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On the Entire Solutions of a Class of Differential Equations Related to Fermat-Type Equations

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Abstract

We study a substantially generalized version of the equation on nonlinear monomial differential equations, and obtain two-sided inequalities for the growth of entire solutions. We also study the Fermat-type differential equation with two exponential terms, and obtain almost all of the entire solutions.

Keywords: entire solution; nonlinear differential equation; Nevanlinna theory; Fermat-type equation

MSC: 34M05; 30D35; 34M10

1. Introduction and Main Results

In this paper, we use the basic results and standard notations in Nevanlinna theory (see, e.g., [1–3]), such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$ and the counting function of poles $N(r, f)$. We denote by $S(r, f)$ a quantity satisfying $S(r, f) = o(T(r, f))$ for $r \rightarrow \infty$ outside a possible exceptional set of finite linear measure, not necessarily the same at each occurrence.

We also need the order and type of entire functions to estimate the growth of solutions. The order $\rho(g)$ of an entire function g is defined by

$$\rho(g) := \limsup_{r \rightarrow \infty} \frac{\log \log M(r, g)}{\log r},$$

where $M(r, g)$ is the maximum modulus $\max_{|z|=r} |g(z)|$. If $0 < \rho(g) < \infty$, then the type $\sigma(g)$ of g is defined by

$$\sigma(g) := \limsup_{r \rightarrow \infty} \frac{\log M(r, g)}{r^{\rho(g)}}.$$

Recently, there have been many results on entire solutions of complex differential equations using Nevanlinna theory and related methods, for example, [4–12]. Among them, motivated by previous results on equations $ff'' = p(z) \sin^2 z$ and $ff' = \frac{1}{2} \sin 2z$, Gundersen, Lü, Ng and Yang [6] considered entire solutions of nonlinear monomial differential equations of the form

$$f^{n_0} (f')^{n_1} (f'')^{n_2} \dots (f^{(k)})^{n_k} = H(z), \quad (1)$$

where $H \not\equiv 0$ is entire, $k \geq 1$, $n_0 \geq 1$ and $n_k \geq 1$, and obtained the following two-sided inequalities for the growth of the entire solution of (1):



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Theorem 1 ([6]). If f is an entire solution of a monomial differential Equation (1), then we have

$$\frac{1}{q}T(r, H) + S(r, f) \leq T(r, f) \leq \frac{1}{n_0}T(r, H) + S(r, f),$$

where $q = n_0 + n_1 + \cdots + n_k$. Hence, $\rho(f) = \rho(H)$.

However, in ref. [6], Gundersen et al. showed Theorem A cannot hold for other equations that are not of the form (1), since $f(z) = \sin z$ solves $f^2 + (f')^2 = 1$. So in this paper, we substantially extend Equation (1) to

$$L_1(f)^{n_1}L_2(f)^{n_2} \cdots L_k(f)^{n_k} = H(z), \quad (2)$$

where $L_i(f) = a_{i,0}f + a_{i,1}f' + \cdots + a_{i,m_i}f^{(m_i)}$ are nonzero linear differential polynomials with constant coefficients, $H(z)$ is an entire function, and $n_i \geq 1$ for all $i = 1, \dots, k$. This generalization introduces substantial difficulties not present in the monomial case: the zero structure of $L_i(f)$ is no longer governed solely by a single derivative, which leads to a more complex growth behavior of solutions. This is already evident from the example discussed in Gundersen et al., and consequently, new analytical tools are needed to handle the general case. We obtain the following estimate for the growth of entire solutions of (2):

Theorem 2. Let f be an entire solution of (2); then for sufficiently large r ,

$$\frac{1}{n_1 + \cdots + n_k}T(r, H) + S(r, f) \leq T(r, f) \leq K_2 \max\{r, T((2r)^{1+\epsilon}, H)\} + S(r, f)$$

holds for some $K_2 > 0$ and any $\epsilon > 0$. Moreover, if $\rho(H) < \infty$, then for sufficiently large r ,

$$\frac{1}{n_1 + \cdots + n_k}T(r, H) + S(r, f) \leq T(r, f) \leq K_2 \max\{r, r^{\lfloor \rho(H) \rfloor}, T(r, H)\} + S(r, f)$$

holds for some $K_2 > 0$. Hence, $\rho(f) = \rho(H)$ when $1 \leq \rho(H) \leq \infty$, and $\rho(f) \leq 1$ when $\rho(H) < 1$.

Theorem 2 has many applications in estimating growth of entire solutions; one example is the Fermat-type differential equation

$$f(z)^2 + a(L(f))^2 = H(z) \quad (3)$$

for some complex constant a , linear differential polynomial $L(f)$ in f with constant coefficients, and entire function $H(z)$. Note that Equation (3) was first discussed by Yang and Li in 2004 [13] (Theorem 1), where $a = 1$ and H is a polynomial. By Theorem 2 and under the conditions above, we have $T(r, f) = O(r)$ for all entire solutions f . In fact, the obtained solutions must be of the form $f(z) = \frac{1}{2}(P(z)e^{R(z)} + Q(z)e^{-R(z)})$, where P, Q, R are polynomials, $PQ = H$ and $R(z) = \lambda z$, so $T(r, \frac{1}{2}(P(z)e^{\lambda z} + Q(z)e^{-\lambda z})) = O(r)$ by [14,15].

Since the work of [13], numerous works have studied Fermat-type differential equations of this type as well as their extensions (such as difference-type equations), for example, [7,9,11,16–21]. Among them, in 2007, Tang and Liao [9] studied the case where a, H are polynomials and $L(f) = f^{(n)}$. In 2023, Guo and Liu [7] considered some special cases where $L(f) = f'$, $a = 1$ and $H = e^{2h(z)}$. Recently in 2024, Xu and Haldar [10] studied the existence and explicit forms of finite-order transcendental entire solutions to several classes of Fermat-type difference and partial differential-difference equations in $\mathbb{C}^n$.

Note that the above results were obtained under the condition that H is a polynomial or a single exponential term. We introduce some results obtained when H has two expo-

nential terms. In 2022, motivated by the trigonometric identity $\cos^2 z - \sin^2 z = \cos 2z$, Gundersen et al. [6] considered the equation

$$(f(z))^2 - (f'(z))^2 = \cos 2z, \quad (4)$$

and obtained the following result:

Theorem 3 ([6]). *The only entire solutions of differential Equation (4) are the four solutions $f(z) = \pm \cos z, \pm i \sin z$.*

Note that $\cos 2z$ is a special case of $p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}$, where $p_1, p_2, \lambda_1, \lambda_2$ are constants with $p_1 p_2 \neq 0$ and $\lambda_1 \neq \lambda_2$. So in 2023, Gao and Gao [5] considered the equation

$$(f(z))^2 + a(f'(z))^2 = p_1 e^{\lambda z} + p_2 e^{-\lambda z}, \quad (5)$$

and then obtained the following result:

Theorem 4 ([5]). *Let a, p_1, p_2 , and λ be nonzero constants satisfying $9a\lambda^2 + 4 \neq 0$. Then Equation (5) has entire solutions if and only if the condition $a\lambda^2 + 1 = 0$ or $a\lambda^2 - 4 = 0$ holds. Moreover, we have*

(i) *If $a\lambda^2 + 1 = 0$, then the entire solutions of the above equation are*

$$f(z) = r_i e^{\lambda z} + s_i e^{-\lambda z} + t_i,$$

where t_i ($i = 1, 2, 3, 4$) are the four roots of $t^4 + p_1 p_2 = 0$, $r_i = \frac{p_1}{2t_i}$ and $s_i = \frac{p_2}{2t_i}$.

(ii) *If $a\lambda^2 - 4 = 0$, then the entire solutions of (5) are*

$$f(z) = a_i e^{\frac{\lambda z}{2}} + b_j e^{-\frac{\lambda z}{2}},$$

where a_i ($i = 1, 2$) is the square root of $\frac{p_1}{2}$, and b_j ($j = 1, 2$) is the square root of $\frac{p_2}{2}$.

