

Article

Finite Comparison Algebras for Motivic Projectors and Triangular Stability

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Abstract

We study a finite-stage criterion for representing auxiliary or motivated idempotents by ordinary motivic projectors. To a fixed finite list of operations on a motive M we attach a finite comparison algebra $E_{\Gamma}(M)$, a finite-dimensional ordinary comparison subring R , and a surjective comparison map $R \rightarrow E_{\Gamma}(M)$; its kernel is the defect ideal recording the ordinary relations that become true after comparison. If this defect ideal is nilpotent, then every idempotent in the chosen finite comparison algebra has an ordinary idempotent representative in R . The same nilpotence gives unipotent conjugacy of representatives after passage to a common finite refinement, and hence a refinement-independent isomorphism class of the resulting Karoubi summand. We isolate filtered, support-theoretic, and Peirce triangular criteria for verifying defect nilpotence. In a block upper-triangular comparison algebra, diagonal nilpotence exponents a_i give an exponent at most $a_1 + \cdots + a_m$ for the full triangular defect ideal. We include an explicit calculation for the standard Chow–Künneth projectors of $\mathbb{P}^n$, explain how known motivic nilpotence principles can supply non-zero finite defects, and apply the triangular criterion to projective bundles, smooth blow-ups, and finite towers.

Keywords: finite comparison algebras; motivic projectors; motivic representability; nilpotent defect ideals; idempotent lifting; triangular stability; Chow–Künneth decompositions; projective bundles and blow-ups

MSC: 14C15; 14C25; 18M05; 16N40

1. Introduction

Projectors arising in the theory of algebraic cycles are often first constructed in a category that is larger, or more flexible, than the ordinary motivic category in which one ultimately wants to work. Examples include projectors coming from the decomposition theorem, from semismall maps, from motivated cycles, or from Chow–Künneth and multiplicative Chow–Künneth decompositions. In such situations one is led to a finite presentation problem: after fixing the correspondences and operations which describe a particular auxiliary projector, when can the resulting idempotent be represented by an ordinary motivic idempotent? If two finite presentations are used, how are the resulting summands compared?

The answer developed here is formulated in terms of finite comparison algebras. Starting from an object M and a finite collection of operations on it, we form an auxiliary algebra $E_{\Gamma}(M)$. Chosen ordinary lifts are placed inside a finite-dimensional subring R of the ordinary endomorphism ring, and the kernel of the comparison map $R \rightarrow E_{\Gamma}(M)$ is the defect ideal. Nilpotence of this ideal puts the problem in the classical situation where



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idempotents lift modulo nilpotent ideals. The same nilpotent comparison gives unipotent conjugacy of two lifts after a common finite refinement.

1.1. Scope and Contribution

The ring-theoretic input is classical; see, for instance, Lam's treatment of idempotents modulo nil ideals ([1], §21). Nilpotence methods in motives are also standard, appearing in Voevodsky's nilpotence theorem, Kimura finite-dimensionality, Rost-type nilpotence principles, and O'Sullivan's distinguished cycles [2–5]. We use these results as inputs for a finite nilpotent-defect reduction principle, rather than as a new global nilpotence theorem.

The contribution is the finite organization of the representability problem. The comparison algebra records only the operations used to define the auxiliary projector, while the defect ideal records the ordinary relations that become true after comparison. Thus, representability is reduced to the nilpotence of a single ideal in an explicitly chosen finite-dimensional algebra. The subsequent results give ways to verify or propagate this nilpotence: filtered support data reduce the question to graded kernels, finite CK/MCK presentations reduce it to ordinary relation defects, and triangular decompositions give explicit exponent bounds from diagonal blocks. The geometric statements below are conditional in this sense: once the indicated finite comparison maps and nilpotent defect ideals are available, the hypotheses are stable in the stated block-stable settings.

1.2. Main Results

The first main result, Theorem 2, gives the finite nilpotent-defect representability criterion. Let

$$R \subseteq \operatorname{End}_{\operatorname{Kar}(\mathcal{A})}(M)$$

be an ordinary comparison subring mapping onto the finite auxiliary algebra attached to the chosen operations. If the kernel of this map is nilpotent, then each compatible idempotent in the auxiliary algebra has an idempotent representative in R . The nilpotence needed here is checked in two elementary ways: by a finite filtration in Lemma 5, and by an upper-triangular Peirce decomposition in Proposition 2.

The same comparison also gives a uniqueness statement. In Theorem 3, two finite constructions of the same auxiliary projector are compared after passage to a common finite stage with nilpotent defect ideal. The resulting idempotents are conjugate by a unit congruent to 1 modulo the defect ideal. Consequently, the associated Karoubi summands are isomorphic at that common stage. No canonical comparison isomorphism is claimed.

The geometric applications amount to verifying the same nilpotence hypotheses in familiar situations. Support filtrations give a filtered argument once the support-defined subspaces have been checked to be two-sided ideals. CK/MCK decompositions give finite presentations whose lifted relation defects generate the comparison kernel. For projective bundles and smooth blow-ups, the standard motivic decompositions lead, after compatible upper-triangular comparison algebras have been chosen, to the triangular stability theorem, Theorem 7. If the diagonal defect exponents are a_i , the full triangular defect ideal has an exponent at most $a_1 + \cdots + a_m$.

1.3. Relation with Motivic Constructions

The motivation comes from familiar constructions in the theory of motives and algebraic cycles. Decomposition-theorem and semismall arguments provide support summands in many cases [6–8]; Chow–Künneth and multiplicative Chow–Künneth decompositions have their own extensive theory [9–14]; finite-dimensionality and nilpotence enter naturally into this circle of ideas [3,5,15]; and the general categorical background includes pure,

numerical, and motivated motives [16–18]. The projective-bundle and blow-up examples below use the standard motivic decompositions of Manin and Fulton [19,20].

These constructions supply projectors and decompositions by geometric means. After a finite list of auxiliary projectors, correspondences, and relations has been fixed, the present argument gives a common algebraic test for when the corresponding idempotents admit ordinary motivic representatives. Thus, Voevodsky-, Kimura-, Rost-, or O’Sullivan-type results may supply nilpotence in concrete motivic categories, and the results here use it when it appears as a finite defect ideal for a selected presentation. Non-zero defects arise from the inclusion of the selected finite kernel in an ambient ideal already controlled by such a nilpotence theorem. The paper also records how defect nilpotence behaves under filtrations and upper-triangular geometric decompositions, where the exponent estimates are purely finite-dimensional.

