

Article

Propagation Behaviors of Rayleigh Waves in Partially Saturated Poroviscoelastic Half-Space

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Abstract

This study investigates the propagation characteristics of Rayleigh waves in partially saturated poroviscoelastic half-spaces based on fundamental theories of elastic wave propagation. Firstly, a poroviscoelastic model tailored to unsaturated porous media is developed, integrating constitutive relations and dynamic equilibrium equations with fractional derivatives to effectively capture the complex, history-dependent viscoelastic behavior of materials. Secondly, the potential function method is employed to derive the dispersion equation for Rayleigh waves in the proposed medium. Lastly, numerical simulations are conducted to analyze the effects of liquid saturation and fractional order parameters on Rayleigh wave velocity, with results presented through graphical illustrations. The analysis reveals that both liquid saturation and fractional order significantly influence the propagation features of Rayleigh waves. This work provides valuable theoretical support for seismic exploration and related engineering applications involving unsaturated porous formations.

Keywords: partially saturated medium; Rayleigh wave; poroviscoelasticity; fractional order derivative; wave propagation

MSC: 86-10

1. Introduction

A Rayleigh wave is a typical elastic surface wave with energy concentrated near the medium surface [1,2], and it plays an essential role in geophysical exploration, geotechnical nondestructive testing, and seismic early warning [3–5]. First discovered by Lord Rayleigh in 1885, Rayleigh waves exhibit constant wave velocity and no dispersion in ideal homogeneous elastic half-spaces, where their velocity can be explicitly determined by the compressional and shear wave velocities of the medium. However, actual geological media are generally inhomogeneous, anisotropic, viscoelastic, and fluid-saturated porous materials. In such complex media, Rayleigh wave velocity is no longer constant but strongly dependent on frequency, liquid saturation, and viscosity. This induces significant wave dispersion and energy damping, making it difficult to accurately calculate wave velocity and attenuation [6].

The classical Biot theory is only applicable to saturated media and ignores the viscosity of the solid skeleton, making it difficult to accurately describe the actual wave propagation characteristics. With the continuous improvement of Biot's elastic wave propagation theory, numerous scholars have studied the propagation characteristics of Rayleigh waves across poroelastic media [7–9]. White et al. [10] extended the classic Biot theory and



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first proposed the wave-induced fluid flow (WIFF) loss mechanism, which clarifies the wave energy dissipation mechanism of periodic structures composed of two thin layers of single-phase fluid-saturated porous media. On this basis, Lo et al. [11] developed a wave propagation model for water-saturated soils containing gas or oil. Regarding capillary pressure as a single-valued function of saturation, this model derives the stress–strain constitutive relations and establishes the governing dynamic equations based on the mass conservation principles of the solid skeleton and two-phase fluids. To further characterize the macroscopic wave responses of multiphase porous media, Albers [12] combined the two-phase Biot model with mixture theory and constructed a macroscopic mechanical model for multiphase saturated media. In addition, many researchers have successfully proposed wave analysis models for porous rocks saturated with immiscible fluids, and relevant findings have been widely reported [13,14].

Viscoelasticity acts as a core factor governing the propagation properties of Rayleigh waves in complex media. Most natural media combine elastic and viscous behaviors, and the coupling between energy dissipation and deformation recovery greatly alters the propagation patterns of Rayleigh waves. Numerous studies have been conducted on this topic worldwide. He et al. [15] adopted the finite difference method to compare the propagation of Rayleigh waves in elastic and viscoelastic media, and analyzed the impacts of the quality factor and Poisson's ratio upon wave field energy. Using a linear viscoelastic model, Akbarov et al. [6] interpreted the dispersion characteristics and the relevant mechanisms of generalized Rayleigh waves in a viscoelastic half-space and clarified the law of viscous action via rheological parameters. Their results indicate that viscoelasticity reduces the velocity of Rayleigh waves. Taking rigid and yielding foundation conditions into account, Verma et al. [16] revealed the evolutionary features of Rayleigh wave velocity and damping in transversely isotropic viscoelastic layers, and verified that the viscoelastic response of the media exhibits prominent time-memory effects. Integer-order viscoelastic models were widely adopted in early research. However, such models cannot accurately present the deformation-history-dependent mechanical features of materials, and fail to characterize viscoelastic media with time-memory effects. Accordingly, they have inherent limitations in practical applications [17,18]. Related experiments combined with molecular dynamics simulations on viscoelastic dampers further prove that the interfacial bonding mode between viscoelastic and rigid components strongly affects dynamic modulus and energy dissipation, and obvious differences exist between Chemlok and epoxy resin bonding schemes [19].

In recent years, fractional derivative theory has become well established. Fractional order models are widely used to describe the complex deformation-history-dependent mechanical responses of media, and many scholars have developed corresponding fractional analysis models for complex medium conditions [20,21]. At present, fractional order viscoelastic theory has been successfully applied to studies on wave propagation in structures, including plates [21–25], cylinders [26,27], and sand half-spaces [28,29]. Kang et al. [30] discussed the energy transfer, reflection, transmission, and other issues of body waves in sandwich structures using the fractional order viscoelastic model. Zhao et al. [31] investigated the transient responses in multi-layered partially saturated porous media with the fractional order viscoelastic model. However, most existing studies apply fractional viscoelastic theory to transient problems and body waves. Meanwhile, most investigations on Rayleigh waves either assume elastic media or that multiple immiscible fluids in pores are not taken into consideration [32,33].

This paper takes the unsaturated poroviscoelastic half-space as the research object and explores the velocity characteristics of Rayleigh wave propagation. A fractional order viscoelastic model is applied to describe the mechanical behavior of the solid skeleton, with

emphasis on analyzing the impacts of saturation and fractional viscoelastic parameters of the solid framework on wave velocity. Referring to Reference [30], Section 2 derives the fractional order constitutive equations and dynamic equilibrium equations. Under free boundary conditions, potential functions are introduced in Section 3 to solve for the Rayleigh wave velocity. In Section 4, numerical results are adopted to explore the effects of the above parameters.

2. Poroviscoelastic Model for Partially Saturated Porous Medium

2.1. Constitutive Model

In this paper, we consider the partially saturated poroviscoelastic medium consisting of three phases, i.e., the viscoelastic solid skeleton, gas, and liquid phases. Let ϑ_s , ϑ_1 and ϑ_2 represent volumetric proportions for the solid skeleton, pore gas, and pore fluid, respectively, with $\vartheta_s + \vartheta_1 + \vartheta_2 = 1$, $\varphi = \vartheta_1 + \vartheta_2$ is the porosity. Let $S_1 = \vartheta_1/\varphi$ and $S_2 = \vartheta_2/\varphi$.