Then very recently in 2026, Gao, Jiang, Liu and Song [4] considered the remaining case $9a\lambda^2 + 4 = 0$ of Theorem C, and obtained the following:

Theorem 5 ([4]). *If $a = -\frac{4}{9\lambda^2}$, then the entire solutions of (5) are*

$$f_i(z) = c_i e^{\frac{\lambda}{2} z} + d_i e^{-\frac{3\lambda}{2} z}$$

and

$$f_j(z) = c_j e^{-\frac{\lambda}{2} z} + d_j e^{\frac{3\lambda}{2} z},$$

where c_i is the square root of $\frac{9}{8} p_1$ and $d_i = \frac{p_2}{3p_1} c_i$ for $i = 1, 2$, and c_j is the square root of $\frac{9}{8} p_2$ and $d_j = \frac{p_1}{3p_2} c_j$ for $j = 3, 4$.

In fact, Gao et al. [4] considered the equation

$$(f(z))^n + a(f'(z))^n = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z},$$

where n is a positive integer; a, p_i ($i = 1, 2$) are nonzero constants; and λ_j ($j = 1, 2$) are nonzero distinct constants. They provided the exact form solutions to the above equation except the case $n = 2, \lambda_1 \neq \pm \lambda_2$. In this case, Gao et al. [4] only proved a special case when $a = \frac{1}{2\lambda_1 \lambda_2}$.

Theorem 6 ([4]). If $a = \frac{1}{2\lambda_1\lambda_2}$, $\lambda_1 \neq -\lambda_2$ with $|\lambda_1| \leq |\lambda_2|$. Suppose that f is an entire solution of

$$(f(z))^2 + a(f'(z))^2 = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}. \quad (6)$$

Then $\lambda_1 = -\frac{1}{2}\lambda_2$ and

$$f(z) = c_i e^{-\lambda_1 z} + d_i e^{2\lambda_1 z},$$

where c_i is the square root of $\frac{4}{3}p_2$ and $d_i = \frac{p_1}{4p_2}c_i$ for $i = 5, 6$.

However, the case $a \neq \frac{1}{2\lambda_1\lambda_2}$ was not treated in [4], as the methods employed there are not applicable in this situation. Consequently, handling this case requires substantially different approaches and new technical tools. In the present paper, we do not restrict ourselves to the tools used in [4]; we also employ Wiman–Valiron theory, results on functions of exponential type, and a lexicographic ordering on complex numbers. These new tools enable us to determine the explicit form of entire solutions of (6) for the general parameter a , up to a few exceptional cases, which are discussed separately. Our result is the following theorem:

Theorem 7. Let a , p_1 , p_2 , and λ_1 , λ_2 be nonzero complex constants with $\lambda_1 \neq \pm\lambda_2$, and without loss of generality, $|\lambda_1| \leq |\lambda_2|$. Suppose $f(z)$ is an entire solution of Equation (6); then f must be of the form $f(z) = \sum_{n=1}^N a_n e^{b_n z}$ with $a_n \neq 0$ and $b_i \neq b_j$ for $i \neq j$. Moreover, if $N < 5$, a and f must satisfy one of the following conditions:

(i) $N = 2$, $a = -\frac{4}{\lambda_1\lambda_2}$; then

$$f(z) = \pm \sqrt{\frac{p_1}{1 + a\lambda_1^2/4}} e^{\lambda_1 z/2} \pm \sqrt{\frac{p_2}{1 + a\lambda_2^2/4}} e^{\lambda_2 z/2}.$$

(ii) $N = 2$, $a = -\frac{4}{(2\lambda_1 - \lambda_2)^2}$; then

$$f(z) = \frac{p_1}{2a_2(1 + ab_1b_2)} e^{b_1 z} + a_2 e^{b_2 z}, \quad a_2 = \pm \sqrt{\frac{p_2}{1 + ab_2^2}}, \quad b_1 = \lambda_1 - \frac{\lambda_2}{2}, \quad b_2 = \frac{\lambda_2}{2}.$$

(iii) $N = 2$, $a = -\frac{4}{(2\lambda_2 - \lambda_1)^2}$; then

$$f(z) = \frac{p_1}{2a_2(1 + ab_1b_2)} e^{b_1 z} + a_2 e^{b_2 z}, \quad a_2 = \pm \sqrt{\frac{p_2}{1 + ab_2^2}}, \quad b_1 = \lambda_2 - \frac{\lambda_1}{2}, \quad b_2 = \frac{\lambda_1}{2}.$$

(iv) $N = 3$, $\lambda_2 = -8\lambda_1$, $a = -\frac{1}{4\lambda_1^2}$; then

$$f(z) = a_1 e^{2\lambda_1 z} + a_2 e^{-\lambda_1 z} + a_3 e^{-4\lambda_1 z}, \quad 3a_1 a_2 = p_1, \quad \frac{3}{4}a_2^2 + 6a_1 a_3 = 0, \quad -3a_3^2 = p_2.$$

There are some examples to illustrate that Cases (i)–(iv) in Theorem 7 do have solutions.

Example 1 (Case (i) of Theorem 2). Let $\lambda_1 = 1$, $\lambda_2 = 4$, $p_1 = 3$, $p_2 = -3$ and $a = -1$; then $f_{1,1} = 2e^{\frac{z}{2}} + e^{2z}$, $f_{1,2} = 2e^{\frac{z}{2}} - e^{2z}$, $f_{1,3} = -f_{1,1}$ and $f_{1,4} = -f_{1,2}$ satisfy the equation

$$(f(z))^2 - (f'(z))^2 = 3e^z - 3e^{4z}.$$

Example 2 ([4] (Example 5), Case (ii) of Theorem 2). Let $\lambda_1 = 1$, $\lambda_2 = -2$, $p_1 = 6$, $p_2 = 3$ and $a = -1/4$; then $f_{2,1} = 2e^{-z} + e^{2z}$ and $f_{2,2} = -f_{2,1}$ satisfy the equation

$$(f(z))^2 - \frac{1}{4}(f'(z))^2 = 6e^z + 3e^{-2z}.$$

Example 3 (Another example for Case (ii) of Theorem 2). Let $\lambda_1 = 3$, $\lambda_2 = 4$, $p_1 = 1$, $p_2 = -3$ and $a = -1$; then $f_{3,1} = -\frac{1}{2}e^z + e^{2z}$ and $f_{3,2} = -f_{3,1}$ satisfy the equation

$$(f(z))^2 - (f'(z))^2 = e^{3z} - 3e^{4z}.$$

Example 4 (Case (iii) of Theorem 2). Let $\lambda_1 = i$, $\lambda_2 = 2i$, $p_1 = 1$, $p_2 = 1$ and $a = 4/9$; then $f_{4,1} = \frac{\sqrt{2}}{2}e^{\frac{3i}{2}z} + \frac{3\sqrt{2}}{4}e^{\frac{i}{2}z}$ and $f_{4,2} = -f_{4,1}$ satisfy the equation

$$(f(z))^2 + \frac{4}{9}(f'(z))^2 = e^{iz} + e^{2iz}.$$

Example 5 (Case (iv) of Theorem 2). Let $\lambda_1 = 1$, $\lambda_2 = -8$, $p_1 = 12$, $p_2 = -12$ and $a = -1/4$; then $f_{5,1} = -e^{2z} - 4e^{-z} + 2e^{-4z}$, $f_{5,2} = -\omega^2 e^{2z} - 4\omega e^{-z} + 2e^{-4z}$, $f_{5,3} = -\omega e^{2z} - 4\omega^2 e^{-z} + 2e^{-4z}$, $f_{5,4} = -f_{5,1}$, $f_{5,5} = -f_{5,2}$ and $f_{5,6} = -f_{5,3}$ satisfy the equation

$$(f(z))^2 - \frac{1}{4}(f'(z))^2 = 12e^z - 12e^{-8z},$$

where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Remark 1. In the proof of Theorem 7, when $N = 4$, f is not an entire solution of (6). Unfortunately, we have not found an example for $N \geq 5$ of Theorem 7. However, in our proof, we have the following result:

Under the same conditions as in Theorem 7 but with $N \geq 5$, the coefficients satisfy the following condition for a positive integer $K \geq N - 1$:

$$\frac{\lambda_1}{\lambda_2} = \frac{-2K^2 + 4K - 1}{2K^2}, \quad a = -\left[\frac{2K}{(K-1)\lambda_2}\right]^2.$$

In fact, for any fixed λ_1/λ_2 (i.e., for any integer $K \geq 4$), we can determine all the a_n and b_n coefficients of $f(z) = \sum_{n=1}^N a_n e^{b_n z}$ according to the last part of the proof of Theorem 7. In fact, when $K = 3$, there are five equations that the four a_n must satisfy through calculation, which leads to $a_n = 0$ for all $n = 1, \dots, N$. If K continues to increase, the number of equations that a_n satisfies will far exceed the number of a_n , and a_n can almost only be zero. So f in this case may not be an entire solution of Equation (6). Therefore, we propose the following conjecture:

Conjecture. Under the same conditions as in Theorem 7, Equation (6) has no entire solutions of the form $f(z) = \sum_{n=1}^N a_n e^{b_n z}$ with $N \geq 5$. Equivalently, every entire solution of Equation (6) must be a sum of at most four exponential terms; in fact, based on the classification in Theorem 7, only the cases $N = 2$ and $N = 3$ are expected to occur.

