs a stator-flux oriented reference frame where the d -axis aligns with the stator flux vector, yielding $\lambda_{qs} = 0$ by definition. In this frame, the stator terminal voltage simplifies to $v_{ds} \approx 0$ and $v_{qs} \approx V_{\text{grid}}$ under balanced grid conditions. This orientation provides natural decoupling between active and reactive power control, with stator reactive power primarily determined by i_{ds} and active power by i_{qs} [15]. The electromagnetic torque for an induction machine in per-unit is given by:

$$T_e = \frac{3}{2}(\lambda_{ds}i_{qs} - \lambda_{qs}i_{ds}) \quad (67)$$

In the stator-flux oriented reference frame where the d -axis aligns with the stator flux vector, $\lambda_{qs} = 0$ by definition. The flux orientation constraint $\lambda_{qs} = L_s i_{qs} + L_m i_{qr} = 0$ yields $i_{qs} = -(L_m/L_s)i_{qr}$. Substituting these relationships:

$$T_e = \frac{3}{2}\lambda_{ds}i_{qs} = \frac{3}{2}\lambda_{ds}\left(-\frac{L_m}{L_s}i_{qr}\right) = -\frac{3}{2}\frac{L_m}{L_s}\lambda_{ds}i_{qr} \quad (68)$$

where the sign convention for generator operation (motor convention) is accounted for. The d -axis stator flux can be expressed in terms of the magnetizing current $i_{ms} = i_{ds} + i_{dr}$. For typical operating conditions where leakage flux is small, $\lambda_{ds} \approx L_m i_{ms}$, yielding:

$$T_e \approx -\frac{3}{2}\frac{L_m^2}{L_s}i_{ms}i_{qr} \quad (69)$$

This form explicitly shows that electromagnetic torque is proportional to the product of magnetization level (through i_{ms}) and rotor q -axis current, making i_{qr} the natural control variable for torque regulation through the speed controller [30]. The magnetizing current is maintained approximately constant through the reactive power controller's regulation of i_{dr} .

The GSC utilizes a grid-voltage oriented reference frame established by the PLL, where the d -axis aligns with the grid voltage vector at the point of common coupling [31]. In this frame, proper synchronization yields $v_{qg} \approx 0$ and $v_{dg} \approx V_{PCC}$ at steady-state. This orientation facilitates independent control of active and reactive power flow through the d - and q -axis current components, respectively [32].

3.2. Rotor-Side Converter Control

3.2.1. Speed Controller-Outer Loop

The speed controller forms the primary path for active power regulation below rated wind speed. It receives a speed reference ω_{ref} typically generated by a maximum power point tracking algorithm and produces a q -axis rotor current reference. The speed error must be converted to electrical radians per second to maintain dimensional consistency in the per-unit system:

$$e_{\omega} = (\omega_{\text{ref}} - \omega_r) \quad (70)$$

A proportional-integral controller processes this error to generate the current reference:

$$i_{qr,\text{ref}} = K_{p,\omega} e_{\omega} + K_{i,\omega} \xi_{\omega} \quad (71)$$

$$\frac{d\xi_{\omega}}{dt} = e_{\omega} \quad (72)$$

3.2.2. Reactive Power Controller-Outer Loop

The reactive power controller operates in parallel with the speed controller, managing stator reactive power by commanding the d -axis rotor current. The controller employs a feedforward-plus-feedback structure to improve steady-state accuracy and transient response. The magnetizing current in the stator-flux frame is:

$$i_{ms} = \frac{v_{qs}}{\omega_s L_m} \quad (73)$$

From the stator reactive power expression $Q_s = \frac{3}{2} v_{qs} i_{ds}$ and the flux linkage relationship in the stator-flux frame $\lambda_{ds} = L_s i_{ds} + L_m i_{dr}$, the d -axis stator current can be expressed as:

$$i_{ds} = \frac{L_m}{L_s} (i_{ms} - i_{dr}) \quad (74)$$

Substituting into the reactive power expression and solving for i_{dr} :

$$i_{dr} = i_{ms} - \frac{2Q_s L_s}{3v_{qs} L_m} \quad (75)$$

Using the steady-state voltage relationship $v_{qs} \approx \omega_s L_m i_{ms}$ (neglecting stator resistance for feedforward design), the feedforward current becomes:

$$i_{dr,\text{ff}} = i_{ms} - \frac{2Q_{s,\text{ref}} L_s}{3\omega_s L_m^2 i_{ms}} \quad (76)$$

This feedforward term provides the approximate steady-state d -axis rotor current required to achieve the desired reactive power, with the feedback term (79) correcting for modeling errors and disturbances. The reactive power error is:

$$e_Q = Q_{s,\text{ref}} - Q_s \quad (77)$$

The feedback term employs a PI controller with a scaling gain to account for the nonlinear relationship:

$$K_{\text{scale}} = -\frac{2L_s}{3\omega_s L_m^2 i_{ms}} \quad (78)$$

$$i_{dr,\text{fb}} = K_{\text{scale}}(K_{p,Q}e_Q + K_{i,Q}\xi_Q) \quad (79)$$

$$\frac{d\xi_Q}{dt} = e_Q \quad (80)$$

The total d -axis current reference is:

$$i_{dr,\text{ref}} = i_{dr,\text{ff}} + i_{dr,\text{fb}} \quad (81)$$

3.2.3. Inner Current Controllers

The inner current controllers regulate rotor dq -axis currents with high bandwidth. The current errors are:

$$e_d = i_{dr,\text{ref}} - i_{dr} \quad (82)$$

$$e_q = i_{qr,\text{ref}} - i_{qr} \quad (83)$$

Each axis employs a PI controller with identical gains for symmetry:

$$v_{dr,1} = K_{p,i}e_d + K_{i,i}\xi_d \quad (84)$$

$$v_{qr,1} = K_{p,i}e_q + K_{i,i}\xi_q \quad (85)$$

$$\frac{d\xi_d}{dt} = e_d \quad (86)$$

$$\frac{d\xi_q}{dt} = e_q \quad (87)$$

To improve dynamic performance and reduce cross-coupling, feedforward compensation terms are added based on the rotor voltage equations. The feedforward voltages compensate for resistive drops and slip-frequency coupling:

$$v_{dr,2} = r_r i_{dr} - \omega_{sl} \sigma L_r i_{qr} \quad (88)$$

$$v_{qr,2} = r_r i_{qr} + \omega_{sl} \left(\sigma L_r i_{dr} + \frac{L_m^2 i_{ms}}{L_s} \right) \quad (89)$$

The total rotor voltage commands are:

$$v_{dr} = v_{dr,1} + v_{dr,2} \quad (90)$$

$$v_{qr} = v_{qr,1} + v_{qr,2} \quad (91)$$

(see Figure 4).

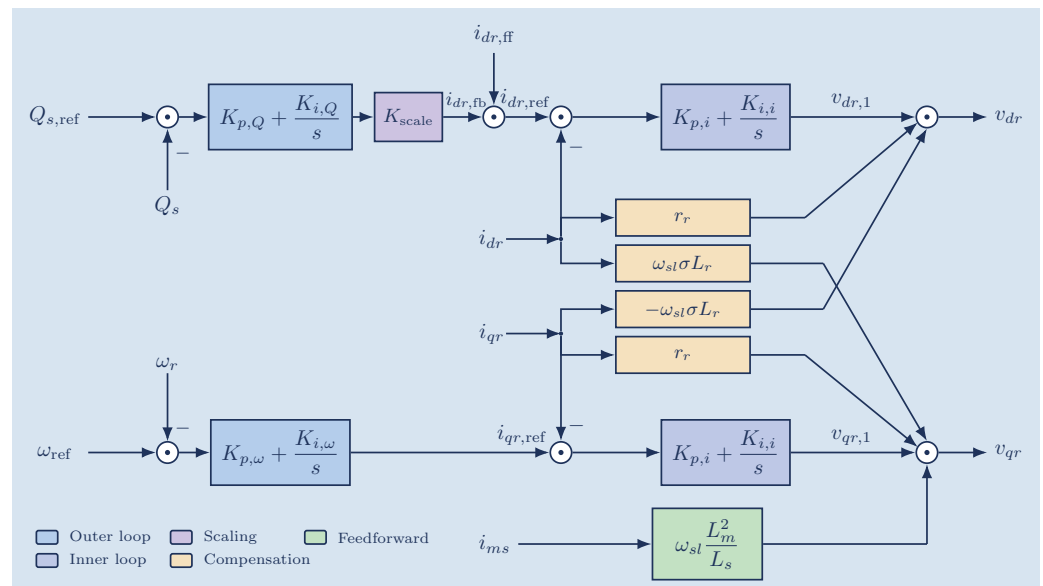


Figure 4. Cascaded control structure of the rotor-side converter. The outer loops regulate reactive power (d-axis) and rotor speed (q-axis), while the inner loops track the corresponding rotor current references.

3.3. Grid-Side Converter Control

3.3.1. DC-Link Voltage Controller

The DC-link voltage controller regulates V_{dc} by commanding the d -axis grid current [33]. The DC-link capacitor stores energy $E = \frac{1}{2}C_{dc}V_{dc}^2$, and its dynamics are governed by power balance. In actual units and actual time:

$$\frac{dE_{\text{actual}}}{dt_{\text{actual}}} = P_{r,\text{actual}} - P_{g,\text{actual}} \quad (92)$$

Converting to per-unit quantities where $E_{\text{pu}} = E_{\text{actual}}/E_{\text{base}}$ with energy base $E_{\text{base}} = P_{\text{base}}/\omega_{\text{base}}$, and differentiating with respect to actual time:

$$E_{\text{base}} \frac{dE_{\text{pu}}}{dt_{\text{actual}}} = P_{\text{base}}(P_{r,\text{pu}} - P_{g,\text{pu}}) \quad (93)$$

Substituting $E_{\text{base}} = P_{\text{base}}/\omega_{\text{base}}$ and simplifying:

$$\frac{dE_{\text{pu}}}{dt_{\text{actual}}} = \omega_{\text{base}}(P_{r,\text{pu}} - P_{g,\text{pu}}) \quad (94)$$

Expressing the per-unit energy as $E_{\text{pu}} = \frac{1}{2}C_{dc}V_{dc}^2$ and taking the time derivative:

$$C_{dc}V_{dc} \frac{dV_{dc}}{dt_{\text{actual}}} = \omega_{\text{base}}(P_r - P_g) \quad (95)$$

yielding the DC-link voltage dynamics in per-unit with time in seconds:

$$\frac{dV_{dc}}{dt} = \frac{\omega_{\text{base}}(P_r - P_g)}{C_{dc}V_{dc}} \quad (96)$$

