

Article

Complete Optimal Cost Design for RIFs, Assuming That the Contact Area Is Partially Compressed with Eccentric Column

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Abstract

This article presents a complete cost-optimized design for rectangular isolated footings (RIFs) under biaxial bending with a partially compressed contact area (PCCA) and an eccentric column, i.e., part of the base is in contact with the soil (generating compression), and the other part is not in contact with the soil (generating neither compression nor tension), and the soil pressure behaves linearly. The proposed model is presented by integration to find the moments in the critical sections, the bending shears in the critical sections and the punching shear acting on the RIF, and these are compared with those that must be resisted according to the standards (ACI 318-19). Some articles show complete designs for RIFs with a fully compressed contact area (FCCA) and an eccentric column, and other articles show only the minimum area (A_{min}) for RIFs with a PCCA and an eccentric column. The main contributions of this article are as follows: (1) the A_{min} is determined from the loads and moments, assuming that the loads and moments it resists must be greater than or equal to those acting on the RIF; (2) the minimum cost (C_{min}) is determined. Three numerical studies are described to demonstrate the advantages of this study over other models. The results indicate that, according to the studies presented in this article, savings of up to 45.99% for A_{min} and 93.62% for C_{min} can be achieved using the model proposed in this article (MPA); this occurs when the pressure acting on the foundation does not reach the maximum available soil pressure (p_{max}). Therefore, this paper could be of great use to engineers dedicated to the construction of foundations.

Keywords: complete cost-optimized design; rectangular isolated footings; eccentric column; partially compressed contact area

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1. Introduction

A RIF is a shallow foundation used to transmit the point loads from structures, such as columns, to the ground. These foundations are designed to efficiently distribute the load to the soil, preventing differential settlement. They are generally made of concrete and can be square or rectangular, depending on the load they must support. Their design takes into account factors such as the p_{max} and the characteristics of the terrain to guarantee the stability and durability of the structure.

Several researchers have presented information in various publications on the p_{max} for different foundations, such as Lee et al. [1], Omid and Hosseiniinia [2], Hamed and Ehsan [3], Yang et al. [4], Zhang et al. [5], Turedi et al. [6], Khennouf and Baheddi [7], Surya and Manash [8], Das et al. [9], Elkafrawy et al. [10], Banu et al. [11], and Demirdogen and Han [12].

Studies have found that the column-loading conditions of the foundation and its response can be verified through the capacity of the structural system. For example, Huang et al. [13] estimated the bearing capacity of reinforced-concrete (RC) columns subjected to combined loadings, reinforced with high-performance ferrocement laminate and bonded steel plates. Guo et al. [14] investigated the capacity of an RC frame to withstand the loss of a corner column. Sun et al. [15] presented the behavior of the compressive response and pore structure of self-compacting concrete at 3, 7, 14, and 28 days for three representative curing regimes, such as 20 °C–95% RH, 10 °C–75% RH, and 3 °C–50% RH.

The most significant studies to find the A_{min} for RIFs, assuming a PCCA, have been conducted by several researchers, such as Algin [16,17], Özmen [18], Rodriguez-Gutierrez and Aristizabal-Ochoa [19,20], Aydogdu [21], Girgin [22], Rawat et al. [23], Lezgy-Nazargah et al. [24], Himeur et al. [25], Vela-Moreno et al. [26], and Moreno-Landeros et al. [27].

The most relevant research on the design of reinforced-concrete RIFs includes the studies by Camp and Assadollahi [28], Rawat and Mittal [29], Nigdeli et al. [30], Galvis and Smith-Pardo [31], Chaudhuri and Maity [32], Al-Ansari and Afzal [33], Solorzano and Plevris [34], Waheed et al. [35], and Luévanos-Rojas [36,37].

The articles most related to the topic discussed in this work include that by Vela-Moreno et al. [26], who determined the A_{min} , assuming a PCCA with a concentric column. Moreno-Landeros et al. [27] investigated the A_{min} , assuming a PCCA with an eccentric column. Luévanos-Rojas [36] developed a complete design for RIFs, assuming an FCCA with an eccentric column. Luévanos-Rojas [37] investigated the complete design for RIFs, assuming a PCCA with a concentric column.

This model allows finding the complete and cost-optimized design for RIFs, assuming a PCCA, i.e., a part of the contact zone of the foundation is compressed and another part has no pressure. The formulation is performed by integration to determine the effects on the critical sections acting on the RIF, and these are compared with those provided by the standards (ACI 318-19) [38]. Some researchers have presented complete designs of RIFs for an FCCA with an eccentric column [36], and other researchers have presented only the A_{min} of a RIF for a PCCA with an eccentric column [27]. The contribution of this paper is to determine the A_{min} (RIF sides) from the unfactored loads and moments, assuming that the loads and moments it resists must be greater than or equal to those acting on the RIF, and to determine the C_{min} , the factored loads and moments are used. Three comparative studies are presented to observe the differences: (1) the MPA and the model proposed by Luévanos-Rojas (MPLR) for C_{min} [36]; (2) the MPA and the model proposed by Moreno-Landeros (MPML) for A_{min} [27]; and (3) the MPA of the complete design from the A_{min} data obtained in the previous study and the PCCA for C_{min} .

2. Mathematical Model

This paper starts from the following premise, according to Bowles [39]: the foundation is completely rigid and rests on an elastic and homogeneous soil, meaning the soil pressure on the RIF is linear.

Figure 1 represents the complete resultant eccentricities diagram of the base for the RIF, depending on the location of P .

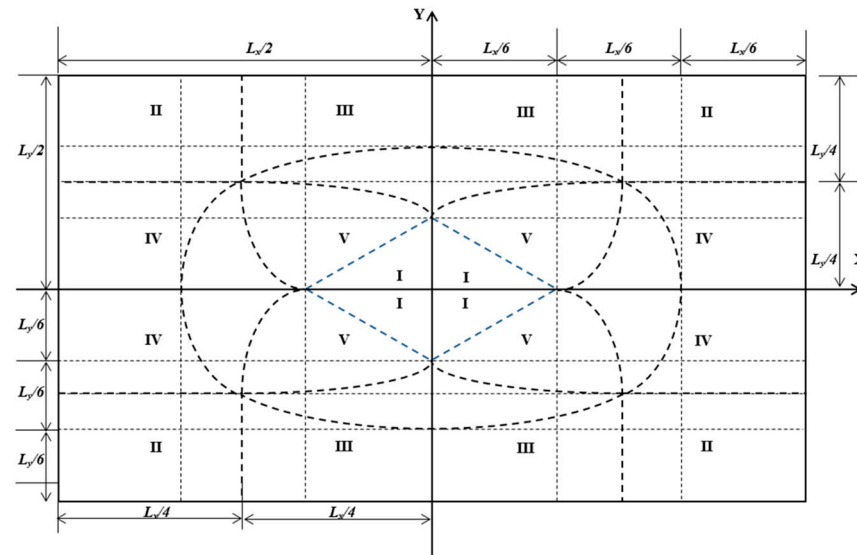


Figure 1. Complete diagram of the resultant eccentricities for RIF.

Table 1 indicates the total eccentricity relationships for each case that must be met [26].

Table 1. Relationships of eccentricities for each case [26].

Case	Eccentricity Relationship
I	$\frac{e_{xT}}{L_x} + \frac{e_{yT}}{L_y} \leq \frac{1}{6}$
II	$\frac{e_{xT}}{L_x} \geq \frac{1}{4}; \frac{e_{yT}}{L_y} \geq \frac{1}{4}$
III	$\frac{e_{xT}}{L_x} \leq \frac{1}{4}; \frac{e_{yT}}{L_y} \geq \frac{1}{4} + \frac{(4e_{xT} - L_x)^3}{12L_x^3}$
IV	$\frac{e_{xT}}{L_x} \geq \frac{1}{4} + \frac{(4e_{yT} - L_y)^3}{12L_y^3}; \frac{e_{yT}}{L_y} \leq \frac{1}{4}$
V	$\frac{e_{xT}}{L_x} + \frac{e_{yT}}{L_y} \geq \frac{1}{6}; \frac{e_{xT}}{L_x} \leq \frac{1}{4} + \frac{(4e_{yT} - L_y)^3}{12L_y^3}; \frac{e_{yT}}{L_y} \leq \frac{1}{4} + \frac{(4e_{xT} - L_x)^3}{12L_x^3}$

where e_{xT} is the total eccentricity on the X-axis (m) = M_{yA}/P , and e_{yT} is the total eccentricity on the Y-axis (m) = M_{xA}/P , $M_{xA} = Pe_y + M_x$, and $M_{yA} = Pe_x + M_y$. Loads and moments are not factored to determine the area.

Figure 2 represents a RIF subjected to biaxial bending with an eccentric column.

The biaxial bending equation, which applies when the ultimate axial load P_u is within the central nucleus (FCCA), is [36]

$$p_{us} = \frac{P_u}{A} + \frac{(P_u e_y + M_{ux})y}{I_x} + \frac{(P_u e_x + M_{uy})x}{I_y}, \quad (1)$$

where p_{us} is the ultimate soil pressure in any region of the base (kN/m²); P_u is the ultimate axial load (kN); A is the contact area with the soil at the base (m²); M_{uxA} ($M_{uxA} = P_u e_y + M_{ux}$)

and M_{uyA} ($M_{uyA} = P_u e_x + M_{uy}$) are the ultimate resultant moments acting on both axes, the main axes (kN-m); M_{ux} and M_{uy} are the ultimate moments acting on the column on both axes, the column axes (kN-m); e_x is the eccentricity of the column on the X-axis (m); e_y is the eccentricity of the column on the Y-axis (m); I_x and I_y are the moments of inertia with respect to the X- and Y-axes (m⁴); and x and y are the coordinates of both axes at the base where the point of study is located (m).

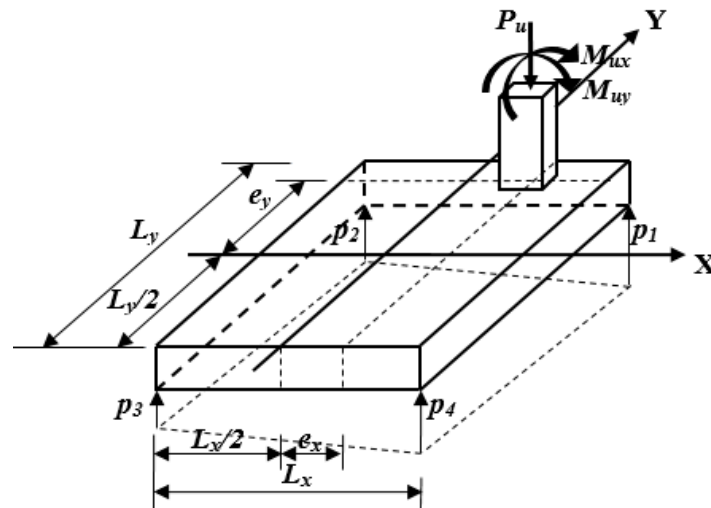


Figure 2. RIF subjected to biaxial bending.

The biaxial bending equation is not functional when the axial load P_u is not within the central nucleus (PCCA). The biaxial bending equation is [27]

$$p_{uz} = \frac{p_{umax} [L_{x1}(2y - L_y) + L_{y1}(2x - L_x) + 2L_{x1}L_{y1}]}{2L_{x1}L_{y1}}, \quad (2)$$

where p_{uz} is the ultimate ground pressure at any point of the footing (kN/m²), p_{umax} is the ultimate maximum pressure generated by the ground (kN/m²), L_{x1} is the X distance from p_1 to where it crosses NA, L_{y1} is the Y distance from p_1 to where it crosses NA, and L_x and L_y are the sides of the RIF.

The critical sections of moments occur on the faces of the column; the critical sections for bending shears appear at a distance d (the effective depth of footing) from the column faces; and the critical sections for punching shears occur at a perimeter formed at a distance $d/2$ from the column faces in the two directions [38].

2.1. A_{min} for RIF

Figure 3 represents a RIF under biaxial bending with five possible cases to determine the A_{min} . To determine the A_{min} , the unfactored loads and moments must be used.

A_{min} (objective function) is determined by

$$A_{min} = L_x L_y, \quad (3)$$

Table 2 indicates the constraints for P_R , M_{xR} and M_{yR} for the five cases [27].

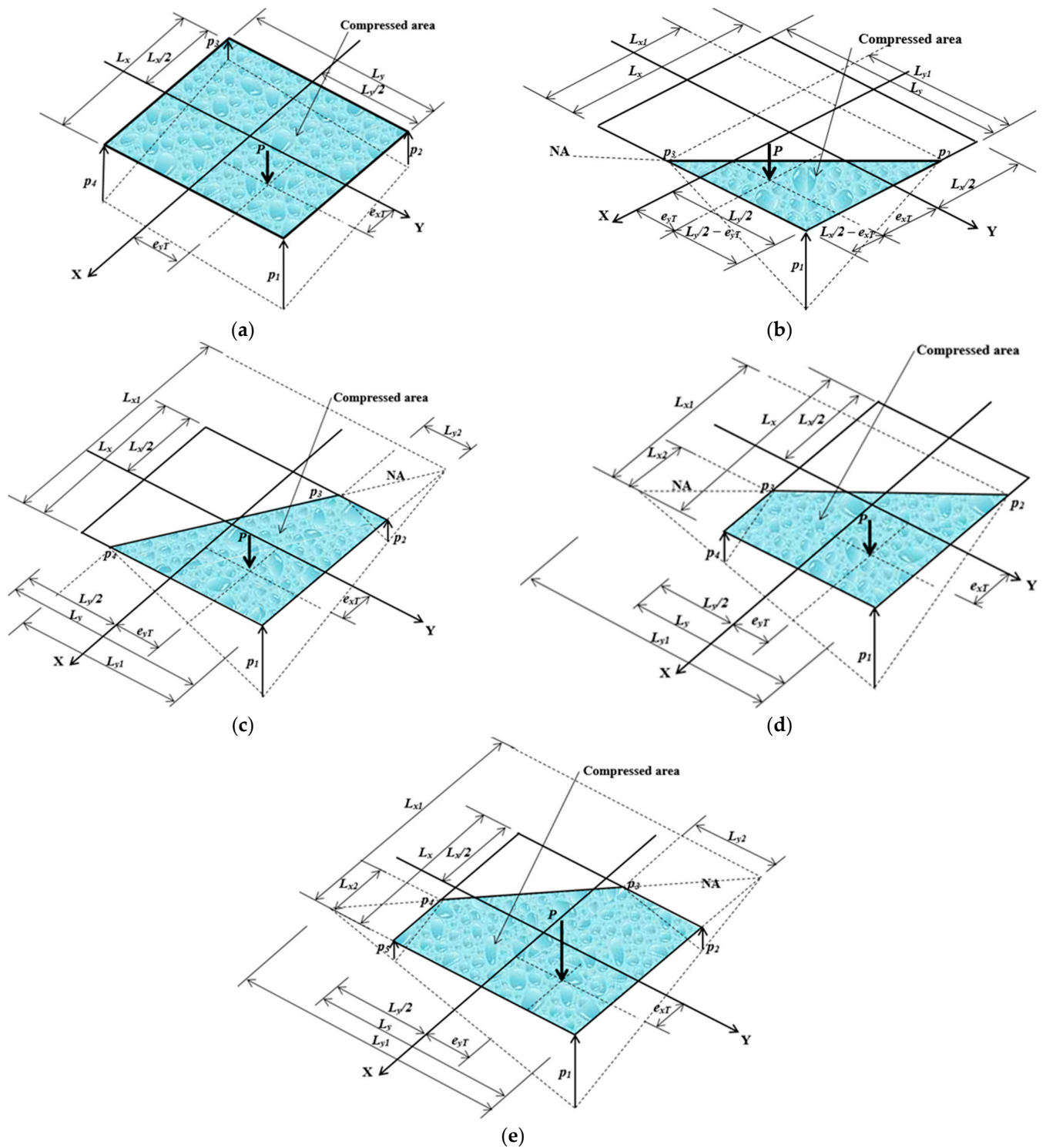


Figure 3. RIF with five possible cases: (a) Case I; (b) Case II; (c) Case III; (d) Case IV; and (e) Case V.