1.4. Organization

Sections 2 and 3 introduce the finite comparison algebras, defect ideals, filtered nilpotence, Peirce triangular nilpotence, and the main lifting theorem. Section 5 proves refinement rigidity, and Section 6 treats enlargements that preserve the fixed comparison data. Section 7 begins with explicit examples, then explains how external motivic nilpotence inputs produce non-zero finite defect ideals, and finally gives the support, CK/MCK, projective-bundle, blow-up, and finite-tower verifications. Appendix A supplies the detailed ring-theoretic calculations behind the lifting, filtered, and triangular arguments, keeping the main text focused on the motivic comparison problem.

2. Finite Comparison Data

Throughout, k denotes a field of characteristic 0 and all motives are taken with rational coefficients. We work in a fixed ordinary motivic category $\mathcal{M}$, assumed to be pseudo-abelian and tensor, using endomorphism rings, tensor products, and Karoubi envelopes. The reader may keep in mind either a suitable subcategory of $\mathbf{DM}(k, \mathbb{Q})$, or the category of Chow motives in the parts concerning Chow–Künneth projectors.

The word *finite* refers to the chosen stage of the construction. Thus, a finite comparison datum is generated by a fixed finite list of objects, correspondences, and relations; the comparison algebra which appears in the lifting theorem is assumed to be finite-dimensional over $\mathbb{Q}$. This convention separates finite-stage algebra from any broader ambient motivic category in which the construction may sit.

Unless otherwise stated, rings and algebras are associative and unital, subalgebras of endomorphism rings contain the ambient identity, homomorphisms are unital, and ideals are two-sided. Quotient and graded pieces may be non-unital when this is explicitly indicated.

Definition 1 (Finite ordinary ambient category). *A finite ordinary ambient category, in the sense used here, is a rigid pseudo-abelian tensor subcategory*

$$\mathcal{A} \subset \mathcal{M}$$

together with a chosen finite stage inside it. The finite stage consists of a finite family of objects

$$\mathcal{G} = \{G_1, \dots, G_r\}$$

and a finite list of operations on these objects, such as direct sums, selected direct summands, Tate twists, duals, tensor products, and, in triangulated settings, shifts and finitely many distinguished triangles, subject to the following conditions:

1. All objects and morphisms used in the projector construction belong to the ordinary category generated at this finite stage.
2. For every object $M \in \mathcal{A}$ relevant to the construction under consideration, and similarly for every finite direct sum or tensor-envelope object explicitly introduced by the datum, the endomorphisms generated by the chosen finite operations form a finite-dimensional $\mathbb{Q}$ -algebra inside the corresponding endomorphism ring in $\text{Kar}(\mathcal{A})$.
3. The projector data to be studied are compatible with the chosen operations used at this stage.

Thus, only a controlled part of the ambient category is used: only the part reached by the chosen finite stage enters the comparison argument, together with the endomorphisms relevant to the projector under consideration.

Remark 1. The word *finite* does not mean that the tensor subcategory has only finitely many objects. Once one closes under tensor products, twists, and summands, the surrounding category may of course be infinite. The finiteness assumption is on the selected construction and on the finite-dimensional endomorphism algebra used in the lifting argument.

Definition 2 (Finite comparison datum and finite comparison algebra). Fix a pair $(\mathcal{A}, M)$ with $M \in \mathcal{A}$. A finite comparison datum on $(\mathcal{A}, M)$ consists of the following:

1. A finite-dimensional $\mathbb{Q}$ -algebra

$$E_{\Gamma}(M)$$

whose elements are the motivated endomorphisms of M generated by the chosen finite datum.

2. A finite generating set

$$\Gamma_{\text{gen}} = \{\gamma_1, \dots, \gamma_t\}$$

of $E_{\Gamma}(M)$ as a $\mathbb{Q}$ -algebra.

3. Chosen ordinary lifts

$$\tilde{\Gamma}_{\text{gen}} = \{\tilde{\gamma}_1, \dots, \tilde{\gamma}_t\} \subseteq \mathcal{E}_{\mathcal{A}}(M) := \text{End}_{\text{Kar}(\mathcal{A})}(M).$$

4. A finite-dimensional ordinary word algebra

$$\mathcal{W}_{\Gamma, \mathcal{A}}(M) \subseteq \mathcal{E}_{\mathcal{A}}(M)$$

generated, as a unital $\mathbb{Q}$ -algebra, by the chosen lifts $\tilde{\Gamma}_{\text{gen}}$.

The symbol Γ denotes the full finite comparison datum; the distinguished generating set is denoted Γ_{gen} .

Definition 3 (Words and word algebra). Given a finite comparison datum as in Definition 2, a word is a noncommutative polynomial expression in the lifted generators $\tilde{\Gamma}_{\text{gen}}$. The word algebra is the finite-dimensional ordinary algebra $\mathcal{W}_{\Gamma, \mathcal{A}}(M)$ fixed in the datum. Thus, the finiteness of the algebra generated by the chosen lifts is part of the finite-stage hypothesis.

Definition 4 (Comparison homomorphism). A comparison homomorphism for the chosen finite comparison datum is a surjective $\mathbb{Q}$ -algebra homomorphism,

$$\rho_{\Gamma}: \mathcal{W}_{\Gamma, \mathcal{A}}(M) \twoheadrightarrow E_{\Gamma}(M)$$

sending each lifted generator $\tilde{\gamma}_i$ to its motivated shadow γ_i .

The existence of this homomorphism is part of the datum. It means that the algebraic relations among the chosen ordinary words have prescribed images in the finite comparison

algebra. Once the comparison homomorphism is fixed, the defect ideal below is simply its kernel. The comparison map is attached only to the chosen finite word algebra; additional operations in a larger ambient motivic category are not included unless they are added to a new finite datum.

Definition 5 (Motivated shadows and ordinary representatives). Let ρ_Γ be a comparison homomorphism for a fixed finite comparison datum. If $\tilde{x} \in \mathcal{W}_{\Gamma, \mathcal{A}}(M)$, its motivated shadow is the element

$$\rho_\Gamma(\tilde{x}) \in E_\Gamma(M).$$

More generally, let R be a comparison subring and let $\varphi: R \rightarrow E_\Gamma(M)$ be a comparison map extending ρ_Γ on the chosen word algebra. A chosen element $\tilde{x} \in R$ with $\varphi(\tilde{x}) = x \in E_\Gamma(M)$ is called an ordinary lift of x relative to φ . If $x = \pi^{\text{mot}}$ is an idempotent and $\tilde{x} = e$ can be chosen idempotent in R , then e is an ordinary motivic representative of π^{mot} relative to (R, φ) .