In the case of isotropic, fluid-saturated poroviscoelastic materials containing two immiscible fluids, the overall stress tensor $\boldsymbol{\tau}$ acting across a unit area can be separated into partial stress components, i.e., $\boldsymbol{\tau}_s$, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$. This decomposition explicitly considers the full viscoelastic behavior of the solid framework.

$$\boldsymbol{\tau} = \boldsymbol{\tau}_s + \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 \quad (1)$$

where $\boldsymbol{\tau}_s$, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ denote the partial stress tensors of the solid skeleton, gas, and liquid phases under drained conditions. The following constitutive relations are obtained after revising the equations in Reference [30]:

$$\boldsymbol{\tau}_s = 2\mu\boldsymbol{\varepsilon} + \left[\left(c_{11} - \frac{2}{3}\bar{\mu} \right) \nabla \cdot \mathbf{u} + c_{12} \nabla \cdot \mathbf{U}_1 + c_{13} \nabla \cdot \mathbf{U}_2 \right] \mathbf{I} \quad (2)$$

$$\boldsymbol{\tau}_1 = (c_{12} \nabla \cdot \mathbf{u} + c_{22} \nabla \cdot \mathbf{U}_1 + c_{23} \nabla \cdot \mathbf{U}_2) \mathbf{I} \quad (3)$$

$$\boldsymbol{\tau}_2 = (c_{13} \nabla \cdot \mathbf{u} + c_{23} \nabla \cdot \mathbf{U}_1 + c_{33} \nabla \cdot \mathbf{U}_2) \mathbf{I} \quad (4)$$

$\boldsymbol{\varepsilon}$ refers to the strain tensor of the solid framework; $\mathbf{u}$, $\mathbf{U}_1$ and $\mathbf{U}_2$ describe the displacement vectors of solid matrix, gaseous, and liquid phases under drained environments, and $\mathbf{I}$ is the unit tensor. The complex bulk and shear moduli, together with six material coefficients c_{ij} , can be written in the forms below [30]:

$$\bar{K}_b = K_{b0} \frac{1 + (i\omega\tau_{\epsilon_1})^{\gamma_1}}{1 + (i\omega\tau_{\zeta_1})^{\gamma_1}}, \quad 0 \leq \gamma_1 \leq 1 \quad (5a)$$

$$\bar{\mu} = \mu_0 \frac{1 + (i\omega\tau_{\epsilon_2})^{\gamma_2}}{1 + (i\omega\tau_{\zeta_2})^{\gamma_2}}, \quad 0 \leq \gamma_2 \leq 1 \quad (5b)$$

where K_{b0} and μ_0 represent asymptotic relaxed bulk and shear modulus in the limit of $t \rightarrow \infty$, respectively. Parameters γ_1 and γ_2 correspond to the fractional orders associated with the bulk and shear moduli of the drained solid skeleton, respectively. In this paper, the fractional order exponent characterizes the magnitude of the memory effect of viscoelastic materials. When it approaches zero, the influence of loading history only persists for a short period, the stress relaxation follows nearly exponential decay, the material rapidly forgets previous loading history, and behaves similarly to an ordinary viscous fluid. When it approaches unity, relaxation decays slowly following a power law; the loading history has a long-lasting impact, corresponding to damping materials with high damping capacity and severe hysteresis. The measurement method for this parameter is presented by Pawlak

Z M [32], $\tau_{\epsilon 1}$ and $\tau_{\sigma 1}$ represent the relaxation constants of volumetric strain and bulk stress, and $\tau_{\epsilon 2}$ and $\tau_{\sigma 2}$ refer to the relaxation times of shear strain and shear stress. Expressions for the six physical parameters c_{ij} can be found in Ref. [30]:

$$\begin{aligned} c_{11} &= \bar{K}_b, \quad c_{12} = c_{21} = K_1 \theta_s S_1 (A_1 + K_2) / D, \quad c_{13} = c_{31} = K_2 \theta_s S_2 (A_1 + K_1) / D \\ c_{22} &= K_1 \theta_1 (S_1 K_2 + A_1) / D, \quad c_{23} = c_{32} = K_1 K_2 S_1 S_2 \phi / D, \quad c_{33} = K_2 \theta_2 (S_2 K_1 + A_1) / D \\ D &= K_1 S_2 + A_1 + K_2 S_1, \quad A_1 = \frac{dp_c}{dS_1} S_1 S_2 \end{aligned} \quad (6)$$

The K_1 and K_2 represent the bulk modulus of gaseous and liquid phases, respectively. When two immiscible fluids coexist in a porous medium, a pressure discontinuity occurs at their interface due to fluid interaction. Under isothermal conditions and neglecting hysteresis effects, the capillary pressure can be expressed as $p_c = p_c(S_1)$.

2.2. Dynamic Equilibrium Equations

Under the assumptions of infinitesimal deformation and negligible external body forces, this paper adopts the momentum conservation equations presented in Reference [30]:

$$\rho_s \theta_s \ddot{\mathbf{u}} = \nabla \cdot \boldsymbol{\tau} + d_1 (\dot{\mathbf{U}}_1 - \dot{\mathbf{u}}) + d_2 (\dot{\mathbf{U}}_2 - \dot{\mathbf{u}}) \quad (7)$$

$$\rho_1 \theta_1 \ddot{\mathbf{U}}_1 = \nabla \cdot \boldsymbol{\tau}_1 - d_1 (\dot{\mathbf{U}}_1 - \dot{\mathbf{u}}) \quad (8)$$

$$\rho_2 \theta_2 \ddot{\mathbf{U}}_2 = \nabla \cdot \boldsymbol{\tau}_2 - d_2 (\dot{\mathbf{U}}_2 - \dot{\mathbf{u}}) \quad (9)$$

where ρ_s , ρ_1 and ρ_2 denote the densities of solid, gas, and liquid phases, respectively; $\dot{\mathbf{u}}$, $\dot{\mathbf{U}}_1$ and $\dot{\mathbf{U}}_2$ represent the first time derivative of displacement vectors for the solid skeleton, gaseous, and liquid phases; $\ddot{\mathbf{u}}$, $\ddot{\mathbf{U}}_1$ and $\ddot{\mathbf{U}}_2$ are the second-order temporal derivative of their relevant displacement vectors; and d_1 and d_2 are the viscous drag coefficients reported in [30].

By substituting Equations (2)–(4) into Equations (7)–(9), the dynamic displacement equations are derived.

$$\begin{aligned} &\rho_s \theta_s \ddot{\mathbf{u}} - d_1 (\dot{\mathbf{U}}_1 - \dot{\mathbf{u}}) - d_2 (\dot{\mathbf{U}}_2 - \dot{\mathbf{u}}) \\ &= \nabla \left[(c_{11} + \bar{\mu}/3) \nabla \cdot \mathbf{u} + c_{12} \nabla \cdot \dot{\mathbf{U}}_1 + c_{13} \nabla \cdot \dot{\mathbf{U}}_2 \right] + \bar{\mu} \nabla^2 \mathbf{u} \end{aligned} \quad (10)$$

$$\rho_1 \theta_1 \ddot{\mathbf{U}}_1 + d_1 (\dot{\mathbf{U}}_1 - \dot{\mathbf{u}}) = \nabla (c_{12} \nabla \cdot \mathbf{u} + c_{22} \nabla \cdot \dot{\mathbf{U}}_1 + c_{23} \nabla \cdot \dot{\mathbf{U}}_2) \quad (11)$$

$$\rho_2 \theta_2 \ddot{\mathbf{U}}_2 + d_2 (\dot{\mathbf{U}}_2 - \dot{\mathbf{u}}) = \nabla (c_{13} \nabla \cdot \mathbf{u} + c_{23} \nabla \cdot \dot{\mathbf{U}}_1 + c_{33} \nabla \cdot \dot{\mathbf{U}}_2) \quad (12)$$

3. Rayleigh Waves

As illustrated in Figure 1, the numerical analysis employs a Cartesian coordinate system. Rayleigh waves propagate within the x-y plane, which corresponds to the surface of the half-space medium composed of partially saturated poroviscoelastic material. The positive z-direction extends inward into the poroviscoelastic medium.