The rotor power P_r enters the DC-link from the back-to-back converter, and grid power $P_g = \frac{3}{2}(v_{dg}i_{dg} + v_{qg}i_{qg})$ is delivered to the grid. In the PLL-synchronized dq frame where

$v_{qg} \approx 0$, this simplifies to $P_g \approx \frac{3}{2}v_{dg}i_{dg}$. The rotor power is filtered through a first-order low-pass filter to remove switching harmonics and provide smooth feedforward compensation:

$$\frac{dP_{r,\text{filt}}}{dt} = \frac{1}{\tau_f}(P_r - P_{r,\text{filt}}) \quad (97)$$

where τ_f is the filter time constant, selected to attenuate high-frequency components while maintaining adequate bandwidth for power tracking. The voltage controller employs a squared-voltage error formulation to eliminate square root operations and align with the energy-based dynamics:

$$e_{V_{dc}} = V_{dc,\text{ref}}^2 - V_{dc}^2 \quad (98)$$

This formulation is physically motivated: the stored energy is proportional to V_{dc}^2 , making the squared error a natural choice for energy regulation. The controller gains are designed specifically for this squared-error formulation and cannot be directly compared with controllers using linear voltage error. A proportional-integral controller processes the squared voltage error:

$$u_{dc} = K_{p,dc}e_{V_{dc}} + K_{i,dc}\xi_{dc} \quad (99)$$

$$\frac{d\xi_{dc}}{dt} = e_{V_{dc}} \quad (100)$$

The d -axis current reference combines the PI controller output with feedforward compensation from the filtered rotor power:

$$i_{dg,\text{ref}} = -\frac{2(u_{dc} + P_{r,\text{filt}})}{3v_{sm}} \quad (101)$$

where v_{sm} is the grid voltage magnitude. The feedforward term $P_{r,\text{filt}}$ provides instantaneous compensation for rotor power variations, significantly reducing DC voltage deviation during transients and improving disturbance rejection.

3.3.2. Grid Current Controllers

The grid current controllers regulate d - and q -axis currents with high bandwidth to track references generated by the outer DC voltage and reactive power loops. The AC filter dynamics in the PLL-synchronized grid frame are governed by:

$$\frac{di_{dg}}{dt} = \frac{\omega_{\text{base}}}{L_f}(u_{dg} - v_{dg} - R_f i_{dg} + \omega_{\text{pll}} L_f i_{qg}) \quad (102)$$

$$\frac{di_{qg}}{dt} = \frac{\omega_{\text{base}}}{L_f}(u_{qg} - v_{qg} - R_f i_{qg} - \omega_{\text{pll}} L_f i_{dg}) \quad (103)$$

where u_{dg} and u_{qg} are the converter voltage commands, v_{dg} and v_{qg} are the grid voltages measured in the PLL frame, and ω_{pll} is the PLL frequency output. The cross-coupling terms ($\pm\omega_{\text{pll}}L_f i_{dg,qg}$) arise from the rotation of the coordinate frame and must be compensated to achieve independent control of the two axes. The current tracking errors are defined as:

$$e_{dg} = i_{dg,\text{ref}} - i_{dg} \quad (104)$$

$$e_{qg} = i_{qg,\text{ref}} - i_{qg} \quad (105)$$

Proportional-integral controllers process these errors with identical gains for both axes:

$$u_{dg,1} = K_{p,ig}e_{dg} + K_{i,ig}\xi_{dg} \quad (106)$$

$$u_{qg,1} = K_{p,ig}e_{qg} + K_{i,ig}\xi_{qg} \quad (107)$$

$$\frac{d\tilde{\zeta}_{dg}}{dt} = e_{dg} \quad (108)$$

$$\frac{d\tilde{\zeta}_{qg}}{dt} = e_{qg} \quad (109)$$

(see Figure 5).

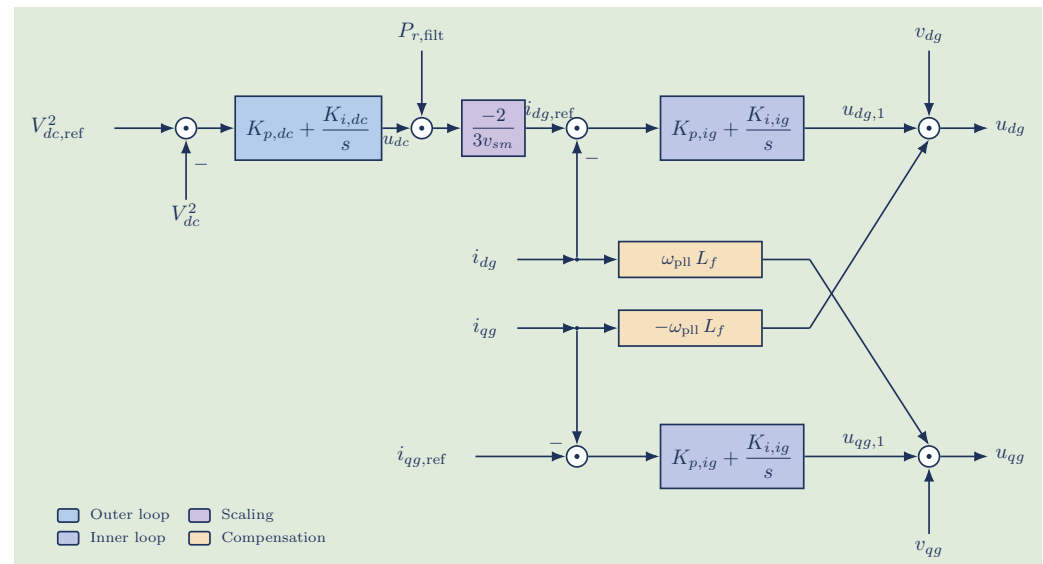


Figure 5. Grid-side converter control with squared DC-link voltage error formulation. Filtered rotor power $P_{r,flt}$ provides feedforward compensation on the d-axis, and the q-axis directly regulates reactive current.

Feedforward compensation terms are added to cancel known disturbances and decouple the two control axes:

$$u_{dg,2} = v_{dg} - \omega_{pll} L_f i_{qg} \quad (110)$$

$$u_{qg,2} = v_{qg} + \omega_{pll} L_f i_{dg} \quad (111)$$

These terms cancel the grid voltage and cross-coupling disturbances, allowing the PI controllers to focus solely on tracking errors. The total control voltages combining feedback and feedforward are:

$$u_{dg} = u_{dg,1} + u_{dg,2} \quad (112)$$

$$u_{qg} = u_{qg,1} + u_{qg,2} \quad (113)$$

Substituting these into (102) and (103) yields the compensated dynamics:

$$\frac{di_{dg}}{dt} = \frac{\omega_{base}}{L_f} (u_{dg,1} - R_f i_{dg}) \quad (114)$$

$$\frac{di_{qg}}{dt} = \frac{\omega_{base}}{L_f} (u_{qg,1} - R_f i_{qg}) \quad (115)$$

Note that the resistive voltage drops ($R_f i_{dg,qg}$) remain as disturbances that the PI controllers must reject. For improved performance, these could be added to the feedforward terms, but the small resistance makes this unnecessary in practice. The current controller bandwidth (50 rad/s) is sufficiently high to provide effective disturbance rejection. The reactive power reference is converted to a q-axis current command through the relationship derived from

three-phase power equations. In the PLL-synchronized frame where $v_{qg} \approx 0$, the reactive power is:

$$Q_g = \frac{3}{2}(v_{qg}i_{dg} - v_{dg}i_{qg}) \approx -\frac{3}{2}v_{dg}i_{qg} \quad (116)$$

Solving for the current reference:

$$i_{qg,\text{ref}} = -\frac{2Q_{g,\text{ref}}}{3v_{sm}} \quad (117)$$

where v_{sm} is the nominal grid voltage magnitude. The negative sign ensures that positive reactive power reference (capacitive support to grid) commands negative i_{qg} under motor convention. The hierarchical control structure maintains $\omega_{cl,i} \gg \omega_{cl,dc}$ with the current loop bandwidth approximately 5 times faster than the DC voltage loop, ensuring the inner loops settle before outer loop adjustments propagate through the system.

3.4. Phase-Locked Loop Synchronization

PLL Structure

The phase-locked loop tracks the phase angle of the PCC voltage to establish the grid-voltage oriented reference frame for GSC control. The three-phase voltages are transformed to the rotating dq frame using the current angle estimate θ_{pll} . The synchronization error is detected through the q -axis voltage component. The PLL frequency output is:

$$\omega_{pll} = \omega_s + K_{p,pll}v_{qg} + K_{i,pll}\tilde{\zeta}_{pll} \quad (118)$$

$$\frac{d\tilde{\zeta}_{pll}}{dt} = v_{qg} \quad (119)$$

The angle is obtained by integration:

$$\frac{d\theta_{pll}}{dt} = \omega_{pll} \quad (120)$$

When locked to the grid, $v_{qg} = 0$ at steady-state, so $\omega_{pll} = \omega_s$ and θ_{pll} tracks the grid voltage angle with zero error.

For stiff grid operation where voltage phase remains nearly constant, high PLL bandwidth provides fast synchronization with minimal phase error. Under weak grid conditions ($\text{SCR} < 5$), high PLL bandwidth can lead to instability through interaction with DFIG dynamics. This phenomenon has been widely reported in recent literature as a major challenge for converter-based generation interfacing to weak grids [34]. The PLL attempts to track voltage phase fluctuations caused by the generator's own current injection through substantial line impedance, creating a positive feedback loop. When the generator injects significant current, the voltage angle at the PCC differs from the infinite bus angle and varies dynamically with current changes, destabilizing the synchronization mechanism [35].

To mitigate this instability, the PLL bandwidth must be reduced for weak grid operation [36]. These reduced gains effectively filter high-frequency voltage angle fluctuations, allowing stable operation despite significant voltage phase variations. The reduced bandwidth means slower synchronization following grid disturbances but provides robust stability under weak grid conditions (see Figure 6).