Table 2. A_{min} constraints for biaxial bending with loads and moments without factoring.

Case	Constraints
I	$p_1 = \frac{P}{L_x L_y} + \frac{6(M_x + P e_y)}{L_x L_y^2} + \frac{6(M_y + P e_x)}{L_x^2 L_y}$
	$p_2 = \frac{P}{L_x L_y} + \frac{6(M_x + P e_y)}{L_x L_y^2} - \frac{6(M_y + P e_x)}{L_x^2 L_y}$
	$p_3 = \frac{P}{L_x L_y} - \frac{6(M_x + P e_y)}{L_x L_y^2} - \frac{6(M_y + P e_x)}{L_x^2 L_y}$

	$p_4 = \frac{P}{L_x L_y} - \frac{6(M_x + P e_y)}{L_x L_y^2} + \frac{6(M_y + P e_x)}{L_x^2 L_y}$
	$0 \leq p_1, p_2, p_3, p_4 \leq p_{max}$
II	$P_R = \frac{p_{max} L_{x1} L_{y1}}{6}$ $M_{xR} = \frac{p_{max} L_{x1} L_{y1} (2L_y - L_{y1})}{24}$ $M_{yR} = \frac{p_{max} L_{x1} L_{y1} (2L_x - L_{x1})}{24}$ $P_A \leq P_R, M_{xA} \leq M_{xR}, M_{yA} \leq M_{yR}, L_x \geq L_{x1} \text{ and } L_y \geq L_{y1}$
III	$P_R = \frac{p_{max} L_{y1} [L_{x1}^3 - (L_{x1} - L_x)^3]}{6 L_{x1}^2}$ $M_{xR} = \frac{p_{max} L_{y1} \{L_{x1}^4 (2L_y - L_{y1}) + (L_x - L_{x1})^3 [L_{x1} (2L_y - L_{y1}) + L_x L_{y1}]\}}{24 L_{x1}^3}$ $M_{yR} = \frac{p_{max} L_{y1} L_x^3 (2L_{x1} - L_x)}{24 L_{x1}^2}$ $P_A \leq P_R, M_{xA} \leq M_{xR}, M_{yA} \leq M_{yR}, L_x \leq L_{x1} \text{ and } L_y \geq L_{y1}$
IV	$P_R = \frac{p_{max} L_{x1} [L_{y1}^3 - (L_{y1} - L_y)^3]}{6 L_{y1}^2}$ $M_{yR} = \frac{p_{max} L_{x1} \{L_{y1}^4 (2L_x - L_{x1}) + (L_y - L_{y1})^3 [L_{y1} (2L_x - L_{x1}) + L_y L_{x1}]\}}{24 L_{y1}^3}$ $M_{xR} = \frac{p_{max} L_{x1} L_y^3 (2L_{y1} - L_y)}{24 L_{y1}^2}$ $P_A \leq P_R, M_{xA} \leq M_{xR}, M_{yA} \leq M_{yR}, L_x \geq L_{x1} \text{ and } L_y \leq L_{y1}$
V	$P_R = \frac{p_{max} [L_{x1}^3 L_{y1}^3 + L_{y1}^3 (L_x - L_{x1})^3 + L_{x1}^3 (L_y - L_{y1})^3]}{6 L_{x1}^2 L_{y1}^2}$ $M_{xR} = \frac{p_{max} \{L_{x1}^4 L_y^3 (2L_{y1} - L_y) - L_{y1}^3 (L_{x1} - L_x)^3 [L_{x1} (2L_y - L_{y1}) + L_x L_{y1}]\}}{24 L_{x1}^3 L_{y1}^2}$ $M_{yR} = \frac{p_{max} \{L_{y1}^4 L_x^3 (2L_{x1} - L_x) - L_{x1}^3 (L_{y1} - L_y)^3 [L_{y1} (2L_x - L_{x1}) + L_y L_{x1}]\}}{24 L_{x1}^2 L_{y1}^3}$ $P_A \leq P_R, M_{xA} \leq M_{xR}, M_{yA} \leq M_{yR}, L_x \leq L_{x1} \text{ and } L_y \leq L_{y1}$

P_R is the resisting vertical load; P_A is the unfactored vertical load that acts; M_{xR} is the resisting moment with respect to the X-axis; M_{xA} is the unfactored moment acting with respect to the X-axis; M_{yR} is the resisting moment with respect to the Y-axis; M_{yA} is the unfactored moment acting with respect to the Y-axis; p_1, p_2, p_3 and p_4 are the soil pressures at each corner of the base (kN/m²); p_{max} is the maximum available soil pressure (kN/m²); and L_x, L_y, L_{x1} and L_{y1} are limited to satisfying the conditions of each case.

2.2. C_{min} for RIF

To determine C_{min} , the factored loads and moments must be used.

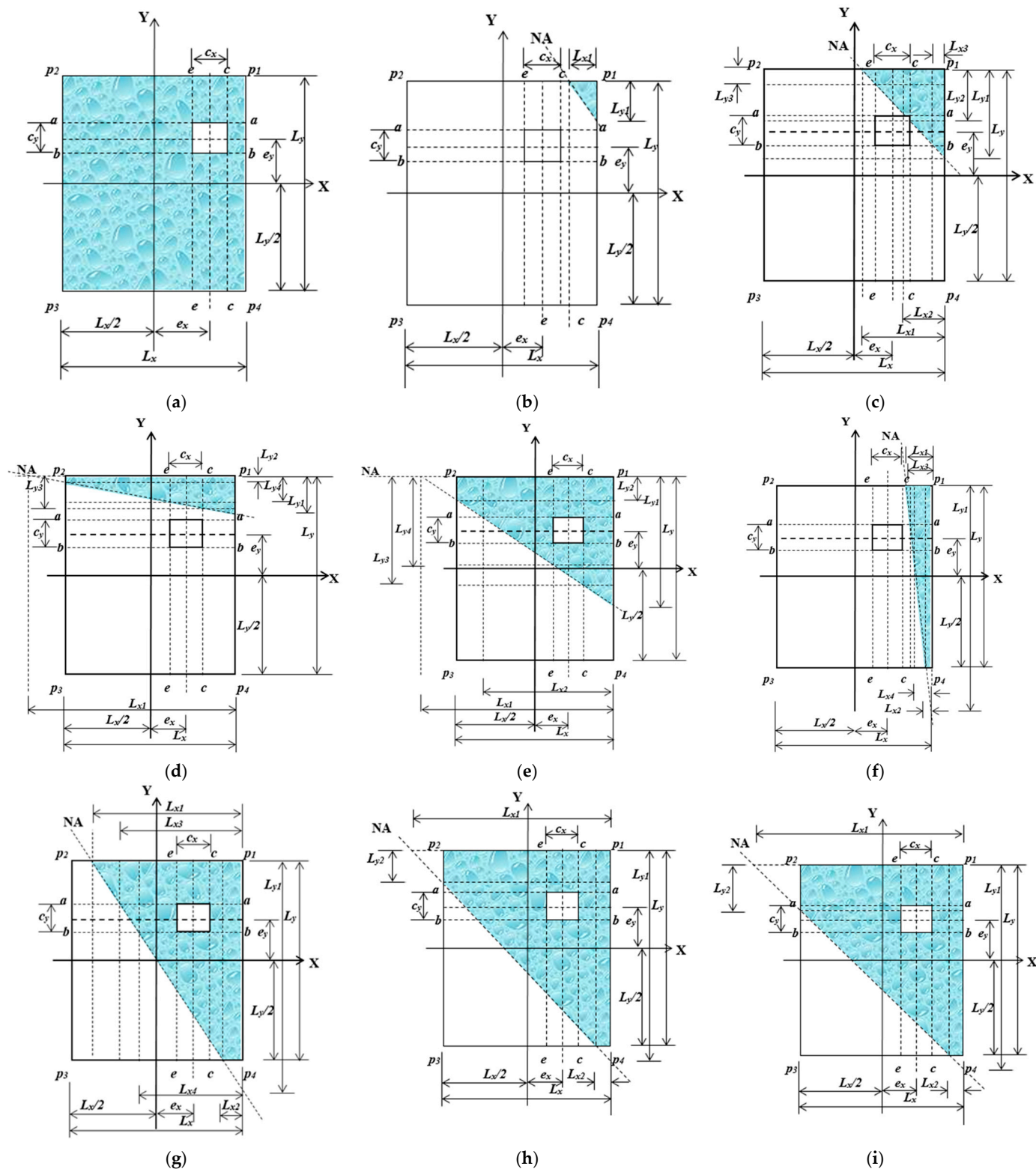
Case I assumes an FCCA, and the p_{us} is determined using Equation (1). Cases II to V assume a PCCA, and the p_{uz} is determined using Equation (2).

2.2.1. Moments

Figure 4 represents the critical sections of moments for an RIF with an eccentric column (axes $a-a$, $b-b$, $c-c$ and $e-e$).

The ultimate moments are determined from the positive axes to the axis where the moment is to be calculated (shaded area by the lever arm to the axis where the moment is to be determined).

The ultimate moments in each section for each case are shown in Appendix A.



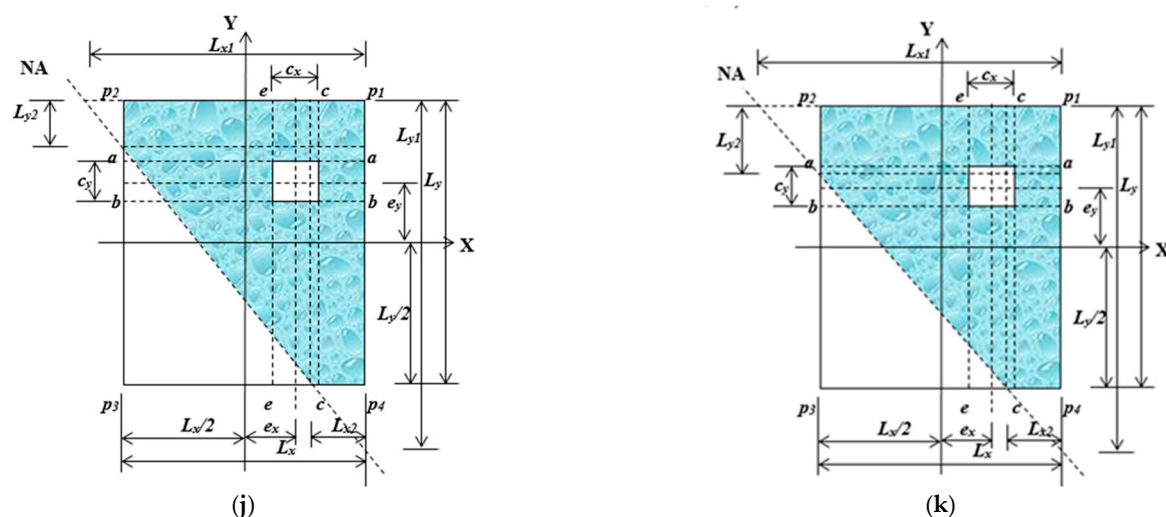
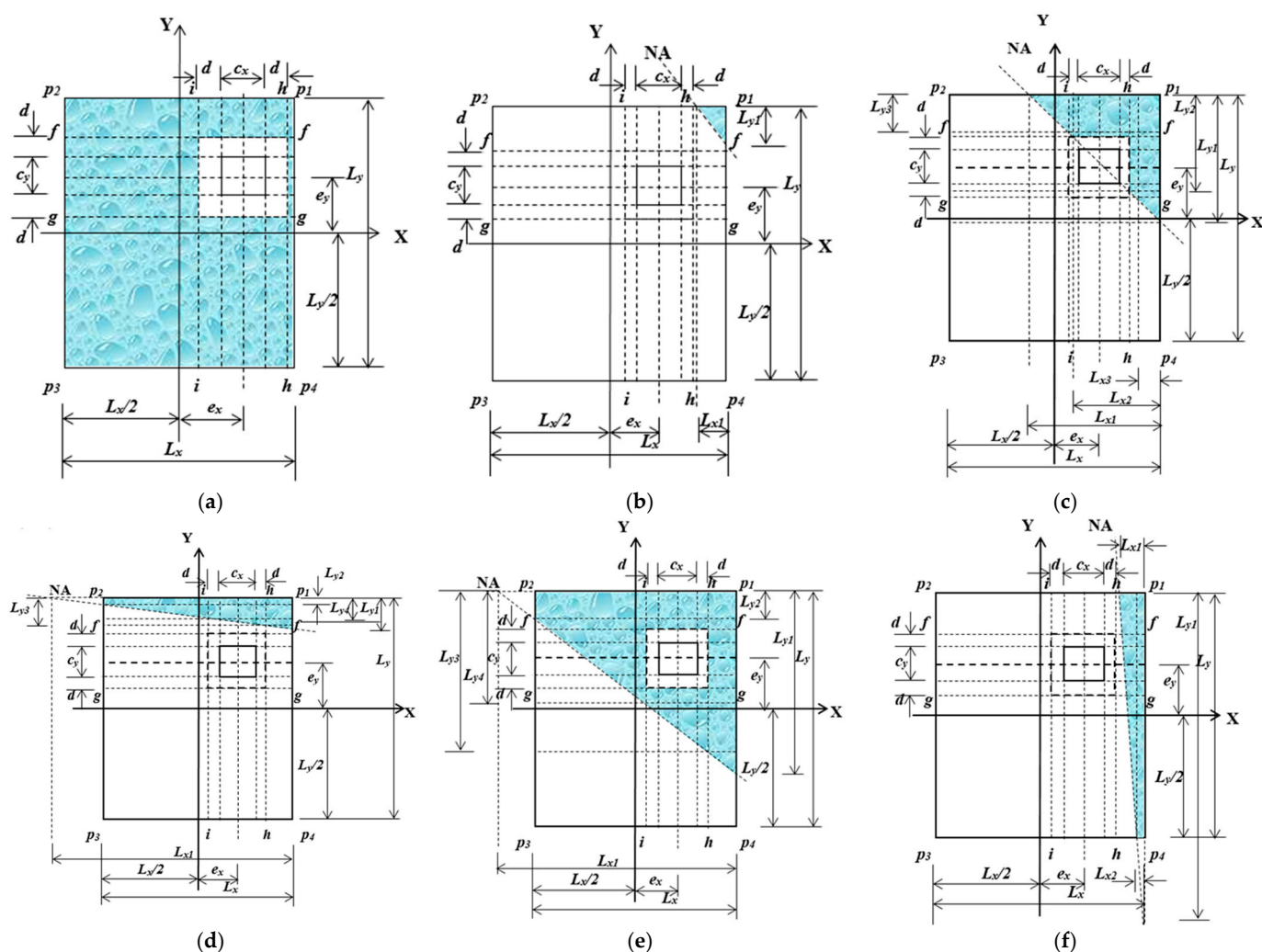


Figure 4. Moments for biaxial bending: (a) Case I; (b) Case IIA; (c) Case IIB; (d) Case IIIA; (e) Case IIIB; (f) Case IVA; (g) Case IVB; (h) Case VA; (i) Case VB; (j) Case VC; and (k) Case VD.

2.2.2. Bending Shears

Figure 5 indicates the critical sections of bending shears for an RIF with an eccentric column (axes f - f , g - g , h - h and i - i).