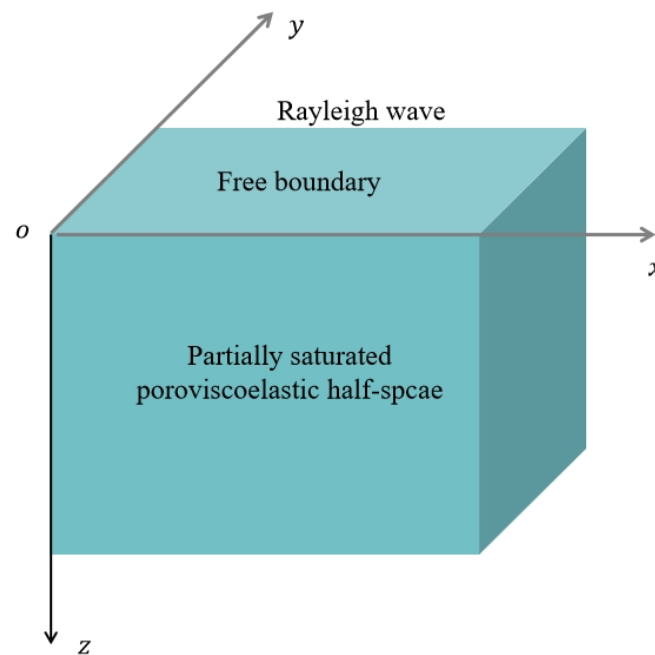


Figure 1. Partially saturated poroviscoelastic half-space.

3.1. Field Equations

By the Helmholtz vector decomposition theorem, a vector field can be decomposed into scalar and vector potentials:

$$\mathbf{u} = \nabla \varphi_s + \nabla \times \psi_s \mathbf{i}_2 \quad (13)$$

$$\mathbf{U}_1 = \nabla \varphi_1 + \nabla \times \psi_1 \mathbf{i}_2 \quad (14)$$

$$\mathbf{U}_2 = \nabla \varphi_2 + \nabla \times \psi_2 \mathbf{i}_2 \quad (15)$$

where φ_s , φ_1 and φ_2 denote the scalar compression potentials and ψ_s , ψ_1 and ψ_2 represent scalar shear potentials for the solid skeleton, liquid phase, and gas phase, respectively; the term $\psi_s \mathbf{i}_2$, $\psi_1 \mathbf{i}_2$ and $\psi_2 \mathbf{i}_2$ form the vector potential after multiplying by the unit vector $\mathbf{i}_2$.

Substituting Equations (13)–(15) into Equations (10)–(12), taking the divergence of both sides of the equation, and considering that the curl field is divergence-free, we obtain

$$\begin{aligned} (c_{11} + 4\bar{\mu}/3)\nabla^2 \varphi_s + c_{12}\nabla^2 \varphi_1 + c_{13}\nabla^2 \varphi_2 \\ = \rho_s \theta_s \ddot{\varphi}_s - d_1(\dot{\varphi}_1 - \dot{\varphi}_s) - d_2(\dot{\varphi}_2 - \dot{\varphi}_s) \end{aligned} \quad (16a)$$

$$c_{12}\nabla^2 \varphi_s + c_{22}\nabla^2 \varphi_1 + c_{23}\nabla^2 \varphi_2 = \rho_1 \theta_1 \ddot{\varphi}_1 + d_1(\dot{\varphi}_1 - \dot{\varphi}_s) \quad (16b)$$

$$c_{13}\nabla^2 \varphi_s + c_{23}\nabla^2 \varphi_1 + c_{33}\nabla^2 \varphi_2 = \rho_2 \theta_2 \ddot{\varphi}_2 + d_2(\dot{\varphi}_2 - \dot{\varphi}_s) \quad (16c)$$

With the same decomposition method, combining Equations (10)–(12), the curl is operated on both sides of the equations. Considering the irrotational property of the gradient field yields the following formulation:

$$\rho_s \theta_s \ddot{\psi}_s - d_1(\dot{\psi}_1 - \dot{\psi}_s) - d_2(\dot{\psi}_2 - \dot{\psi}_s) = \bar{\mu} \nabla^2 \psi_s \quad (17a)$$

$$\rho_1 \theta_1 \ddot{\psi}_1 + d_1(\dot{\psi}_1 - \dot{\psi}_s) = 0 \quad (17b)$$

$$\rho_2 \theta_2 \ddot{\psi}_2 + d_2(\dot{\psi}_2 - \dot{\psi}_s) = 0 \quad (17c)$$

In the present work, we adopt the potential functions given below:

$$\varphi_m = A_m \exp[-k_1 z + i(k_R x - \omega t)], \quad m = s, 1, 2 \quad (18a)$$

$$\psi_m = B_m \exp[-k_2 z + i(k_R x - \omega t)], \quad m = s, 1, 2 \quad (18b)$$

where $i^2 = -1$, k_R denotes the complex wave numbers associated with the Rayleigh wave, and ω represents the angular frequency, denoted by $\omega = 2\pi f$ (f stands for the frequency). A_m and B_m are the amplitude coefficients associated with the relevant potential functions. The decay exponents k_1 and k_2 are given by $k_1 = \sqrt{k_R^2 - k_p^2}$ and $k_2 = \sqrt{k_R^2 - k_s^2}$, where k_p and k_s are the complex wave numbers of the P-wave and S-wave.

Substituting Equation (18) into Equation (16), we obtain the dispersion equations as follows:

$$\begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} \begin{pmatrix} A_s \\ A_1 \\ A_2 \end{pmatrix} = 0 \quad (19)$$

where

$$\begin{aligned} E_{11} &= \rho_s \theta_s \omega^2 + i\omega(d_1 + d_2) - \left(c_{11} + \frac{4}{3}\bar{\mu}\right)k_p^2, \quad E_{12} = -i\omega d_1 - c_{12}k_p^2 \\ E_{13} &= i\omega d_2 - c_{13}k_p^2, \quad E_{21} = -i\omega d_1 - c_{12}k_p^2, \quad E_{22} = \rho_1 \theta_1 \omega^2 + i\omega d_1 - c_{22}k_p^2 \\ E_{23} &= -c_{23}k_p^2, \quad E_{31} = -i\omega d_2 - c_{13}k_p^2, \quad E_{32} = -c_{23}k_p^2 \\ E_{33} &= \rho_2 \theta_2 \omega^2 + i\omega d_2 - c_{33}k_p^2 \end{aligned}$$

Similarly, substituting Equation (18) into Equation (17), we obtain

$$\begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix} \begin{pmatrix} B_s \\ B_1 \\ B_2 \end{pmatrix} = 0 \quad (20)$$

where

$$\begin{aligned} G_{11} &= \rho_s \theta_s \omega^2 + i\omega(d_1 + d_2) + \bar{\mu}k_s^2, \quad G_{12} = -i\omega d_1, \quad G_{13} = -i\omega d_2 \\ G_{21} &= -i\omega d_1, \quad G_{22} = \rho_1 \theta_1 \omega^2 + i\omega d_1, \quad G_{23} = 0 \\ G_{31} &= -i\omega d_2, \quad G_{32} = 0, \quad G_{33} = \rho_2 \theta_2 \omega^2 + i\omega d_2 \end{aligned}$$

To ensure nontrivial solutions for Equations (19) and (20), the determinant of their respective coefficient matrices must be zero. For a specified angular frequency ω , we obtain six complex roots together with another two complex roots, expressed as $k_p = \text{Re}(k_p) + i\text{Im}(k_p)$ and $k_s = \text{Re}(k_s) + i\text{Im}(k_s)$, with Re and Im standing for real imaginary components. The real part stands for the wave number, whereas the imaginary part characterizes body wave attenuation. When $z \rightarrow \infty$, i.e., at infinity, the potential function defined by Equation (18) vanishes if and only if the imaginary part of the complex wave number of the body wave is positive. Therefore, only the valid solutions of k_p and k_s with positive imaginary parts of the complex wave number are retained. Accordingly, only three physically valid complex roots exist for Equation (19), with six solutions representing the complex wave numbers of compression waves (P1, P2, P3 waves sorted from high to low wave velocity). Likewise, Equation (20) yields only one physically valid root representing the complex wave number of the transverse wave. Generally, for a fixed angular frequency ω , the velocity of compression waves and transverse waves varies with frequency and is defined as $v_p = \omega/\text{Re}(k_p)$ and $v_s = \omega/\text{Re}(k_s)$, respectively.