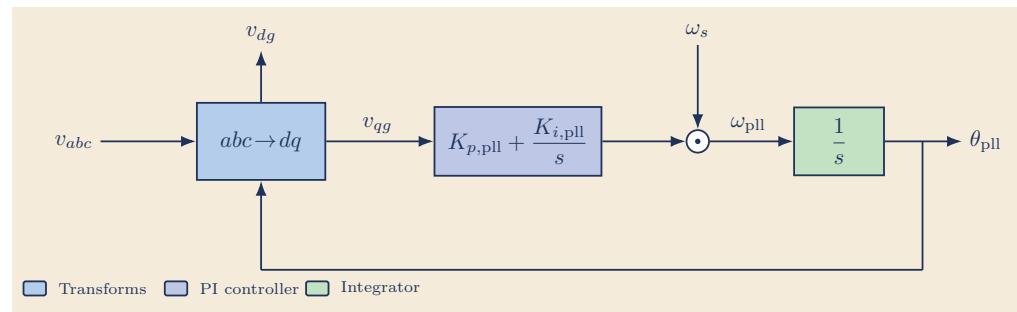


Figure 6. Synchronous reference frame phase-locked loop used to extract the grid voltage angle θ_{pll} .

3.5. Pitch Angle Control

Cascaded Control Structure

The pitch angle controller activates when wind speed exceeds rated conditions, limiting mechanical power and preventing generator over-speeding. While various advanced optimization algorithms and controller architectures have been benchmarked for this subsystem, the cascaded structure employed here uses analytically derived PI gains for each of its three nested loops. The first outer loop computes a pitch command from speed error:

$$\beta_{cmd,1} = K_{p,\beta,\omega}(\omega_r - \omega_{rated}) + K_{i,\beta,\omega}\zeta_{speed} \quad (121)$$

$$\frac{d\zeta_{speed}}{dt} = \omega_r - \omega_{rated} \quad (122)$$

The second outer loop computes a pitch command from power error:

$$\beta_{cmd,2} = K_{p,\beta,P}(P_e - P_{rated}) + K_{i,\beta,P}\zeta_{power} \quad (123)$$

$$\frac{d\zeta_{power}}{dt} = P_e - P_{rated} \quad (124)$$

where $P_e = P_s + P_r$ is the total electrical power and $P_{rated} = 1.0$ pu. The composite pitch angle command is:

$$\beta_{cmd} = \beta_{cmd,1} + \beta_{cmd,2} \quad (125)$$

The third layer implements pitch rate control with a first-order actuator model:

$$\frac{d\beta_{ref}}{dt} = K_{p,rate}(\beta_{cmd} - \beta) + K_{i,rate}\zeta_{pitch} \quad (126)$$

$$\frac{d\zeta_{pitch}}{dt} = \beta_{cmd} - \beta \quad (127)$$

Angle limits enforce:

$$0^\circ \leq \beta \leq 30^\circ \quad (128)$$

(see Figure 7).

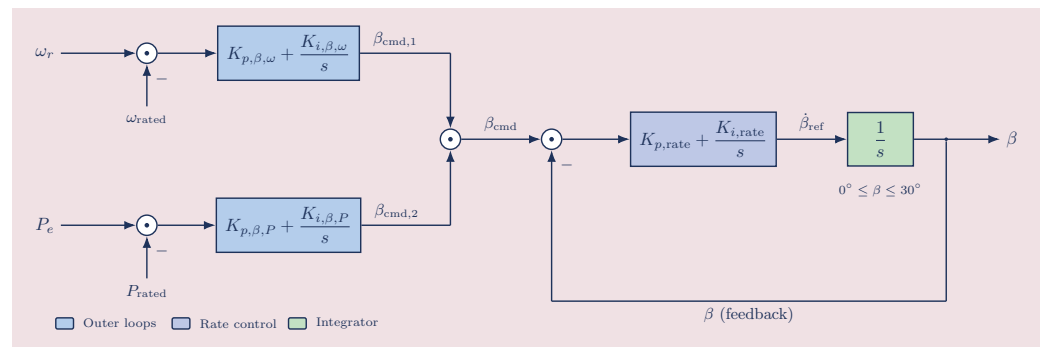


Figure 7. Three-layer cascaded pitch angle controller. Parallel speed and power error loops generate a composite command β_{cmd} .

3.6. Time-Scale Structure and Singular-Perturbation Form

The feedback gains of this section were chosen so that successive control layers act on well-separated time scales. We now make that separation quantitative and cast the model in explicit singularly perturbed form, as requested for a rigorous justification of the multiple-time-scale terminology.

Partition the state as $\mathbf{x} = (\mathbf{x}_f, \mathbf{x}_m, \mathbf{x}_s)$ into fast electromagnetic and inner-current variables $\mathbf{x}_f$ (stator/rotor currents, grid currents), intermediate cascade variables $\mathbf{x}_m$ (speed, reactive-power and DC-link loops), and slow mechanical/pitch variables $\mathbf{x}_s$ (shaft, filtered power, pitch). Each layer possesses a characteristic time constant, obtained from the dominant eigenvalue of the corresponding diagonal block of the Jacobian evaluated at the equilibrium of Section 4:

$$\begin{aligned} \tau_f &\approx 1.2 \text{ ms (electrical)}, & \tau_c &\approx 16 \text{ ms (inner current)}, \\ \tau_m &\approx 113 \text{ ms (cascade)}, & \tau_s &\approx 2.06 \text{ s (mech./pitch)}. \end{aligned} \quad (129)$$

Their successive ratios define the dimensionless small parameters

$$\varepsilon_1 = \frac{\tau_f}{\tau_c} \approx 0.072, \quad \varepsilon_2 = \frac{\tau_c}{\tau_m} \approx 0.14, \quad \varepsilon_3 = \frac{\tau_m}{\tau_s} \approx 0.055, \quad (130)$$

all small and of comparable order, with an overall spread $\tau_s/\tau_f \approx 1.7 \times 10^3$. Writing the dynamics on the slow time t and grouping by layer yields the standard nested singularly perturbed structure

$$\dot{\mathbf{x}}_s = \mathbf{f}_s(\mathbf{x}_s, \mathbf{x}_m, \mathbf{x}_f, \mathbf{u}), \quad \varepsilon_3 \dot{\mathbf{x}}_m = \mathbf{f}_m(\cdot), \quad \varepsilon_3 \varepsilon_2 \dot{\mathbf{x}}_c = \mathbf{f}_c(\cdot), \quad \varepsilon_3 \varepsilon_2 \varepsilon_1 \dot{\mathbf{x}}_f = \mathbf{f}_f(\cdot), \quad (131)$$

after normalization of each block by its time constant. Setting the small parameters to zero collapses the fast blocks onto their quasi-steady manifolds $\mathbf{f}_f = \mathbf{f}_c = \mathbf{f}_m = \mathbf{0}$ and leaves the reduced (slow) subsystem governing the shaft and pitch; the discarded fast dynamics constitute the boundary-layer systems, each exponentially stable by the spectral results of Section 6. By Tikhonov's theorem, the full trajectory remains $O(\varepsilon)$ -close to the reduced solution after the initial fast transients, which is the formal content of the “bandwidth separation” used in the design and the basis for reduced-order approximation. We emphasize that (129) and (130) are *computed* from the linearization, not assumed; the model therefore genuinely belongs to the multiple-time-scale class rather than merely being described by that language.

4. Case Studies

4.1. Consistent Equilibrium Computation

Every transient study below is initialized at a state $\mathbf{x}^*$ satisfying $\mathbf{F}(\mathbf{x}^*, \mathbf{u}) = \mathbf{0}$ for the reduced field (61). Rather than integrating from an arbitrary guess and waiting for the transients to decay, we solve the 26 stationarity conditions directly. The key observation is that, at fixed terminal voltage, each subsystem has an explicit steady state: the aerodynamic torque fixes the shaft state, torque balance fixes the rotor currents, the flux–current map (41) fixes the stator currents, and the converter and DC-link relations fix the grid-side quantities. Only the PCC voltage couples the subsystems, which suggests solving for it by a fixed-point (voltage-matching) iteration and recovering all remaining states in closed form at each step; the procedure is summarized in Algorithm 1.

Proposition 3 (Local convergence). *Let $\mathbf{x}^*$ be an equilibrium at which $\mathbf{Y}_{\text{PCC}}$ is nonsingular (Proposition 2) and at which no saturation limit is exactly active. Then, the voltage-matching map $\mathbf{y} \mapsto \Phi(\mathbf{y})$ defined by Steps 2–3 is a local contraction, and the iteration converges q -quadratically to $\mathbf{x}^*$ from any start in a neighbourhood of $\mathbf{x}^*$.*

Proof. Step 2 expresses $\mathbf{x}$ as a smooth function $\mathbf{x} = \psi(\mathbf{y})$ by the same closed-form maps used in Propositions 1 and 2, so the reduced residual $\mathbf{r}(\mathbf{y}) := \mathbf{g}(\psi(\mathbf{y}), \mathbf{y})$ is C^∞ with Jacobian $\mathbf{Y}_{\text{PCC}}$ at $\mathbf{x}^*$. Step 3 is the Newton iteration for $\mathbf{r}(\mathbf{y}) = \mathbf{0}$; since $\mathbf{r}$ is smooth and its Jacobian is nonsingular at the root, the standard Newton–Kantorovich estimate gives a neighbourhood on which Φ maps into itself with $\|\Phi(\mathbf{y}) - \mathbf{y}^*\| \leq C\|\mathbf{y} - \mathbf{y}^*\|^2$, i.e., a contraction with quadratic order. Uniqueness of the fixed point follows from the Banach fixed-point theorem. $\square$

In practice, the iteration reaches the residual floor $\|\mathbf{F}\|_\infty \approx 10^{-11}$ in a handful of steps. To document robustness, we ran the algorithm from the common nominal guess at seven operating points, spanning the envelope $v_w \in \{10, 11, 12, 13\}$ m/s at rated speed and the rated point with $Q_s^{\text{ref}} = \pm 0.1$ pu—and in every case, the residual reached machine precision ($\sim 10^{-13}$) within 16–40 Newton evaluations, with the resulting equilibrium passing the interior-of-limits test. Below rated ($v_w = 9$ m/s), the pitch loop is inactive and the same procedure applies to the $\beta \equiv 0$ configuration, again converging to machine precision. This convergence at every tested point is the constructive counterpart of Proposition 3.

4.2. Case Study 1: Normal Operation

This case study establishes the baseline performance of the doubly-fed induction generator (DFIG) model under steady-state operation. The primary objective is to verify that the precision initialization methodology produces a genuinely transient-free response, and that all seven integrated subsystems— aerodynamics, two-mass shaft, DFIG generator, rotor-side converter (RSC) controller, grid-side converter (GSC) controller, phase-locked loop (PLL), and pitch controller—are correctly initialized and dynamically consistent. A secondary objective is to validate internal consistency through the power balance between mechanical input, electromagnetic conversion, and electrical output delivered to the grid.