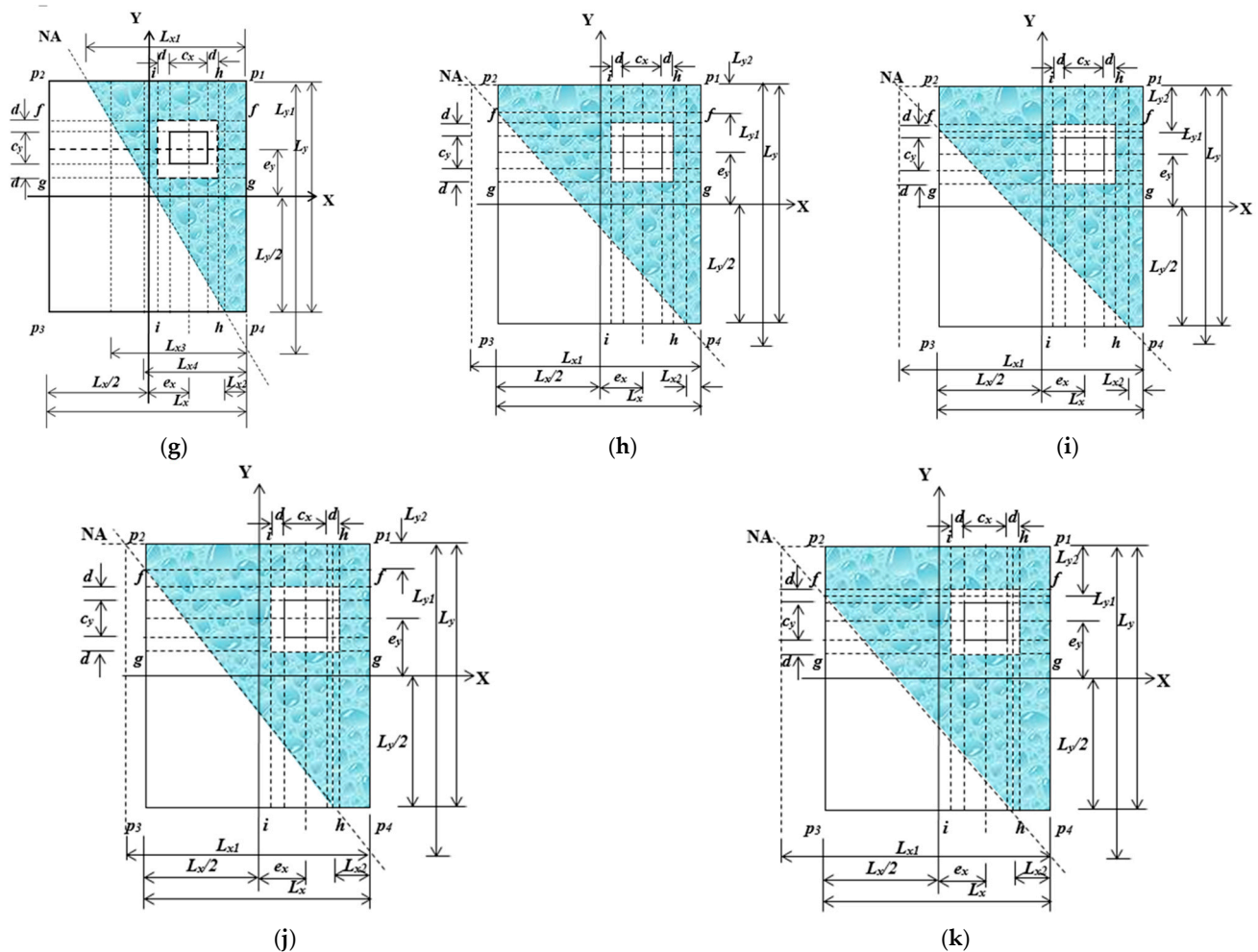


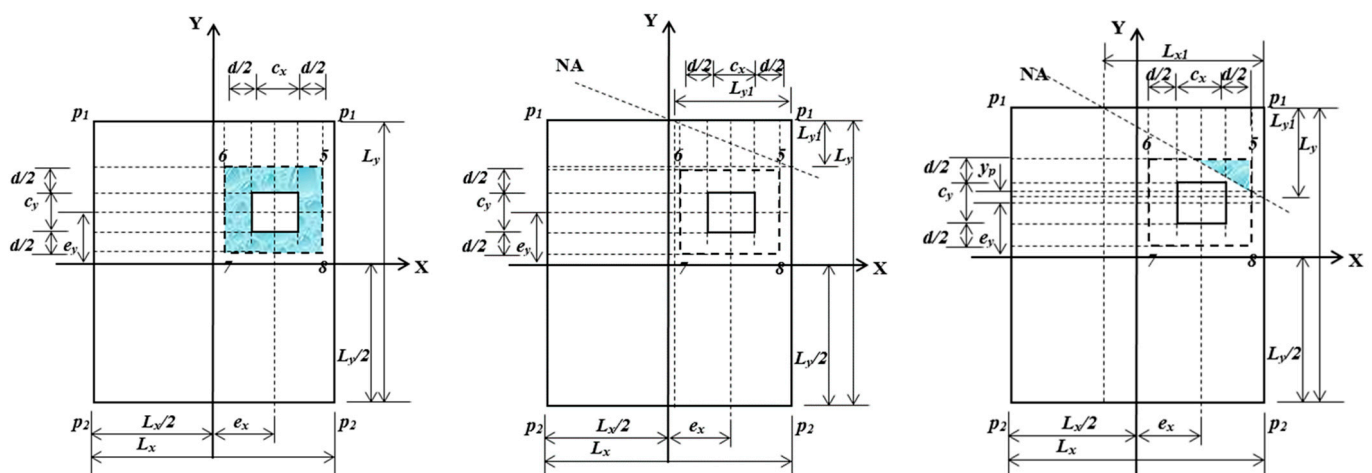
Figure 5. Bending shears for biaxial bending: (a) Case I; (b) Case IIA; (c) Case IIB; (d) Case IIIA; (e) Case IIIB; (f) Case IVA; (g) Case IVB; (h) Case VA; (i) Case VB; (j) Case VC; and (k) Case VD.

The ultimate bending shears are obtained from the positive axes to the axis where the bending shear is to be calculated (shaded area).

The ultimate bending shears in each section for each case are shown in Appendix B.

2.2.3. Punching Shears

Figure 6 represents the critical perimeter of punching shears for an RIF with an eccentric column (perimeter formed by points 5, 6, 7 and 8). Here, seven different cases appear, because only the critical perimeter is considered.



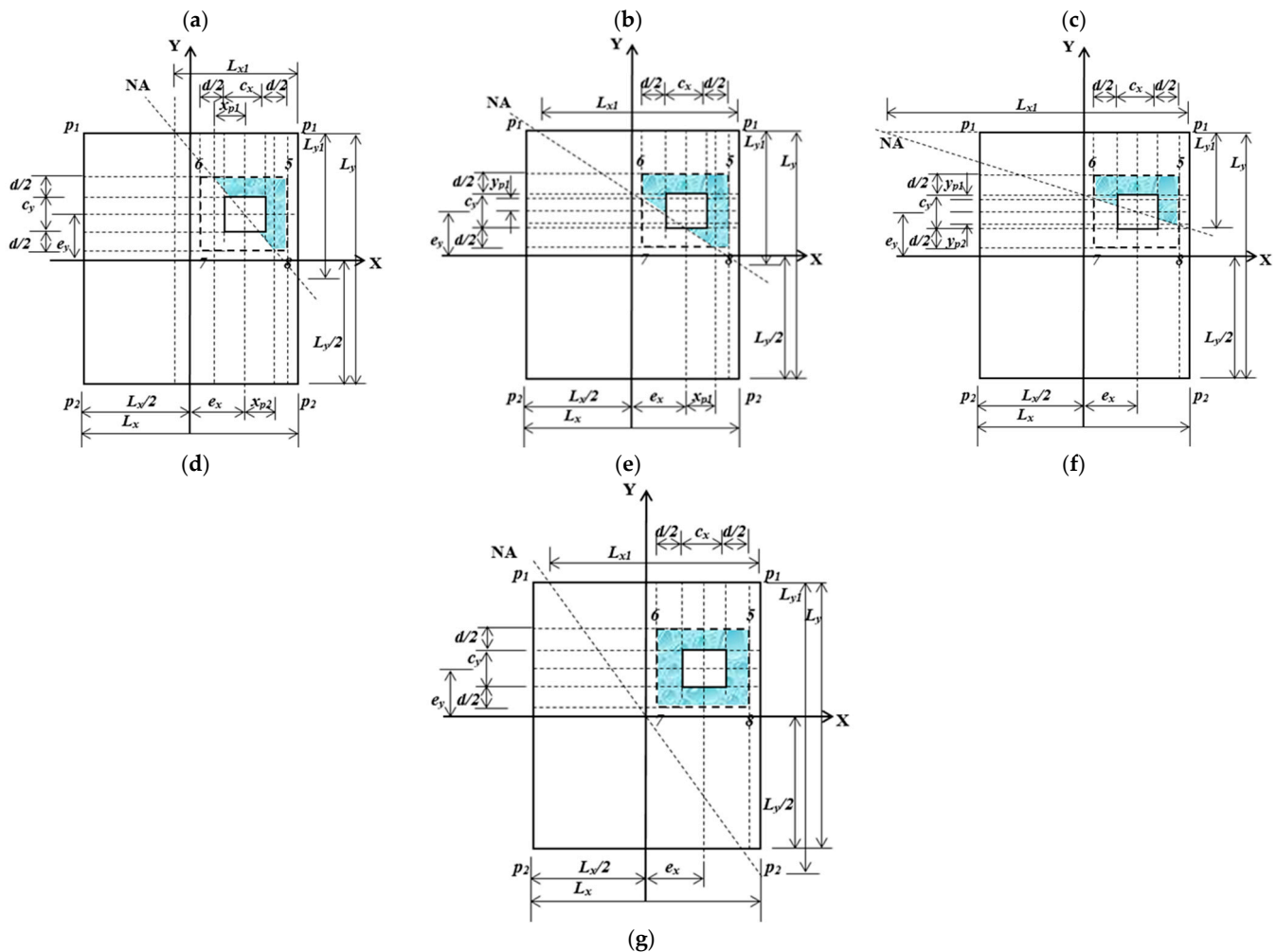


Figure 6. Punching shears for biaxial bending: (a) Case I; (b) Case IIA; (c) Case IIB; (d) Case IIC; (e) Case IID; (f) Case IIIA; and (g) Case IVA.

The ultimate punching shears are determined from the positive axes to the axis where the punching shear is to be obtained, but within the critical perimeter (shaded area).

The ultimate punching shears in each section for each case are shown in Appendix C.

2.2.4. Main Equations for C_{min}

The equation of C_{min} is indicated below:

$$C_{min} = V_c C_c + \delta_s V_s C_s, \quad (4)$$

where C_{min} is in USD, C_c is the concrete cost (USD/m³), V_c is the concrete volume (m³), C_s is the steel cost (USD/kN), V_s is the steel volume (m³), and δ_s is the steel density equal to 78 kN/m³.

The volumes for a RIF are

$$V_s = A_{sx} L_x + A_{sy} L_y, \quad (5)$$

$$V_c = L_x L_y (d + r) - (A_{sx} L_x + A_{sy} L_y), \quad (6)$$

where A_{sx} is the steel area in the X direction (m²), A_{sy} is the steel area in the Y direction (m²), and r is the concrete cover (m).

By substituting $\delta_s C_s = \alpha C_c$ (where $\alpha = \delta_s C_s / C_c$) and Equations (5) and (6) into Equation (4), the following equation for C_{min} is obtained:

$$C_{min} = C_c [L_x L_y (d + r) + (\alpha - 1)(A_{sxB} L_x + A_{syB} L_y)], \quad (7)$$

The constraints are indicated below [38].

For moments,

$$|M_{ua}|, |M_{ub}| \leq \phi_f f_y d A_{syB} \left(1 - \frac{A_{syB} f_y}{1.7 L_x d f'_c} \right), \quad (8)$$

$$|M_{uc}|, |M_{ue}| \leq \phi_f f_y d A_{sxB} \left(1 - \frac{A_{sxB} f_y}{1.7 L_y d f'_c} \right) \quad (9)$$

where f_y is the specified yield strength of the reinforcement of steel (MPa), f'_c is the specified compressive strength of the concrete at 28 days (MPa), and ϕ_f is the bending strength reduction factor (0.90).

For bending shears,

$$|V_{uf}|, |V_{ug}| \leq 0.17 \phi_v \sqrt{f'_c} L_x d, \quad (10)$$

$$|V_{uh}|, |V_{ui}| \leq 0.17 \phi_v \sqrt{f'_c} L_y d. \quad (11)$$

where ϕ_v is the shear strength reduction factor (0.85).

For the punching shear,

$$|V_{up}| \leq \begin{cases} 0.17 \phi_v \left(1 + \frac{2}{\beta_c} \right) \sqrt{f'_c} b_0 d \\ 0.083 \phi_v \left(\frac{\alpha_s d}{b_0} + 2 \right) \sqrt{f'_c} b_0 d' \\ 0.33 \phi_v \sqrt{f'_c} b_0 d \end{cases} \quad (12)$$

where β_c is the relationship between the long side and the short side of the column; b_0 is the perimeter of the punching shear (m); and $\alpha_s = 20$ (corner column), $\alpha_s = 30$ (edge column), and $\alpha_s = 40$ (interior column).

For percentages of reinforcing steel are

$$\left. \begin{array}{c} \frac{0.25 \sqrt{f'_c}}{f_y} \\ \frac{1.4}{f_y} \end{array} \right\} \leq \rho_x, \rho_y \leq 0.75 \left[\frac{0.85 \beta_1 f'_c}{f_y} \left(\frac{600}{600 + f_y} \right) \right], \quad (13)$$

where

$$0.65 \leq \beta_1 = \left(1.05 - \frac{f'_c}{140} \right) \leq 0.85,$$

where ρ_x and ρ_y are the percentages of reinforcing steel in the X and Y directions, and β_1 is the factor relating the depth of the equivalent rectangular compressive stress block to the neutral axis depth.

For the reinforcing steel areas,

$$A_{syB} = \rho_y L_x d, \quad (14)$$

$$A_{sxB} = \rho_x L_y d. \quad (15)$$

Figure 7 represents the flowchart of the algorithm for the C_{min} procedure of a RIF under biaxial bending (a genetic algorithm to optimize the C_{min}).

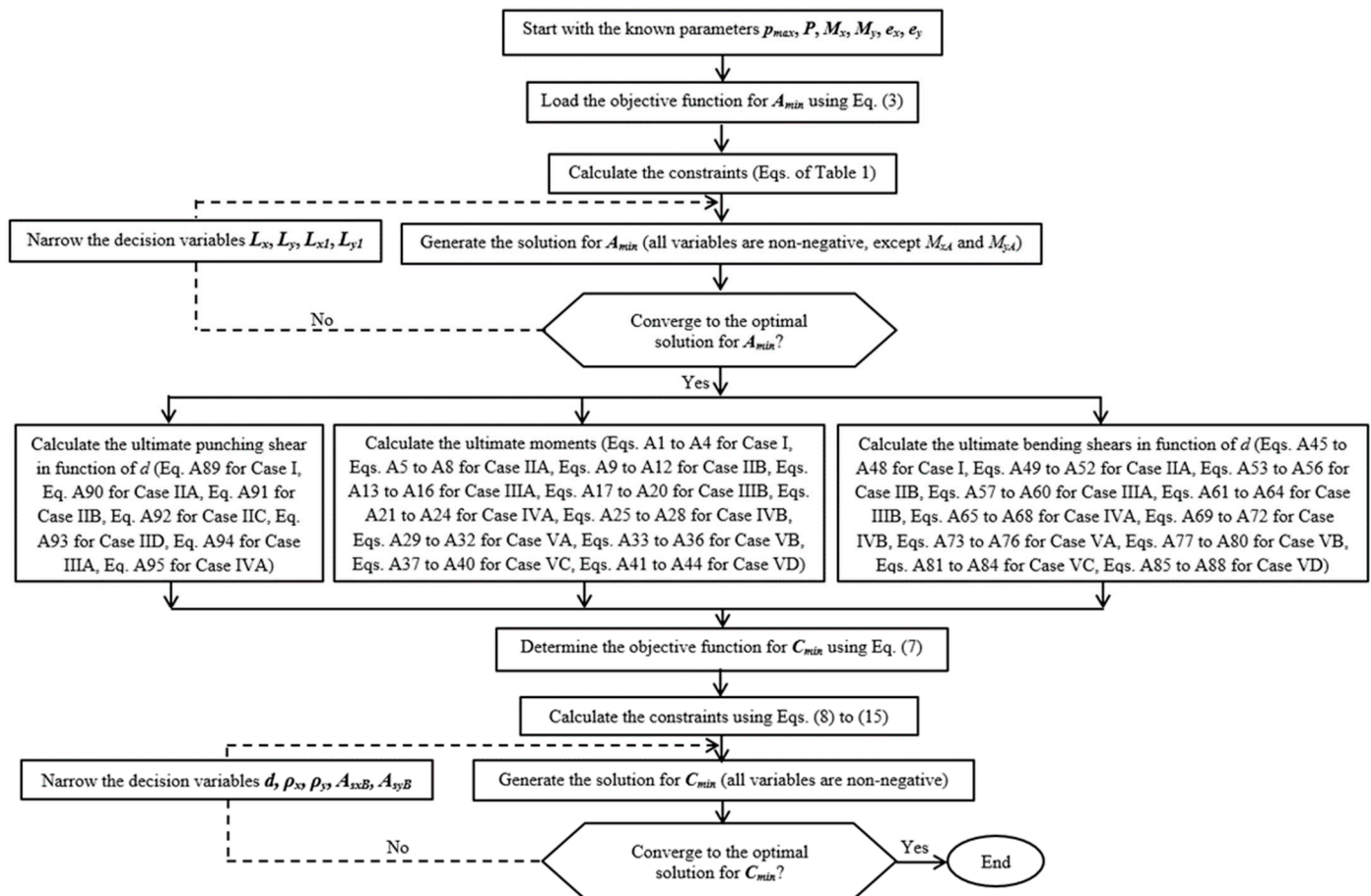


Figure 7. Flowchart for procedure of C_{min} for RIF under biaxial bending.