A Rayleigh wave is generated by the superposition of the P-wave and S-wave at the free surface boundary of a half-space medium in a partially saturated poroviscoelastic medium. Equation (18) can be further rewritten as follows:

$$\varphi_s = \sum_{n=1}^3 \left[A_s^{(n)} \exp(-\zeta_1^{(n)} z) \right] \exp[i(k_R x - \omega t)] \quad (21a)$$

$$\varphi_1 = \sum_{n=1}^3 \left[A_1^{(n)} \exp(-\zeta_1^{(n)} z) \right] \exp[i(k_R x - \omega t)] \quad (21b)$$

$$\varphi_2 = \sum_{n=1}^3 \left[A_2^{(n)} \exp(-\zeta_1^{(n)} z) \right] \exp[i(k_R x - \omega t)] \quad (21c)$$

$$\psi_s = B_s \exp(-\zeta_2 z) \exp[i(k_R x - \omega t)] \quad (22a)$$

$$\psi_1 = B_1 \exp(-\zeta_2 z) \exp[i(k_R x - \omega t)] \quad (22b)$$

$$\psi_2 = B_2 \exp(-\zeta_2 z) \exp[i(k_R x - \omega t)] \quad (22c)$$

where $\zeta_1^{(n)} = \sqrt{k_R^2 - k_{pn}^2}$, ($n = 1, 2, 3$).

Using Equations (19) and (20), we obtain the following proportional coefficients for potential function amplitudes:

$$\delta_{f1p}^{(n)} = \frac{A_1^{(n)}}{A_s^{(n)}} = \frac{E_{21}^{(n)} E_{33}^{(n)} - E_{31}^{(n)} E_{23}^{(n)}}{E_{32}^{(n)} E_{23}^{(n)} - E_{22}^{(n)} E_{33}^{(n)}} \quad (23a)$$

$$\delta_{f2p}^{(n)} = \frac{A_2^{(n)}}{A_s^{(n)}} = \frac{E_{12}^{(n)} E_{21}^{(n)} - E_{11}^{(n)} E_{22}^{(n)}}{E_{13}^{(n)} E_{22}^{(n)} - E_{23}^{(n)} E_{12}^{(n)}} \quad (23b)$$

$$\delta_{f1s} = \frac{B_1}{B_s} = -\frac{G_{12}}{G_{22}} \quad (23c)$$

$$\delta_{f2s} = \frac{B_2}{B_s} = \frac{G_{12} G_{21} - G_{11} G_{22}}{G_{13} G_{22}} \quad (23d)$$

in which $E_{ij}^{(n)}$, ($n = 1, 2, 3$) can be expressed as

$$E_{11}^{(n)} = \rho_s \theta_s \omega^2 + i\omega(d_1 + d_2) - \left(c_{11} + \frac{4}{3}\bar{\mu}\right) k_{pn}^2$$

$$E_{12}^{(n)} = -i\omega d_1 - c_{12} k_{pn}^2, \quad E_{13}^{(n)} = -i\omega d_2 - c_{13} k_{pn}^2$$

$$E_{21}^{(n)} = -i\omega d_1 - c_{12} k_{pn}^2, \quad E_{22}^{(n)} = \rho_1 \theta_1 \omega^2 + i\omega d_1 - c_{22} k_{pn}^2, \quad E_{23}^{(n)} = -c_{23} k_{pn}^2, \quad E_{31}^{(n)} = -i\omega d_2 - c_{13} k_{pn}^2$$

$$E_{32}^{(n)} = -c_{23} k_{pn}^2, \quad E_{33}^{(n)} = \rho_2 \theta_2 \omega^2 + i\omega d_2 - c_{33} k_{pn}^2$$

3.2. Boundary Conditions

The plane $z = 0$ is considered a traction-free surface, Terzaghi boundary conditions are adopted, and the permeable and gas-permeable boundaries are represented by the zero deviatoric stress condition of the two-phase fluid [34], and the boundary conditions at $z = 0$ can be written as

$$\tau_s^{zz} = 0, \tau_s^{xz} = 0, \tau_1 = 0, \tau_2 = 0 \quad (24)$$

where

$$\begin{aligned} \tau_s^{zz} = & \left(c_{11} - \frac{2}{3}\bar{\mu} \right) \left(\frac{\partial^2 \varphi_s}{\partial x^2} + \frac{\partial^2 \varphi_s}{\partial z^2} \right) + 2\bar{\mu} \left(\frac{\partial^2 \varphi_s}{\partial z^2} + \frac{\partial^2 \psi_s}{\partial x \partial z} \right) \\ & + a_{12} \left(\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial z^2} \right) + a_{13} \left(\frac{\partial^2 \varphi_2}{\partial x^2} + \frac{\partial^2 \varphi_2}{\partial z^2} \right) \end{aligned} \quad (25a)$$

$$\tau_s^{xz} = \bar{\mu} \left(2 \frac{\partial^2 \varphi_s}{\partial x \partial z} + \frac{\partial^2 \psi_s}{\partial x^2} - \frac{\partial^2 \psi_s}{\partial z^2} \right) \quad (25b)$$

$$\tau_1 = c_{12} \left(\frac{\partial^2 \varphi_s}{\partial x^2} + \frac{\partial^2 \varphi_s}{\partial z^2} \right) + c_{22} \left(\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial z^2} \right) + c_{23} \left(\frac{\partial^2 \varphi_2}{\partial x^2} + \frac{\partial^2 \varphi_2}{\partial z^2} \right) \quad (25c)$$

$$\tau_2 = c_{13} \left(\frac{\partial^2 \varphi_s}{\partial x^2} + \frac{\partial^2 \varphi_s}{\partial z^2} \right) + c_{23} \left(\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial z^2} \right) + c_{33} \left(\frac{\partial^2 \varphi_2}{\partial x^2} + \frac{\partial^2 \varphi_2}{\partial z^2} \right) \quad (25d)$$

Using Equations (22)–(25), we derive the equations regarding potential amplitudes:

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{pmatrix} \begin{pmatrix} A_s^{(1)} \\ A_s^{(2)} \\ A_s^{(3)} \\ B_s \end{pmatrix} = 0 \quad (26)$$

the explicit expressions for parameters f_{ij} are given in Appendix A.

For Equation (26) to have nontrivial solutions, the determinant of the coefficient matrix, denoted by $\mathbf{F}$, must vanish. Hence, the dispersion equation of Rayleigh waves in this medium is expressed as

$$\mathbf{F} = \begin{vmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{vmatrix} = 0 \quad (27)$$

From Equation (27), we derive the complex wave number k_R . Its velocity is given by $v_R = \omega / \text{Re}(k_R)$.

4. Numerical Results and Discussion

To further analyze the propagation characteristics and mechanical response laws of Rayleigh waves in a partially saturated poroviscoelastic half-space, numerical simulations are carried out in this paper. The physical parameters adopted in the subsequent numerical examples are listed in Table 1.

Table 1. Physical properties of the partially saturated porous media [30,35].