Definition 6 (Compatible motivated idempotent). A motivated idempotent

$$\pi^{\text{mot}} \in E_\Gamma(M)$$

is called compatible with the finite datum if it is one of the idempotents expressible in the finite comparison algebra generated by that datum. Thus, compatibility is relative to the chosen finite list of generators and relations; it is not a statement about all idempotents in a larger motivated or realization category. When the datum is fixed, we simply speak of a compatible idempotent.

Definition 7 (Representability). A compatible motivated idempotent $\pi^{\text{mot}} \in E_\Gamma(M)$ is representable in $\mathcal{A}$ if the following exist:

1. A finite-dimensional comparison subring $R \subseteq \text{End}_{\text{Kar}(\mathcal{A})}(M)$ equipped with a surjective $\mathbb{Q}$ -algebra homomorphism

$$\varphi: R \rightarrow E_\Gamma(M)$$

extending the comparison map on $\mathcal{W}_{\Gamma, \mathcal{A}}(M)$.

2. An idempotent $e \in R$ such that $\varphi(e) = \pi^{\text{mot}}$.

Such an e is the ordinary motivic representative of π^{mot} in the finite comparison subring R . When one can take $R = \mathcal{W}_{\Gamma, \mathcal{A}}(M)$ and $\varphi = \rho_\Gamma$, we say that π^{mot} is word-representable.

Remark 2. The point of including R in the definition is to remember where the lifted idempotent lives. The word-representable case is the minimal one, while the general definition permits a finite comparison subring containing the fixed word algebra. The comparison situation is summarized by the commutative diagram below:

$$\begin{array}{ccc} \mathcal{W}_{\Gamma, \mathcal{A}}(M) & \hookrightarrow & R \\ & \searrow \rho_\Gamma & \downarrow \varphi \\ & & E_\Gamma(M) \end{array}$$

Here, the diagonal arrow is the original word comparison map, while the vertical arrow is the chosen comparison map on the finite subring where the idempotent will be lifted.

3. Defect Ideals and Nilpotence Hypotheses

The finite-dimensional subring R is the object on which all lifting arguments take place.

Definition 8 (Comparison subring and defect ideal). Let $R \subseteq \mathcal{E}_{\mathcal{A}}(M)$ be a finite-dimensional $\mathbb{Q}$ -subalgebra containing $\mathcal{W}_{\Gamma, \mathcal{A}}(M)$, and let

$$\varphi: R \twoheadrightarrow E_{\Gamma}(M)$$

be a surjective $\mathbb{Q}$ -algebra homomorphism extending the comparison homomorphism on $\mathcal{W}_{\Gamma, \mathcal{A}}(M)$. The defect ideal of (R, φ) is the two-sided ideal

$$\mathfrak{d}(R, \varphi) := \ker(\varphi).$$

When $R = \mathcal{W}_{\Gamma, \mathcal{A}}(M)$ and $\varphi = \rho_{\Gamma}$, we write simply $\mathfrak{d}_{\Gamma}(M)$.

Remark 3. The defect ideal is the ordinary part of the relations that vanish in the auxiliary algebra. In examples, the essential geometric point is to prove this nilpotence.

Definition 9 (Basic finite defect-nilpotence datum). Let $\mathcal{A}$ be a finite ordinary ambient category and let $M \in \mathcal{A}$. We say that $(\mathcal{A}, M)$ satisfies the basic finite defect-nilpotence datum if the following exist:

1. A finite comparison datum on $(\mathcal{A}, M)$, with comparison homomorphism

$$\rho_{\Gamma}: \mathcal{W}_{\Gamma, \mathcal{A}}(M) \twoheadrightarrow E_{\Gamma}(M).$$

2. A finite-dimensional comparison subring $R \subseteq \mathcal{E}_{\mathcal{A}}(M)$ and a surjective homomorphism

$$\varphi: R \twoheadrightarrow E_{\Gamma}(M)$$

extending ρ_{Γ} .

3. Nilpotence of the defect ideal $\mathfrak{d}(R, \varphi) = \ker(\varphi)$.

These three items give precisely the setting in which the idempotent-lifting argument will be applied.

Proposition 1 (Reduction to a finite-dimensional comparison subring). Let $(\mathcal{A}, M)$ be equipped with a finite comparison datum. Suppose there exists a finite-dimensional comparison subring

$$R \subseteq \mathcal{E}_{\mathcal{A}}(M)$$

containing $\mathcal{W}_{\Gamma, \mathcal{A}}(M)$, together with a surjective homomorphism

$$\varphi: R \twoheadrightarrow E_{\Gamma}(M)$$

extending ρ_{Γ} , and assume that $\ker(\varphi)$ is nilpotent. Then, every compatible motivated idempotent on M is representable in $\mathcal{A}$.

Proof. Every compatible idempotent lies in $E_{\Gamma}(M)$ by definition. Since φ is surjective with nilpotent kernel, each such idempotent lifts to an idempotent of R by Lemma 4. Thus, the pair (R, φ) together with the lifted idempotent $e \in R$ witnesses representability in the sense of Definition 7. $\square$

In the special case $R = \mathcal{W}_{\Gamma, \mathcal{A}}(M)$ with $\varphi = \rho_{\Gamma}$, nilpotence of $\mathfrak{d}_{\Gamma}(M)$ gives word representability directly from Proposition 1.

Lemma 1 (Nilpotent extensions of nilpotent ideals). Let $J \subseteq K$ be two-sided ideals in a ring R . If $J^a = 0$ and $(K/J)^b = 0$, then $K^{ab} = 0$.

Proof. The condition $(K/J)^b = 0$ means precisely that $K^b \subseteq J$. Hence,

$$K^{ab} = (K^b)^a \subseteq J^a = 0.$$

This proves nilpotence of K . $\square$

Lemma 2 (Subideals and quotients of nilpotent ideals). *Let I be a nilpotent two-sided ideal in a ring R .*

1. *Every two-sided ideal $J \triangleleft R$ with $J \subseteq I$ is nilpotent.*
2. *For every two-sided ideal $J \triangleleft R$ with $J \subseteq I$, the quotient ideal I/J is nilpotent in R/J .*

Proof. If $I^n = 0$, then $J^n \subseteq I^n = 0$ and $(I/J)^n = (I^n + J)/J = 0$ in R/J . The expanded form of this elementary calculation is also recorded in Appendix A.2. $\square$

Lemma 3 (Idempotents lift across nilpotent ideals). *Let R be a ring, let $I \subseteq R$ be a nilpotent ideal, and let $\bar{e} \in R/I$ be an idempotent. Then, $\bar{e}$ lifts to an idempotent $e \in R$.*

Proof. This is the standard lifting lemma for idempotents modulo nilpotent ideals; see ([1], §21). The elementary reduction and the explicit correction formula in the square-zero case are recalled in Appendix A.1. $\square$

Lemma 4 (Lifting through a surjective map with nilpotent kernel). *Let*

$$\rho: R \twoheadrightarrow S$$

be a surjective ring homomorphism with nilpotent kernel. Then, every idempotent in S lifts to an idempotent in R .