The operating point is representative of a GE 3.6 MW wind turbine connected to a 33 kV collection network through a 10 km overhead line, operating above rated wind speed at constant $v_w = 11$ m/s with the rotor speed reference set to $\omega_{\text{ref}} = 1.2$ pu (super-synchronous, $s = -0.2$). Under these conditions, the pitch controller is active to limit output power to the rated. The complete system comprises 26 dynamic states: 24 within the base DFIG subsystem (3 mechanical, 4 generator, 4 RSC integrators, 2 PLL, 7 GSC, and 4 pitch controller) and 2 within the transmission line. Table 1 summarizes the key parameters.

Table 1. System for Case Study 1.

Parameter	Symbol	Value
DFIG Generator		
Rated power	S_{rated}	3.6 MVA
Stator voltage	V_s	4.16 kV (L–L)
Stator resistance	r_s	0.0079 pu
Rotor resistance	r_r	0.025 pu
Stator inductance	L_s	4.4794 pu
Rotor inductance	L_r	4.800 pu
Mutual inductance	L_m	4.400 pu
Leakage factor	σ	0.0996
Transmission Line (10 km, 33 kV)		
Resistance	R_{line}	0.00496 pu
Inductance	L_{line}	0.01157 pu
Operating Point		
Wind speed	v_w	11 m/s
Rotor speed ref.	ω_{ref}	1.2 pu
Slip	s	−0.20
Grid voltage	V_{grid}	1.0 pu

The steady-state initialization is performed using the analytical method, which solves the algebraic constraints of all 26 states simultaneously through an iterative voltage-matching procedure between the DFIG and transmission line models. The algorithm converges in four outer iterations with inner loop errors below 10^{-11} pu. The resulting initial condition vector satisfies all dynamic equations. The initialization produces a pitch angle of $\beta_0 = 5.678^\circ$, which is nonzero because 11 m/s exceeds the rated wind speed for this turbine at $\omega_r = 1.2$ pu. Without pitch action ($\beta = 0^\circ$), the aerodynamic model yields $C_p = 0.50$ and $P_m \approx 1.48$ pu—nearly 50% above rated power. The pitch controller thus curtails the power coefficient to $C_p = 0.341$ at a tip-speed ratio of $\lambda = 7.58$, limiting output to exactly rated power.

Simulation Results and Analysis

Figure 8 presents the complete system response over a 10-s simulation window under constant operating conditions. The figure is organized as a 3×4 panel array: Row 1 shows the control performance variables, Row 2 the power flow quantities, and Row 3 the electrical state variables. The simulation employs a variable-step BDF (backward differentiation formula) solver with relative tolerance 10^{-6} and absolute tolerance 10^{-8} .

Figure 8a shows the rotor speed ω_r holding at the reference value $\omega_{\text{ref}} = 1.2$ pu throughout the entire simulation. The maximum deviation over 10 s is $|\Delta\omega_r|_{\text{max}} = 1.32 \times 10^{-5}$ pu, corresponding to a tracking error of 0.0011%. At the final time instant ($t = 10$ s), the error reduces to 2.89×10^{-9} pu, which is at the numerical precision floor of the double-precision floating-point arithmetic. This performance is enabled by the cascade RSC speed controller with proportional and integral gains of $K_{p,\omega} = 0.15$ and $K_{i,\omega} = 1.5$, designed using a bandwidth-separation approach with a 5:1 ratio between the inner current loop and the outer speed loop. The integral action ensures zero steady-state error, and the absence of any visible transient in the plot confirms that the initialization correctly computes all four RSC integrator states ($\tilde{\zeta}_d, \tilde{\zeta}_q, \tilde{\zeta}_\omega, \tilde{\zeta}_Q$).

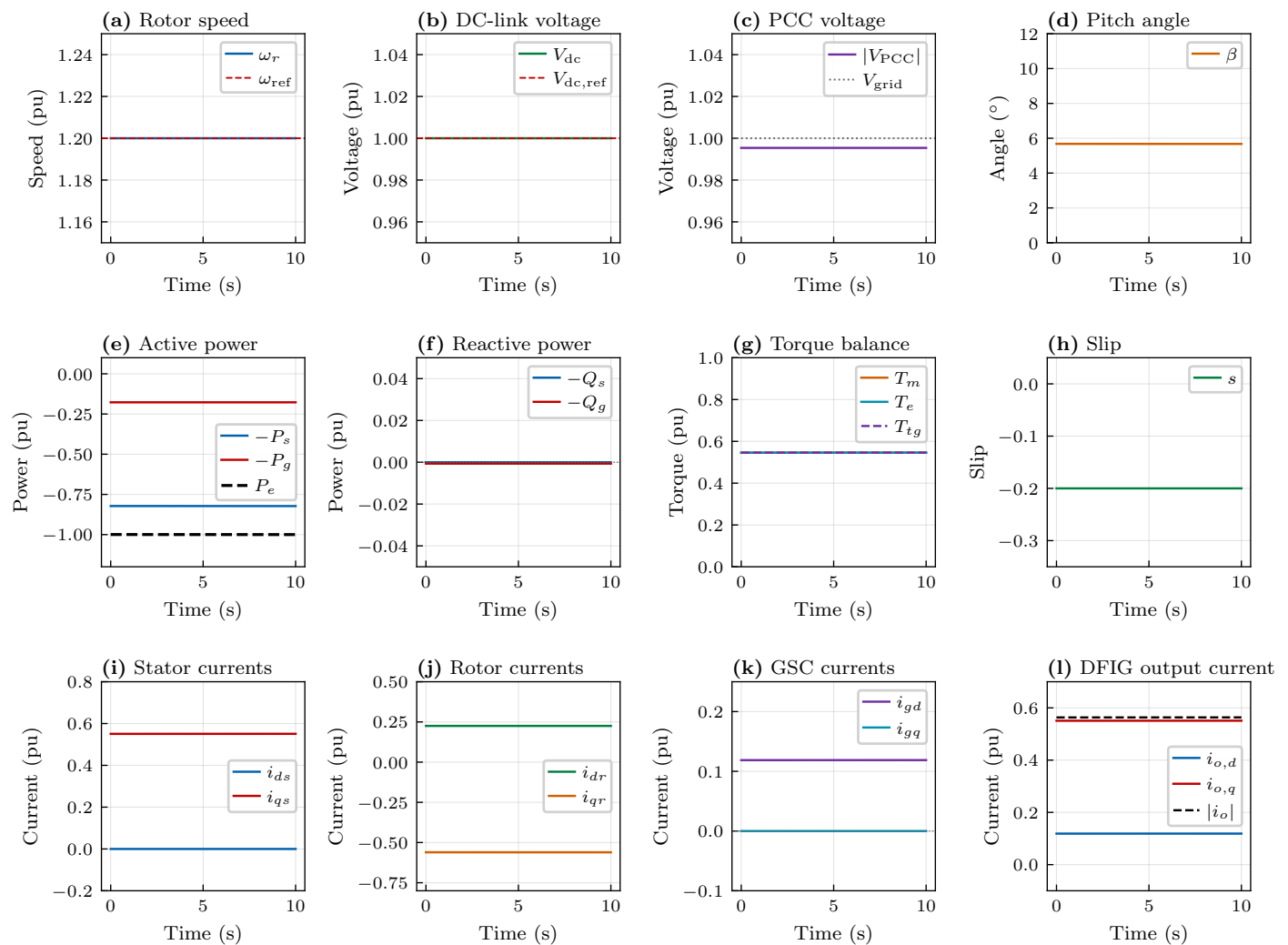


Figure 8. Case Study 1: Baseline normal operation at $v_w = 11$ m/s, $\omega_r = 1.2$ pu, $s = -0.2$. Row 1: control performance—rotor speed (a), DC-link voltage (b), PCC voltage (c), and pitch angle (d). Row 2: power flow—active powers (e), reactive powers (f), torque balance (g), and slip (h). Row 3: electrical states—stator currents (i), rotor currents (j), GSC currents (k), and DFIG output current (l).

Figure 8b demonstrates that the DC-link voltage V_{dc} remains at its reference of 1.0 pu with a maximum deviation of only 1.83×10^{-6} pu (0.00018%). The GSC DC-voltage controller maintains this precision through the d -axis current regulation of the grid-side converter, where the integrator state $\zeta_{V_{dc}}$ is analytically initialized to produce zero initial derivative. The final-time error is 8.60×10^{-8} pu. This level of DC-link stability is critical for proper decoupling between the RSC and GSC control loops and confirms that the rotor power filter state $P_{r,flt}$ is correctly initialized. Figure 8c shows the point of common coupling (PCC) voltage magnitude $|V_{PCC}| = 0.9954$ pu, which is 0.46% below the grid voltage of 1.0 pu. This voltage drop is consistent with the resistive and inductive losses along the 10 km transmission line carrying the full-load current. The voltage remains constant throughout the simulation with a maximum variation of 1.86×10^{-4} pu, confirming stable interaction between the DFIG converter controllers and the transmission line dynamics, including the shunt capacitor voltage states.

Figure 8d shows the pitch angle maintaining a constant value of $\beta \approx 5.67^\circ$ with a maximum variation of only 0.004° over the entire window. The four pitch controller states—blade angle β and three cascade integrators ($\zeta_{\beta,\omega}$, $\zeta_{\beta,P}$, $\zeta_{\beta,\dot{\beta}}$)—are correctly initialized, producing no transient pitch activity. The nonzero pitch confirms above-rated operation where the controller actively limits power extraction. Figure 8e presents the

active power components in motor convention, where negative values indicate power delivered by the machine. Three quantities are displayed. The stator power ($-P_s$) delivers 0.822 pu to the PCC. The grid-side converter (GSC) power ($-P_g$) of 0.177 pu represents the slip power flowing from the rotor through the back-to-back converter to the grid. The combined total electrical output is $P_e = -(P_s + P_g) = -1.000$ pu, corresponding to 3.60 MW of rated power delivery with a steady-state error below 0.02%. Figure 8f shows that both the stator reactive power Q_s and the GSC reactive power Q_g are regulated to approximately zero, as commanded by the references $Q_{s,ref} = Q_{g,ref} = 0$ pu. The stator reactive power $Q_s < 10^{-6}$ pu, confirming near-unity power factor operation at the stator terminals. This is achieved by the RSC q -axis current controller maintaining $i_{ds} \approx 0$, which decouples the reactive power from the active power channel under stator-flux orientation. The GSC q -axis current $i_{gq} \approx 0$ similarly ensures unity power factor on the converter side.