$\rho_s (\text{kg} \cdot \text{m}^{-3})$	$\rho_1 (\text{kg} \cdot \text{m}^{-3})$	$\rho_2 (\text{kg} \cdot \text{m}^{-3})$	$K_1 (\text{Pa})$	$K_2 (\text{Pa})$	$\mu_0 (\text{Pa})$
2650	103	990	3.7×10^6	2.3×10^9	9×10^9
ϕ	$K_{b0} (\text{Pa})$	S_1	$\frac{dP_c}{dS_1} (\text{Pa})$	m	γ
0.2	12×10^9	0.5	0.05×10^6	0.8	0.5
$\tau_{e1} (\text{s})$	$\tau_{\sigma 1} (\text{s})$	$f (\text{Hz})$			
2×10^{-6}	1×10^{-6}	1000			

For the numerical examples in this section, we set $\gamma_1 = \gamma_2 = \gamma$ to simplify the parametric analysis. When $\gamma = 1$, the fractional order viscoelastic model degenerates

into the classical integer-order counterpart in terms of complex modulus. We use this special case as a control to verify the computational accuracy of the proposed fractional model. This simplified form is employed as a control case to validate the computational accuracy of the presented fractional model. The variations in Rayleigh wave velocity against excitation frequency are compared for $\gamma = 1$ (integer-order model) and $\gamma = 0.98$ (fractional order model).

The solution procedure for wave velocities in this paper is as follows:

Step 1. Solve body wave dispersion Equations (19) and (20) using the Newton method, and filter out physically meaningful roots by imposing the condition that the imaginary part is positive. The three sets of solutions corresponding to P-waves are distinguished based on the wave velocity relationship of the three longitudinal waves, while the shear wave solution is the unique root obtained by solving Equation (20) and applying the aforementioned screening criterion.

Step 2. Substitute the solved complex wave numbers of shear and longitudinal waves into Equation (27), and solve the complex wave number of Rayleigh waves via the Newton method. Subsequently, the wave velocities of all waves can be separately calculated from the relation between wave velocity and complex wave number.

Numerical results show that the two velocity curves share nearly identical overall trends (Figure 2). In the low-frequency range, the curves almost overlap, and the Rayleigh wave velocity remains nearly constant. As frequency increases, the wave velocity rises rapidly and gradually levels off at high frequencies. Notably, obvious discrepancies exist between the wave velocities for $\gamma = 1$ and $\gamma = 0.98$ within the high-frequency stable region. It demonstrates that even a slight change in fractional order can induce non-negligible influences on Rayleigh wave propagation via the time-memory effect of viscoelastic media, which further highlights the necessity and superiority of adopting the fractional order viscoelastic model.

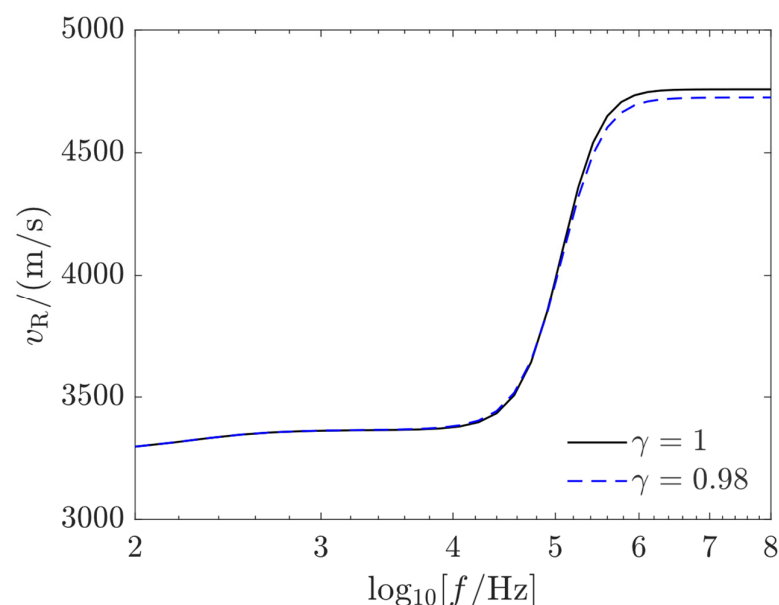


Figure 2. Variation comparison of Rayleigh wave velocity under different fractional orders.

Figure 3 presents the variation in Rayleigh wave velocity with liquid saturation at each frequency (10 Hz, 100 Hz, 500 Hz, and 1000 Hz) for a fixed viscoelastic coefficient of $\gamma_1 = \gamma_2 = 0.5$. When S_2 approaches zero, the medium approaches a dry condition where the pore space is predominantly occupied by gas. Gas possesses extremely high compressibility, and the drag loss between solid and gas phases is weaker than that between solid and liquid phases. This results in a higher overall stiffness of the medium and,

consequently, a greater wave velocity. Owing to the viscoelastic characteristics of the porous medium, the wave velocity increases markedly as the excitation frequency rises. Furthermore, when the frequency exceeds 100 Hz, the saturation variation exhibits a negligible influence on the Rayleigh wave velocity.

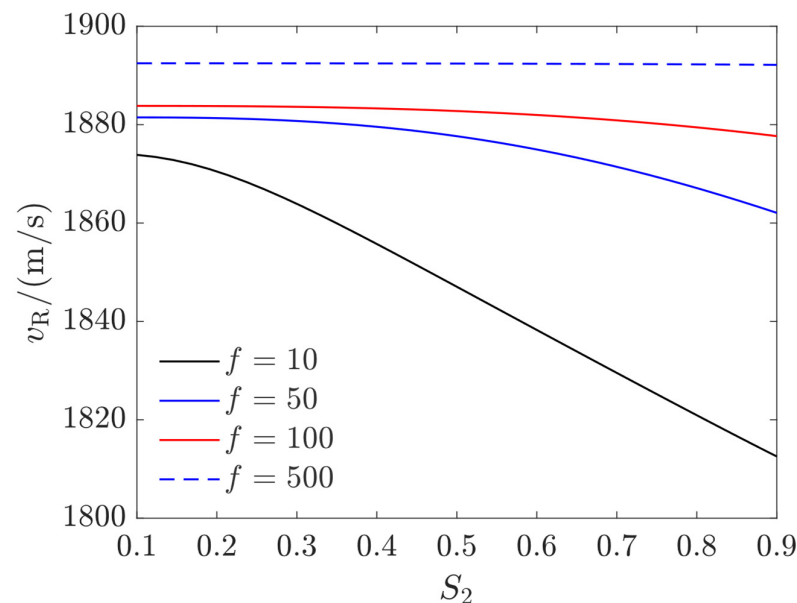


Figure 3. Variation in Rayleigh wave velocity versus liquid saturation at different frequencies.

Since Rayleigh waves are surface waves formed by the interfacial coupling and superposition of compressional and shear waves, this study investigates the variations in velocities of compressional and shear waves with fractional order parameters. Furthermore, the relationship between Rayleigh wave velocity and the viscoelastic properties of a half-space consisting of partially saturated porous viscoelastic media is derived.

Figure 4 illustrates the effect of the fractional derivative order on the propagation velocities of three compressive waves (P1, P2, P3) and shear waves. The velocities of P2 and P3 are minimally affected by changes in the fractional order, whereas this parameter significantly impacts the velocities of P1 and shear waves. This is attributed to the fact that P2 and P3 waves are inter-fluid relative motion waves dominated by fluid compressibility, solid-fluid drag and capillary interfacial forces. Since the fractional exponent describes the viscoelasticity of the solid skeleton, its impact on P2 and P3 wave velocities is comparatively minor. For a given fractional order, P1 and shear waves exhibit nearly identical variation trends.