Proof. Apply Lemma 3 to the quotient presentation $S \cong R/\ker(\rho)$. $\square$

Definition 10 (Filtered defect-nilpotence datum). *Let $\varphi: R \twoheadrightarrow S$ be a surjective comparison homomorphism. A filtered defect-nilpotence datum for φ consists of the following:*

1. *A finite increasing filtration by two-sided ideals:*

$$0 = F^{-1}R \subseteq F^0R \subseteq \cdots \subseteq F^NR = R.$$

2. *A finite increasing filtration by two-sided ideals on the target algebra:*

$$0 = F^{-1}S \subseteq F^0S \subseteq \cdots \subseteq F^NS = S.$$

3. *Compatibility of φ with these filtrations: $\varphi(F^iR) = F^iS$ for all i , and hence each induced graded map,*

$$\mathrm{gr}_F^i \varphi: \mathrm{gr}_F^i R \twoheadrightarrow \mathrm{gr}_F^i S$$

is surjective.

4. *For every i , the kernel of $\mathrm{gr}_F^i \varphi$ is a nilpotent ideal of $\mathrm{gr}_F^i R$.*

The quotient rings $\mathrm{gr}_F^i R = F^iR/F^{i-1}R$ and $\mathrm{gr}_F^i S = F^iS/F^{i-1}S$ are regarded as possibly non-unital rings. Thus, the kernel of $\mathrm{gr}_F^i \varphi$ is a two-sided ideal in the corresponding quotient ring.

Lemma 5 (Filtered nilpotence criterion). *Let $\varphi: R \rightarrow S$ be a surjective ring homomorphism equipped with a filtered defect-nilpotence datum in the sense of Definition 10. Then, $\ker(\varphi)$ is nilpotent.*

Proof. Let $K = \ker(\varphi)$ and put $K^{(i)} = K \cap F^i R$. The ideals $K^{(i)}$ form a finite filtration of K . For each i , the quotient $K^{(i)} / K^{(i-1)}$ embeds as a two-sided subideal of the graded kernel $\ker(\text{gr}_F^i \varphi)$. Nilpotence of that graded kernel says concretely that a sufficiently long product of elements of $K^{(i)}$ has graded class zero, hence lies in $K^{(i)} \cap F^{i-1} R = K^{(i-1)}$. Thus, $K^{(i)} / K^{(i-1)}$ is nilpotent. Starting with $K^{(-1)} = 0$ and applying Lemma 1 at each step gives nilpotence of $K^{(N)} = K$. The induction is written out in Appendix A.3. $\square$

For example, if a finite word-controlled filtration has been chosen on the ordinary comparison algebra, if its levels are two-sided ideals, and if its graded kernels are nilpotent, then the preceding lemma makes the defect ideal nilpotent.

Proposition 2 (Peirce triangular test). *Let*

$$\rho: R \twoheadrightarrow S$$

be a surjective ring homomorphism. Assume that R and S carry Peirce block decompositions indexed by the same finite ordered set

$$1_R = e_1 + \cdots + e_m, \quad 1_S = \bar{e}_1 + \cdots + \bar{e}_m,$$

by pairwise orthogonal idempotents such that $\rho(e_i) = \bar{e}_i$. Put

$$R_{ij} := e_i R e_j, \quad S_{ij} := \bar{e}_i S \bar{e}_j.$$

Assume further the following:

1. $R_{ij} = 0$ and $S_{ij} = 0$ for $i > j$;
2. $\rho(R_{ij}) \subseteq S_{ij}$ for all i, j ;
3. For each i , the induced diagonal map

$$\rho_{ii}: R_{ii} \twoheadrightarrow S_{ii}$$

has nilpotent kernel.

Then, $\ker(\rho)$ is nilpotent.

Proof. Let $N_R = \bigoplus_{i < j} R_{ij}$ and $N_S = \bigoplus_{i < j} S_{ij}$. These are strictly upper-triangular two-sided ideals, hence $N_R^m = N_S^m = 0$. The map induced by ρ on the diagonal quotients is

$$R/N_R \cong \prod_i R_{ii} \longrightarrow \prod_i S_{ii} \cong S/N_S,$$

and its kernel is the finite product of the diagonal kernels, so it is nilpotent. If $K = \ker(\rho)$, then $K \cap N_R$ is nilpotent and $K/(K \cap N_R)$ embeds into this diagonal kernel. Lemma 1 now gives nilpotence of K . The block calculation is expanded in Appendix A.4. $\square$

Proposition 2, followed by Lemma 4, is the triangular form of the idempotent-lifting argument used below.

Theorem 1 (Filtered representability theorem). *Let $(\mathcal{A}, M)$ be equipped with a finite comparison datum and a comparison subring (R, φ) . Assume there exists a filtered defect-nilpotence datum in the sense of Definition 10. Then, every compatible motivated idempotent on M is representable in $\mathcal{A}$.*

Proof. By Lemma 5, the map

$$\varphi: R \longrightarrow E_{\Gamma}(M)$$

has nilpotent kernel. It is surjective by construction. Therefore, Proposition 1 applies. $\square$

4. The Finite Nilpotent-Defect Representability Theorem

We can now state the finite reduction theorem.

Theorem 2 (Finite nilpotent-defect representability theorem). *Let $\mathcal{A}$ be a finite ordinary ambient category and let $M \in \mathcal{A}$. Assume that $(\mathcal{A}, M)$ satisfies the basic finite defect-nilpotence datum of Definition 9. Then, every compatible motivated idempotent*

$$\pi^{\text{mot}} \in E_{\Gamma}(M)$$

is representable in $\mathcal{A}$.