2. Preliminary Results and Some Lemmas

2.1. Nevanlinna Theory

The first is the lemma on logarithmic derivatives, which can be found in any book about Nevanlinna theory.

Lemma 1 (Lemma on Logarithmic Derivatives, [1–3]). Let $f(z)$ be a transcendental meromorphic function and $k \geq 1$ be an integer. Then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

and if $f(z)$ is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

We use the following Borel-type theorem to obtain the explicit solutions of Equation (6):

Lemma 2 ([3]). Let $f_j(z)$ ($j = 1, 2, \dots, n$) ($n \geq 2$) be a system of meromorphic functions and $g_k(z)$ ($k = 1, 2, \dots, n$) be a system of entire functions, satisfying the following conditions:

- (i) $\sum_{j=1}^n f_j(z) e^{g_j(z)} \equiv 0$.
 - (ii) For $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ is non-constant.
 - (iii) For $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j) = S(r, e^{g_h - g_k})$.
- Then $f_j(z) \equiv 0$ ($j = 1, 2, \dots, n$).

We need the following lemma to estimate the growth between $T(r, g)$ and $M(r, g)$:

Lemma 3 ([2,3]). Let g be an entire function and assume that $0 < r < R < \infty$ and that the maximum modulus satisfies $M(r, g) \geq 1$. Then

$$T(r, g) \leq \log M(r, g) \leq \frac{R+r}{R-r} T(R, g).$$

2.2. Wiman–Valiron Theory

We also need the Wiman–Valiron theory (see, e.g., [2,22,23]), which plays an important role in considering entire solutions of complex differential equations. Let g be an entire function whose Taylor expansion is $g(z) = \sum_{n=0}^{\infty} a_n z^n$. For a given $r > 0$, we define $\mu(r) := \max_{n \leq 0} \{|a_n| r^n\}$ as the maximum term, and $\nu(r) := \max\{m, \mu(r) = |a_m| r^m\}$ as the central index.

We use the following lemmas in this paper, which can all be found in [22,23]:

Lemma 4. If $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function, $a_0 \neq 0$, then

$$\log \mu(r) = \log |a_0| + \int_0^r \frac{\nu(t) dt}{t}.$$

Lemma 5. If $g(z)$ is an entire function of order $\rho(g)$, then

$$\rho(g) = \limsup_{r \rightarrow \infty} \frac{\log \log \mu(r)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log \nu(r)}{\log r}.$$

Lemma 6. If $g(z)$ is an entire function of finite order $\rho(g)$, then

$$\lim_{r \rightarrow \infty} \frac{\log M(r, g)}{\log \mu(r)} = 1.$$

The following lemma plays an important role in analyzing the growth of the derivative of functions:

Lemma 7 ([2,22,23]). Let g be a transcendental entire function, let $0 < \delta < 1/4$ and let z be such that $|z| = r$ and that

$$|g(z)| > M(r, g)v(r)^{-\frac{1}{4}+\delta}$$

holds. Then there exists a set $F \subset \mathbb{R}_+$ of finite logarithmic measure such that

$$g^{(m)}(z) = \left(\frac{v(r)}{z}\right)^m (1 + o(1))g(z)$$

holds for all $m \geq 0$ and all $r \notin F$.

2.3. Entire Functions of Exponential Type

If an entire function's order is less than 1 or its order is 1 but its type is finite, then it is called an entire function of exponential type.

We will prove the following lemma to estimate the growth of entire functions:

Lemma 8. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function with $v(r) = O(r)$, then $f(z)$ is of exponential type.

Proof. By our assumption, $\log v(r) \leq \log r + M_0$ for some finite $M_0 > 0$ and sufficiently large r . Then by Lemma 5, we obtain $\rho(f) \leq 1$. When $\rho(f) < 1$, the result is clear, so we just discuss the condition $\rho(f) = 1$. We will discuss the following two cases:

Case 1. $a_0 \neq 0$. Since $v(r) = O(r)$, we have $v(r) \leq Mr$ for a finite $M > 0$ and sufficiently large r . By Lemma 4, $\log \mu(r) = \log |a_0| + \int_0^r \frac{v(t)dt}{t} \leq \log |a_0| + Mr$; hence, $\log \mu(r) \leq M_1 r$ for finite $M_1 = M + 1 > 0$ and sufficiently large r . By Lemma 6, $\log M(r, f) \leq M_2 r$ for finite $M_2 = 2M_1 > 0$ and sufficiently large r . From the definition of type when $\rho(f) = 1$, we obtain

$$\sigma(f) = \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r} \leq \limsup_{r \rightarrow \infty} \frac{M_2 r}{r} = M_2 < \infty,$$

so the conclusion holds.

Case 2. $a_j = 0$ for all $j = 0, 1, \dots, k-1$ and $a_k \neq 0$. Set $g(z) = \frac{f(z)}{z^k}$; then $g(z)$ is an entire function with $v(r, g) = v(r, f) - k \leq Mr$. Therefore, by Case 1, $\sigma(g) \leq M$. Notice that $M(r, f) \leq r^k M(r, g)$. We obtain

$$\sigma(f) = \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r} \leq \limsup_{r \rightarrow \infty} \frac{k \log r + \log M(r, g)}{r} \leq M,$$

and so the conclusion holds. $\square$

Before proving Lemma 11, we need the following preliminary results: We first introduce the Polya representation theorem, which can be found in [24], pp. 26–30.

Lemma 9 ([24]). If $f(z)$ is an entire function of exponential type σ , then there exists a function $F(w)$ which is analytic outside some finite disk d about the origin such that

$$f(z) = \frac{1}{2\pi i} \int_C F(w) e^{zw} dw$$

for every curve C containing d .

From the proof of Lemma 9 in [24], C can be chosen to be $|w| = \sigma + \epsilon$, $\epsilon > 0$. We need the following preliminary result:

Lemma 10 ([24]). If $F(w)$ is analytic on the contour C and

$$\int_C F(w)e^{zw}dw = 0,$$

then $F(w)$ is analytic inside C .

Now we are able to prove the following lemma:

Lemma 11. Let $f(z)$ be an entire function of exponential type σ , satisfying $f(z+c) = kf(z)$, where c and k are nonzero constants. Then $f(z)$ has the form

$$f(z) = \sum_{|w_i| \leq \sigma} a_i e^{zw_i},$$

where a_i are constants, and w_i are zeros of $e^{cw} - k$.

Proof. By Lemma 9, we obtain

$$0 = f(z+c) - kf(z) = \int_C [e^{(z+c)w} - ke^{zw}]F(w)dw = \int_C e^{zw}F(w)(e^{cw} - k)dw,$$

where C contains $|w| < \sigma + \epsilon$, $\epsilon > 0$. By Lemma 10, $F(w)(e^{cw} - k)$ must be analytic inside C . Since $(e^{cw} - k)' = ce^{cw}$ has no zeros, all zeros of $e^{cw} - k$ are simple. Since the poles of $F(w)$ must be the zeros of $e^{cw} - k$, $F(w)$ must be analytic except for simple poles at w_i , where w_i are zeros of $e^{cw} - k$, which are inside C . Therefore, we write

$$F(w) = \sum_{|w_i| \leq \sigma} \frac{A_i}{w - w_i} + g(w),$$

where $g(w)$ is analytic on C and inside it. Hence,

$$f(z) = \frac{1}{2\pi i} \int_C \left[\sum_{|w_i| \leq \sigma} \frac{A_i}{w - w_i} + g(w) \right] e^{zw} dw.$$

By Cauchy's theorem, $\int_C g(w)e^{zw}dw = 0$, while

$$\int_C \frac{e^{zw}}{w - w_i} dw = e^{zw_i} \int_C \frac{e^{z(w-w_i)}}{w - w_i} dw = 2\pi i e^{zw_i} \operatorname{Res}_{w=w_i} \frac{e^{z(w-w_i)}}{w - w_i} = 2\pi i e^{zw_i}.$$

This completes the proof. $\square$

2.4. Other Lemmas

The following lemma is useful when exceptional sets in Nevanlinna theory and Wiman–Valiron theory have to be avoided:

Lemma 12 ([2]). Let $g : (0, +\infty) \rightarrow \mathbb{R}$ and $h : (0, +\infty) \rightarrow \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(r^\alpha)$ holds for all $r > r_0$.