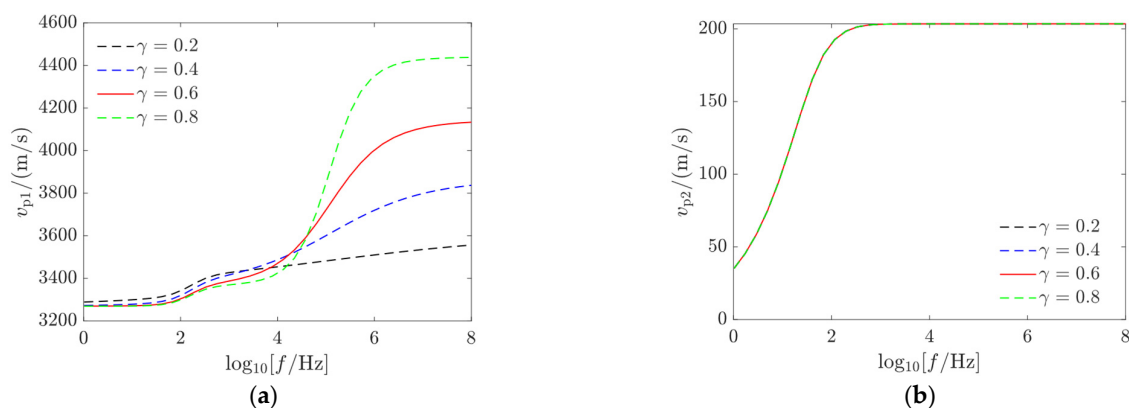


Figure 4. Cont.

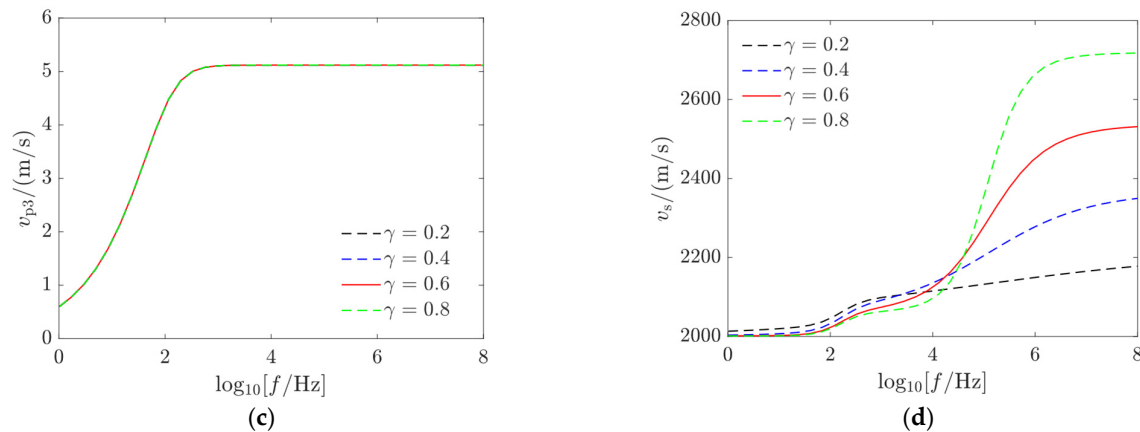


Figure 4. Velocities of (a) P_1 , (b) P_2 , (c) P_3 , and (d) S waves as functions of frequency at different fractional orders.

As shown in Figure 4b,c, all curves overlap during the initial stage. Similarly, in Figure 4a,d, the wave velocities corresponding to the four fractional orders are nearly identical at low frequencies (around 1 Hz). With increasing frequency, the curves show a gradual divergence, and the wave velocities tend to stabilize at a constant value in the high-frequency range. In the low-frequency region, a smaller fractional order γ results for larger wave velocities; conversely, in the high-frequency range, a larger γ leads to lower wave velocities. Overall, wave velocity increases rapidly at low frequencies, then exhibits slight variations and stabilizes at a steady value at higher frequencies. A larger fractional order corresponds to a higher steady-state velocity. For the fractional viscoelastic model depicted in Figure 5, the frequency dispersion of the Rayleigh wave velocity is primarily governed by the superposition of multiple wave components.

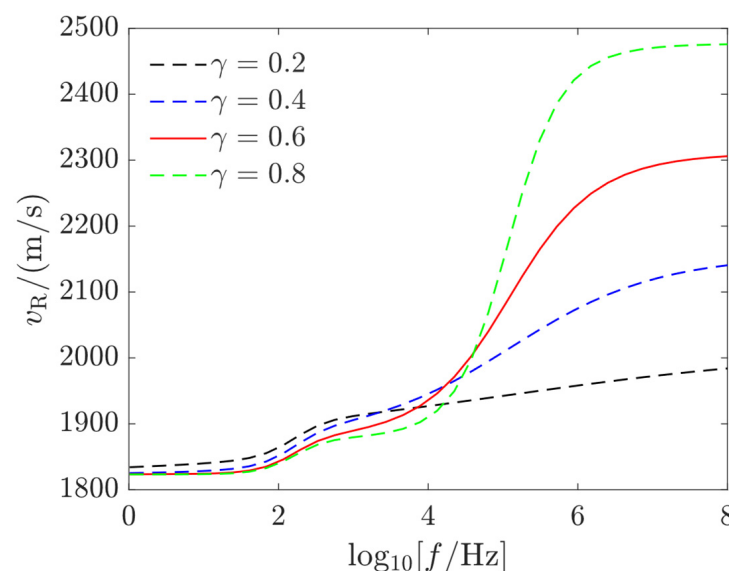


Figure 5. Variation in Rayleigh wave speed with frequency across different fractional derivative parameters.

Figure 6 shows the variation characteristics of Rayleigh wave velocity with intrinsic permeability in a half-space semi-saturated porous viscoelastic medium under different liquid saturations ($S_2 = 0.01, 0.6, 0.7, 0.8$). The results demonstrate that the velocity of Rayleigh waves remains stable in low and high permeability regions, and rises rapidly within the medium permeability range. Additionally, wave velocities under different liquid saturations tend to converge at high permeability levels. Furthermore, when the liquid saturation is extremely low, the permeability has a very limited influence on the wave

velocity, which is the reason that the effects of viscous and inertial coupling between the gas–liquid two-phase and the solid particles are very small when the soil is close to drying.

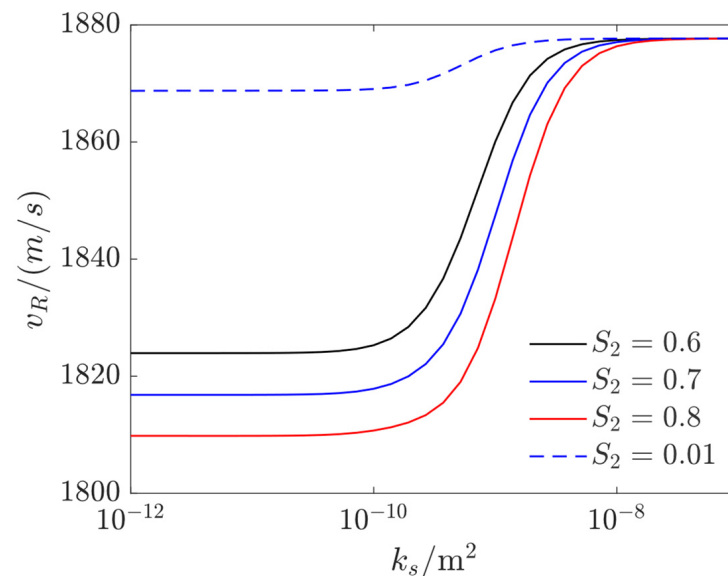


Figure 6. Variation in Rayleigh wave velocity with intrinsic permeability for various saturations.

Notably, at $S_2 = 0.01$, the medium is nearly dry, and pore spaces are dominated by gas. Since viscous and inertial coupling between the solid matrix and pore fluids can be ignored, intrinsic permeability has a negligible impact on Rayleigh wave velocity.