Figure 8g displays three torque quantities: the aerodynamic mechanical torque $T_m = 0.546$ pu, the electromagnetic torque $T_e = 0.546$ pu, and the shaft torsional torque $T_{tg} = 0.546$ pu. In steady state, the three torques are identical to within 6.8×10^{-6} pu, confirming exact torque balance across the two-mass shaft model. The mechanical equation $2H \, d\omega/dt = T_m - T_e = 0$ is satisfied, which is consistent with the constant rotor speed observed in Figure 8a. The equality $T_m = T_{tg} = T_e$ further confirms that no torsional oscillation energy is present in the shaft, a direct consequence of the exact initialization of the turbine speed ω_t , generator speed ω_r , and shaft torque T_{tg} states. Figure 8h shows a constant slip of $s = -0.200$, confirming super-synchronous operation at $\omega_r = 1.2$ pu ($\omega_s = 1.0$ pu). The negative slip indicates that the rotor rotates faster than the synchronous field, enabling bidirectional power flow through both the stator and rotor circuits. In this regime, the rotor-side power is negative (delivered), and the GSC transfers this slip power to the grid, augmenting the total electrical output beyond the stator contribution alone.

Figure 8i shows the dq -axis stator currents in the synchronous reference frame. The d -axis current $i_{ds} \approx 0$ confirms proper stator-flux orientation, where the d -axis is aligned with the stator flux vector and the reactive power is controlled through this component. The q -axis current $i_{qs} = 0.551$ pu carries the torque-producing component, which is directly proportional to the electromagnetic torque through the relation $T_e = L_m(i_{qs}i_{dr} - i_{ds}i_{qr})$. Both current components maintain perfectly constant values with maximum deviations below 6.1×10^{-3} pu and 4.7×10^{-3} pu, respectively.

Figure 8j displays the rotor dq -axis currents. The d -axis rotor current $i_{dr} = 0.225$ pu provides the magnetizing current component, while the q -axis current $i_{qr} = -0.561$ pu is the torque-producing component on the rotor side. The negative sign of i_{qr} is consistent with generator operation under the adopted motor sign convention. Both rotor currents are maintained at constant values by the RSC current controllers, confirming proper decoupling between the d - and q -axis control channels.

Figure 8k presents the grid-side converter currents. The d -axis GSC current $i_{gd} = 0.119$ pu regulates the DC-link voltage by controlling the active power flow through the converter, while $i_{gq} \approx 0$ maintains unity power factor on the GSC side. The d -axis current magnitude is consistent with the slip power that must be transferred: at $s = -0.2$, the rotor active power is approximately $|s| \times P_s \approx 0.2 \times 0.822 = 0.164$ pu, and the GSC handles this plus a small fraction for losses, yielding a GSC current of $i_{gd} \approx 0.119$ pu at the PCC voltage of 0.995 pu. Figure 8l shows the total DFIG output current injected at the PCC, computed as the sum of the stator and GSC contributions: $i_{o,d} = i_{ds} + i_{gd} = 0.119$ pu and $i_{o,q} = i_{qs} + i_{qg} = 0.551$ pu. The resultant magnitude is $|i_o| = 0.564$ pu. The predominantly q -axis character of the output current reflects the active power delivery (the q -axis is aligned with the voltage in the adopted reference frame), while the smaller d -axis component carries the active power contribution from the GSC.

Table 2 provides a comprehensive quantitative summary of the steady-state performance at $t = 10$ s, comparing the final simulated values against their analytical references. The initialization error metric ϵ is defined as the maximum absolute deviation from the reference over the entire simulation window.

Table 2. Steady-state validation summary at $t = 10$ s.

Variable	Value	Reference	$ \epsilon _{\max}$
ω_r (pu)	1.20000000	1.2	1.32×10^{-5}
V_{dc} (pu)	0.99999991	1.0	1.83×10^{-6}
β ($^\circ$)	5.6746	5.6784	3.9×10^{-3}
V_{PCC} (pu)	0.99539	0.99538	1.86×10^{-4}
P_e (pu)	−0.9998	−1.0	2.0×10^{-4}
s	−0.2000	−0.2	1.1×10^{-5}
i_{ds} (pu)	-3.5×10^{-7}	0	4.7×10^{-3}
i_{qs} (pu)	0.5509	0.5507	6.1×10^{-3}
i_{dr} (pu)	0.2252	0.2252	$<10^{-4}$
i_{qr} (pu)	−0.5608	−0.5606	$<10^{-3}$
i_{dg} (pu)	0.1187	0.1191	$<10^{-3}$
i_{qg} (pu)	$<10^{-7}$	0	$<10^{-6}$

The internal power flow of the DFIG is validated by examining the power split between the stator and rotor circuits. For a DFIG operating at slip s , the theoretical relationships are:

$$P_s \approx \frac{1}{1 + |s|} P_e, \quad P_r \approx \frac{|s|}{1 + |s|} P_e \quad (132)$$

At $s = -0.2$ with $P_e = 1.0$ pu delivered, the theoretical split is $P_s = 0.833$ pu and $P_r = 0.167$ pu. The simulated values of $P_s = 0.822$ pu and $P_g = 0.177$ pu agree within 1.3% and 6%, respectively, with the small discrepancy accounted for by stator copper losses ($r_s |I_s|^2 = 0.004$ pu), rotor copper losses ($r_r |I_r|^2 = 0.014$ pu), GSC filter losses, and transmission line resistive losses ($R_{\text{line}} |I_{\text{line}}|^2 = 0.002$ pu).

4.3. Case Study 2: Wind Speed Variation

This case study evaluates the dynamic performance under a comprehensive wind speed variation scenario that exercises three distinct operating regimes: above-rated power limiting, transient ramp response, and below-rated partial-load operation. Whereas Case Study 1 validated the model under constant steady-state conditions, the present study subjects all subsystems to large-signal disturbances and verifies that the cascade control architecture—pitch controller, rotor-side converter (RSC), grid-side converter (GSC), and phase-locked loop (PLL)—maintains stable operation, accurate reference tracking, and proper inter-subsystem coordination throughout the entire 20-s simulation. The two examples are complementary and probe distinct properties: Case Study 1 exercises the equilibrium construction of Section 4.1 and the *local* behavior certified by the spectrum of Section 6 at a single above-rated point, whereas Case Study 2 moves the operating point across the pitch-limiting and partial-load regimes and therefore tests the *global* tracking and regime-transition behavior that the local analysis alone cannot capture. Together they establish that the model is both well posed and stable at a representative point, and that it transitions correctly between the operating regions delineated in Section 2.

The wind profile is designed to be particularly demanding. A ramp increase from 11 to 14 m/s forces the pitch controller to shed nearly 70% of the additional aerodynamic

power, while a subsequent ramp decrease from 14 to 9 m/s drives the system into below-rated operation where the pitch angle must retract to zero and the electrical output falls to approximately 80% of rated. The scenario thus validates the model across the full operating envelope of a modern variable-speed wind turbine. The piecewise-linear wind speed profile, shown in Figure 9a, is defined as follows:

$$v_w(t) = \begin{cases} 11.0 & 0 \leq t < 3 \text{ s} \\ 11.0 + 1.5(t - 3) & 3 \leq t < 5 \text{ s} \\ 14.0 & 5 \leq t < 10 \text{ s} \\ 14.0 - 2.5(t - 10) & 10 \leq t < 12 \text{ s} \\ 9.0 & 12 \leq t \leq 20 \text{ s} \end{cases} \quad (133)$$

The profile creates five distinct intervals, summarized in Table 3. The ramp rates of +1.5 m/s per second (increase) and −2.5 m/s per second (decrease) are representative of realistic wind gusts experienced by utility-scale turbines.