Proof. By the definition of the basic finite defect-nilpotence datum, there exists a comparison subring $R \subseteq \mathcal{E}_{\mathcal{A}}(M)$ and a surjective homomorphism

$$\varphi: R \twoheadrightarrow E_{\Gamma}(M)$$

extending the comparison map, and the same datum requires its kernel to be nilpotent. Since π^{mot} is compatible, it belongs to $E_{\Gamma}(M)$ by definition. Applying Lemma 4 to φ produces an idempotent

$$e \in R \subseteq \mathcal{E}_{\mathcal{A}}(M) = \text{End}_{\text{Kar}(\mathcal{A})}(M)$$

mapping to π^{mot} . Therefore, the pair (R, φ) and the idempotent $e \in R$ witness representability in the sense of Definition 7. $\square$

Remark 4. *The theorem is a reduction principle. Its hypotheses include the finite-dimensional ordinary comparison subring, the surjective comparison map, and nilpotence of the defect ideal. Once these data have been verified, the conclusion is the formal idempotent-lifting consequence of that verification.*

5. Refinement Rigidity of Represented Idempotents

We now compare two lifts of the same auxiliary projector. The needed algebra is again standard: idempotent lifts modulo a nilpotent ideal are conjugate by a unit congruent to 1.

Definition 11 (Represented compatible idempotent). *Fix a finite ordinary ambient category $\mathcal{A}$ and an object $M \in \mathcal{A}$. A represented compatible idempotent consists of a finite-dimensional comparison subring*

$$R_{\mathcal{A}} \subseteq \text{End}_{\text{Kar}(\mathcal{A})}(M),$$

a surjective comparison map

$$\varphi_{\mathcal{A}}: R_{\mathcal{A}} \twoheadrightarrow E_{\Gamma}(M),$$

and an idempotent $e_{\mathcal{A}} \in R_{\mathcal{A}}$, whose image is a fixed compatible idempotent

$$\varphi_{\mathcal{A}}(e_{\mathcal{A}}) = \pi^{\text{mot}} \in E_{\Gamma}(M).$$

Here, $E_{\Gamma}(M)$ denotes the finite comparison algebra common to the presentations under comparison.

Definition 12 (Common compatible refinement). Let $(\mathcal{A}, M, e_{\mathcal{A}})$ and $(\mathcal{B}, M, e_{\mathcal{B}})$ be represented compatible idempotents with the same motivated shadow π^{mot} . A common compatible refinement consists of the following:

1. A finite ordinary ambient category $\mathcal{C}$ into which the finite data from $\mathcal{A}$ and $\mathcal{B}$ map, together with a finite-dimensional comparison subring $R_{\mathcal{C}}$.
2. Injective ring homomorphisms

$$\iota_{\mathcal{A}}: R_{\mathcal{A}} \hookrightarrow R_{\mathcal{C}}, \quad \iota_{\mathcal{B}}: R_{\mathcal{B}} \hookrightarrow R_{\mathcal{C}}.$$

3. A surjective comparison map

$$\varphi_{\mathcal{C}}: R_{\mathcal{C}} \twoheadrightarrow E_{\Gamma}(M)$$

satisfying

$$\varphi_{\mathcal{C}} \circ \iota_{\mathcal{A}} = \varphi_{\mathcal{A}}, \quad \varphi_{\mathcal{C}} \circ \iota_{\mathcal{B}} = \varphi_{\mathcal{B}}.$$

4. Nilpotence of the common defect ideal

$$\mathfrak{d}_{\mathcal{C}} := \ker(\varphi_{\mathcal{C}}).$$

Equivalently, the maps in a common refinement form the diagram below:

$$\begin{array}{ccccc} R_{\mathcal{A}} & \xhookrightarrow{\iota_{\mathcal{A}}} & R_{\mathcal{C}} & \xhookleftarrow{\iota_{\mathcal{B}}} & R_{\mathcal{B}} \\ \varphi_{\mathcal{A}} \downarrow & & \downarrow \varphi_{\mathcal{C}} & & \downarrow \varphi_{\mathcal{B}} \\ E_{\Gamma}(M) & \xlongequal{\quad} & E_{\Gamma}(M) & \xlongequal{\quad} & E_{\Gamma}(M) \end{array}$$

Both squares commute, and both representatives therefore become lifts of the same idempotent in a single finite comparison algebra.

Remark 5. The common refinement is not required to be minimal. For rigidity it is enough that both finite presentations map to one finite stage and that the comparison map at that stage has nilpotent kernel.

Proposition 3 (Unipotent conjugacy of idempotent lifts). Let R be a ring, let $I \triangleleft R$ be a nilpotent two-sided ideal, and let $e, f \in R$ be idempotents with the same image in R/I . Then, there exists a unit

$$u \in 1 + I$$

such that

$$f = ueu^{-1}.$$

Consequently, the Karoubi summands cut out by e and f are isomorphic after applying the corresponding unit.

Proof. Since e and f have the same image in R/I , one has $f - e \in I$. Put

$$a := fe + (1 - f)(1 - e).$$

Modulo I , the two idempotents become equal, and therefore

$$a \equiv e^2 + (1 - e)^2 = e + (1 - e) = 1 \pmod{I}.$$

Thus, $a \in 1 + I$, hence a is a unit because I is nilpotent. A direct computation, not using commutativity, gives

$$ae = fe = fa.$$

Therefore, $ae = fa$, and multiplying on the right by a^{-1} gives

$$f = aea^{-1}.$$

Taking $u = a$ proves the assertion. In a Karoubi category, the isomorphism is represented by

$$fue: (M, e) \longrightarrow (M, f),$$

with the inverse represented by

$$eu^{-1}f: (M, f) \longrightarrow (M, e).$$

The relations $f = ueu^{-1}$ and $e = u^{-1}fu$ show that these two morphisms compose to the identity idempotents on the corresponding Karoubi summands. $\square$

Theorem 3 (Refinement rigidity in a common compatible refinement). *Let $(\mathcal{A}, M, e_{\mathcal{A}})$ and $(\mathcal{B}, M, e_{\mathcal{B}})$ be represented compatible idempotents with the same motivated shadow π^{mot} . Suppose they admit a common compatible refinement $\mathcal{C}$. Then,*

1. *The images $\iota_{\mathcal{A}}(e_{\mathcal{A}})$ and $\iota_{\mathcal{B}}(e_{\mathcal{B}})$ are conjugate by a unit in $1 + \mathfrak{d}_{\mathcal{C}}$.*
2. *The Karoubi summands*

$$(M, \iota_{\mathcal{A}}(e_{\mathcal{A}})), \quad (M, \iota_{\mathcal{B}}(e_{\mathcal{B}}))$$

are isomorphic in $\text{Kar}(\mathcal{C})$.