The more detailed procedure for determining C_{min} is as follows:

- (1) The known parameters for obtaining A_{min} are p_{max} , P , M_x , M_y , e_x and e_y .
- (2) Determine the objective function for A_{min} .
- (3) Obtain the constraints listed in Table 1 for each case.
- (4) Generate the optimal solution for A_{min} in each case separately, assuming all variables are non-negative, except M_{xA} and M_{yA} .
- (5) Determine the A_{min} for each case; if the A_{min} is not found, the decision variables must be shortened (L_x , L_y , L_{x1} and L_{y1}) and then adjusted.
- (6) In all studies, Case I consists of a FCCA, and the smallest area in Cases II to V corresponds to the PCCA.
- (7) Once the dimensions of the RIF are known, determine the moments, bending shears, and punching shear. Determine the bending shears and punching shear as a function of " d ".
- (8) Determine the objective function for C_{min} .
- (9) Obtain the constraints listed in Equations (8)–(15) for each case.
- (10) Generate the optimal solution for C_{min} in each case, assuming all variables are non-negative.
- (11) Determine the C_{min} for each case; if the C_{min} is not found, the decision variables must be shortened (d , ρ_x , ρ_y , A_{sx} and A_{sy}) and then adjusted.

3. Numerical Studies

This section shows three types of numerical studies developed by the MPA: the first study is the one presented by Luévanos-Rojas [36], which shows the column located anywhere in the footing, but with the FCCA and, here, the PCCA; the second study is that presented by Moreno-Landeros et al. [27], which shows the column located anywhere in the footing and only the A_{min} with the PCCA, but it does not show the C_{min} ; and the third study consists of presenting the C_{min} from the second study.

Tables 3, 5, 7, 9 and 11 indicate the data for the different examples to determine A_{min} , assuming that the loads and moments are not factored, where P_D is the dead load, P_L is the live load, P (total load) = $P_D + P_L$, M_{xD} is the dead moment on the X-axis, M_{xL} is the live moment on the X-axis, M_x (total moment) = $M_{xD} + M_{xL}$, M_{yD} is the dead moment on the Y-axis, M_{yL} is the live moment on the Y-axis, and M_y (total moment) = $M_{yD} + M_{yL}$.

Tables 4, 6, 8, 10 and 12–15 indicate the data for the different examples to determine C_{min} , assuming that P_u (ultimate load) = $1.2 P_D + 1.6 P_L$, M_{ux} (ultimate moment on the X-axis) = $1.2 M_{xD} + 1.6 M_{xL}$, and M_{uy} (ultimate moment on the Y-axis) = $1.2 M_{yD} + 1.6 M_{yL}$ [38].

In all studies, Case I consists of an FCCA, and the smallest area in Cases II to V corresponds to the PCCA.

The general data for study 1 of the four examples for C_{min} are c_x (side of the column in the X direction) = 40 cm, c_y (side of the column in the Y direction) = 40 cm, r (concrete cover) = 8 cm, p_{max} = 180 kN/m², f'_c = 21 MPa, f_y = 420 MPa, and α = 90 [36]. Each example presents four different problems.

Table 3 indicates the data of example 1 for A_{min} ($e_x = 0$, and $e_y = 0$). Table 4 represents the data of example 1 for C_{min} . Table 5 indicates the data of example 2 for A_{min} ($e_x = 0$, and $e_y = L_y/2 - c_y/2$). Table 6 represents the data of example 2 for C_{min} . Table 7 indicates the data of example 3 for A_{min} ($e_x = L_x/2 - c_x/2$, and $e_y = 0$). Table 8 represents the data of example 3 for C_{min} . Table 9 indicates the data of example 4 for A_{min} ($e_x = L_x/2 - c_x/2$, and $e_y = L_y/2 - c_y/2$). Table 10 represents the data of example 4 for C_{min} .

Table 3. A_{min} of study 1 in example 1 [36].

Problem	Input Data									Case	Output Data				
	P_D kN	P_L kN	P kN	M_{xD} kN-m	M_{xL} kN-m	M_x kN-m	M_{yD} kN-m	M_{yL} kN-m	M_y kN-m		M_{xA} kN-m	M_{yA} kN-m	L_x m	L_y m	A_{min} m ²
1.1	500	500	1000	150	75	225	100	50	150	I	225	150	2.55	3.80	9.69
										V	225	150	2.30	4.00	9.20
1.2	400	450	850	150	75	225	100	50	150	I	225	150	2.45	3.65	8.94
										V	225	150	2.45	3.65	8.94
1.3	500	250	750	150	75	225	100	50	150	I	225	150	2.40	3.60	8.64
										V	225	150	2.40	3.55	8.52
1.4	400	200	600	150	75	225	100	50	150	I	225	150	3.00	4.50	13.50
										V	225	150	2.30	3.40	7.82

Table 4. C_{min} of study 1 in example 1 [36].

Problem	Case	Input Data							Output Data						
		L_x m	L_y m	A_{min} m ²	P_u kN	M_{ux} kN-m	M_{uy} kN-m	M_{uxA} kN-m	M_{uyA} kN-m	A_{sxB} cm ²	A_{syB} cm ²	d cm	ρ_x	ρ_y	C_{min} USD
1.1	I	2.55	3.80	9.69	1400	300	200	300	200	45.19	52.36	36.00	0.00333	0.00575	7.03Cc
	V	2.30	4.00	9.20	1400	300	200	300	200	46.62	86.96	35.00	0.00333	0.01080	8.01Cc
1.2	I	2.45	3.65	8.94	1200	300	200	300	200	41.90	45.88	34.00	0.00333	0.00543	6.20Cc
	V	2.45	3.65	8.94	1200	300	200	300	200	37.68	54.84	31.00	0.00333	0.00722	6.09Cc
1.3	I	2.40	3.60	8.64	1000	300	200	300	200	35.94	45.86	30.00	0.00333	0.00637	5.52Cc
	V	2.40	3.55	8.52	1000	300	200	300	200	33.10	50.46	28.00	0.00333	0.00751	5.37Cc
1.4	I	3.00	4.50	13.50	800	300	200	300	200	35.51	62.77	24.00	0.00333	0.00883	7.74Cc

V	2.30	3.40	7.82	800	300	200	300	200	30.57	40.90	27.00	0.00333	0.00659	4.60	Cc
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Table 5. A_{min} of study 1 in example 2 [36].

Problem	Input Data									Case	Output Data				
	P_D kN	P_L kN	P kN	M_{xD} kN-m	M_{xL} kN-m	M_x kN-m	M_{yD} kN-m	M_{yL} kN-m	M_y kN-m		M_{xA} kN-m	M_{yA} kN-m	L_x m	L_y m	A_{min} m ²
2.1	600	450	1025	−150	−75	−225	100	50	150	I	82.50	150	9.05	1.00	9.05
										V	108.12	150	9.00	1.05	9.45
2.2	520	360	880	−150	−75	−225	100	50	150	I	39	150	6.95	1.00	6.95
										V	105	150	7.15	1.15	8.22
2.3	500	250	750	−150	−75	−225	100	50	150	I	0	150	5.15	1.00	5.15
										V	93.75	150	5.55	1.25	6.94
2.4	400	200	600	−150	−75	−225	100	50	150	I	0	150	4.00	1.15	4.60
										V	75	150	3.90	1.40	5.46

Table 6. C_{min} of study 1 in example 2 [36].

Problem	Case	Input Data									Output Data					
		L_x m	L_y m	A_{min} m ²	P_u kN	M_{ux} kN-m	M_{uy} kN-m	M_{uxA} kN-m	M_{uyA} kN-m	A_{sxB} cm ²	A_{syB} cm ²	d cm	ρ_x	ρ_y	C_{min} USD	
2.1	I	9.05	1.00	9.05	1400	−300	200	120	200	50.26	262.19	87.00	0.00578	0.00333	14.98C _C	
	V	9.00	1.05	9.45	1400	−300	200	155	200	92.98	260.74	87.00	0.010178	0.00333	18.86C _C	
2.2	I	6.95	1.00	6.95	1200	−300	200	60	200	37.06	178.20	77.00	0.00481	0.00333	9.79C _C	
	V	7.15	1.15	8.22	1200	−300	200	150	200	67.47	171.43	72.00	0.00815	0.00333	12.63C _C	
2.3	I	5.15	1.00	5.15	1000	−300	200	0	200	28.71	106.33	62.00	0.00463	0.00333	5.87C _C	
	V	5.55	1.25	6.94	1000	−300	200	125	200	61.71	96.10	52.00	0.00949	0.00333	8.28C _C	
2.4	I	4.00	1.15	4.60	800	−300	200	0	200	24.32	62.60	47.00	0.00450	0.00333	4.04C _C	
	V	3.90	1.40	5.46	800	−300	200	100	200	49.03	54.55	42.00	0.00834	0.00333	5.11C _C	

Table 7. A_{min} of study 1 in example 3 [36].

Problem	Input Data									Case	Output Data				
	P_D kN	P_L kN	P kN	M_{xD} kN-m	M_{xL} kN-m	M_x kN-m	M_{yD} kN-m	M_{yL} kN-m	M_y kN-m		M_{xA} kN-m	M_{yA} kN-m	L_x m	L_y m	A_{min} m ²
3.1	600	450	1025	100	50	150	−150	−75	−225	I	150	82.50	1.00	9.05	9.05
										V	150	108.12	1.05	9.00	9.45
3.2	520	360	880	100	50	150	−150	−75	−225	I	150	39	1.00	6.95	6.95
										V	150	105	1.15	7.15	8.22
3.3	500	250	750	100	50	150	−150	−75	−225	I	150	0	1.00	5.15	5.15
										V	150	93.75	1.25	5.55	6.94
3.4	400	200	600	100	50	150	−150	−75	−225	I	150	0	1.15	4.00	4.60
										V	150	75	1.40	3.90	5.46

Table 8. C_{min} of study 1 in example 3 [36].

Problem	Case	Input Data								Output Data					
		L_x m	L_y m	A_{min} m ²	P_u kN	M_{ux} kN-m	M_{uy} kN-m	M_{uxA} kN-m	M_{uyA} kN-m	A_{sxB} cm ²	A_{syB} cm ²	d cm	ρ_x	ρ_y	C_{min} USD
3.1	I	1.00	9.05	9.05	1400	200	−300	200	120	262.19	50.26	87.00	0.00333	0.00578	14.98Cc
	V	1.05	9.00	9.45	1400	200	−300	200	155	260.74	92.98	87.00	0.00333	0.010178	18.86Cc
3.2	I	1.00	6.95	6.95	1200	200	−300	200	60	178.20	37.06	77.00	0.00333	0.00481	9.79Cc
	V	1.15	7.15	8.22	1200	200	−300	200	150	171.43	67.47	72.00	0.00333	0.00815	12.63Cc
3.3	I	1.00	5.15	5.15	1000	200	−300	200	0	106.33	28.71	62.00	0.00333	0.00463	5.87Cc
	V	1.25	5.55	6.94	1000	200	−300	200	125	96.10	61.71	52.00	0.00333	0.00949	8.28Cc
3.4	I	1.15	4.00	4.60	800	200	−300	200	0	62.60	24.32	47.00	0.00333	0.00450	4.04Cc
	V	1.40	3.90	5.46	800	200	−300	200	100	54.55	49.03	42.00	0.00333	0.00834	5.11Cc

Table 9. A_{min} of study 1 in example 4 [36].

Problem	Input Data									Case	Output Data				
	P_D kN	P_L kN	P kN	M_{xD} kN-m	M_{xL} kN-m	M_x kN-m	M_{yD} kN-m	M_{yL} kN-m	M_y kN-m		M_{xA} kN-m	M_{yA} kN-m	L_x m	L_y m	A_{min} m ²
4.1	500	250	750	-500	-250	-750	-400	-200	-600	I	-18.75	-3.55	2.00	2.35	4.70
										V	168.75	150	2.40	2.85	6.84
4.2	420	185	605	-500	-250	-750	-400	-200	-600	I	-84.50	-70.62	2.15	2.60	5.59
										V	-39.12	-116	2.00	2.75	5.50
4.3	300	150	450	-500	-250	-750	-400	-200	-600	I	-120	-93.75	2.65	3.20	8.48
										V	-311.25	-251.25	1.95	2.35	4.58
4.4	200	100	300	-500	-250	-750	-400	-200	-600	I	-112.50	-90	3.80	4.65	17.67
										V	-292.50	-240	2.80	3.45	9.66

Table 10. C_{min} of study 1 in example 4 [36].

Problem	Case	Input Data									Output Data				
		L_x m	L_y m	A_{min} m ²	P_u kN	M_{ux} kN-m	M_{uy} kN-m	M_{uxA} kN-m	M_{uyA} kN-m	A_{sxB} cm ²	A_{syB} cm ²	d cm	ρ_x	ρ_y	C_{min} USD
4.1	I	2.00	2.35	4.70	1000	-1000	-800	-20	0	42.26	42.76	54.00	0.00333	0.00396	4.56Cc
	V	2.40	2.85	6.84	1000	-1000	-800	65	-20	62.01	79.07	59.00	0.00369	0.00558	7.91Cc
4.2	I	2.15	2.60	5.59	800	-1000	-800	80	-96	41.56	50.10	48.00	0.00333	0.00485	5.08Cc
	V	2.00	2.75	5.50	800	-1000	-800	-220	-380	54.03	41.97	59.00	0.00333	0.00356	5.67Cc
4.3	I	2.65	3.20	8.48	600	-1000	-800	40	-128	47.77	63.18	40.00	0.00373	0.00596	7.00Cc
	V	1.95	2.35	4.58	600	-1000	-800	-575	-555	36.78	31.31	47.00	0.00333	0.00342	3.81Cc
4.4	I	3.80	4.65	17.67	400	-1000	-800	48	-120	57.15	74.78	35.00	0.00351	0.00562	12.63Cc
	V	2.80	3.45	9.66	400	-1000	-800	-550	180	61.52	62.57	41.00	0.00435	0.00545	8.19Cc

The general data for study 2 of the four examples for A_{min} are $c_x = 40$ cm, $c_y = 40$ cm, and $p_{max} = 200$ kN/m² [27]. Each example presents three different problems.