Figure 7 illustrates the combined effects of liquid saturation S_2 and fractional order γ on the evolution of Rayleigh wave velocity v_R at four typical excitation frequencies ($f = 10$ Hz, 100 Hz, 1000 Hz, 10^5 Hz). At low frequencies, liquid saturation significantly suppresses Rayleigh wave velocity, which declines as saturation increases. When the frequency enters the high-frequency region, the influence of saturation nearly disappears, and the velocity curves for different saturation values overlap entirely. Regarding the fractional order: in the low-frequency range, Rayleigh wave velocity follows a unimodal trend, rising first and then falling with the increase in fractional order; in contrast, the wave velocity is monotonically positively correlated with fractional order at high frequency.

To quantify the relative magnitudes of the influences exerted by fluid saturation S_2 and viscoelasticity characterized by the fractional order exponent on wave velocity, a sensitivity analysis is conducted as follows. With the other parameter held constant, identical small perturbations are applied to S_2 and γ , and the resulting variations in Rayleigh wave velocity induced by the two perturbations are compared. As illustrated in Figure 8, the Rayleigh wave velocity exhibits nearly equal sensitivity to the two parameters at low frequencies. At high frequencies, however, the wave velocity is far more sensitive to viscoelasticity than to saturation. This phenomenon can be explained as follows: low-frequency pore fluids possess sufficient time for dynamic response, so the two-phase seepage effect associated with saturation can be fully activated, rendering saturation one of the dominant non-negligible variables. In contrast, pore fluids are inertially trapped within pores under high-frequency excitation, and macroscopic cross-pore seepage barely takes place. The equivalent modulus of the medium remains nearly unchanged, which leads to negligible variations in wave velocity.

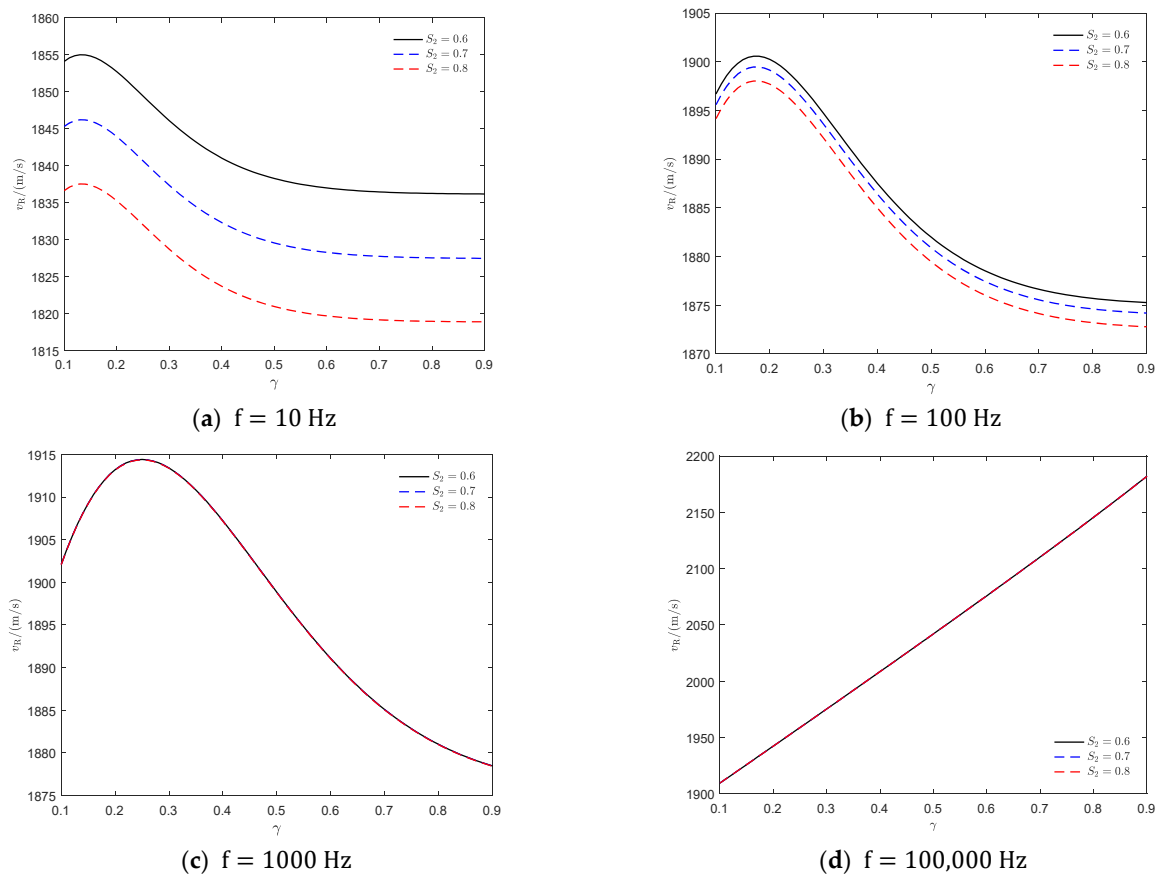


Figure 7. Variation in the velocity of the Rayleigh wave with fractional order under different saturations.

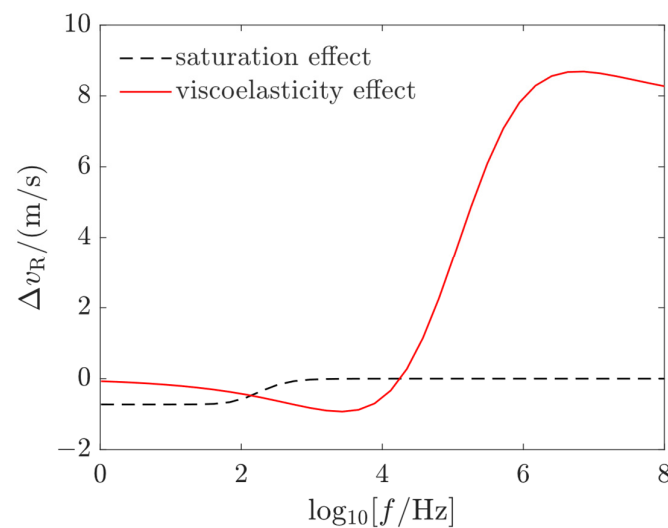


Figure 8. Frequency-dependent sensitivity of Rayleigh wave velocity to saturation and viscoelastic parameters.

To verify the validity of the model established in this paper, the viscoelastic constitutive relation adopted herein is degraded into an elastic one and compared with numerical examples from published literature. As shown in Figure 9, the results show that the predictions of this work are in good agreement with those reported in the reference within an acceptable error range. For instance, at a frequency of $f = 1000$ Hz, the wave velocity calculated by the proposed model is 1231.8 m/s, while the value given in Ref. [30] is 1232.1 m/s, yielding a relative error of only 0.024%.

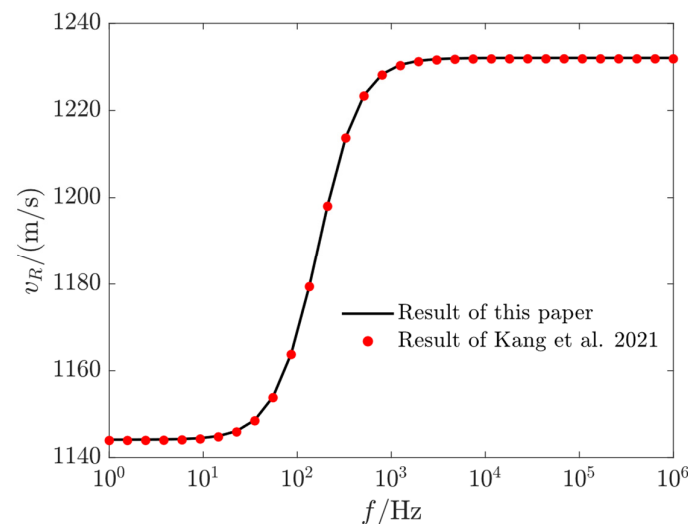


Figure 9. Comparison of dispersion curves [30].