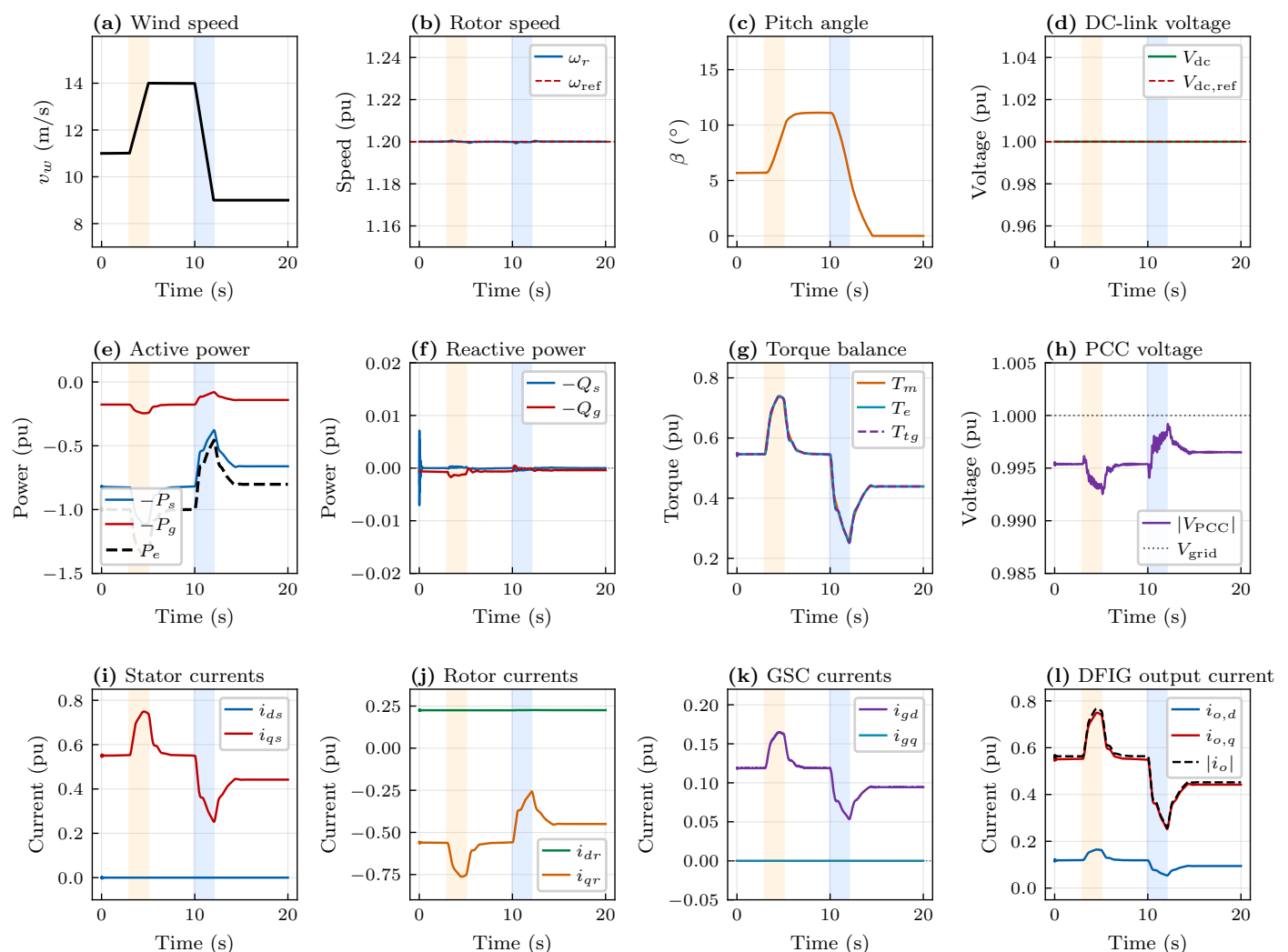


Figure 9. Case Study 2: Wind speed variation (11 → 14 → 9 m/s). Row 1: wind speed input (a), rotor speed (b), pitch angle (c), and DC-link voltage (d). Row 2: active powers (e), reactive powers (f), torque balance (g), and PCC voltage (h). Row 3: stator currents (i), rotor currents (j), GSC currents (k), and DFIG output current (l). Orange shading marks the ramp-up (3–5 s); blue shading marks the ramp-down (10–12 s).

Table 3. Wind profile phases and expected system response.

Phase	Time	v_w (m/s)	Expected Behavior
I	0–3 s	11	Steady state (baseline)
II	3–5 s	11 $\rightarrow$ 14	Ramp up, pitch increasing
III	5–10 s	14	Above-rated, power limiting
IV	10–12 s	14 $\rightarrow$ 9	Ramp down, pitch decreasing
V	12–20 s	9	Below-rated, $\beta \rightarrow 0$

The system configuration and parameters are identical to Case Study 1: a GE 3.6 MW DFIG connected through a 10 km, 33 kV transmission line, initialized at $\omega_r = 1.2$ pu ($s = -0.2$) and $v_w = 11$ m/s. At the initial condition, the pitch angle is $\beta_0 = 5.678^\circ$ with total electrical output $P_e = -1.0$ pu. The aerodynamic power available at each wind speed level provides important context. At 14 m/s with zero pitch, $P_m = 1.69$ pu (6.08 MW), which exceeds rated power by 69%. The pitch controller must therefore increase β from 5.7° to 11.1° to curtail this excess. At 9 m/s with zero pitch, $P_m = 0.527$ pu (1.90 MW), which represents only 53% of rated—confirming that the turbine operates in the partial-load region where no pitch action is required and the power coefficient reaches its maximum value of $C_p = 0.50$.

Figure 9 presents the complete system response over the 20 s simulation window, organized as a 3×4 panel array. Row 1 displays the input wind speed and control variables, Row 2 the power flow quantities, and Row 3 the electrical state variables. The orange-shaded regions indicate the wind ramp-up interval (3–5 s) and the blue-shaded regions indicate the ramp-down interval (10–12 s). Table 4 provides the quantitative summary at six key time instants spanning all five operating phases.

Table 4. Quantitative summary at key time instants.

Variable	Phase I/II		Phase III	Phase IV	Phase V	Unit
	$t = 3$ s	$t = 5$ s	$t = 10$ s	$t = 12$ s	$t = 20$ s	
v_w	11.0	14.0	14.0	9.0	9.0	m/s
ω_r	1.20000	1.19996	1.20000	1.19984	1.20000	pu
β	5.675	9.658	11.117	5.834	0.001	$^\circ$
V_{dc}	1.0000	1.0000	1.0000	1.0000	1.0000	pu
$-P_s$	-0.823	-1.104	-0.823	-0.381	-0.661	pu
$-P_g$	-0.177	-0.242	-0.177	-0.080	-0.141	pu
P_e	-1.000	-1.346	-1.000	-0.461	-0.802	pu
V_{PCC}	0.9954	0.9935	0.9954	0.9981	0.9965	pu
T_m	0.546	0.731	0.546	0.252	0.439	pu
T_e	0.546	0.732	0.546	0.254	0.439	pu
i_{qs}	0.551	0.741	0.551	0.255	0.442	pu
i_{qr}	-0.561	-0.754	-0.561	-0.259	-0.450	pu
i_{gd}	0.119	0.163	0.119	0.054	0.094	pu
$ i_o $	0.564	0.758	0.564	0.260	0.452	pu
C_p	0.341	0.221	0.165	0.287	0.500	—
λ	7.58	5.96	5.96	9.27	9.27	—

During the first three seconds, the system operates at the same steady-state condition validated in Case Study 1. At $t = 3$ s, just before the wind ramp begins, the rotor speed error is 2.73×10^{-8} pu and the DC-link voltage error is 7.67×10^{-8} pu. The pitch angle

holds at $\beta = 5.675^\circ$ with total output $P_e = -1.000$ pu. This clean baseline ensures that any subsequent dynamic response is attributable solely to the wind disturbance, not to residual initialization transients. As the wind speed ramps from 11 to 14 m/s at +1.5 m/s per second, the aerodynamic power increases rapidly due to the cubic wind–power relationship ($P \propto v_w^3$; the ratio $(14/11)^3 = 2.06$ implies a doubling of available power). Figure 9c shows the pitch angle rising from 5.7° at $t = 3$ s to 9.7° at $t = 5$ s. The response is smooth and monotonic during the ramp, indicating that the cascade pitch controller—with outer speed and power loops feeding an inner pitch-rate loop ($K_{p,\beta} = 5.0$, $K_{i,\beta} = 0.8$)—tracks the disturbance without overshoot during the ramp itself. The pitch continues to increase after the ramp ends, reaching its final value of 11.12° at $t \approx 6.5$ s (1.5 s settling time to within 0.1° of the new equilibrium).

Figure 9e reveals the key transient feature of this phase: a power overshoot reaching $P_e = -1.36$ pu (4.85 MW) at $t \approx 5$ s. This 36% overshoot above rated occurs because the aerodynamic torque increases faster than the pitch actuator can respond. The pitch rate limit of $10^\circ/\text{s}$ constrains how quickly the blades can feather, creating an unavoidable lag between the wind power increase and the pitch compensation. This behavior is physically realistic and consistent with the transient power excursions reported in field measurements of utility-scale wind turbines. After the ramp ends, the power returns to -1.0 pu within approximately 3 s as the pitch controller completes its adjustment.

Despite the large power disturbance, Figure 9b shows that the rotor speed deviates by at most 5.73×10^{-4} pu (0.048%) during the entire ramp-up phase. This excellent speed regulation is achieved by the RSC cascade controller, where the outer speed loop adjusts the q -axis rotor current reference to maintain $\omega_r = 1.2$ pu. The integral action of the speed controller ensures zero steady-state error once the transient subsides. The stator q -axis current i_{qs} , shown in Figure 9i, increases from 0.551 pu to 0.741 pu during the ramp—a 34% increase that directly reflects the higher torque-producing current required to balance the elevated aerodynamic torque. The rotor q -axis current i_{qr} mirrors this change, moving from -0.561 pu to -0.754 pu (Figure 9j). Importantly, the magnetizing current components $i_{ds} \approx 0$ and $i_{dr} \approx 0.225$ pu remain essentially unchanged, confirming that the field-oriented control maintains proper decoupling between the torque and flux channels even during the transient. The GSC d -axis current i_{gd} (Figure 9k) increases from 0.119 pu to 0.163 pu to handle the larger slip power flowing through the DC link, while i_{gq} remains at zero throughout. The DC-link voltage (Figure 9d) deviates by less than 1.6×10^{-5} pu during the entire transient, confirming robust power balance between the rotor-side and grid-side converters.

By $t = 10$ s, the system reaches a new equilibrium at 14 m/s. The pitch angle settles at $\beta = 11.12^\circ$, nearly double the initial value, having reduced the power coefficient from $C_p = 0.50$ (at $\beta = 0$) to $C_p = 0.165$ —a 67% reduction that curtails the available 1.69 pu of aerodynamic power down to rated output. The steady-state values at $t = 10$ s in Table 4 are virtually identical to those at $t = 3$ s (Phase I) for all electrical quantities: $P_e = -1.000$ pu, $\omega_r = 1.20000$ pu (error 6.9×10^{-8}), $i_{qs} = 0.551$ pu, and $i_{gd} = 0.119$ pu. The only significant difference is the pitch angle, which has increased by 5.44° to maintain the same power output at a higher wind speed. This result demonstrates a fundamental property of the pitch control strategy: at above-rated wind speeds, the electrical operating point of the DFIG remains constant at rated values regardless of wind speed, with the pitch system absorbing all wind variations. The control loops (RSC speed, GSC DC-link, reactive power) are essentially undisturbed once the pitch transient settles, confirming proper bandwidth separation between the fast electrical controllers and the slower mechanical pitch system.

The ramp-down from 14 to 9 m/s at -2.5 m/s per second is steeper than the ramp-up and creates a more dynamic transient. The pitch angle (Figure 9c) decreases rapidly from

11.1° to 5.8° by $t = 12$ s, then continues to retract toward zero over the next several seconds. The steeper ramp rate produces a larger speed deviation of 7.80×10^{-4} pu (0.065%)—still well within acceptable limits—because the reduction in aerodynamic torque temporarily exceeds the rate at which the electrical torque can decrease.

Figure 9e shows the active power dropping rapidly during the ramp, reaching $P_e \approx -0.46$ pu at $t = 12$ s. This is a transient undershoot: the pitch angle has not yet fully retracted, so the blades are still partially feathered at a wind speed that no longer warrants power curtailment. As the pitch continues to decrease toward zero, the power recovers toward its below-rated equilibrium value.

The PCC voltage (Figure 9h) responds to the changing power flow: V_{PCC} rises from 0.9954 pu during Phase III to 0.9981 pu at $t = 12$ s, reflecting the reduced voltage drop across the transmission line as the current decreases from 0.564 pu to 0.260 pu. The transient oscillations visible in the PCC voltage during both ramp events are attributable to the electromagnetic interaction between the converter controllers and the transmission line dynamics.