Table 11 shows the input and output data for the comparison of A_{min} from the MPML [27] and the MPA. Problem 1.A, 1.B and 1.C are for $e_x = 0$ and $e_y = 0$; Problem 2.A, 2.B and 2.C are for $e_x = 0$ and $e_y = L_y/2 - c_y/2$; Problem 3.A, 3.B and 3.C are for $e_x = L_x/2 - c_x/2$ and $e_y = 0$; and Problem 4.A, 4.B and 4.C are for $e_x = L_x/2 - c_x/2$ and $e_y = L_y/2 - c_y/2$.

Table 11. Comparison of A_{min} between MPML and MPA [27].

Problem	Input Data			Model	Output Data		
	P kN	M_x kN-m	M_y kN-m		L_x m	L_y m	A_{min} m ²
1.A	1200	800	400	MPML	3.05	5.00	15.25
				MPA	2.75	5.50	15.12
1.B	800	800	400	MPML	2.75	5.45	14.99
				MPA	2.70	5.45	14.72
1.C	400	800	400	MPML	3.25	6.45	20.96
				MPA	2.90	5.80	16.82
2.A	1200	400	-800	MPML	1.75	4.95	8.66
				MPA	1.80	5.60	10.08
2.B	800	400	-800	MPML	2.40	3.25	7.80
				MPA	1.90	3.55	6.75
2.C	400	400	-800	MPML	4.45	2.65	11.79
				MPA	3.70	2.70	9.99
3.A	1200	-800	400	MPML	4.95	1.75	8.66
				MPA	5.60	1.80	10.08
3.B	800	-800	400	MPML	3.25	2.40	7.80
				MPA	3.55	1.90	6.75
3.C	400	-800	400	MPML	2.65	4.45	11.79

				MPA	2.70	3.70	9.99
4.A	800	-800	-700	MPML	2.10	2.35	4.94
				MPA	2.45	2.70	6.62
4.B	600	-800	-700	MPML	2.35	2.65	6.23
				MPA	2.15	2.40	5.16
4.C	400	-800	-700	MPML	2.80	3.15	8.82
				MPA	2.75	3.10	8.52

The general data for study 3 of the four examples for C_{min} are $c_x = 40$ cm, $c_y = 40$ cm, $r = 8$ cm, $p_{max} = 200$ kN/m², $f'_c = 21$ MPa, $f_y = 420$ MPa, and $\alpha = 90$ [27]. Each example presents three different problems.

Table 12 indicates the data of example 1 for C_{min} ($e_x = 0$, and $e_y = 0$). Table 13 indicates the data of example 2 for C_{min} ($e_x = L_x/2 - c_x/2$, and $e_y = 0$). Table 14 indicates the data of example 3 for C_{min} ($e_x = 0$, and $e_y = L_y/2 - c_y/2$). Table 15 indicates the data of example 4 for C_{min} ($e_x = L_x/2 - c_x/2$, and $e_y = L_y/2 - c_y/2$).

Table 12. C_{min} of study 3 in example 1 [27].

Problem	Case	Input Data								Output Data					
		L_x m	L_y m	A_{min} m ²	P_u kN	M_{ux} kN-m	M_{uy} kN-m	M_{uxA} kN-m	M_{uyA} kN-m	A_{sxB} cm ²	A_{syB} cm ²	d cm	ρ_x	ρ_y	C_{min} USD
1.A	I	4.00	8.00	32.00	1760	1160	580	1160	580	111.89	149.56	42.00	0.00333	0.00890	30.63C _C
	V	2.75	5.50	15.12	1760	1160	580	1160	580	93.41	85.86	51.00	0.00333	0.00612	15.41C _C
1.B	I	6.00	12.00	72.00	1200	1160	580	1160	580	131.87	203.76	33.00	0.00333	0.01029	58.32C _C
	V	2.70	5.45	14.72	1200	1160	580	1160	580	76.22	80.33	42.00	0.00333	0.00708	13.09C _C
1.C	I	12.00	24.00	288.00	600	1160	580	1160	580	191.81	288.23	24.00	0.00333	0.01001	174.21C _C
	II	2.90	5.80	16.82	600	1160	580	1160	580	50.22	79.51	26.00	0.00333	0.01068	11.12C _C

Table 13. C_{min} of study 3 in example 2 [27].

Problem	Case	Input Data								Output Data					
		L_x m	L_y m	A_{min} m ²	P_u kN	M_{ux} kN-m	M_{uy} kN-m	M_{uxA} kN-m	M_{uyA} kN-m	A_{sxB} cm ²	A_{syB} cm ²	d cm	ρ_x	ρ_y	C_{min} USD
2.A	I	1.75	4.95	8.66	1760	580	-1160	580	28	110.44	48.79	67.00	0.00333	0.00416	10.37C _C
	V	1.80	5.60	10.08	1760	580	-1160	580	72	124.94	69.72	67.00	0.00333	0.00578	13.04C _C
2.B	I	2.40	3.25	7.80	1200	580	-1160	580	40	64.22	40.44	42.00	0.00470	0.00401	6.44C _C
	V	1.90	3.55	6.75	1200	580	-1160	580	-260	73.29	47.13	62.00	0.00333	0.00400	7.45C _C
2.C	I	4.40	6.00	26.40	600	580	-1160	580	40	92.18	56.68	32.00	0.00480	0.00403	17.20C _C
	V	3.70	2.70	9.99	600	580	-1160	580	-170	83.89	45.59	37.00	0.00840	0.00333	8.35C _C

Table 14. C_{min} of study 3 in example 3 [27].

Problem	Case	Input Data								Output Data					
		L_x m	L_y m	A_{min} m ²	P_u kN	M_{ux} kN-m	M_{uy} kN-m	M_{uxA} kN-m	M_{uyA} kN-m	A_{sxB} cm ²	A_{syB} cm ²	d cm	ρ_x	ρ_y	C_{min} USD
3.A	I	4.95	1.75	8.66	1760	-1160	580	28	580	48.79	110.44	67.00	0.00416	0.00333	10.37C _C
	V	5.60	1.80	10.08	1760	-1160	580	72	580	69.72	124.94	67.00	0.00578	0.00333	13.04C _C
3.B	I	3.25	2.40	7.80	1200	-1160	580	40	580	40.44	64.22	42.00	0.00401	0.00470	6.44C _C
	V	3.55	1.90	6.75	1200	-1160	580	-260	580	47.13	73.29	62.00	0.00400	0.00333	7.45C _C
3.C	I	6.00	4.40	26.40	600	-1160	580	40	580	56.68	92.18	32.00	0.00403	0.00480	17.20C _C
	V	2.70	3.70	9.99	600	-1160	580	-170	580	45.59	83.89	37.00	0.00333	0.00840	8.35C _C

Table 15. C_{min} of study 3 in example 4 [27].

Problem	Case	Input Data								Output Data					
		L_x m	L_y m	A_{min} m ²	P_u kN	M_{ux} kN-m	M_{uy} kN-m	M_{uxA} kN-m	M_{uyA} kN-m	A_{sxB} cm ²	A_{syB} cm ²	d cm	ρ_x	ρ_y	C_{min} USD

4.A	I	2.10	2.35	4.94	1160	-1160	-1020	-29.00	-34.00	46.95	44.41	60.00	0.00333	0.00352	5.16C _c
	V	2.45	2.70	6.62	1160	-1160	-1020	174.00	169.00	59.95	69.21	56.00	0.00397	0.00504	7.20C _c
4.B	I	2.35	2.65	6.23	880	-1160	-1020	-170.00	-162.00	45.89	53.66	52.00	0.00333	0.00439	5.96C _c
	V	2.15	2.40	5.16	880	-1160	-1020	0	-5.00	50.35	45.10	63.00	0.00333	0.00333	5.59C _c
4.C	I	3.35	3.80	12.73	580	-1160	-1020	-174.00	-164.50	63.40	71.49	40.00	0.00417	0.00534	10.42C _c
	V	2.75	3.10	8.52	580	-1160	-1020	-377.00	-338.00	56.21	66.27	24.00	0.00412	0.00548	7.64C _c

4. Results

Tables 3–10 represent the results for C_{min} that considered the PCCA of the study presented by Luevanos-Rojas [36].

Table 3 indicates the following:

1. When P decreases in Case I, L_x , L_y and A_{min} decrease up to $P = 750$ kN and then increase.
2. When P decreases in Case V, L_x increases up to $P = 850$ kN and then decreases; L_y and A_{min} decrease.
3. The relationship between the A_{min} in both cases is such that it is smaller in Case V in most problems, except at $P = 850$ kN, where the values are equal. The greatest savings in Case V occur at $P = 600$ kN, being 42.07% compared with Case I (problem 1.4).
4. $M_{xA} = M_x$ and $M_{yA} = M_y$ are equal for the two cases because P has no eccentricities (e_x and $e_y = 0$).

Table 4 indicates the following:

1. When P_u decreases in Case I, A_{sxB} and d decrease; A_{syB} and C_{min} decrease up to $P_u = 1000$ kN and then increase.
2. When P_u decreases in Case V, A_{sxB} , A_{syB} , d and C_{min} decrease.
3. The relationship between the C_{min} in both cases is such that it is lower in Case V, except at $P_u = 1400$ kN, where it is lower in Case I. The greatest savings in Case I occur at $P_u = 1400$, being 12.23% compared with Case V (problem 1.1). The greatest savings in Case V occur at $P_u = 800$ kN, being 40.57% compared with Case I (problem 1.4).

Table 5 indicates the following:

1. When P decreases in Case I, L_x and A_{min} decrease; L_y is equal to one, except at $P = 600$ kN, where it is greater than one; and M_{xA} tends to decrease until $P = 750$ kN and then remains constant and equal to zero.
2. When P decreases in Case V, L_y increases; M_{xA} , L_x and A_{min} decrease.
3. The relationship between A_{min} in both cases is such that it is smaller in Case I in all problems. The greatest savings occur at $P = 750$ kN, being 25.79% compared with Case V (problem 2.3).
4. $M_{xA} \neq M_x$ is different in the two cases because P has eccentricity ($e_y \neq 0$), and $M_{yA} = M_y$ is equal in the two cases because P has no eccentricity ($e_x = 0$).

Table 6 indicates the following:

1. When P_u decreases in Case I, A_{sxB} , A_{syB} , d and C_{min} decrease.
2. When P_u decreases in Case V, A_{sxB} , A_{syB} , d and C_{min} decrease.
3. The relationship between C_{min} in both cases is such that it is lower in Case I in all problems. The greatest savings in Case I occur at $P_u = 1000$ kN, being 29.11% compared with Case V (problem 2.3).

Table 7 indicates the following:

1. When P decreases in Case I, L_y and A_{min} decrease; L_x is equal to one, except at $P = 600$ kN, where it is greater than one; M_{yA} tends to decrease until $P = 750$ kN and then remains constant and equal to zero.
2. When P decreases in Case V, L_x increases; M_{yA} , L_y and A_{min} decrease.

3. The relationship between A_{min} in both cases is such that it is smaller in Case I in all problems. The greatest savings occur at $P = 750$ kN, being 25.79% compared with Case V (problem 3.3).
4. $M_{xA} = M_x$ is equal in the two cases because P has no eccentricity ($e_y = 0$), and $M_{yA} \neq M_y$ is different in the two cases because P has eccentricity ($e_x \neq 0$).

Table 8 indicates the following:

1. When P_u decreases in Case I, A_{sxB} , A_{syB} , d and C_{min} decrease.
2. When P_u decreases in Case V, A_{sxB} , A_{syB} , d and C_{min} decrease.
3. The relationship between C_{min} in both cases is such that it is lower in Case I in all problems. The greatest savings in Case I occur at $P_u = 1000$ kN, being 29.11% compared with Case V (problem 3.3).

Table 9 indicates the following:

1. When P decreases in Case I, L_x , L_y and A_{min} increase; M_{xA} and M_{yA} (absolute value) increase until $P = 450$ kN and then decrease.
2. When P decreases in Case V, L_x , L_y and A_{min} decrease up to $P = 450$ kN and then increase; M_{xA} and M_{yA} (absolute value) decrease until $P = 605$ kN and then increase until $P = 450$ kN and subsequently decrease.
3. The relationship between A_{min} in both cases is such that it is lower in Case V, except at $P = 750$ kN, where it is lower in Case I. The greatest savings in Case I occur at $P = 750$, being 31.29% compared with Case V (problem 4.1). The greatest savings in Case V occur at $P = 450$ kN, being 45.99% compared with Case I (problem 4.3).
4. $M_{xA} \neq M_x$ and $M_{yA} \neq M_y$ are different in the two cases because P has eccentricities (e_x and $e_y \neq 0$).

Table 10 indicates the following:

1. When P_u decreases in Case I, A_{syB} and C_{min} increase; d decreases; and A_{sxB} decreases up to $P_u = 800$ kN and then increases.
2. When P_u decreases in Case V, d decreases; A_{sxB} , A_{syB} and C_{min} decrease up to $P_u = 600$ kN and then increase.
3. The relationship between C_{min} in both cases is such that it is lower in Case I at $P_u = 1000$ and 800 kN, and in Case V at $P_u = 600$ and 400 kN. The greatest savings in Case I occur at $P_u = 1000$ kN, being 42.35% compared with Case V (problem 4.1). The greatest savings in Case V occur at $P_u = 600$ kN, being 45.57% compared with Case I (problem 4.3).

Table 11 represents the results for the A_{min} for the RIF in the study presented by Moreno-Landeros et al. [27].

Table 11 indicates the following:

1. When P decreases in the MPML (problem 1), L_y increases; L_x and A_{min} decrease up to $P = 800$ kN and then increase.
2. When P decreases in the MPA (problem 1), L_x , L_y and A_{min} decrease up to $P = 800$ kN and then increase.
3. The relationship between A_{min} in the two models is such that it is lower in the MPA in all problems. The greatest savings occur at $P = 400$ kN, being 19.75% compared with the MPML (problem 1.C).
4. When P decreases in the MPML (problem 2), L_x increases; L_y decreases; and A_{min} decreases up to $P = 800$ kN and then increases.
5. When P decreases in the MPA (problem 2), L_x increases; L_y decreases; and A_{min} decreases up to $P = 800$ kN and then increases.
6. The relationship between A_{min} in the two models is such that it is lower for MPML at $P = 1200$ kN and for MPA at $P = 800$ and 400 kN. The greatest savings for MPML

occur at $P = 1200$ kN, being 14.09% compared with MPA (problem 2.A). The greatest savings for MPA occur at $P = 400$ kN, being 15.27% compared with MPML (problem 2.C).

7. When P decreases in the MPML (problem 3), L_x decreases; L_y increases; and A_{min} decreases up to $P = 800$ kN and then increases.
8. When P decreases in the MPA (problem 3), L_x decreases; L_y increases; and A_{min} decreases up to $P = 800$ kN and then increases.
9. The relationship between A_{min} in the two models is lower for MPML at $P = 1200$ kN and for MPA at $P = 800$ and 400 kN. The greatest savings for MPML occur at $P = 1200$ kN, being 14.09% compared with MPA (problem 3.A). The greatest savings for MPA occur at $P = 400$ kN, being 15.27% compared with MPML (problem 3.C).
10. When P decreases in the MPML (problem 4), L_x , L_y and A_{min} increase.
11. When P decreases in the MPA (problem 4), L_x , L_y and A_{min} decrease up to $P = 800$ kN and then increase.
12. The relationship between A_{min} in the two models is such that it is lower for MPML at $P = 800$ kN and for MPA at $P = 600$ and 400 kN. The greatest savings for MPML occur at $P = 800$ kN, being 25.38% compared with MPA (problem 4.A). The greatest savings for MPA occur at $P = 400$ kN, being 17.17% compared with MPML (problem 4.B).

Tables 12–15 represent the results for the C_{min} design that considers the complete design for the RIF in the study presented by Moreno-Landeros et al. [27].