3. *The resulting isomorphism class at the stage $\mathcal{C}$ depends only on the common motivated idempotent π^{mot} and not on the chosen representatives of its two finite lifts.*

Proof. The compatibility conditions in Definition 12 imply

$$\varphi_{\mathcal{C}}(\iota_{\mathcal{A}}(e_{\mathcal{A}})) = \pi^{\text{mot}} = \varphi_{\mathcal{C}}(\iota_{\mathcal{B}}(e_{\mathcal{B}})).$$

Thus, the two common-stage idempotents have the same image modulo the nilpotent ideal $\mathfrak{d}_{\mathcal{C}} = \ker(\varphi_{\mathcal{C}})$. Proposition 3 gives a unit $u \in 1 + \mathfrak{d}_{\mathcal{C}}$ with

$$\iota_{\mathcal{B}}(e_{\mathcal{B}}) = u \iota_{\mathcal{A}}(e_{\mathcal{A}}) u^{-1}.$$

This proves the conjugacy and gives the asserted isomorphism of Karoubi summands. If one changes either finite lift while keeping the same image π^{mot} , another application of Proposition 3 in $R_{\mathcal{C}}$ gives the same common-stage object-isomorphism class. $\square$

Definition 13 (Filtered refinement system). *A filtered refinement system for a compatible motivated idempotent π^{mot} consists of the following:*

1. *A filtered index category $\mathcal{I}$.*
2. *For each $i \in \mathcal{I}$, a represented compatible idempotent $(\mathcal{A}_i, M, e_i)$, with comparison ring R_i and comparison map φ_i , whose motivated shadow is π^{mot} .*
3. *For every morphism $\alpha: i \rightarrow j$ in $\mathcal{I}$, a datum-preserving finite ambient enlargement*

$$u_{\alpha}: \mathcal{A}_i \hookrightarrow \mathcal{A}_j$$

together with a compatible injective unital homomorphism $u_{\alpha,R}: R_i \hookrightarrow R_j$ commuting with the comparison maps and transporting represented idempotents to represented idempotents with the same motivated shadow π^{mot} .

4. The transition functors and the homomorphisms $u_{\alpha,R}$ are compatible with composition, after the identifications fixed in the system.
5. For every pair $i, j \in \mathcal{I}$, a stage $k \in \mathcal{I}$ receiving maps from i and j such that the transported images of $(\mathcal{A}_i, M, e_i)$ and $(\mathcal{A}_j, M, e_j)$ in $\mathcal{A}_k$ form a common compatible refinement with nilpotent defect ideal.

Theorem 4 (Rigidity along a filtered refinement system). *Let a compatible motivated idempotent π^{mot} admit a filtered refinement system. Then, the represented ordinary summands attached to the stages of the system determine a single object-isomorphism class after passage to sufficiently high common-refinement stages. In this sense the ordinary summand is independent, up to isomorphism at a common later stage, of the finite ambient presentation used to represent π^{mot} .*

Proof. For any two stages i, j , choose a common dominating stage k , as in Definition 13. The transition maps transport e_i and e_j to two idempotents in the comparison ring R_k , and these idempotents have the same image π^{mot} under the comparison map at k . Theorem 3 gives an isomorphism between the corresponding summands in $\text{Kar}(\mathcal{A}_k)$. If two choices of k are made, filteredness provides a further stage dominating both choices; compatibility of the transition maps transports the two comparisons to that stage, where another application of Theorem 3 puts them in the same object-isomorphism class. For any finite chain of comparisons, one passes to a stage dominating all terms in the chain and applies the same argument there. Thus, the represented summands associated with π^{mot} have a single object-isomorphism class after passage to sufficiently high common-refinement stages. No canonical or unique comparison isomorphism is asserted. $\square$

6. Ambient Enlargements

The preceding results are finite: the data are a finite-dimensional comparison subring, a surjective comparison map, a nilpotent defect ideal, and, for rigidity, a common finite refinement. This section records the elementary compatibility needed when the same finite data are viewed in a larger ambient category.

Definition 14 (Datum-preserving finite ambient enlargement). *Fix a finite comparison datum Γ on $M \in \mathcal{A}$. A datum-preserving finite ambient enlargement relative to Γ ,*

$$i: \mathcal{A} \hookrightarrow \mathcal{B}$$

is a fully faithful tensor functor of finite ordinary ambient categories that transports the chosen generators, their ordinary lifts, and their motivated shadows to the same finite comparison datum in $\mathcal{B}$. In triangulated settings, the functor is also required to preserve the chosen shifts and distinguished triangles used in the datum. Equivalently, the comparison algebra used after enlargement is the transported algebra $E_{i_\Gamma}(i_*M)$ identified with $E_\Gamma(M)$, not the larger algebra generated by any additional operations that may exist in $\mathcal{B}$.*

Proposition 4 (Persistence of representability under datum-preserving enlargement). *Let $i: \mathcal{A} \hookrightarrow \mathcal{B}$ be a datum-preserving finite ambient enlargement relative to a fixed comparison datum Γ on $M \in \mathcal{A}$. If a compatible idempotent belonging to this fixed comparison datum is representable in $\mathcal{A}$, then the same idempotent is representable in $\mathcal{B}$ relative to the transported comparison datum $i_*\Gamma$.*

Proof. Choose a witnessing pair (R, φ) for representability in $\mathcal{A}$ together with an idempotent $e \in R$ lifting the given compatible idempotent. The fully faithful tensor functor sends R to a subring

$$i_*(R) \subseteq \text{End}_{\text{Kar}(\mathcal{B})}(i_*M)$$

and carries e to an idempotent $i_*(e) \in i_*(R)$. Since the enlargement is datum-preserving relative to Γ , the fixed finite comparison algebra is transported as the same algebraic target:

$$E_\Gamma(M) \cong E_{i_*\Gamma}(i_*M).$$

Therefore, φ transports to a surjective comparison homomorphism:

$$\varphi_{\mathcal{B}}: i_*(R) \twoheadrightarrow E_{i_*\Gamma}(i_*M)$$

with $\varphi_{\mathcal{B}}(i_*(e))$ equal to the same compatible idempotent viewed in $\mathcal{B}$. This proves representability in the enlarged ambient category for the transported finite datum. $\square$

The same observation applies to word algebras. Words built from the fixed generators of Γ remain the same words after applying i , and so the word algebra in $\mathcal{A}$ identifies with the transported word algebra in $\mathcal{B}$. If a directed family of finite ordinary ambient categories carries the same fixed datum from some stage onward, Theorem 2 applies at that stage and Proposition 4 transports the representing idempotents along the directed system. Different choices of lifts are compared by the rigidity theorem of Section 5.

7. Examples and Geometric Verifications

We now record several standard situations in which the finite algebra can be identified or propagated. They are organized according to the form of the nilpotence check: explicit comparison algebras, external sources of non-zero defect nilpotence, support filtrations, CK/MCK relation defects, and triangular block decompositions.