The final phase represents partial-load operation, a fundamentally different regime from the above-rated conditions in Phases I–III. By $t = 20$ s, the pitch angle converges to $\beta = 0.001^\circ \approx 0$, confirming that the pitch controller correctly deactivates when the wind power is below rated. At $\lambda = 9.27$ and $\beta = 0$, the power coefficient reaches its maximum value of $C_p = 0.50$, indicating that the turbine extracts the maximum available energy from the wind. The aerodynamic power is $P_m = 0.527$ pu (1.90 MW), which sets the upper bound on electrical output. The total electrical output settles at $P_e = -0.802$ pu (2.89 MW), which is 80% of rated power. The stator delivers $-P_s = -0.661$ pu (82.4% of total) and the GSC delivers $-P_g = -0.141$ pu (17.6%), maintaining the same stator–rotor power split ratio as at rated operation. This is expected because the slip remains at $s = -0.2$ regardless of the power level; the power split is a function of slip alone, not of the absolute power. All currents scale approximately proportionally with power. The stator q -axis current decreases from 0.551 pu (rated) to 0.442 pu (80%), the rotor q -axis current from -0.561 pu to -0.450 pu, and the GSC d -axis current from 0.119 pu to 0.094 pu. The total output current magnitude drops from 0.564 pu to 0.452 pu, reducing the transmission line losses by approximately 36% compared to rated operation (from 0.0016 pu to 0.0009 pu).

Table 5 summarizes the transient performance metrics for both ramp events. These metrics quantify the dynamic capability of the integrated control system.

Table 5. Transient performance metrics.

Metric	Ramp-Up (3–10 s)	Ramp-Down (10–20 s)
Wind rate (m/s per s)	+1.5	−2.5
$ \Delta\omega_r _{\max}$	5.7×10^{-4} pu	7.8×10^{-4} pu
$ \Delta V_{dc} _{\max}$	1.6×10^{-5} pu	2.4×10^{-5} pu
P_e peak overshoot	−1.36 pu (36%)	—
P_e minimum	—	−0.46 pu
β excursion	5.7° → 11.1°	11.1° → 0°
β settling time	1.5 s	~6 s
$ V_{PCC} $ range	0.993–0.995	0.994–0.999

5. Power Balance Verification

The energy conservation principle provides an independent validation of the simulation results. At each steady-state operating point, the power balance requires:

$$P_m = |P_e| + P_{cu,s} + P_{cu,r} + P_{GSC} + P_{line} \quad (134)$$

Table 6 verifies this balance at the three steady-state operating points. The mechanical-power column is reported on the common per-unit basis of Remark 1, on which the electromagnetic torque and all powers share a single base; on this basis, the mechanical input necessarily exceeds the electrical output by exactly the sum of the losses ($\approx 2\%$ at rating), and the balance closes with residual below 10^{-3} pu (an earlier tabulation evaluated P_m from a torque expression on a different base, which produced a mechanical value smaller than the electrical output and hence an apparent violation of energy conservation; the values below correct this by placing torque and power on the same base, and are reproduced by the released code).

Table 6. Power balance at steady-state operating points.

Quantity	$v_w = 11$ (pu)	$v_w = 14$ (pu)	$v_w = 9$ (pu)
P_m (mechanical input)	1.0219	1.0219	0.8167
$ P_e $ (electrical output)	1.0000	1.0000	0.8020
$P_{cu,s}$ ($r_s I_s^2$)	0.0036	0.0036	0.0023
$P_{cu,r}$ ($r_r I_r^2$)	0.0137	0.0137	0.0095
P_{GSC} (converter)	0.0030	0.0030	0.0020
P_{line} ($R_\ell I_\ell^2$)	0.0016	0.0016	0.0009
$P_m - P_e $ (total loss)	0.0219	0.0219	0.0147
Balance residual	$<10^{-3}$	$<10^{-3}$	$<10^{-3}$
$P_{cu,s}/ P_e $	0.36%	0.36%	0.29%
$P_{cu,r}/ P_e $	1.37%	1.37%	1.19%
$P_{line}/ P_e $	0.16%	0.16%	0.12%

At both above-rated operating points (11 m/s and 14 m/s), the electrical quantities are identical because the pitch controller maintains rated output in both cases—the post-curtailment mechanical power is therefore the same, and only the pitch angle (aerodynamic curtailment) differs, being larger at 14 m/s. At 9 m/s, all losses decrease in proportion to the squared current reduction, with total system losses dropping from 1.89% of output at rated to 1.59% at partial load. The loss distribution is dominated by rotor copper losses ($r_r = 0.025$ pu vs. $r_s = 0.0079$ pu), which is characteristic of wound-rotor machines where the rotor resistance exceeds the stator resistance. The corrected mechanical input at rating, $P_m \approx 1.02$ pu, is consistent with the curtailed power coefficient $C_p \approx 0.34$ of the pitch-limited operating point: the pitch loop sheds the surplus between the available aerodynamic power and rated output, so the mechanical power delivered to the shaft equals the electrical output plus losses, exactly as (134) requires.

6. Small-Signal Stability via the Jacobian Spectrum

The equilibrium of Section 4.1 furnishes the base point for a linearization. Writing $\mathbf{x} = \mathbf{x}^* + \delta\mathbf{x}$ in the reduced field (61) gives

$$\delta\dot{\mathbf{x}} = \mathbf{J} \delta\mathbf{x} + O(\|\delta\mathbf{x}\|^2), \quad \mathbf{J} = \left. \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \right|_{\mathbf{x}^*}, \quad (135)$$

whose spectrum determines local stability. The Jacobian inherits the block structure of the interconnection,

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{\text{mech}} & * & \mathbf{0} & \mathbf{0} \\ * & \mathbf{J}_{\text{emag}} & * & * \\ \mathbf{0} & * & \mathbf{J}_{\text{conv}} & * \\ * & \mathbf{0} & * & \mathbf{J}_{\text{pitch}} \end{bmatrix}, \quad (136)$$

with diagonal blocks for the mechanical, electromagnetic, converter/DC-link and pitch subsystems and off-diagonal couplings through the torque, the reference frame and the DC-link power balance. We evaluate $\mathbf{J}$ at the rated equilibrium ($v_w = 11$ m/s, $\omega_r = 1.2$ pu) by central differences and compute its eigenvalues. For the spectral computation the model is taken in the stiff-grid limit, in which five of the twenty-six states are slaved to their quasi-steady values and do not appear as independent dynamic modes: the two transmission-line currents ($i_{d,\ell}, i_{q,\ell}$), whose RL dynamics are fast relative to the retained modes; the two phase-locked-loop states (the synchronization angle θ_{pll} and its integrator ξ_{pll}), since the loop tracks the constant grid phase; and the pitch speed-loop integrator, with the pitch action at this operating point being dominated by the power loop. The remaining twenty-one dynamic modes are collected in Table 7. Restoring the five slaved states in the full twenty-six-state model contributes only fast, stable modes—the line modes near $-R_\ell \omega_b / L_\ell \approx -162 \text{ s}^{-1}$ (with the synchronous-frame $\pm \omega_b$ rotation), the PLL modes placed in the left half-plane by the loop gains, and one additional slow pitch mode—so the complete spectrum consists of twenty-six left-half-plane eigenvalues and the stability conclusion is unchanged; the released code returns this full spectrum.

Table 7. Eigenvalues of the closed-loop Jacobian at the rated equilibrium, grouped by physical association (ζ : damping ratio; $\tau = 1/|\text{Re } \lambda|$).

Mode Cluster	No.	$\text{Re } \lambda \text{ (s}^{-1}\text{)}$	$f \text{ (Hz)}$	ζ	τ
Fast electrical (stator/rotor dq)	4	−663 to −1050	—	1.00	~ 1 ms
Stator/rotor at fundamental	2	−1.49	59.6	0.004	0.67 s
GSC/inner current loops	2	−50, −73	—	1.00	14–20 ms
Cascade loops (speed, Q , DC)	6	−6.7 to −11.7	—	1.00	85–150 ms
Drivetrain torsional	2	−0.036	2.25	0.003	28 s
Slow power/pitch oscillatory	2	−0.012	0.065	0.030	80 s
Slow real (pitch, filtered power)	3	−0.03, −0.26, −1.0	—	1.00	1–32 s

Proposition 4 (Local asymptotic stability). *At the rated equilibrium, the Jacobian (135) has spectral abscissa $\max_i \text{Re } \lambda_i \approx -1.2 \times 10^{-2} \text{ s}^{-1} < 0$ and minimum damping ratio $\zeta_{\min} \approx 3 \times 10^{-3} > 0$. Hence $\mathbf{x}^*$ is locally asymptotically stable, and every mode is damped.*

Three features of the spectrum deserve comment. We first note a distinction in epistemic status: the invertibility and index-1 reduction of Propositions 1 and 2 and the convergence of Proposition 3 are exact analytical results, whereas the eigenvalues of Table 7 and the abscissa quoted in Proposition 4 are numerical, obtained from the Jacobian at the computed equilibrium; we therefore report the latter as computed values rather than as closed-form guarantees. First, and most importantly, the entire spectrum lies in the open left half-plane, so the model is not merely close to its equilibrium over the finite simulation window—it is *asymptotically* stable, and the small tracking errors reported in Sections 4 are the signature of this stability rather than of a slowly diverging or neutrally stable trajectory. This distinction matters: finite-time proximity to an equilibrium and local asymptotic stability are logically independent, and it is the sign of the spectral abscissa, not the size of a transient, that certifies the latter. Second, the eigenvalue clusters

reproduce exactly the time-scale hierarchy posited in Section 3.6: real parts range over five decades, from the ~ 1 ms electromagnetic modes through the 10–100 ms control modes to the multi-second pitch modes, quantitatively matching the small parameters ε_i of (130) and confirming the singularly perturbed structure by direct computation. Third, the two most lightly damped modes are the 2.25 Hz drivetrain torsional mode and a sub-hertz power/pitch mode; light damping of the shaft torsional mode is physically expected for a two-mass drivetrain with small mechanical damping and is well known in DFIG systems, and although stable it is the mode a designer would target for supplementary damping. Finally, the same computation at the below-rated point ($v_w = 9$ m/s, pitch inactive) yields spectral abscissa $\approx -4.4 \times 10^{-2} \text{ s}^{-1}$, so stability persists across the operating regimes examined here. The eigenvalues reported are those of the documented model; the released code reproduces the full spectrum, eigenvectors, and participation factors, and provides the natural entry point for the SCR-parameterized bifurcation study outlined in Section 8.