Table 12 indicates the following:

1. When P_u decreases in Case I, A_{sxB} , A_{syB} and C_{min} increase; d decreases.
2. When P_u decreases in Cases II and V, A_{sxB} , A_{syB} , d and C_{min} decrease.
3. The relationship between C_{min} in both cases is such that it is lower in Cases II and V in all problems. The greatest savings in Case V occur at $P_u = 600$ kN, being 93.62% compared with Case I (problem 1.C).

Table 13 indicates the following:

1. When P_u decreases in Case I, A_{sxB} , A_{syB} and C_{min} decrease up to $P_u = 1200$ kN and then increase; d decreases.
2. When P_u decreases in Case V, A_{sxB} and C_{min} decrease up to $P_u = 1200$ kN and then increase; A_{syB} and d decrease.
3. The relationship between C_{min} in both cases is such that it is lower in Case I at $P_u = 1760$ and 1200 kN, and in Case V at $P_u = 600$ kN. The greatest savings in Case I occur at $P_u = 1760$ kN, being 20.48% compared with Case V (problem 2.A). The greatest savings in Case V occur at $P_u = 600$ kN, being 51.45% compared with Case I (problem 2.C).

Table 14 indicates the following:

1. When P_u decreases in Case I, A_{sxB} , A_{syB} and C_{min} decrease up to $P_u = 1200$ kN and then increase; d decreases.
2. When P_u decreases in Case V, A_{sxB} and C_{min} decrease up to $P_u = 1200$ kN and then increase; A_{syB} and d decrease.
3. The relationship between C_{min} in both cases is such that it is lower in Case I at $P_u = 1760$ and 1200 kN, and in Case V at $P_u = 600$ kN. The greatest savings in Case I occur at $P_u = 1760$ kN, being 20.48% compared with Case V (problem 3.A). The greatest savings in Case V occur at $P_u = 600$ kN, being 51.45% compared with Case I (problem 3.C).

Table 15 indicates the following:

1. When P_u decreases in Case I, A_{sxB} decreases up to $P_u = 880$ kN and then increases; A_{syB} and C_{min} increase; and d decreases.
2. When P_u decreases in Case V, A_{sxB} , A_{syB} and C_{min} decrease up to $P_u = 880$ kN and then increase; A_{syB} and d decrease.
3. The relationship between C_{min} in both cases is such that it is lower in Case I at $P_u = 1160$ kN, and in Case V at $P_u = 880$ and 580 kN. The greatest savings in Case I occur at $P_u = 1160$ kN, being 28.33% compared with Case V (problem 4.A). The greatest savings in Case V occur at $P_u = 580$ kN, being 26.67% compared with Case I (problem 3.C).

Figure 8 represents A_{min} and C_{min} (study 1) of the MPA for the PCCA and of the MPLR for the FCCA [36]. Figure 8a indicates the A_{min} values, and the greatest savings occur in the MPA, with 45.99%, compared with the MPLR of problem 4.3. Figure 8b indicates the C_{min} values, and the greatest savings occur in the MPA, with 45.57%, compared with the MPLR of problem 4.3.

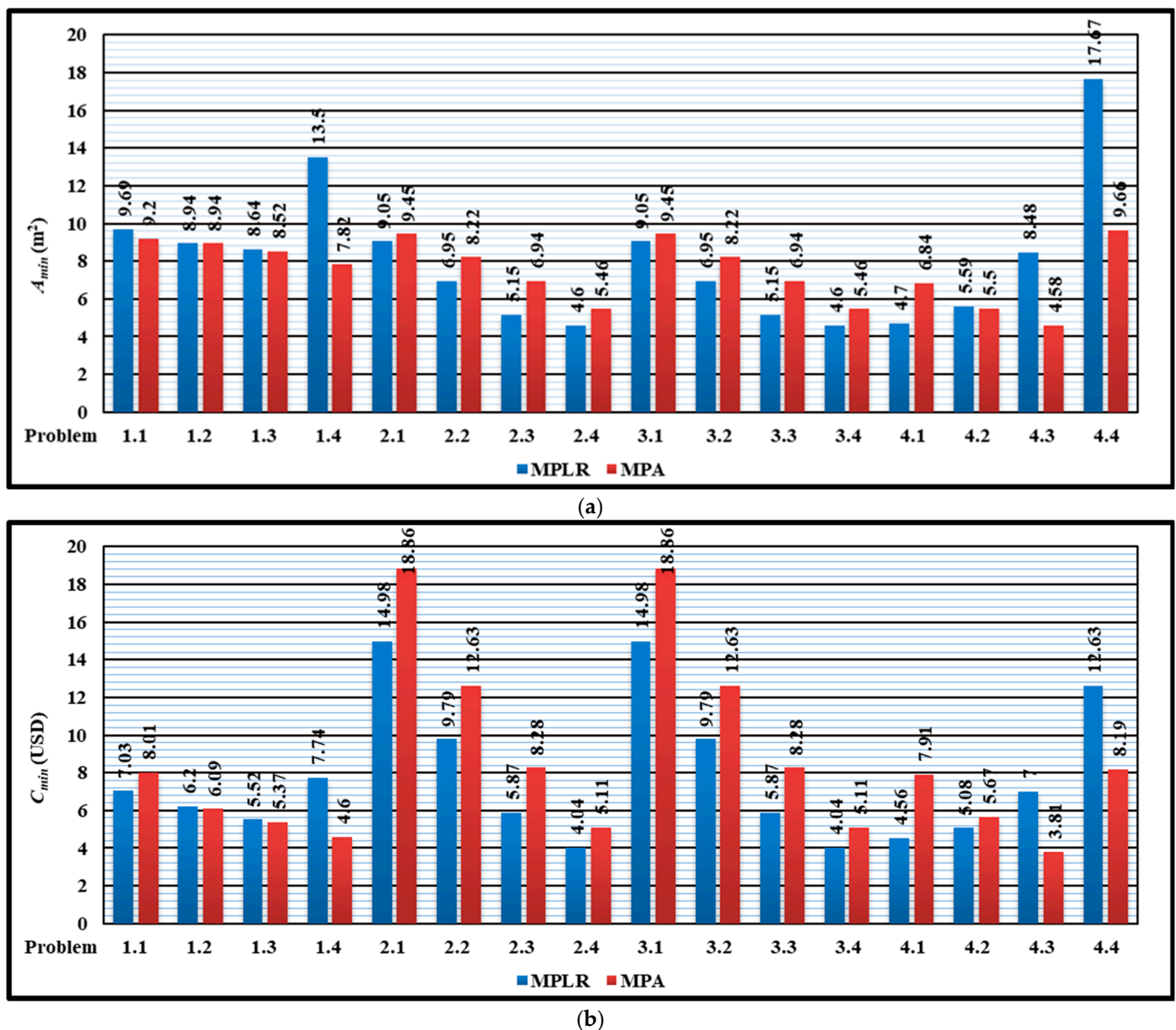


Figure 8. Comparison of study 1 between MPA and MPLR: (a) A_{min} ; (b) C_{min} .

Figure 9 represents the A_{min} (study 2) of the MPA (assuming that the loads and moments it resists must be greater than or equal to those acting on the RIF) and of the MPML (assuming that the loads and moments it resists must be equal to those acting on the RIF), both for the PCCA [27]. The greatest savings in the MPA for A_{min} occur at 19.75% compared with the MPML of problem 1.C, and the greatest savings in the MPML for A_{min} occur at 25.38% compared with the MPA of problem 4.A.

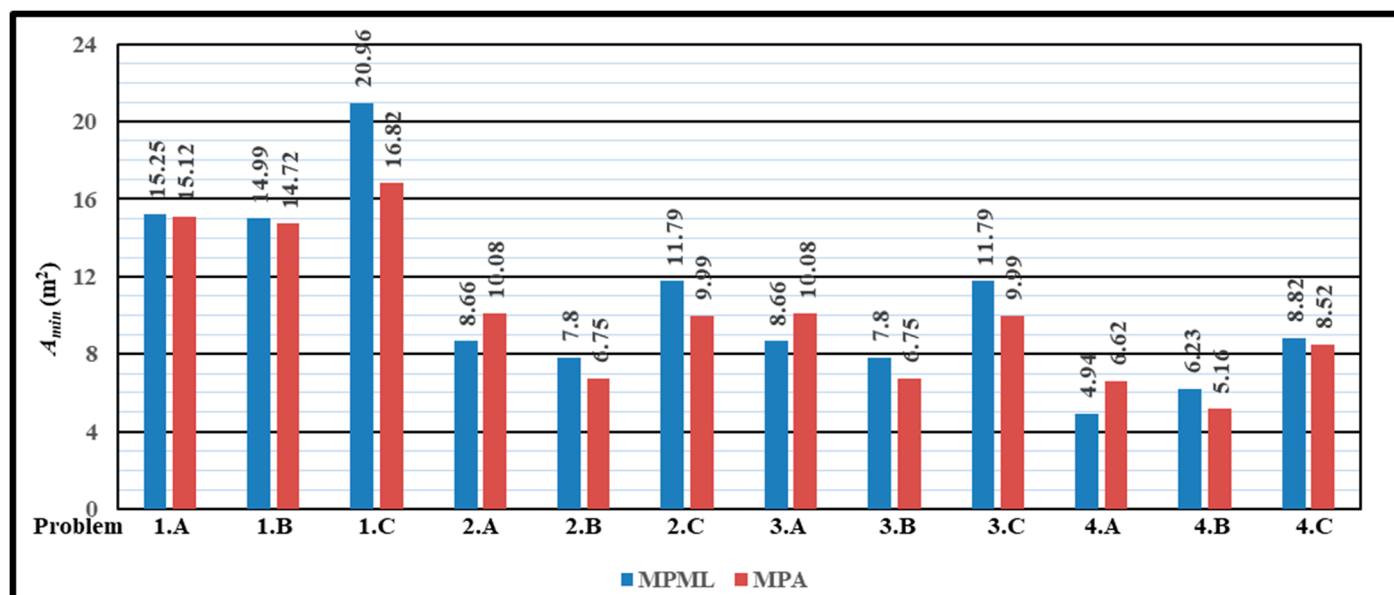


Figure 9. Comparison of study 2 between MPA and MPLR for A_{min} .

Figure 10 represents C_{min} (study 3), taking into account that the sides of the foundation are known from the second study in the MPA for the PCCA. This is shown for the FCCA and PCCA. The greatest savings in the PCCA for C_{min} occur at 93.62% compared with the FCCA of problem 1.C, and the greatest savings in the FCCA for C_{min} occur at 28.33% compared with the PCCA of problem 4.A.

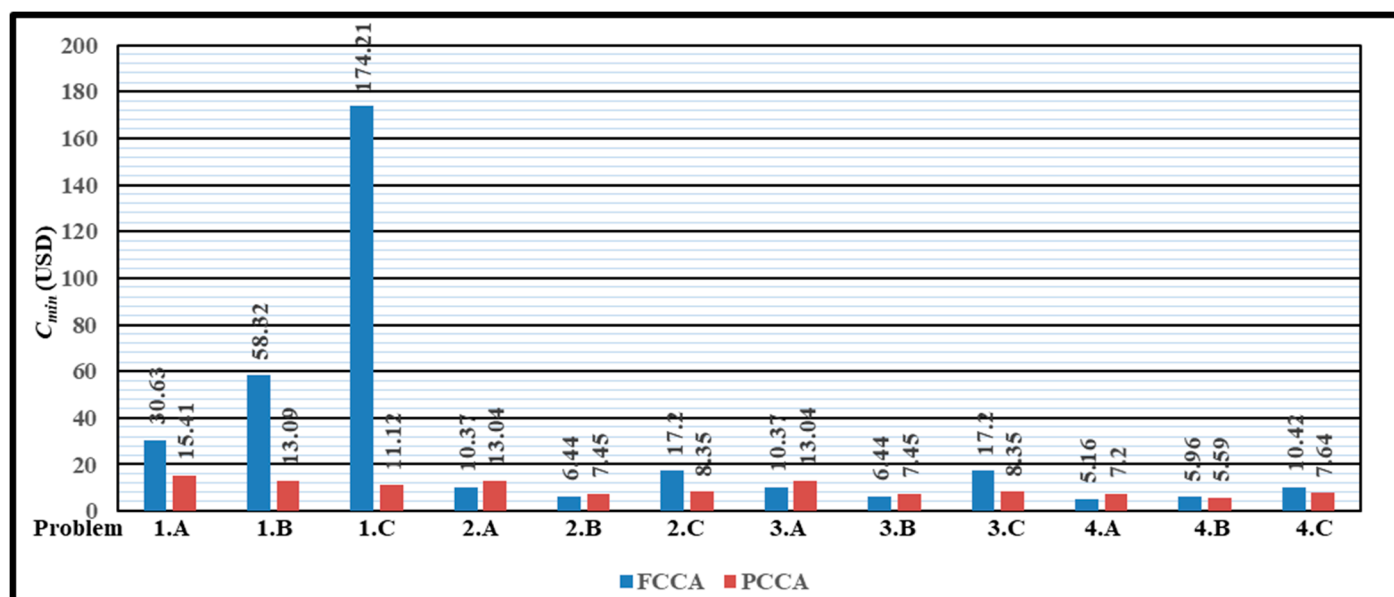


Figure 10. Comparison of study 3 between FCCA and PCCA for C_{min} .

5. Conclusions

The work described in this article applies only to the A_{min} and C_{min} for a RIF. The assumptions of this study are as follows: the foundation is rigid, and the soil is elastic and homogeneous, so the condition of biaxial bending is met, i.e., the soil pressure is linear.

This article concludes the following:

1. Some articles show the equations for finding the sides and A_{min} of a RIF with a PCCA, assuming the loads and moments it resists must be equal to those acting on the RIF.
2. The MPA shows the equations for finding the sides and A_{min} of a RIF with a PCCA, assuming that the loads and moments it resists must be greater than or equal to those acting on the RIF.
3. When the soil pressure on the RIF does not reach p_{max} , significant savings in A_{min} and in C_{min} can be achieved as follows:
 - 3.1. The MPA (PCCA) shows savings of 42.07% in A_{min} and 40.57% in C_{min} compared with the MPLR (FCCA) for $e_x = 0$ and $e_y = 0$ (see Tables 3 and 4).
 - 3.2. The MPA (PCCA) shows savings of 45.99% in A_{min} and 45.57% in C_{min} compared with the MPLR (FCCA) for $e_x = L_x/2 - c_x/2$ and $e_y = L_y/2 - c_y/2$ (see Tables 9 and 10).
 - 3.3. The MPA presents savings of 19.75% in A_{min} compared with the MPML for $e_x = 0$ and $e_y = 0$ (see Table 11).
 - 3.4. The MPA (PCCA) shows savings of 93.62% in C_{min} compared with the FCCA for $e_x = 0$ and $e_y = 0$ (see Table 12).
 - 3.5. The MPA (PCCA) presents savings of 51.45% in C_{min} compared with the FCCA for $e_x = L_x/2 - c_x/2$ and $e_y = 0$ (see Table 13).
 - 3.6. The MPA (PCCA) presents savings of 51.45% in C_{min} compared with the FCCA for $e_x = 0$ and $e_y = L_y/2 - c_y/2$ (see Table 14).
 - 3.7. The MPA (PCCA) presents savings of 26.67% in C_{min} compared with the FCCA for $e_x = L_x/2 - c_x/2$ and $e_y = L_y/2 - c_y/2$ (see Table 15).
4. The MPA could be used to revise p_{max} , assuming the objective function p_{max} , with the same constraints.
5. The MPA could be used for another type of code, since here, the loads and moments acting on the RIF are developed, and these are compared with those that must be resisted according to the standards (ACI 318-19) [38].

The next investigations can determine (1) the C_{min} for a RIF with a variable thickness, assuming a PCCA; (2) the C_{min} for an EIF, assuming a PCCA.

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Appendix A

The ultimate moments for each case are as follows.