7.1. An Explicit Comparison Datum

We first spell out a case in which all ingredients of the finite comparison algebra can be written down. This elementary example shows exactly what the comparison algebra, the comparison map, and the defect ideal are in a concrete motivic situation.

Example 1 (The standard projectors of projective space). Let $X = \mathbb{P}_k^n$ and work in Chow motives with rational coefficients. Let $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$, and write $p_1, p_2: X \times X \rightarrow X$ for the projections. For $0 \leq a \leq n$, put

$$\pi_a := p_1^*(h^{n-a})p_2^*(h^a) \in CH^n(X \times X)_{\mathbb{Q}}.$$

For decomposable correspondences, the composition rule gives

$$(p_1^*\alpha p_2^*\beta) \circ (p_1^*\gamma p_2^*\delta) = \deg(\beta\gamma) p_1^*\alpha p_2^*\delta.$$

Since the pushforward to $\text{Spec } k$ of h^m vanishes unless $m = n$, and $\deg(h^n) = 1$, it follows that

$$\pi_b \circ \pi_a = \delta_{ab} \pi_a, \quad \sum_{a=0}^n \pi_a = \Delta_X.$$

Thus, the π_a are the $n+1$ non-zero standard Chow–Künneth projectors of $\mathbb{P}^n$, indexed here by Tate degree. In the usual cohomological indexing, these are $\pi_{2a} = p_1^*(h^{n-a})p_2^*(h^a)$, while $\pi_{2a+1} = 0$.

Define the finite auxiliary algebra

$$S := \prod_{a=0}^n \mathbb{Q}\epsilon_a, \quad \epsilon_a \epsilon_b = \delta_{ab} \epsilon_a,$$

and define the ordinary comparison algebra

$$R := \bigoplus_{a=0}^n \mathbb{Q}\pi_a \subseteq \text{End}(h(X)).$$

The comparison homomorphism

$$\rho: R \longrightarrow S, \quad \pi_a \longmapsto \epsilon_a,$$

is an isomorphism of finite-dimensional $\mathbb{Q}$ -algebras. Hence, the defect ideal is

$$\ker(\rho) = 0,$$

which is nilpotent. A compatible idempotent of S is exactly a sum $\sum_{a \in I} \epsilon_a$ for some subset $I \subseteq \{0, \dots, n\}$, and its ordinary motivic representative is the idempotent

$$\sum_{a \in I} \pi_a \in R.$$

This gives a completely explicit finite comparison datum, with zero defect.

Example 2 (A zero-defect blow-up instance). Let $\tilde{X} = \text{Bl}_{\text{Spec } k} \mathbb{P}^2$ be the blow-up of the projective plane at a k -rational point. The standard blow-up formula gives

$$h(\tilde{X}) \simeq h(\mathbb{P}^2) \oplus h(\text{Spec } k)(1)$$

in Chow motives. Take on $h(\mathbb{P}^2)$ the explicit algebra from Example 1, and take the one-dimensional exact algebra $\mathbb{Q}\eta$ on the Tate summand. With respect to this direct-sum decomposition, choose the diagonal finite comparison algebra

$$R_{\tilde{X}} := R_{\mathbb{P}^2} \times \mathbb{Q}\eta, \quad S_{\tilde{X}} := S_{\mathbb{P}^2} \times \mathbb{Q}\eta,$$

and the evident comparison isomorphism $R_{\tilde{X}} \xrightarrow{\sim} S_{\tilde{X}}$. Its defect ideal is again zero. Thus, the blow-up formula supplies a completely checked finite comparison datum in this elementary case. Allowing off-diagonal transition blocks leads to the triangular nilpotence situation treated in Corollary 1 and Corollary 3.

7.2. External Sources of Non-Zero Defect Nilpotence

The preceding examples have zero defect, so every ingredient can be checked without using an external nilpotence theorem. For non-zero defects, nilpotence is an additional motivic input, not a consequence of idempotent lifting. The next elementary observation records the way such input enters the finite comparison formalism.

Proposition 5 (Ambient nilpotence produces finite defect nilpotence). Let $R \subseteq \text{End}(M)$ be a finite-dimensional ordinary comparison algebra and let

$$\theta: R \longrightarrow T$$

be an algebra homomorphism to a finite-dimensional algebra. Put $S := \theta(R)$ and let $\rho: R \rightarrow S$ be the induced surjection. Suppose there is a two-sided ideal $J \triangleleft R$ such that

$$\ker(\rho) \subseteq J, \quad J^N = 0.$$

Then, the defect ideal $K := \ker(\rho)$ is nilpotent and satisfies $K^N = 0$. Consequently, the finite nilpotent-defect representability theorem applies to the comparison datum defined by $R \rightarrow S$.

Proof. Since K is a two-sided ideal of R contained in the nilpotent two-sided ideal J , every product of N elements of K is also a product of N elements of J . Hence, $K^N \subseteq J^N = 0$. The final sentence is Theorem 2 applied to the surjection ρ . $\square$

Remark 6 (Motivic sources of the ambient ideal). Proposition 5 is the finite algebraic form of several familiar inputs. For a Kimura finite-dimensional Chow motive, Kimura's nilpotence theorem gives nilpotence for numerically trivial endomorphisms; hence, any selected finite-dimensional defect ideal contained in the numerical kernel is nilpotent [3,15]. In contexts where a Rost nilpotence principle applies, elements in the relevant base-change kernel are nilpotent; for the selected finite-dimensional comparison algebra this gives nilpotence of any finite defect ideal contained in that kernel, since a nil ideal in a finite-dimensional algebra is nilpotent [4]. Similarly, distinguished-cycle or motivated-cycle constructions may give an exact finite comparison on a preferred subalgebra; it must still be checked that any remaining ordinary relation defects lie in a known nilpotent kernel. These nilpotence principles therefore enter here as external inputs controlling the selected finite defect ideal.

Remark 7 (Non-zero defects versus exact examples). If $\ker(\rho)$ is non-zero in Proposition 5, the proposition gives a non-zero nilpotent defect ideal. The projective-space example is the exact case $K = 0$: it shows how the finite comparison datum is constructed, while the non-zero case requires an external theorem placing the selected defect kernel inside a nilpotent motivic ideal.

7.3. Support Filtrations

The decomposition theorem and its semismall refinements naturally produce finitely many support strata, local summands, and a filtration indexed by closure order on supports [6–8]. The support estimate below is used only after the corresponding support-defined subspaces have been verified to be two-sided ideals in the chosen finite comparison algebra. The point for the present argument is precisely this multiplicative compatibility, as required in Definition 10.