We need the following lemma related to Equation (6) to estimate the number of zeros of f and f' :

Lemma 13 ([4]). If f is an entire solution of (6), then

- (i) $N(r, \frac{1}{f}) \neq S(r, f)$ and $N(r, \frac{1}{f'}) \neq S(r, f)$;

(ii) All zeros of f are simple.

3. Proof of Theorem 2

If for all $i = 1, \dots, k$, $L_i(f) = a_{i,0}f \neq 0$, then $H = a_0f^{n_1+\dots+n_k} \neq 0$, and the conclusion follows immediately from the standard estimate $T(r, H) = (n_1 + \dots + n_k)T(r, f) + S(r, f)$. Next we consider the case that at least one $L_i(f)$ contains a derivative term $f^{(m_i)}$ with $m_i > 0$.

From the elementary properties of $T(r, f)$ and the well-known inequality $T(r, f') \leq T(r, f) + S(r, f)$ for all entire functions f , we can derive that

$$\begin{aligned} T(r, H) &= T(r, L_1(f)^{n_1} L_2(f)^{n_2} \dots L_k(f)^{n_k}) \\ &\leq T(r, L_1(f)^{n_1}) + \dots + T(r, L_k(f)^{n_k}) \\ &= n_1 T(r, L_1(f)) + \dots + n_k T(r, L_k(f)) \end{aligned}$$

holds. Since $L_1(f), \dots, L_k(f)$ are entire functions, by Lemma 1, we have

$$\begin{aligned} T(r, L_i(f)) &= m(r, L_i(f)) \\ &\leq m(r, f) + m\left(r, \frac{L_i(f)}{f}\right) \\ &\leq m(r, f) + m\left(r, \frac{f'}{f}\right) + \dots + m\left(r, \frac{f^{(m_i)}}{f}\right) + S(r, f) \\ &\leq T(r, f) + S(r, f). \end{aligned}$$

Therefore, we have $T(r, H) \leq (n_1 + \dots + n_k)T(r, f) + S(r, f)$, and so we have $T(r, f) \geq \frac{1}{n_1 + \dots + n_k} T(r, H) + S(r, f)$.

We will prove the estimate of $T(r, f)$ in the other direction. We assume that at least one $a_{i,j} \neq 0$ for $j > 0$; otherwise the conclusion is clear. Let $\nu(r)$ be the central index of f . If $\nu(r) = O(r)$, then by Lemma 8, f is of exponential type. Then $T(r, f) = O(r)$ by the definition of exponential type and Lemma 3, and so the conclusion holds. We now discuss the case $r = o(\nu(r))$.

By Lemma 7, let $F \subset \mathbb{R}_+$ be a set of finite logarithmic measure such that

$$f^n \prod_{i=1}^k \left[a_{i,0} + a_{i,1} \left(\frac{\nu(r)}{z} \right) (1 + o(1)) + \dots + a_{i,m_i} \left(\frac{\nu(r)}{z} \right)^{m_i} (1 + o(1)) \right]^{n_i} = H(z)$$

holds for $r = |z| \notin F$, with z chosen as in Lemma 7, and $n = n_1 + n_2 + \dots + n_k$. Since $r = o(\nu(r))$ for $r \notin F$, we obtain $|\frac{\nu(r)}{z}| \rightarrow \infty$ as $r = |z| \rightarrow \infty$, which implies that for at least one i ,

$$\lim_{r \rightarrow \infty, r \notin F} |a_{i,0} + a_{i,1} \left(\frac{\nu(r)}{z} \right) (1 + o(1)) + \dots + a_{i,m_i} \left(\frac{\nu(r)}{z} \right)^{m_i} (1 + o(1))| = \infty,$$

and for $L_j(f) = a_{j,0}f$, the limit is $|a_{j,0}| > 0$. Therefore, $|\frac{H(z)}{f^n(z)}| \rightarrow \infty$ as $r \rightarrow \infty$ and $r \notin F$. Let $|f(z_0)| = M(r, f)$ for some sufficiently large $r \notin F$; then $M(r, H) \geq |H(z_0)| \geq |f(z_0)|^n |\frac{H(z_0)}{f^n(z_0)}| \geq M(r, f)$. Since the maximum modulus function $M(r, f)$ of an entire function is monotone increasing in r , by Lemma 12, $M(r, f) \leq M(r^{1+\epsilon}, H)$ for all $r > r_0$ and $\epsilon > 0$. By Lemma 3, we have

$$T(r, f) \leq \log M(r, f) \leq \log M(r^{1+\epsilon}, H) \leq 3T((2r)^{1+\epsilon}, H)$$

for sufficiently large r and any $\epsilon > 0$, which completes the first part of the theorem.

Now we assume that $\rho(H) < \infty$. By $\frac{1}{n_1+\dots+n_k}T(r, H) + S(r, f) \leq T(r, f)$, we have $\rho(f) \geq \rho(H)$. By the proof above, we get

$$\begin{aligned}\rho(H) &= \limsup_{r \rightarrow \infty} \frac{\log T(r, H)}{\log r} \\ &= \limsup_{r \rightarrow \infty} \frac{\log T((2r)^{1+\epsilon}, H)}{(1+\epsilon) \log 2r} \\ &\geq \frac{1}{1+\epsilon} \limsup_{r \rightarrow \infty} \frac{\log T(r, f) - \log 3}{\log r + \log 2} \\ &= \frac{1}{1+\epsilon} \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} \\ &= \frac{1}{1+\epsilon} \rho(f).\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $\rho(f) \leq \rho(H)$; hence, $\rho(f) = \rho(H)$.

Set $L_1(f) = \Pi_1(z)e^{P_1(z)}$, where $P_1(z)$ is a polynomial whose degree is at most $\lfloor \rho(H) \rfloor$, and $\Pi_1(z)$ is a canonical product formed of the zeros of $L_1(f)$. From the properties of factorization, we have $T(r, e^{P_1(z)}) = O(r^{\lfloor \rho(H) \rfloor})$, and

$$T(r, \Pi_1) = N(r, \frac{1}{\Pi_1}) \leq N(r, \frac{1}{H}) \leq T(r, H);$$

hence, we have, for some $K_2 > 0$ and sufficiently large r ,

$$T(r, L_1(f)) \leq K_2 \max\{r, r^{\lfloor \rho(H) \rfloor}, T(r, H)\} + S(r, f).$$

From the form of L_1 , we can determine that

$$f^{(m)} + a_{m-1}f^{(m-1)} + \dots + a_1f' + a_0f = L_1 \quad (7)$$

holds for some coefficients and $m \geq 0$. By [2], all entire solutions of (7) are $f = f_0 + C_1f_1 + \dots + C_mf_m$, where $C_1, \dots, C_m \in \mathbb{C}$, $f_1, \dots, f_m$ is a solution base of the corresponding homogeneous equation of (7), $f_0 = A_1f_1 + \dots + A_mf_m$, $A'_j = L_1G_j(f_1, \dots, f_m)e^\Phi$ for $j = 1, \dots, m$, each G_j is a differential polynomial of $f_1, \dots, f_m$, and Φ is a primitive function of a_{m-1} . Since $a_0, \dots, a_{m-1}$ are constants, we determine that $T(r, e^\Phi) = O(r)$, $T(r, f_j) = O(r)$, $T(r, G_j) = O(r)$, and $T(r, A_j) = O(\max\{T(r, L_1), r\})$ hold for all $j = 1, \dots, m$. Therefore, we have

$$T(r, f) \leq K_2 \max\{r, r^{\lfloor \rho(H) \rfloor}, T(r, H)\} + S(r, f),$$

and so the conclusion holds.