Case I

$$M_{ua} = - \int_{e_y + \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{us} \left(y - e_y - \frac{c_y}{2} \right) dx dy, \quad (A1)$$

$$M_{ub} = \frac{P_u c_y}{2} + M_{ux} - \int_{e_y - \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{us} \left(y - e_y + \frac{c_y}{2} \right) dx dy, \quad (A2)$$

$$M_{uc} = - \int_{e_x + \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{us} \left(x - e_x - \frac{c_x}{2} \right) dy dx, \quad (A3)$$

$$M_{ue} = \frac{P_u c_x}{2} + M_{uy} - \int_{e_x - \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{us} \left(x - e_x + \frac{c_x}{2} \right) dy dx, \quad (A4)$$

Case IIA

For $L_y/2 - L_{y1} \geq e_y + c_y/2$,

$$M_{ua} = - \int_{\frac{L_y}{2} - L_{y1}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy, \quad (A5)$$

$$M_{ub} = \frac{P_u c_y}{2} + M_{ux} - \int_{\frac{L_y}{2} - L_{y1}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy, \quad (A6)$$

$$M_{uc} = - \int_{\frac{L_x}{2} - L_{x1}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx, \quad (A7)$$

$$M_{ue} = \frac{P_u c_x}{2} + M_{uy} - \int_{\frac{L_x}{2} - L_{x1}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx, \quad (A8)$$

Case IIB

For $L_y/2 - L_{y1} \leq e_y + c_y/2$,

$$M_{ua} = - \int_{e_y + \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} - L_{x2}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy - \int_{e_y + \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2} - L_{x2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy, \quad (A9)$$

$$M_{ub} = \frac{P_u c_y}{2} + M_{ux} - \int_{e_y - \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} - L_{x3}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy - \int_{e_y - \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2} - L_{x3}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy, \quad (A10)$$

$$M_{uc} = - \int_{e_x + \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} - L_{y2}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx - \int_{e_x + \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2} - L_{y2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx, \quad (A11)$$

$$M_{ue} = \frac{P_u c_x}{2} + M_{uy} - \int_{e_x - \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} - L_{y3}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx - \int_{e_x - \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2} - L_{y3}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx, \quad (A12)$$

where L_{x1} is the horizontal distance from p_1 to where it crosses NA, L_{y1} is the vertical distance from p_1 to where it crosses NA, L_{x2} is the horizontal distance from face p_1-p_4 on the “a-a”-axis to where it crosses NA, L_{y2} is the vertical distance from face p_1-p_2 on the “c-c”-axis to where it crosses NA, L_{x3} is the horizontal distance from face p_1-p_4 on the “b-b”-axis to where it crosses NA, and L_{y3} is the vertical distance from face p_1-p_2 on the “e-e”-axis to where it crosses NA.

$$L_{x2} = \frac{L_{x1}(2e_y + c_y - L_y + 2L_{y1})}{2L_{y1}},$$

$$L_{y2} = \frac{L_{y1}(2e_x + c_x - L_x + 2L_{x1})}{2L_{x1}},$$

$$L_{x3} = \frac{L_{x1}(2e_y - c_y - L_y + 2L_{y1})}{2L_{y1}},$$

$$L_{y3} = \frac{L_{y1}(2e_x - c_x - L_x + 2L_{x1})}{2L_{x1}}.$$

Case IIIA

For $L_y/2 - L_{y1} \geq e_y + c_y/2$,

$$M_{ua} = - \int_{\frac{L_y}{2} - L_{y2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy - \int_{\frac{L_y}{2} - L_{y1}}^{\frac{L_y}{2} - L_{y2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy, \quad (A13)$$

$$M_{ub} = \frac{P_u c_y}{2} + M_{ux} - \int_{\frac{L_y}{2} - L_{y2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy - \int_{\frac{L_y}{2} - L_{y1}}^{\frac{L_y}{2} - L_{y2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy, \quad (A14)$$

$$M_{uc} = - \int_{e_x + \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} - L_{y3}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx - \int_{e_x + \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2} - L_{y3}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx, \quad (A15)$$

$$M_{ue} = \frac{P_u c_x}{2} + M_{uy} - \int_{e_x - \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} - L_{y4}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx - \int_{e_x - \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2} - L_{y4}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx, \quad (A16)$$

Case IIIB

For $L_y/2 - L_{y1} \leq e_y + c_y/2$,

$$M_{ua} = - \int_{e_y + \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy - \int_{e_y + \frac{c_y}{2}}^{\frac{L_y}{2} - L_{y2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy, \quad (A17)$$

$$M_{ub} = \frac{P_u c_y}{2} + M_{ux} - \int_{\frac{L_y}{2} - L_{y2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy - \int_{e_y - \frac{c_y}{2}}^{\frac{L_y}{2} - L_{y2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy, \quad (A18)$$

$$M_{uc} = - \int_{e_x + \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} - L_{y3}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx - \int_{e_x + \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2} - L_{y3}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx, \quad (A19)$$

$$M_{ue} = \frac{P_u c_x}{2} + M_{uy} - \int_{e_x - \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} - L_{y4}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx - \int_{e_x - \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2} - L_{y4}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx, \quad (A20)$$

where L_{y2} is the vertical distance from face p_1 - p_2 on face p_2 - p_3 to where it crosses NA, L_{y3} is the vertical distance from face p_1 - p_2 on the “ c - c ”-axis to where it crosses NA, and L_{y4} is the vertical distance from face p_1 - p_2 on the “ e - e ”-axis to where it crosses NA.

$$L_{y2} = \frac{L_{y1}(L_{x1} - L_x)}{L_{x1}},$$

$$L_{y3} = \frac{L_{y1}(2e_x + c_x - L_x + 2L_{x1})}{2L_{x1}},$$

$$L_{y4} = \frac{L_{y1}(2e_x - c_x - L_x + 2L_{x1})}{2L_{x1}}.$$

Case IVA

For $L_x/2 - L_{x1} \geq e_x + c_x/2$,

$$M_{ua} = - \int_{e_y + \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} - L_{x3}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy - \int_{e_y + \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2} - L_{x3}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy, \quad (A21)$$

$$M_{ub} = \frac{P_u c_y}{2} + M_{ux} - \int_{e_y - \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} - L_{x4}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy - \int_{e_y - \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2} - L_{x4}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy, \quad (A22)$$

$$M_{uc} = - \int_{\frac{L_x}{2} - L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx - \int_{\frac{L_x}{2} - L_{x1}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx, \quad (A23)$$

$$M_{ue} = \frac{P_u c_x}{2} + M_{uy} - \int_{\frac{L_x}{2} - L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx - \int_{\frac{L_x}{2} - L_{x1}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx, \quad (A24)$$

Case IVB

For $L_x/2 - L_{x1} \leq e_x + c_x/2$,

$$M_{ua} = - \int_{e_y + \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} - L_{x3}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy - \int_{e_y + \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2} - L_{x3}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy, \quad (A25)$$

$$M_{ub} = \frac{P_u c_y}{2} + M_{ux} - \int_{e_y - \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} - L_{x4}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy - \int_{e_y - \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2} - L_{x4}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy, \quad (A26)$$

$$M_{uc} = - \int_{\frac{L_x}{2} - L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx - \int_{e_x + \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx, \quad (A27)$$

$$M_{ue} = \frac{P_u c_x}{2} + M_{uy} - \int_{\frac{L_x}{2} - L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx - \int_{e_x - \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx, \quad (A28)$$

where L_{x2} is the horizontal distance from face p_1 - p_4 on face p_3 - p_4 to where it crosses NA, L_{x3} is the horizontal distance from face p_1 - p_4 on the “ a - a ”-axis to where it crosses NA, and L_{x4} is the horizontal distance from face p_1 - p_4 on the “ b - b ”-axis to where it crosses NA.

$$L_{x2} = \frac{L_{x1}(L_{y1} - L_y)}{L_{y1}},$$

$$L_{x3} = \frac{L_{x1}(2e_y + c_y - L_y + 2L_{y1})}{2L_{y1}},$$

$$L_{x4} = \frac{L_{x1}(2e_y - c_y - L_y + 2L_{y1})}{2L_{y1}}.$$

Case VA

For $L_x/2 - L_{x2} \geq e_x + c_x/2$ and $L_y/2 - L_{y2} \geq e_y + c_y/2$,

$$M_{ua} = - \int_{\frac{L_y}{2}-L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy - \int_{e_y+\frac{c_y}{2}}^{\frac{L_y}{2}-L_{y2}} \int_{\frac{L_x}{2}+\frac{L_{x1}(L_y-2L_{y1}-2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy, \quad (A29)$$

$$M_{ub} = \frac{P_u c_y}{2} + M_{ux} - \int_{\frac{L_y}{2}-L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy - \int_{e_y+\frac{c_y}{2}}^{\frac{L_y}{2}-L_{y2}} \int_{\frac{L_x}{2}+\frac{L_{x1}(L_y-2L_{y1}-2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy, \quad (A30)$$

$$M_{uc} = - \int_{\frac{L_x}{2}-L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx - \int_{e_x+\frac{c_x}{2}}^{\frac{L_x}{2}-L_{x2}} \int_{\frac{L_y}{2}+\frac{L_{y1}(L_x-2L_{x1}-2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx, \quad (A31)$$

$$M_{ue} = \frac{P_u c_x}{2} + M_{uy} - \int_{\frac{L_x}{2}-L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx - \int_{e_x+\frac{c_x}{2}}^{\frac{L_x}{2}-L_{x2}} \int_{\frac{L_y}{2}+\frac{L_{y1}(L_x-2L_{x1}-2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx, \quad (A32)$$

Case VB

For $L_x/2 - L_{x2} \geq e_x + c_x/2$ and $L_y/2 - L_{y2} \leq e_y + c_y/2$,

$$M_{ua} = - \int_{e_y+\frac{c_y}{2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy, \quad (A33)$$

$$M_{ub} = \frac{P_u c_y}{2} + M_{ux} - \int_{\frac{L_y}{2}-L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy - \int_{e_y+\frac{c_y}{2}}^{\frac{L_y}{2}-L_{y2}} \int_{\frac{L_x}{2}+\frac{L_{x1}(L_y-2L_{y1}-2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy, \quad (A34)$$

$$M_{uc} = - \int_{\frac{L_x}{2}-L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx - \int_{e_x+\frac{c_x}{2}}^{\frac{L_x}{2}-L_{x2}} \int_{\frac{L_y}{2}+\frac{L_{y1}(L_x-2L_{x1}-2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx, \quad (A35)$$

$$M_{ue} = \frac{P_u c_x}{2} + M_{uy} - \int_{\frac{L_x}{2}-L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx - \int_{e_x+\frac{c_x}{2}}^{\frac{L_x}{2}-L_{x2}} \int_{\frac{L_y}{2}+\frac{L_{y1}(L_x-2L_{x1}-2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx, \quad (A36)$$

Case VC

For $L_x/2 - L_{x2} \leq e_x + c_x/2$ and $L_y/2 - L_{y2} \geq e_y + c_y/2$,

$$M_{ua} = - \int_{\frac{L_y}{2}-L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy - \int_{e_y+\frac{c_y}{2}}^{\frac{L_y}{2}-L_{y2}} \int_{\frac{L_x}{2}+\frac{L_{x1}(L_y-2L_{y1}-2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy, \quad (A37)$$

$$M_{ub} = \frac{P_u c_y}{2} + M_{ux} - \int_{\frac{L_y}{2}-L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy - \int_{e_y - \frac{c_y}{2}}^{\frac{L_y}{2}-L_{y2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y-2L_{y1}-2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy, \quad (A38)$$

$$M_{uc} = - \int_{e_x + \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx, \quad (A39)$$

$$M_{ue} = \frac{P_u c_x}{2} + M_{uy} - \int_{\frac{L_x}{2}-L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx - \int_{e_x - \frac{c_x}{2}}^{\frac{L_x}{2}-L_{x2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x-2L_{x1}-2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx, \quad (A40)$$

Case VD

For $L_x/2 - L_{x2} \leq e_x + c_x/2$ and $L_y/2 - L_{y2} \leq e_y + c_y/2$,

$$M_{ua} = - \int_{e_y + \frac{c_y}{2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y - \frac{c_y}{2} \right) dx dy, \quad (A41)$$

$$M_{ub} = \frac{P_u c_y}{2} + M_{ux} - \int_{\frac{L_y}{2}-L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy - \int_{e_y - \frac{c_y}{2}}^{\frac{L_y}{2}-L_{y2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y-2L_{y1}-2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} \left(y - e_y + \frac{c_y}{2} \right) dx dy, \quad (A42)$$

$$M_{uc} = - \int_{e_x + \frac{c_x}{2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x - \frac{c_x}{2} \right) dy dx, \quad (A43)$$

$$M_{ue} = \frac{P_u c_x}{2} + M_{uy} - \int_{\frac{L_x}{2}-L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx - \int_{e_x - \frac{c_x}{2}}^{\frac{L_x}{2}-L_{x2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x-2L_{x1}-2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} \left(x - e_x + \frac{c_x}{2} \right) dy dx, \quad (A44)$$

where L_{x2} is the horizontal distance from face p_1 - p_4 on face p_3 - p_4 to where it crosses NA, and L_{y2} is the vertical distance from face p_1 - p_2 on face p_2 - p_3 to where it crosses NA.

$$L_{x2} = \frac{L_{x1}(L_{y1} - L_y)}{L_{y1}},$$

$$L_{y2} = \frac{L_{y1}(L_{x1} - L_x)}{L_{x1}}.$$

Appendix B

The ultimate bending shears for each case are as follows.

Case I

$$V_{uf} = - \int_{e_y + \frac{c_y}{2} + d}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{us} dx dy, \quad (A45)$$

$$V_{ug} = P_u - \int_{e_y - \frac{c_y}{2} - d}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{us} dx dy, \quad (A46)$$

$$V_{uh} = - \int_{e_x + \frac{c_x}{2} + d}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{us} dy dx, \quad (A47)$$

$$V_{ui} = P_u - \int_{e_x - \frac{c_x}{2} - d}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{us} dy dx, \quad (\text{A48})$$

Case IIA

For $L_y/2 - L_{y1} \geq e_y + c_y/2 + d$,

$$V_{uf} = - \int_{\frac{L_y}{2} - L_{y1}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} dx dy, \quad (\text{A49})$$

$$V_{ug} = P_u - \int_{\frac{L_y}{2} - L_{y1}}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} dx dy, \quad (\text{A50})$$

$$V_{uh} = - \int_{\frac{L_x}{2} - L_{x1}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (\text{A51})$$

$$V_{ui} = P_u - \int_{\frac{L_x}{2} - L_{x1}}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (\text{A52})$$

Case IIB

For $L_y/2 - L_{y1} \leq e_y + c_y/2 + d$,

$$V_{uf} = - \int_{e_y + \frac{c_y}{2} + d}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} - L_{x2}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{e_y + \frac{c_y}{2} + d}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2} - L_{x2}} p_{uz} dx dy, \quad (\text{A53})$$

$$V_{ug} = P_u - \int_{e_y - \frac{c_y}{2} - d}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} - L_{x3}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{e_y - \frac{c_y}{2} - d}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2} - L_{x3}} p_{uz} dx dy, \quad (\text{A54})$$

$$V_{uh} = - \int_{e_x + \frac{c_x}{2} + d}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} - L_{y2}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{e_x + \frac{c_x}{2} + d}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2} - L_{y2}} p_{uz} dy dx, \quad (\text{A55})$$

$$V_{ui} = P_u - \int_{e_x - \frac{c_x}{2} - d}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} - L_{y3}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{e_x - \frac{c_x}{2} - d}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2} - L_{y3}} p_{uz} dy dx, \quad (\text{A56})$$

where L_{x1} is the horizontal distance from p_1 to where it crosses NA, L_{y1} is the vertical distance from p_1 to where it crosses NA, L_{x2} is the horizontal distance from face p_1 - p_4 on the “ f - f ”-axis to where it crosses NA, L_{y2} is the vertical distance from face p_1 - p_2 on the “ h - h ”-axis to where it crosses NA, L_{x3} is the horizontal distance from face p_1 - p_4 on the “ g - g ”-axis to where it crosses NA, and L_{y3} is the vertical distance from face p_1 - p_2 on the “ i - i ”-axis to where it crosses NA.