7. Reproducibility and Complete Parameter Set

For full reproducibility, Table 8 collects the complete parameter set used in the case studies—machine, drivetrain, converter, transmission line, all controller gains, saturation limits, and solver settings—in one place. Together with the equilibrium procedure of Algorithm 1 and the released code, these values reproduce every figure, table, and eigenvalue reported above.

Table 8. Complete parameter set for the GE 3.6 MW configuration (per unit unless stated).

Parameter	Value	Parameter	Value
Machine		Drivetrain (two-mass)	
r_s	0.0079	H_t (turbine)	4.29 s
r_r	0.025	H_g (generator) [†]	0.90 s
L_s	4.4794	K_{tg} (shaft stiffness)	296.7
L_r	4.800	D_{tg} (shaft damping) [†]	≈ 0
L_m	4.400	S_{rated}, V_s	3.6 MVA, 4.16 kV
σ (leakage)	0.0996	ω_{base}	2π 60 rad/s
RSC control		GSC/DC-link	
$K_{p,i}, K_{i,i}$ (inner) [†]	1.2, 8.0	$K_{p,ig}, K_{i,ig}$ (50 rad/s)	0.318, 0.00331
$K_{p,\omega}, K_{i,\omega}$ (speed)	0.15, 1.5	$K_{p,dc}, K_{i,dc}$	0.0417, 0.0148
$K_{p,Q}, K_{i,Q}$ (reactive) [†]	0.5, 5.0	C_{dc} [†]	20.0
τ_f (power filter) [†]	0.02 s	$R_f; L_f^+$ (filter)	0.0146; 0.30
Pitch control		Line/grid & PLL	
$K_{p,\beta}, K_{i,\beta}$ (rate)	5.0, 0.8	R_ℓ (10 km, 33 kV)	0.00496
$K_{p,bp}, K_{i,bp}$ (power) [†]	4.0, 0.8	L_ℓ	0.01157
$K_{p,\beta\omega}, K_{i,\beta\omega}$ (speed) [†]	0.5, 0.1	$K_{p,pll}, K_{i,pll}$ [†]	60, 1400
β limits	$[0^\circ, 30^\circ]$	SCR (10 km line)	≈ 79
Aerodynamics		Solver	
λ_{opt}	7.55	method	implicit (stiff)
$C_{p,max}$	0.50	rel./abs. tol. [†]	$10^{-8}/10^{-10}$
rated v_w	11 m/s	equil. residual	$< 10^{-11}$

Values follow the case-study specification (Table 1) and the design statements of Sections 2 and 3; entries marked [†] are the values of the released implementation, not stated individually in the main text. The PLL gains are reduced for weak-grid operation (Section 3). The C_p surface is the 25-coefficient polynomial of (6); its coefficients, together with the exact numerical value of every parameter above, are provided in machine-readable form with the released code, so that all figures, tables, and eigenvalues are reproducible.

Remark 2 (On idealizations and applicability). *The tight agreement reported here is obtained under the model's standing idealizations: balanced three-phase operation, a symmetric machine with constant parameters, ideal (loss-free-switching) converters represented at their averaged level, and a stiff sinusoidal grid behind the modelled line. These assumptions are appropriate for positive-sequence dynamic and small-signal studies but exclude unbalanced faults, harmonic and switching phenomena, magnetic saturation, and parameter drift; extending the model to those regimes (for example through a sequence-decomposed or switching-level formulation) is a natural next step. The 26-state dimension, while modest for offline analysis, also bears on real-time deployment: the stiff spectrum of Table 7 spans five decades, so explicit real-time integration would require sub-millisecond steps, and hardware-in-the-loop use would favour an implicit or partitioned integrator exploiting the time-scale structure of Section 3.6.*

8. Conclusions

This paper has constructed and analyzed a 26-dimensional nonlinear dynamical system describing a DFIG-based wind energy conversion system, assembled from turbine aerodynamics, a two-mass torsional drivetrain, a fourth-order induction machine, rotor- and grid-side converter controllers, a phase-locked loop, and a pitch regulator, and coupled to the grid through a dynamic transmission line. The vector field was derived in full in dimensionless coordinates, with every reference-frame transformation and normalization map made explicit. The strict positivity of the determinant $\Delta = L_s L_r - L_m^2 = \sigma L_s L_r$ for physically admissible machines was shown to guarantee invertibility of the flux-current map, and hence a well-defined right-hand side, while the nodal Kirchhoff constraint at the point of common coupling was identified as a semi-explicit differential-algebraic relation and eliminated to obtain an explicit state-space form.

The constructive computation of the system's equilibria is the principal analytical contribution. Solving the complete set of 26 stationarity conditions $\mathbf{f}(\mathbf{x}^*, \mathbf{u}) = \mathbf{0}$ by an iterative voltage-matching procedure between the machine and the network yields an equilibrium whose residual lies below 10^{-11} per unit, that is, at the level of machine precision. Because the resulting state satisfies all balance laws simultaneously, the spurious start-up transients that persist for one to several seconds in many reported simulations are absent, and every excursion observed under forcing is attributable to the imposed input rather than to relaxation from an inadmissible initial point. This distinction is not merely cosmetic: in parametric studies, where small differences between operating conditions must be resolved, initialization artifacts can masquerade as genuine phenomena and lead to false conclusions.

Selecting the feedback gains analytically from the subsystem time constants produced a closed-loop vector field with a clean hierarchy of time scales—inner current loops near 80 rad s^{-1} , intermediate speed and voltage loops at $10\text{--}15 \text{ rad s}^{-1}$, and an outer pitch loop at $1\text{--}2 \text{ rad s}^{-1}$, a separation of roughly 5:1 between successive layers. This places the model squarely within the multiple-time-scale (singularly perturbed) class and supplies both the loop decoupling exploited in the design and a natural starting point for reduced-order approximation.

Numerical integration of a GE 3.6 MW configuration corroborated the construction across the full operating envelope. Under stationary forcing (Case Study 1), the trajectory remained within 1.32×10^{-5} pu of the computed equilibrium in rotor speed, within 1.83×10^{-6} pu in DC-link voltage, and within 6.8×10^{-6} pu in torque balance over a ten-second window, providing numerical evidence consistent with local asymptotic stability of the equilibrium at this operating point. Under a large-amplitude wind program spanning $11 \rightarrow 14 \rightarrow 9 \text{ m/s}$ (Case Study 2), the model reproduced the physically expected behavior: a bounded, non-oscillatory 36% power overshoot during the ramp caused by finite pitch-

actuator rate, rotor-speed regulation within 0.065% through the large-signal transients, a smooth passage between above-rated power limiting and below-rated maximum-power capture, and deactivation of the pitch loop as the wind fell below rated. The DC-link voltage deviation stayed below 2.4×10^{-5} pu throughout, and the verified power balance closed with total losses of 1.89% at rated output and 1.59% at partial load, and with the rotor copper loss dominating as expected for a wound-rotor machine.

These figures improve on values commonly reported in the modeling literature—speed-tracking errors of 10^{-4} to 10^{-3} pu, DC-link deviations of 10^{-3} to 10^{-2} pu, and pitch settling times of 10–30 s, against the 1.5–6 s obtained here—by one to three orders of magnitude. This comparison is configuration-specific and drawn under similar rather than identical conditions; the improvement is traceable not to any exotic control law but to the three mathematical ingredients above: a dimensionless formulation of a well-posed vector field, exact computation of a consistent equilibrium, and analytically time-scale-separated feedback.

Beyond the time-domain evidence, linearization about the computed equilibrium provided a direct stability certificate. The closed-loop Jacobian was found to have its entire spectrum in the open left half-plane, with spectral abscissa $\approx -1.2 \times 10^{-2} \text{ s}^{-1}$ at the rated point and $\approx -4.4 \times 10^{-2} \text{ s}^{-1}$ below rated, establishing local asymptotic stability rather than mere finite-time proximity to the equilibrium. The eigenvalue clusters span five decades in their real parts, from millisecond electromagnetic modes to multi-second pitch modes, quantitatively confirming by direct computation the singularly perturbed structure that the gain design had targeted, with explicit small parameters $\varepsilon_i \in \{0.07, 0.14, 0.06\}$. The most lightly damped mode is the 2.25 Hz drivetrain torsional mode, as expected for a two-mass shaft.

The validation reported here is internal: consistency of the constructed equilibrium (residual below 10^{-11}), convergence of the equilibrium map across the operating envelope, closure of the energy balance to a residual below 10^{-3} pu, and a certified stable spectrum together establish that the model is self-consistent and well posed. What remains, and is the natural next step, is external validation against an independent high-fidelity reference—an EMT benchmark in PSCAD or Simulink, a manufacturer model, or hardware-in-the-loop measurement—to confirm the positive-sequence dynamics quantitatively under identical disturbances. The stiff, five-decade spectrum makes such a comparison most efficient with an implicit or partitioned real-time integrator that exploits the time-scale separation, and provides the small-signal reference against which the EMT response can be checked mode by mode.

The model is intended as a rigorously initialized and analytically transparent foundation for further study, and several directions follow naturally. The small-signal characterization anticipated in earlier versions of this work is now provided in Section 6: the Jacobian spectrum, and the eigenvectors and participation factors that the released code also returns, isolate the modes responsible for oscillatory behaviour, and the strict invertibility of the constitutive map guarantees that this linearization is well defined. Treating the short-circuit ratio (equivalently the line impedance) as a bifurcation parameter offers a route to locating the onset of PLL-induced instabilities through Hopf and related bifurcations as the grid is progressively weakened, tracking the migration of the eigenvalues of Table 7 toward the imaginary axis. The semi-explicit differential-algebraic structure, analyzed in Section 2.4.3, invites a quantified singular-perturbation reduction with explicit approximation-error bounds built on the small parameters of Section 3.6. Finally, the modular construction extends to multi-machine aggregation for wind-farm studies, to grid-forming control laws, and to hardware-in-the-loop validation platforms, the last of which is the priority for establishing external, quantitative fidelity.

Author Contributions: Conceptualization, A.A.; methodology, A.A.; software, A.A.; validation, A.A. and I.A.; formal analysis, A.A.; investigation, A.A.; writing—original draft preparation, A.A.; writing—review and editing, I.A.; visualization, A.A.; supervision, I.A.; project administration, I.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The raw data supporting the conclusions of this article will be made available by the authors on request.

Conflicts of Interest: The authors declare no conflicts of interest.

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