4. Proof of Theorem 7

In the proof of Theorem 7, we first show that any entire solution f must be of the form

$$f(z) = \sum_{n=1}^N a_n e^{b_n z}, \quad (8)$$

where a_n, b_n are constants with $a_n \neq 0$, $b_i \neq b_j$ for $i \neq j$, and $N \in \mathbb{N}_+$. Firstly, we eliminate the exponential terms on the right-hand side of the equation and obtain a differential identity which implies that every zero of f' is a zero of the product $(f + D_1f'')(f + D_2f'')$, where D_1 and D_2 are constants depending on the coefficients of the equation (their explicit forms are given later in the proof). The rest of the proof is divided according to the

possible values of D_1 and D_2 . In the generic case, we show that there exist two distinct zeros z_1, z_2 of f' such that $f^{(j)}(z_2) = kf^{(j)}(z_1)$ for all $j \geq 0$, with $k = f(z_2)/f(z_1)$. This forces $f(z+c) = kf(z)$ for $c = z_2 - z_1$, which implies that f is a finite sum of exponential functions, giving (8). In the remaining cases, the above recursive relation does not hold; instead, we use the conditions specific to those cases, together with the structure of the equation and Nevanlinna theory, to determine the form of f .

Next, we determine the explicit form of f . For relatively small N , we can directly substitute the expression into the original equation and perform the calculation. However, it is impossible to carry out such computations for all N , particularly when N is large. Hence, we introduce a lexicographic order on the complex plane to compare all exponential terms that arise in the expansion of $f^2 + af'^2$. In fact, we prove that at least four exponential terms cannot be canceled by any other terms in the sum, which implies that two of the corresponding exponential terms must be zero. By induction, we then derive the relations among the exponents. Finally, we classify all possible cases according to which two of the terms vanish, and derive the desired properties of the solution. The details are given in the proof below.

Rewrite Equation (6) as

$$(f + \sqrt{-a}f')(f - \sqrt{-a}f') = p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z}.$$

From Theorem 6 and the fact $p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z}$ is of exponential type, we can determine that all entire solutions of (6) are of exponential type.

By differentiating Equation (6), we obtain

$$2ff' + 2af'f'' = p_1\lambda_1e^{\lambda_1 z} + p_2\lambda_2e^{\lambda_2 z}, \quad (9)$$

Combining (6) and (9), and then eliminating $e^{\lambda_1 z}$, we have

$$\lambda_1 f^2 + a\lambda_1(f')^2 - 2ff' - 2af'f'' = p_2(\lambda_1 - \lambda_2)e^{\lambda_2 z}. \quad (10)$$

Differentiating (10), we obtain

$$2\lambda_1 ff' + 2a\lambda_1 f'f'' - 2(f')^2 - 2ff'' - 2a(f'')^2 - 2af'f''' = \lambda_2 p_2(\lambda_1 - \lambda_2)e^{\lambda_2 z}.$$

Then eliminating $e^{\lambda_2 z}$, we obtain

$$\begin{aligned} & 2\lambda_1 ff' + 2a\lambda_1 f'f'' - 2(f')^2 - 2ff'' - 2a(f'')^2 - 2af'f''' \\ &= \lambda_2 [\lambda_1 f^2 + a\lambda_1(f')^2 - 2ff' - 2af'f'']. \end{aligned}$$

Finally, we have

$$\begin{aligned} & \lambda_1\lambda_2 f^2 - 2(\lambda_1 + \lambda_2)ff' + 2ff'' + (2 + a\lambda_1\lambda_2)(f')^2 \\ & - 2a(\lambda_1 + \lambda_2)f'f'' + 2a(f'')^2 + 2af'f''' = 0. \end{aligned} \quad (11)$$

By Lemma 13 (i) and the fact that f cannot be a polynomial, f' has infinite zeros. Let z_0 be a zero of f' ; by Lemma 13 (ii), we obtain $f(z_0)^2 = p_1e^{\lambda_1 z_0} + p_2e^{\lambda_2 z_0} \neq 0$. Rewrite (11) as

$$\begin{aligned}
& f'[2(\lambda_1 + \lambda_2)f - (a\lambda_1\lambda_2 + 2)f' + 2a(\lambda_1 + \lambda_2)f'' - 2af'''] \\
&= \lambda_1\lambda_2 f^2 + 2ff'' + 2a(f'')^2 \\
&= \lambda_1\lambda_2(f + D_1f'')(f + D_2f'') \\
&= \lambda_1\lambda_2[f^2 + (D_1 + D_2)ff'' + D_1D_2(f'')^2].
\end{aligned} \tag{12}$$

By the equation above, we have $D_1 + D_2 = \frac{2}{\lambda_1\lambda_2}$ and $D_1D_2 = \frac{2a}{\lambda_1\lambda_2}$. Therefore, D_1 and D_2 are the roots of the quadratic equation

$$t^2 - \frac{2}{\lambda_1\lambda_2}t + \frac{2a}{\lambda_1\lambda_2} = 0.$$

Solving this gives

$$D_1 = \frac{1 + \sqrt{1 - 2a\lambda_1\lambda_2}}{\lambda_1\lambda_2}, \quad D_2 = \frac{1 - \sqrt{1 - 2a\lambda_1\lambda_2}}{\lambda_1\lambda_2}.$$

By (12), we obtain $f(z_0) + D_1f''(z_0) = 0$ or $f(z_0) + D_2f''(z_0) = 0$. Note that $D_1D_2 = \frac{2a}{\lambda_1\lambda_2} \neq 0$; we can obtain $f''(z_0) = -\frac{f(z_0)}{D_1}$ or $f''(z_0) = -\frac{f(z_0)}{D_2}$.

Differentiating Equation (11) $n - 2$ times, we obtain

$$2f^{(n)}(f + naf'') + 2af'[-(\lambda_1 + \lambda_2)f^{(n)} + f^{(n+1)}] + \sum_{i,j < n} c_{n,i,j}f^{(i)}f^{(j)} = 0, \tag{13}$$

where $n \geq 3$, $c_{n,i,j}$ are constants related to n, i , and j . Notice that $f(z_0) + naf''(z_0)$ may be zero when $D_1 = na$ or $D_2 = na$. We consider three cases to obtain the form of solutions.

Case 1. Suppose that $D_1 \neq na$ and $D_2 \neq na$ for all $n \geq 3$. Then for any zero z_0 of f' , we have $f(z_0) + naf''(z_0) \neq 0$. Since f' has infinite zeros, there exists at least one D_i , $i = 1, 2$ such that $f(z_1) + D_if''(z_1) = 0$ and $f(z_2) + D_if''(z_2) = 0$ hold for two different zeros z_1, z_2 , of f' . Denote $k = \frac{f(z_2)}{f(z_1)}$ and $c = z_2 - z_1$, then $f(z_2) = kf(z_1)$ and $f''(z_2) = kf''(z_1)$. We now establish the relation $f^{(n)}(z_2) = kf^{(n)}(z_1)$, $n = 0, 1, 2, \dots$ by induction. For $n = 3$, by (13), we get

$$\begin{aligned}
f'''(z_2) &= -\frac{\sum_{i,j < 3} c_{3,i,j}f^{(i)}(z_2)f^{(j)}(z_2)}{2[f(z_2) + 3af''(z_2)]} \\
&= -\frac{k^2 \sum_{i,j < 3} c_{3,i,j}f^{(i)}(z_1)f^{(j)}(z_1)}{2k[f(z_1) + 3af''(z_1)]} = kf'''(z_1).
\end{aligned}$$

Assume that for some $n \geq 3$, we have $f^{(j)}(z_2) = kf^{(j)}(z_1)$ for all $j \leq n$; then we prove the case $n + 1$. Applying Equation (13) with $n + 1$ in place of n , we obtain

$$\begin{aligned}
f^{(n+1)}(z_2) &= -\frac{\sum_{i,j < n+1} c_{n+1,i,j}f^{(i)}(z_2)f^{(j)}(z_2)}{2[f(z_2) + (n+1)af''(z_2)]} \\
&= -\frac{k^2 \sum_{i,j < n+1} c_{n+1,i,j}f^{(i)}(z_1)f^{(j)}(z_1)}{2k[f(z_1) + (n+1)af''(z_1)]} = kf^{(n+1)}(z_1).
\end{aligned}$$