$$L_{x2} = \frac{L_{x1}(2e_y + c_y + d - L_y + 2L_{y1})}{2L_{y1}},$$

$$L_{y2} = \frac{L_{y1}(2e_x + c_x + d - L_x + 2L_{x1})}{2L_{x1}},$$

$$L_{x3} = \frac{L_{x1}(2e_y - c_y - d - L_y + 2L_{y1})}{2L_{y1}},$$

$$L_{y3} = \frac{L_{y1}(2e_x - c_x - d - L_x + 2L_{x1})}{2L_{x1}}.$$

Case IIIA

For $L_{y1}/2 - L_{y1} \geq e_y + c_y/2 + d$,

$$V_{uf} = - \int_{\frac{L_y}{2} - L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{\frac{L_y}{2} - L_{y1}}^{\frac{L_y}{2} - L_{y2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} dx dy, \quad (A57)$$

$$V_{ug} = P_u - \int_{\frac{L_y}{2} - L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{\frac{L_y}{2} - L_{y1}}^{\frac{L_y}{2} - L_{y2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} dx dy, \quad (A58)$$

$$V_{uh} = - \int_{e_x + \frac{c_x}{2} + d}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} - L_{y3}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{e_x + \frac{c_x}{2} + d}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2} - L_{y3}} p_{uz} dy dx, \quad (A59)$$

$$V_{ui} = P_u - \int_{e_x - \frac{c_x}{2} - d}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} - L_{y4}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{e_x - \frac{c_x}{2} - d}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2} - L_{y4}} p_{uz} dy dx, \quad (A60)$$

Case IIIB

For $L_{y1}/2 - L_{y1} \leq e_y + c_y/2 + d$,

$$V_{uf} = - \int_{\frac{L_y}{2} - L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{e_y + \frac{c_y}{2} + d}^{\frac{L_y}{2} - L_{y2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} dx dy, \quad (A61)$$

$$V_{ug} = P_u - \int_{\frac{L_y}{2} - L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz}(x, y) dx dy - \int_{e_y - \frac{c_y}{2} - d}^{\frac{L_y}{2} - L_{y2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz}(x, y) dx dy, \quad (A62)$$

$$V_{uh} = - \int_{e_x + \frac{c_x}{2} + d}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} - L_{y3}}^{\frac{L_y}{2}} p_{uz}(x, y) dy dx - \int_{e_x + \frac{c_x}{2} + d}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2} - L_{y3}} p_{uz}(x, y) dy dx, \quad (A63)$$

$$V_{ui} = P_u - \int_{e_x - \frac{c_x}{2} - d}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} - L_{y4}}^{\frac{L_y}{2}} p_{uz}(x, y) dy dx - \int_{e_x - \frac{c_x}{2} - d}^{\frac{L_x}{2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2} - L_{y4}} p_{uz}(x, y) dy dx, \quad (A64)$$

where L_{y2} is the vertical distance from face p_1 - p_2 on face p_2 - p_3 to where it crosses NA, L_{y3} is the vertical distance from face p_1 - p_2 on the “ h - h ”-axis to where it crosses NA, and L_{y4} is the vertical distance from face p_1 - p_2 on the “ i - i ”-axis to where it crosses NA.

$$L_{y2} = \frac{L_{y1}(L_{x1} - L_x)}{L_{x1}},$$

$$L_{y3} = \frac{L_{y1}(2e_x + c_x - L_x + 2L_{x1})}{2L_{x1}},$$

$$L_{y4} = \frac{L_{y1}(2e_x - c_x - L_x + 2L_{x1})}{2L_{x1}}.$$

Case IVA

For $L_x/2 - L_{x1} \geq e_x + c_x/2 + d$,

$$V_{uf} = - \int_{e_y + \frac{c_y}{2} + d}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} - L_{x3}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{e_y + \frac{c_y}{2} + d}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2} - L_{x3}} p_{uz} dx dy, \quad (A65)$$

$$V_{ug} = P_u - \int_{e_y - \frac{c_y}{2} - d}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} - L_{x4}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{e_y - \frac{c_y}{2} - d}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2} - L_{x4}} p_{uz} dx dy, \quad (A66)$$

$$V_{uh} = - \int_{\frac{L_x}{2} - L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{\frac{L_x}{2} - L_{x1}}^{\frac{L_x}{2} - L_{x2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (A67)$$

$$V_{ui} = P_u - \int_{\frac{L_x}{2} - L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{\frac{L_x}{2} - L_{x1}}^{\frac{L_x}{2} - L_{x2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (A68)$$

Case IVB

For $L_x/2 - L_{x1} \leq e_x + c_x/2 + d$,

$$V_{uf} = - \int_{e_y + \frac{c_y}{2} + d}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} - L_{x3}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{e_y + \frac{c_y}{2} + d}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2} - L_{x3}} p_{uz} dx dy, \quad (A69)$$

$$V_{ug} = P_u - \int_{e_y - \frac{c_y}{2} - d}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} - L_{x4}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{e_y - \frac{c_y}{2} - d}^{\frac{L_y}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2} - L_{x4}} p_{uz} dx dy, \quad (A70)$$

$$V_{uh} = - \int_{\frac{L_x}{2} - L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{e_x + \frac{c_x}{2} + d}^{\frac{L_x}{2} - L_{x2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (A71)$$

$$V_{ui} = P_u - \int_{\frac{L_x}{2} - L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{e_x - \frac{c_x}{2} - d}^{\frac{L_x}{2} - L_{x2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (A72)$$

where L_{x2} is the horizontal distance from face p_1 - p_4 on face p_3 - p_4 to where it crosses NA, L_{x3} is the horizontal distance from face p_1 - p_4 on the “f-f”-axis to where it crosses NA, and L_{x4} is the horizontal distance from face p_1 - p_4 on the “g-g”-axis to where it crosses NA.

$$L_{x2} = \frac{L_{x1}(L_{y1} - L_y)}{L_{y1}},$$

$$L_{x3} = \frac{L_{x1}(2e_y + c_y + d - L_y + 2L_{y1})}{2L_{y1}},$$

$$L_{x4} = \frac{L_{x1}(2e_y - c_y - d - L_y + 2L_{y1})}{2L_{y1}}.$$

Case VA

For $L_x/2 - L_{x2} \geq e_x + c_x/2 + d$ and $L_y/2 - L_{y2} \geq e_y + c_y/2 + d$,

$$V_{uf} = - \int_{L_y - L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{e_y + \frac{c_y}{2} + d}^{\frac{L_y}{2} - L_{y2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} dx dy, \quad (A73)$$

$$V_{ug} = P_u - \int_{L_y - L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{e_y - \frac{c_y}{2} - d}^{\frac{L_y}{2} - L_{y2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} dx dy, \quad (A74)$$

$$V_{uh} = - \int_{\frac{L_x}{2}-L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{e_x+\frac{c_x}{2}+d}^{\frac{L_x}{2}-L_{x2}} \int_{\frac{L_y}{2}+\frac{L_{y1}(L_x-2L_{x1}-2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (\text{A75})$$

$$V_{ui} = P_u - \int_{\frac{L_x}{2}-L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{e_x-\frac{c_x}{2}-d}^{\frac{L_x}{2}-L_{x2}} \int_{\frac{L_y}{2}+\frac{L_{y1}(L_x-2L_{x1}-2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (\text{A76})$$

Case VB

For $L_x/2 - L_{x2} \geq e_x + c_x/2 + d$ and $L_y/2 - L_{y2} \leq e_y + c_y/2 + d$,

$$V_{uf} = - \int_{e_y+\frac{c_y}{2}+d}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} dx dy, \quad (\text{A77})$$

$$V_{ug} = P_u - \int_{\frac{L_y}{2}-L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{e_y-\frac{c_y}{2}-d}^{\frac{L_y}{2}-L_{y2}} \int_{\frac{L_x}{2}+\frac{L_{x1}(L_y-2L_{y1}-2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} dx dy, \quad (\text{A78})$$

$$V_{uh} = - \int_{\frac{L_x}{2}-L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{e_x+\frac{c_x}{2}+d}^{\frac{L_x}{2}-L_{x2}} \int_{\frac{L_y}{2}+\frac{L_{y1}(L_x-2L_{x1}-2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (\text{A79})$$

$$V_{ui} = P_u - \int_{\frac{L_x}{2}-L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{e_x-\frac{c_x}{2}-d}^{\frac{L_x}{2}-L_{x2}} \int_{\frac{L_y}{2}+\frac{L_{y1}(L_x-2L_{x1}-2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (\text{A80})$$

Case VC

For $L_x/2 - L_{x2} \leq e_x + c_x/2 + d$ and $L_y/2 - L_{y2} \geq e_y + c_y/2 + d$,

$$V_{uf} = - \int_{\frac{L_y}{2}-L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{e_y+\frac{c_y}{2}+d}^{\frac{L_y}{2}-L_{y2}} \int_{\frac{L_x}{2}+\frac{L_{x1}(L_y-2L_{y1}-2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} dx dy, \quad (\text{A81})$$

$$V_{ug} = P_u - \int_{\frac{L_y}{2}-L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{e_y-\frac{c_y}{2}-d}^{\frac{L_y}{2}-L_{y2}} \int_{\frac{L_x}{2}+\frac{L_{x1}(L_y-2L_{y1}-2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} dx dy, \quad (\text{A82})$$

$$V_{uh} = - \int_{e_x+\frac{c_x}{2}+d}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (\text{A83})$$

$$V_{ui} = P_u - \int_{\frac{L_x}{2}-L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{e_x-\frac{c_x}{2}-d}^{\frac{L_x}{2}-L_{x2}} \int_{\frac{L_y}{2}+\frac{L_{y1}(L_x-2L_{x1}-2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (\text{A84})$$

Case VD

For $L_x/2 - L_{x2} \leq e_x + c_x/2 + d$ and $L_y/2 - L_{y2} \leq e_y + c_y/2 + d$,

$$V_{uf} = - \int_{e_y+\frac{c_y}{2}+d}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} dx dy, \quad (\text{A85})$$

$$V_{ug} = P_u - \int_{\frac{L_y}{2}-L_{y2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} p_{uz} dx dy - \int_{e_y-\frac{c_y}{2}-d}^{\frac{L_y}{2}-L_{y2}} \int_{\frac{L_x}{2}+\frac{L_{x1}(L_y-2L_{y1}-2y)}{2L_{y1}}}^{\frac{L_x}{2}} p_{uz} dx dy, \quad (\text{A86})$$

$$V_{uh} = - \int_{e_x + \frac{c_x}{2} + d}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (A87)$$

$$V_{ui} = \frac{P_u c_x}{2} + M_{uy} - \int_{\frac{L_x}{2} - L_{x2}}^{\frac{L_x}{2}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} p_{uz} dy dx - \int_{e_x - \frac{c_x}{2} - d}^{\frac{L_x}{2} - L_{x2}} \int_{\frac{L_y}{2} + \frac{L_{y1}(L_x - 2L_{x1} - 2x)}{2L_{x1}}}^{\frac{L_y}{2}} p_{uz} dy dx, \quad (A88)$$

where L_{x2} is the horizontal distance from face p_1 - p_4 on face p_3 - p_4 to where it crosses NA, and L_{y2} is the vertical distance from face p_1 - p_2 on face p_2 - p_3 to where it crosses NA.

$$L_{x2} = \frac{L_{x1}(L_{y1} - L_y)}{L_{y1}},$$

$$L_{y2} = \frac{L_{y1}(L_{x1} - L_x)}{L_{x1}}.$$

Appendix C

The ultimate punching shears for each case are as follows.

Case I

$$V_{up} = P_u - \int_{e_y - \frac{c_y}{2} - \frac{d}{2}}^{e_y + \frac{c_y}{2} + \frac{d}{2}} \int_{e_x - \frac{c_x}{2} - \frac{d}{2}}^{e_x + \frac{c_x}{2} + \frac{d}{2}} p_{us} dx dy, \quad (A89)$$

Case IIA

$$V_{up} = P_u, \quad (A90)$$

Case IIB

$$V_{up} = P_u - \int_{e_y + y_p}^{e_y + \frac{c_y}{2} + \frac{d}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{e_x + \frac{c_x}{2} + \frac{d}{2}} p_{uz} dx dy, \quad (A91)$$

where

$$y_p = \frac{L_y}{2} - L_{y1} - e_y - \frac{L_{y1}(2e_x + c_x + d - L_x)}{2L_{x1}}.$$

Case IIC

$$V_{up} = P_u - \int_{e_y - \frac{c_y}{2} - \frac{d}{2}}^{e_y + \frac{c_y}{2} + \frac{d}{2}} \int_{e_x + x_{p2}}^{e_x + \frac{c_x}{2} + \frac{d}{2}} p_{uz} dx dy - \int_{e_y - \frac{c_y}{2} - \frac{d}{2}}^{e_y + \frac{c_y}{2} + \frac{d}{2}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{e_x + x_{p2}} p_{uz} dx dy, \quad (A92)$$

where

$$x_{p1} = L_{x1} - \frac{L_x}{2} + e_x + \frac{L_{x1}(2e_y + c_y + d - L_y)}{2L_{y1}},$$

$$x_{p2} = \frac{L_x}{2} - L_{x1} - e_x - \frac{L_{x1}(2e_y - c_y - d - L_y)}{2L_{y1}}.$$

Case IID

$$V_{up} = P_u - \int_{e_y + \frac{c_y}{2} + \frac{d}{2}}^{e_y + \frac{c_y}{2} - y_{p1}} \int_{e_x - \frac{c_x}{2} - \frac{d}{2}}^{e_x + \frac{c_x}{2} + \frac{d}{2}} p_{uz} dx dy - \int_{e_y - \frac{c_y}{2} - \frac{d}{2}}^{e_y + \frac{c_y}{2} - y_{p1}} \int_{e_x + x_{p1}}^{e_x + \frac{c_x}{2} + \frac{d}{2}} p_{uz} dx dy - \int_{e_y - \frac{c_y}{2} - \frac{d}{2}}^{e_y + \frac{c_y}{2} - y_{p1}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{e_x + x_{p1}} p_{uz} dx dy, \quad (A93)$$

where

$$y_{p1} = \frac{L_y}{2} - L_{y1} - e_y - \frac{L_{y1}(2e_x - c_x - d - L_x)}{2L_{x1}},$$

$$x_{p1} = \frac{L_x}{2} - L_{x1} - e_x - \frac{L_{x1}(2e_y - c_y - d - L_y)}{2L_{y1}}.$$

Case IIIA

$$V_{up} = P_u - \int_{e_y + \frac{c_y}{2} - y_{p1}}^{e_y + \frac{c_y}{2} + \frac{d}{2}} \int_{e_x - \frac{c_x}{2} - \frac{d}{2}}^{e_x + \frac{c_x}{2} + \frac{d}{2}} p_{uz} dx dy - \int_{e_y - \frac{c_y}{2} + y_{p2}}^{e_y + \frac{c_y}{2} - y_{p1}} \int_{\frac{L_x}{2} + \frac{L_{x1}(L_y - 2L_{y1} - 2y)}{2L_{y1}}}^{e_x + \frac{c_x}{2} + \frac{d}{2}} p_{uz} dx dy, \quad (A94)$$

where

$$y_{p1} = \frac{L_y}{2} - e_y - L_{y1} - \frac{L_{y1}(2e_x - c_x - d - L_x)}{2L_{x1}},$$

$$y_{p2} = L_{y1} - \frac{L_y}{2} + e_y + \frac{L_{y1}(2e_x + c_x + d - L_x)}{2L_{x1}}.$$

Case IVA

$$V_{up} = P_u - \int_{e_y - \frac{c_y}{2} - \frac{d}{2}}^{e_y + \frac{c_y}{2} + \frac{d}{2}} \int_{e_x - \frac{c_x}{2} - \frac{d}{2}}^{e_x + \frac{c_x}{2} + \frac{d}{2}} p_{uz} dx dy, \quad (A95)$$

Note: Case V is equal to Case IV.

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