5. Conclusions

This paper establishes a theoretical model for three-phase partially saturated porous media incorporating fractional order viscoelasticity, derives the dispersion equation of Rayleigh waves, and analyzes the modulation law of various physical parameters on Rayleigh wave velocity.

(1) Mathematical innovations of the model:

The fractional Zener constitutive model is introduced into solid–gas–liquid three-phase porous media, and frequency-dependent complex moduli are constructed to uniformly characterize the memory viscoelasticity of the solid skeleton. The dispersion equations for body waves and Rayleigh surface waves are decoupled layer-by-layer via Helmholtz potential decomposition.

(2) Core physical laws:

This paper investigates the modulation laws of liquid saturation, permeability and viscoelasticity on Rayleigh wave dispersion in incompletely saturated porous media. Numerical results reveal that an increase in liquid saturation reduces the overall equivalent stiffness of the medium, which further lowers the Rayleigh wave velocity. At high frequencies, however, Rayleigh waves exhibit low sensitivity to saturation variations due to fluid inertial effects.

The phase velocity of Rayleigh waves remains nearly constant in both low- and high-permeability regimes, while it rises sharply with increasing permeability within the intermediate permeability range. Moreover, the Rayleigh wave phase velocities under different saturation levels eventually converge to the same value.

Viscoelasticity imposes a pronounced influence on Rayleigh wave velocity. In the low-to-intermediate frequency band, wave velocity first increases and then decreases as the fractional order exponent grows. At sufficiently high frequencies, the vibration period is far shorter than the medium relaxation time. In this case, the delayed additional stiffness induced by the rising fractional order exponent becomes the dominant factor, leading to a positive correlation between wave velocity and the fractional order exponent.

(3) Limitations:

This paper assumes an isotropic medium, small deformations, isothermal conditions, and neglects body forces. The isotropic assumption is inapplicable to media containing bedding planes and fractures. The small-deformation assumption fails to describe large

deformation scenarios induced by strong earthquakes, blasting and other dynamic loads. The isothermal hypothesis cannot be adopted for working conditions such as deep geothermal reservoirs and low-temperature frozen soils. In addition, neglecting body forces is not suitable for deep stratigraphic environments.

(4) Future research directions:

To make the theoretical research more consistent with practical engineering conditions, based on the work of this paper, we will further investigate wave propagation problems in transversely isotropic, anisotropic and inhomogeneous porous media in subsequent studies.

In general, the results indicate that relative to the conventional integer-order model, the fractional order Zener model provides a more accurate representation of the microscopic deformation and viscoelastic behavior of porous media. For practical seismic exploration, it is essential to comprehensively consider the coupled effects of saturation, permeability, excitation frequency, and fractional viscoelastic parameters on Rayleigh wave propagation. The results obtained in this paper provide references for various engineering applications, including seismic exploration, geotechnical site evaluation, hydrocarbon reservoir monitoring, non-destructive testing, and underground structure inspection.

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Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Derivation of Equation (26).

By substituting the potential functions (21) and (22) into Equation (25a) and applying the boundary conditions from Equation (24), we eliminate the common factor on both sides and obtain

$$\begin{aligned} & \left(c_{11} - \frac{2}{3}\bar{\mu}\right) \left[(ik_R)^2 \sum_{n=1}^3 A_s^{(n)} + \sum_{n=1}^3 \left(-\xi_1^{(n)}\right)^2 A_s^{(n)} \right] + 2\bar{\mu} \left[\sum_{n=1}^3 \left(-\xi_1^{(n)}\right)^2 A_s^{(n)} + (ik_R)(-\xi_2) B_s \right] \\ & + a_{12} \left[(ik_R)^2 \sum_{n=1}^3 A_1^{(n)} + \sum_{n=1}^3 \left(-\xi_1^{(n)}\right)^2 A_1^{(n)} \right] + a_{13} \left[(ik_R)^2 \sum_{n=1}^3 A_2^{(n)} + \sum_{n=1}^3 \left(-\xi_1^{(n)}\right)^2 A_2^{(n)} \right] = 0 \end{aligned}$$

Combining Equation (23), we reorganize the amplitude terms in the foregoing expression to derive an amplitude equation:

$$\begin{aligned} & \sum_{n=1}^3 \left[\left(c_{11} - \frac{2}{3}\bar{\mu}\right) k_{pn}^2 + 2\bar{\mu} \left(k_{pn}^2 - k_R^2\right) + a_{12} k_{pn}^2 \delta_{f1p}^{(n)} + a_{13} k_{pn}^2 \delta_{f2p}^{(n)} \right] A_s^{(n)} \\ & + \left(2\bar{\mu} ik_R \sqrt{k_R^2 - k_S^2} \right) B_s = 0 \end{aligned}$$

Similarly, the other three equations in matrix Equation (26) can be obtained.

Expressions for elements of the coefficient matrix in Equation (26) are as follows:

$$\begin{aligned}
 f_{11} &= \left(c_{11} - \frac{2}{3}\bar{\mu}\right)k_{p1}^2 + 2\bar{\mu}(k_{p1}^2 - k_R^2) + a_{12}k_{p1}^2\delta_{f1p}^{(1)} + a_{13}k_{p1}^2\delta_{f2p}^{(1)} \\
 f_{12} &= \left(c_{11} - \frac{2}{3}\bar{\mu}\right)k_{p2}^2 + 2\bar{\mu}(k_{p2}^2 - k_R^2) + a_{12}k_{p2}^2\delta_{f1p}^{(2)} + a_{13}k_{p2}^2\delta_{f2p}^{(2)} \\
 f_{13} &= \left(c_{11} - \frac{2}{3}\bar{\mu}\right)k_{p3}^2 + 2\bar{\mu}(k_{p3}^2 - k_R^2) + a_{12}k_{p3}^2\delta_{f1p}^{(3)} + a_{13}k_{p3}^2\delta_{f2p}^{(3)} \\
 f_{14} &= 2\bar{\mu}ik_R\sqrt{k_R^2 - k_S^2}, \quad f_{21} = 2\bar{\mu}ik_R\sqrt{k_R^2 - k_{p1}^2}, \quad f_{22} = 2\bar{\mu}ik_R\sqrt{k_R^2 - k_{p2}^2} \\
 f_{23} &= 2\bar{\mu}ik_R\sqrt{k_R^2 - k_{p3}^2}, \quad f_{24} = \bar{\mu}(2k_R^2 - k_S^2), \\
 f_{31} &= k_{p1}^2(c_{12} + c_{22}\delta_{f1p}^{(1)} + c_{23}\delta_{f2p}^{(1)}), \quad f_{32} = k_{p2}^2(c_{12} + c_{22}\delta_{f1p}^{(2)} + c_{23}\delta_{f2p}^{(2)}) \\
 f_{33} &= k_{p2}^2(c_{12} + c_{22}\delta_{f1p}^{(3)} + c_{23}\delta_{f2p}^{(3)}), \quad f_{41} = k_{p3}^2(c_{13} + c_{23}\delta_{f1p}^{(1)} + c_{33}\delta_{f2p}^{(1)}) \\
 f_{42} &= k_{p2}^2(c_{13} + c_{23}\delta_{f1p}^{(2)} + c_{33}\delta_{f2p}^{(2)}), \quad f_{43} = k_{p3}^2(c_{13} + c_{23}\delta_{f1p}^{(3)} + c_{33}\delta_{f2p}^{(3)}), \\
 f_{34} &= 0, \quad f_{44} = 0
 \end{aligned}$$

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