To justify the conclusion $f(z+c) = kf(z)$, we compare the Taylor expansions of $f(z+c)$ and $kf(z)$ at z_1 . Since $z_2 = z_1 + c$, we have $z+c-z_2 = z-z_1$. Thus, the Taylor expansion of $f(z+c)$ at z_1 is

$$f(z+c) = \sum_{m=0}^{\infty} \frac{f^{(m)}(z_2)}{m!} (z+c-z_2)^m = \sum_{m=0}^{\infty} \frac{f^{(m)}(z_2)}{m!} (z-z_1)^m.$$

Using the induction result $f^{(m)}(z_2) = kf^{(m)}(z_1)$ for all $m \geq 0$, we obtain

$$f(z+c) = \sum_{m=0}^{\infty} \frac{kf^{(m)}(z_1)}{m!} (z-z_1)^m = k \sum_{m=0}^{\infty} \frac{f^{(m)}(z_1)}{m!} (z-z_1)^m = kf(z).$$

Thus, $f(z+c) = kf(z)$ for all $z \in \mathbb{C}$.

Therefore, by Lemma 11, we have

$$f(z) = \sum_{|w_i| \leq \sigma} a_i e^{zw_i}$$

for some a_i and w_i ; hence, f satisfies (8).

Case 2. Suppose that for $\{i, j\} = \{1, 2\}$, $D_i \neq na$ for all $n \geq 3$, while $D_j = n_j a$ for some $n_j \geq 3$. Without loss of generality, we assume that $D_1 \neq na$ and $D_2 = n_2 a$. We consider two subcases.

Subcase 2.1. There exist at least two zeros, z_1 and z_2 , of f' satisfying $f(z_1) + D_1 f''(z_1) = 0$ and $f(z_2) + D_1 f''(z_2) = 0$. Then by using the same method above, we can obtain (8) in the form of f .

Subcase 2.2. There exists at most one zero, z_0 , of f' satisfying $f(z_0) + D_1 f''(z_0) = 0$. Then we rewrite (11) as

$$\lambda_1 \lambda_2 \frac{f}{f'} - 2(\lambda_1 + \lambda_2) + 2 \frac{f''}{f'} + (a\lambda_1 \lambda_2 + 2) \frac{f'}{f} - 2a(\lambda_1 + \lambda_2) \frac{f''}{f} + 2a \frac{f'''}{f} + 2a \frac{(f'')^2}{ff'} = 0.$$

By Lemma 1 and the fact that f is of exponential type, we obtain $m\left(r, \frac{f}{f'}\right) = O(\log r)$. Denote

$$g(z) := \frac{f + D_2 f''}{f'};$$

then

$$f'' = \frac{g}{D_2} f' - \frac{1}{D_2} f. \quad (14)$$

By assumption, all except one zero of f' are zeros of $f + D_2 f''$. Therefore, $T(r, g) = N(r, g) + m(r, g) = O(\log r)$. By the standard Nevanlinna theory, a meromorphic function f is rational if and only if $T(r, f) = O(\log r)$ (see, e.g., [2] (Theorem 2.2.3)), so g is a rational function.

By differentiating (14) and eliminating f'' , we obtain

$$f''' = \frac{g^2 + D_2(g' - 1)}{D_2^2} f' - \frac{g}{D_2^2} f. \quad (15)$$

Then by combining (12), (14) and (15), we have

$$f'(z) \left[-(a\lambda_1\lambda_2 + 2) + \frac{2a(\lambda_1 + \lambda_2)g(z)}{D_2} - \frac{2ag(z)^2}{D_2^2} + \frac{2a(1 - g'(z))}{D_2} - \frac{\lambda_1\lambda_2 D_1 g(z)^2}{D_2} \right] \\ + f(z) \left[2(\lambda_1 + \lambda_2) - \frac{2a(\lambda_1 + \lambda_2)}{D_2} + \frac{2ag(z)}{D_2^2} - \lambda_1\lambda_2 g(z) \left(1 - \frac{D_1}{D_2} \right) \right] = 0.$$

For simplicity, write the above equation as $f(z)A(z) = f'(z)B(z)$. By Lemma 13, f has infinite simple zeros, so $B(z)$ has infinite zeros. Since g is rational, $B(z)$ is rational. Therefore, $B(z) \equiv 0$ and $A(z) \equiv 0$. Thus, we have

$$2(\lambda_1 + \lambda_2) - \frac{2a(\lambda_1 + \lambda_2)}{D_2} + \frac{2ag(z)}{D_2^2} - \lambda_1\lambda_2 g(z) \left(1 - \frac{D_1}{D_2} \right) = 0 \quad (16)$$

and

$$-(a\lambda_1\lambda_2 + 2) + \frac{2a(\lambda_1 + \lambda_2)g(z)}{D_2} - \frac{2ag(z)^2}{D_2^2} + \frac{2a(1 - g'(z))}{D_2} - \frac{\lambda_1\lambda_2 D_1 g(z)^2}{D_2} = 0.$$

From Equation (16), we obtain

$$\left[\frac{2a}{D_2^2} - \lambda_1\lambda_2 \left(1 - \frac{D_1}{D_2} \right) \right] g = 2 \left(\frac{a}{D_2} - 1 \right) (\lambda_1 + \lambda_2).$$

Multiplying both sides by D_2^2 to clear denominators gives

$$[2a - \lambda_1\lambda_2 D_2(D_2 - D_1)]g = 2D_2(a - D_2)(\lambda_1 + \lambda_2).$$

Since $D_2 = n_2 a$ with $n_2 \geq 3$, a, λ_1, λ_2 are nonzero constants, and $\lambda_1 + \lambda_2 \neq 0$ by our assumption, we have $D_2 \in \mathbb{C} \setminus \{0\}$, $a - D_2 \in \mathbb{C} \setminus \{0\}$, so $2D_2(a - D_2)(\lambda_1 + \lambda_2) \in \mathbb{C} \setminus \{0\}$. If $2a - \lambda_1\lambda_2 D_2(D_2 - D_1) = 0$, then g cannot be a rational function. Therefore,

$$g = \frac{2D_2(\lambda_1 + \lambda_2)(a - D_2)}{2a - \lambda_1\lambda_2 D_2(D_2 - D_1)}$$

is a nonzero constant, and f satisfies a homogeneous second-order linear differential equation with constant coefficients

$$D_2 f'' - g f' + f = 0. \quad (17)$$

If the characteristic equation $D_2 x^2 - g x + 1 = 0$ of (17) satisfies the discriminant $\Delta = 0$, then f must satisfy $f = c_1 e^{a_1 z} + c_2 z e^{a_1 z}$. In this case, we have

$$f^2 + a(f')^2 = [(c_1 + c_2 z)^2 + a(c_1 a_1 + c_2 + c_2 a_1 z)^2] e^{2a_1 z} = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}.$$

By Lemma 2, this case is impossible. Then the discriminant $\Delta \neq 0$, and f has the form $c_1 e^{a_1 z} + c_2 e^{a_2 z}$, where $a_1 \neq a_2$. Therefore, f must satisfy (8) in this subcase.

Case 3. Suppose that $D_1 = n_1 a$ and $D_2 = n_2 a$ hold for some $n_1, n_2 \geq 3$. This implies

$$\begin{cases} 1 + \sqrt{1 - 2a\lambda_1\lambda_2} = n_1 a \lambda_1 \lambda_2, \\ 1 - \sqrt{1 - 2a\lambda_1\lambda_2} = n_2 a \lambda_1 \lambda_2. \end{cases}$$

Let $t = a\lambda_1\lambda_2 \neq 0$; then we obtain

$$1 - 2t = (n_1t - 1)^2 = (n_2t - 1)^2.$$

If $n_1 = n_2$, then $1 - 2t = 0$ and $n_1 = n_2 = 2$, which contradicts $n_1, n_2 \geq 3$. If $n_1 \neq n_2$, then n_1, n_2 solve the equation $tn^2 - 2n + 2 = 0$. Since $n_1, n_2 \geq 3$, we obtain $n_1n_2 = \frac{2}{t} = n_1 + n_2$, which is impossible.

By the discussion above, we determine that f has the form $\sum_{n=1}^N a_n e^{b_n z}$, $a_n \neq 0$. Next, we will discuss the following cases according to the number N of exponential terms:

Case 1. $N = 1$. Then $f = a_1 e^{b_1 z}$ and $f' = a_1 b_1 e^{b_1 z}$. Hence, we obtain $f^2 + af'^2 = a_1^2(1 + ab_1^2)e^{2b_1 z} = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}$, which is impossible by Lemma 2.

Case 2. $N = 2$. Then $f = a_1 e^{b_1 z} + a_2 e^{b_2 z}$ and

$$f^2 + af'^2 = a_1^2(1 + ab_1^2)e^{2b_1 z} + 2a_1a_2(1 + ab_1b_2)e^{(b_1+b_2)z} + a_2^2(1 + ab_2^2)e^{2b_2 z} = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}.$$

By Lemma 2, there exist three subcases.

Subcase 2.1. $1 + ab_1b_2 = 0$. Without loss of generality, we get

$$b_1 = \frac{\lambda_1}{2}, \quad b_2 = \frac{\lambda_2}{2}, \quad a = -\frac{4}{\lambda_1\lambda_2}.$$

Then

$$p_1 = a_1^2 \left(1 + a \frac{\lambda_1^2}{4}\right), \quad p_2 = a_2^2 \left(1 + a \frac{\lambda_2^2}{4}\right),$$

and by solving a_1 and a_2 we obtain

$$a_1 = \pm \sqrt{\frac{p_1}{1 + a\lambda_1^2/4}}, \quad a_2 = \pm \sqrt{\frac{p_2}{1 + a\lambda_2^2/4}}.$$

So the solutions of (6) in this subcase are

$$f(z) = \pm \sqrt{\frac{p_1}{1 + a\lambda_1^2/4}} e^{\lambda_1 z/2} \pm \sqrt{\frac{p_2}{1 + a\lambda_2^2/4}} e^{\lambda_2 z/2}.$$

Subcase 2.2. $1 + ab_1^2 = 0$. We have $\lambda_1 = b_1 + b_2, \lambda_2 = 2b_2$ or $\lambda_2 = b_1 + b_2, \lambda_1 = 2b_2$. By letting the coefficients be zero, we obtain

$$p_2 = a_2^2(1 + ab_2^2), \quad p_1 = 2a_1a_2(1 + ab_1b_2).$$

Solving the two equations above, we obtain

$$a_2 = \pm \sqrt{\frac{p_2}{1 + ab_2^2}}, \quad a_1 = \frac{p_1}{2a_2(1 + ab_1b_2)}.$$

Hence, the solutions of (6) are

$$f(z) = \frac{p_1}{2a_2(1 + ab_1b_2)} e^{b_1 z} + a_2 e^{b_2 z}, \quad b_1 = \lambda_1 - \frac{\lambda_2}{2}, b_2 = \frac{\lambda_2}{2}, a = -\frac{1}{b_1^2}$$

and

$$f(z) = \frac{p_1}{2a_2(1 + ab_1b_2)} e^{b_1 z} + a_2 e^{b_2 z}, \quad b_1 = \lambda_2 - \frac{\lambda_1}{2}, b_2 = \frac{\lambda_1}{2}, a = -\frac{1}{b_1^2}.$$

Subcase 2.3. $1 + ab_2^2 = 0$. By the same method as above, we can obtain the same solutions as for Subcase 2.2.

Case 3. $N \geq 3$. Then

$$f^2 + af'^2 = \sum_{m,n \leq N} a_m a_n (1 + ab_m b_n) e^{(b_m + b_n)z}.$$

Firstly, we need to introduce a lexicographic order between two complex numbers. For any $z_1, z_2 \in \mathbb{C}$, we define $z_1 \prec z_2$ if and only if $\Re z_1 < \Re z_2$, or $\Re z_1 = \Re z_2$ and $\Im z_1 < \Im z_2$. We will prove some properties of this lexicographic order.

1. If $z_1 \prec z_2$, then $z_1 \neq z_2$. This is obvious by definition.
2. If $z_1 \prec z_2$, then $az_1 \prec az_2$ when $a \in \mathbb{R}_+$, and $az_2 \prec az_1$ when $a \in \mathbb{R}_-$. This can be obtained by the fact that $\Re(az) = a\Re z$ and $\Im(az) = a\Im z$ for $a \in \mathbb{R}$.
3. If $z_1 \prec z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$. If $\Re z_1 < \Re z_2$, then $\Re z_1 < \Re z_2 \leq \Re z_3$. If $\Re z_1 = \Re z_2$ and $\Re z_2 < \Re z_3$, then $\Re z_1 < \Re z_3$. If $\Re z_1 = \Re z_2$, $\Re z_2 = \Re z_3$, $\Im z_1 < \Im z_2$, and $\Im z_2 < \Im z_3$, then $\Re z_1 = \Re z_3$ and $\Im z_1 < \Im z_3$. By the discussion above, the conclusion holds.
4. If $z_1 \prec z_2$, then $z_1 + z_3 \prec z_2 + z_3$. Note that if $\Re(z_1 + z_3) = \Re z_1 + \Re z_3 < \Re z_2 + \Re z_3 = \Re(z_2 + z_3)$, or $\Re(z_1 + z_3) = \Re z_1 + \Re z_3 = \Re z_2 + \Re z_3 = \Re(z_2 + z_3)$ and $\Im(z_1 + z_3) = \Im z_1 + \Im z_3 < \Im z_2 + \Im z_3 = \Im(z_2 + z_3)$, by definition, the conclusion holds.
5. If $z_1 \prec z_2$ and $z_3 \prec z_4$, then $z_1 + z_3 \prec z_2 + z_4$. By property 4, we have $z_1 + z_3 \prec z_2 + z_3$ and $z_2 + z_3 \prec z_2 + z_4$. Then by property 3, we obtain the result.

Without loss of generality, we assume $b_1 \prec b_2 \prec \dots \prec b_{N-1} \prec b_N$, and $\lambda_1 \prec \lambda_2$. By the properties above, we obtain $2b_1 \prec b_i + b_j$ for all $i = 1, \dots, N$ and $j = 2, \dots, N$. Note that $b_1 + b_2 \prec b_1 + b_{i'}$ for all $i' = 3, \dots, N$, and $b_1 + b_2 \prec b_2 + b_{j'}$ for all $j' = 2, \dots, N$; this means $b_1 + b_2 \prec b_i + b_j$ holds for all the other terms except $2b_1$ and $b_1 + b_2$. By the above discussion, $2b_1$ and $b_1 + b_2$ cannot be equal to other $b_i + b_j$. By the same argument applied to the reverse lexicographic order, $2b_N$ and $b_{N-1} + b_N$ cannot be canceled. Therefore, the four terms $a_1^2(1 + ab_1^2)e^{2b_1z}$, $2a_1a_2(1 + ab_1b_2)e^{(b_1+b_2)z}$, $2a_{N-1}a_N(1 + ab_{N-1}b_N)e^{(b_{N-1}+b_N)z}$ and $a_N^2(1 + ab_N^2)e^{2b_Nz}$ cannot be eliminated by other terms, and at least two coefficients of the four terms are not zero. In order to obtain the solutions, we have $1 + ab_1^2 = 0$ or $1 + ab_1b_2 = 0$, and $1 + ab_{N-1}b_N = 0$ or $1 + ab_N^2 = 0$. This means $1 + ab_1b_n \neq 0$ for all $n \geq 3$, and $1 + ab_mb_N \neq 0$ for all $m \leq N - 2$. For all $n \geq 3$, $a_1a_n(1 + ab_1b_n)e^{(b_1+b_n)z}$ can only be eliminated by other terms like $a_1a_n(1 + ab_{n_1}b_{n_2})e^{(b_{n_1}+b_{n_2})z}$. For $b_1 + b_k, k = 3, \dots, N$, there exists a different $b_{k_1} + b_{k_2} = b_1 + b_k$ for some $k_1 \leq k_2 = 1, \dots, N$. If $k_1 = 1$, then $k_2 = k$, which is impossible. If $k_2 \geq k$, then $k_1 \leq 1$, which is impossible. So $1 < k_1, k_2 < k$. Set $t_1 = b_1 - b_2$; we have the following relations from $b_1 + b_3$ to $b_1 + b_N$:

$$b_1 + b_3 = 2b_2;$$

...

$$b_1 + b_k = b_{k_1} + b_{k_2}, \quad 1 < k_1, k_2 < k;$$

...

$$b_1 + b_N = b_{N_1} + b_{N_2}, \quad 1 < N_1, N_2 < N.$$

By assumption and the relations above, we can obtain $b_1 - b_2 = 1(b_1 - b_2)$ for $k = 2$. Assume $b_1 - b_j = K_j(b_1 - b_2)$ holds for all $j < k$ and $K_j \in \mathbb{N}_+$; then $b_1 - b_k = b_1 - (b_{k_1} + b_{k_2} - b_1) = 2b_1 + K_{k_1}(b_1 - b_2) - b_1 + K_{k_2}(b_1 - b_2) - b_1 = (K_{k_1} + K_{k_2})(b_1 - b_2)$, and $K_{k_1} + K_{k_2} \in \mathbb{N}_+$. Therefore, $b_1 - b_j = K_j(b_1 - b_2)$ holds for all $2 \leq j \leq N$, $K_j \in \mathbb{N}_+$. For $j_1 < j_2$, we have $b_{j_1} \prec b_{j_2}$, so $b_1 - b_{j_2} \prec b_1 - b_{j_1}$, which yields $b_1 - b_{j_2} = K_{j_2}(b_1 - b_2) \prec K_{j_1}(b_1 - b_2) = b_1 - b_{j_1}$. Since $b_1 - b_2 \prec 0$, we have $K_{j_2} > K_{j_1}$ by the definition of lexicographic order and the fact that $K_j \in \mathbb{N}_+$. Since $K_2 = 1$ and $K_3 = 2$, we have $K_4 \geq 3, K_5 \geq K_4 + 1 \geq 4, \dots, K_j \geq j - 1$. Therefore, $b_1 - b_N = K_N t_1$ and $b_N = (b_1 -$

$b_{N-1}) - (b_1 - b_N) + b_{N-1} = b_{N-1} + (K_{N-1} - K_N)t_1 = b_{N-1} - K_{N1}t_1$ for some integer $K_N \geq N - 1$ and $K_{N1} \geq 1$. Similarly, by the same argument applied to the reverse lexicographic order for $b_j + b_N, j = 1, \dots, N - 2$, set $t_{N-1} = b_{N-1} - b_N$; we obtain $b_1 = b_2 + K'_{N-1}t_{N-1}$ for some integer $K'_{N-1} \geq 1$. Since $K_{N1}t_1 = t_{N-1}$ and $K'_{N-1}t_{N-1} = t_1$, we have $t := b_1 - b_2 = b_{N-1} - b_N$. We will discuss the following subcases:

Subcase 3.1. $1 + ab_1^2 = 0$ and $1 + ab_N^2 = 0$. Then $b_1 + b_2 = \lambda_1$, and $b_{N-1} + b_N = \lambda_2$. By our assumption and the results above, we obtain $\lambda_1 + \lambda_2 = b_1 + b_N + b_2 + b_{N-1} = 2(b_1 + b_N) = 0$, which contradicts $\lambda_1 + \lambda_2 \neq 0$.

Subcase 3.2. $1 + ab_1b_2 = 0$ and $1 + ab_{N-1}b_N = 0$. Then $b_1 = \lambda_1/2$ and $b_N = \lambda_2/2$. From $b_1b_2 = b_{N-1}b_N$, we have $b_1(b_1 - t) = b_N(b_N + t)$ and $(b_1 + b_N)(b_1 - b_N) = t(b_1 + b_N)$. If $b_1 + b_N = 0$, then $\lambda_1 + \lambda_2 = 0$, which is impossible. If $t = b_1 - b_N$, then $N = 2$, which is impossible.

Subcase 3.3. $1 + ab_1^2 = 0$ and $1 + ab_{N-1}b_N = 0$. Then $b_1 + b_2 = \lambda_1$ and $b_N = \lambda_2/2$. By $b_1^2 = b_{N-1}b_N$, we have $(b_N + Kt)^2 = b_N(b_N + t)$; then by computation, we obtain

$$t = -\frac{2K-1}{K^2}b_N, \quad b_1 = -\frac{K-1}{K}b_N, \quad b_{N-1} = \left(\frac{K-1}{K}\right)^2 b_N$$

and

$$\frac{\lambda_1}{\lambda_2} = \frac{-2K^2 + 4K - 1}{2K^2}, \quad a = -\left[\frac{2K}{(K-1)\lambda_2}\right]^2.$$

Subcase 3.4. $1 + ab_1b_2 = 0$ and $1 + ab_N^2 = 0$. Similarly, we obtain

$$\frac{\lambda_2}{\lambda_1} = \frac{-2K^2 + 4K - 1}{2K^2}, \quad a = -\left[\frac{2K}{(K-1)\lambda_1}\right]^2.$$

By the discussion above, if we assume $|\lambda_1| \leq |\lambda_2|$, then the only possible solutions for $N \geq 3$ occur when

$$\frac{\lambda_1}{\lambda_2} = \frac{-2K^2 + 4K - 1}{2K^2}, \quad a = -\left[\frac{2K}{(K-1)\lambda_2}\right]^2.$$

Now we consider some special cases. Let $N = 3$; then from $b_3 + b_1 = 2b_2$, we have $K = 2$, $\lambda_2 = -8\lambda_1$, $a = -\frac{1}{4\lambda_1^2}$, and $f = a_1e^{2\lambda_1z} + a_2e^{-\lambda_1z} + a_3e^{-4\lambda_1z}$ for some $a_1, a_2, a_3 \in \mathbb{C} \setminus \{0\}$. By calculation, we obtain

$$f^2 + a(f')^2 = 3a_1a_2e^{\lambda_1} + \left(\frac{3}{4}a_2^2 + 6a_1a_3\right)e^{-2\lambda_1} - 3a_3^2e^{-8\lambda_1} = p_1e^{\lambda_1z} + p_2e^{\lambda_2z},$$

which implies

$$\begin{cases} 3a_1a_2 = p_1 \\ \frac{3}{4}a_2^2 + 6a_1a_3 = 0 \\ -3a_3^2 = p_2 \end{cases}$$

Next, we consider $N = 4$; then by calculation, we have $K = 3$, $\lambda_2 = -\frac{18}{7}\lambda_1$, $a = -\frac{49}{36\lambda_1^2}$, and $f(z) = a_1e^{\frac{6}{7}\lambda_1z} + a_2e^{\frac{1}{7}\lambda_1z} + a_3e^{-\frac{4}{7}\lambda_1z} + a_4e^{-\frac{9}{7}\lambda_1z}$ for some $a_1, a_2, a_3, a_4 \in \mathbb{C} \setminus \{0\}$. Then the system

$$\begin{cases} \frac{35}{36}a_2^2 + \frac{10}{3}a_1a_3 = 0 \\ \frac{5}{3}a_1a_2 = p_1 \\ 5a_1a_4 + \frac{20}{9}a_2a_3 = 0 \\ \frac{5}{9}a_3^2 + \frac{5}{2}a_2a_4 = 0 \\ -\frac{5}{4}a_4^2 = p_2 \end{cases}$$

holds for coefficients. By the first two equations, we have $a_3 = -\frac{7}{24}\frac{a_2^2}{a_1}$,

$$a_4 = -\frac{2}{9}\frac{a_3^2}{a_2} = -\frac{2}{9a_2}\left(-\frac{7}{24}\frac{a_2^2}{a_1}\right)^2 = -\frac{49}{2592}\frac{a_2^3}{a_1^2},$$

and

$$9a_1\left(-\frac{49}{2592}\frac{a_2^3}{a_1^2}\right) + 4a_2\left(-\frac{7}{24}\frac{a_2^2}{a_1}\right) = -\frac{49}{288}\frac{a_2^3}{a_1} - \frac{7}{6}\frac{a_2^3}{a_1} = 0,$$

which implies $-(385a_2^3)/(288a_1) = 0$. Since $a_1 \neq 0$, we conclude that $a_2 = 0$, contradicting the assumption that all coefficients a_i are non-zero. Therefore, in the case $N = 4$, f cannot be a solution of (6).

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