Definition 15 (Support-filtered representability datum). Let $f: X \rightarrow Y$ be a proper map and let $M = M(X)$ be the object of interest. A support-filtered representability datum consists of the following:

1. A finite stratification $Y = \bigsqcup_{\alpha \in \Sigma} Y_\alpha$ indexing the support pieces relevant to the decomposition data.
2. A filtration by two-sided ideals

$$0 = F^{-1}E_\Gamma(M) \subseteq F^0E_\Gamma(M) \subseteq \cdots \subseteq F^N E_\Gamma(M) = E_\Gamma(M)$$

whose graded pieces are generated by projector data supported on the corresponding strata.

3. A filtration by two-sided ideals on a finite-dimensional ordinary comparison subring,

$$0 = F^{-1}R \subseteq F^0R \subseteq \cdots \subseteq F^N R = R \subseteq \mathcal{E}_A(M)$$

containing the word algebra $\mathcal{W}_{\Gamma, \mathcal{A}}(M)$, such that $F^i R \cdot F^j R \subseteq F^{\max\{i, j\}} R$ and the comparison map

$$\varphi: R \rightarrow E_{\Gamma}(M)$$

extends ρ_{Γ} and satisfies $\varphi(F^i R) = F^i E_{\Gamma}(M)$ for every i , hence induces surjections on graded pieces.

4. For each i , the graded defect ideal

$$\ker(\mathrm{gr}_F^i \varphi: \mathrm{gr}_F^i R \rightarrow \mathrm{gr}_F^i E_{\Gamma}(M))$$

is nilpotent.

Thus, the support stratification supplies parallel two-sided ideal filtrations on the auxiliary and ordinary algebras, and the nilpotence problem is pushed to the associated graded pieces. The requirement that the support-defined pieces be two-sided ideals is part of the datum; the support estimate below verifies multiplicative compatibility only after this algebraic condition has been checked in the chosen finite comparison rings.

Remark 8 (How support filtrations are verified in practice). *In geometric applications one usually orders the strata so that F^i records chosen operations supported on the union of strata of complexity at most i . One must check, inside the chosen finite comparison algebra, both that these support-defined subspaces are two-sided ideals and that composition of correspondences preserves the chosen support order. After this verification, the remaining nilpotence question is local on the associated graded pieces.*

Lemma 6 (Generator test for support ideals). *Let R be a finite-dimensional algebra generated by a finite set $\mathcal{U}$, and let*

$$0 = F^{-1}R \subseteq F^0R \subseteq \cdots \subseteq F^N R = R$$

be an increasing sequence of $\mathbb{Q}$ -subspaces. Suppose that each $F^a R$ is spanned by chosen correspondence representatives whose supports lie in the prescribed support level a . If

$$uF^a R \subseteq F^a R, \quad F^a R u \subseteq F^a R$$

for every generator $u \in \mathcal{U}$ and every a , then each $F^a R$ is a two-sided ideal of R . In particular, it is enough to verify the support condition after left and right composition with the chosen finite generators.

Proof. Every element of R is a finite $\mathbb{Q}$ -linear combination of words in the generators. The assumed left and right stability under each generator implies, by induction on word length, that left and right multiplication by every word preserves $F^a R$. Linearity then gives $RF^a R \subseteq F^a R$ and $F^a RR \subseteq F^a R$. $\square$

Remark 9 (Support criterion in correspondence language). *The preceding lemma is the algebraic check hidden in many support arguments. If a finite generating set has been chosen, one verifies that composing a generator with a correspondence supported over $Z_{\leq a}$, on either side, remains supported over $Z_{\leq a}$. The support-composition estimate below is then used only to justify this stability inside the selected finite comparison algebra. Without this two-sided-ideal property, the filtered nilpotence lemma does not apply.*

Lemma 7 (Support-composition estimate). *Let $\Xi_{12} \subset X_1 \times X_2$ and $\Xi_{23} \subset X_2 \times X_3$ be correspondences for which the composition is defined. Then,*

$$\text{Supp}(\Xi_{23} \circ \Xi_{12}) \subseteq p_{13}(p_{12}^{-1} \text{Supp}(\Xi_{12}) \cap p_{23}^{-1} \text{Supp}(\Xi_{23})) \subseteq X_1 \times X_3.$$

Consequently, once the support-defined subspaces in the chosen finite comparison algebra have been checked to be two-sided ideals, and if the finite support order is chosen so that the right-hand side is contained in the union of support pieces of complexity at most $\max\{i, j\}$ whenever Ξ_{12} has support complexity at most i and Ξ_{23} has support complexity at most j , then the corresponding support filtration is multiplicative:

$$F^i R \cdot F^j R \subseteq F^{\max\{i, j\}} R.$$

Proof. The composition of correspondences is represented by the proper pushforward along p_{13} of the intersection product

$$p_{12}^* \Xi_{12} \cdot p_{23}^* \Xi_{23}$$

on $X_1 \times X_2 \times X_3$. The support of this intersection product is contained in the intersection of the inverse images of the two supports, and proper pushforward cannot create support outside the image under p_{13} . This proves the displayed inclusion. The final assertion follows after combining this support estimate with the assumed two-sided-ideal property of the support-defined levels and the stated compatibility of the chosen support order with the image operation. $\square$

Lemma 8 (Support-filtered data induce filtered defect-nilpotence data). *A support-filtered representability datum canonically determines a filtered defect-nilpotence datum in the sense of Definition 10.*

Proof. The support-filtered datum provides filtered source and target algebras, a filtration-preserving surjection, and nilpotent graded kernels. By assumption, the filtration on R is by two-sided ideals and is multiplicative. This is exactly the content of a filtered defect-nilpotence datum. $\square$

Support-by-support nilpotence is therefore an immediate instance of the filtered argument: if the graded defect ideal at each support level is nilpotent, then Lemma 8 and Lemma 5 imply nilpotence of the total support defect ideal.

Theorem 5 (Support-filtered representability theorem). *Let $M \in \mathcal{A}$ admit a support-filtered representability datum. Then, every compatible motivated idempotent lying in the finite support-decomposition comparison algebra is representable.*

Proof. Let (R, φ) be the filtered comparison system supplied by the datum. By Lemma 8, this is a filtered defect-nilpotence datum. Hence, Lemma 5 shows that the total defect ideal $\ker(\varphi)$ is nilpotent. Theorem 1 then applies and produces an idempotent in R , lifting any compatible motivated idempotent in the finite comparison algebra. This is precisely representability in the sense of Definition 7. $\square$

The next proposition is a criterion. The geometric work in an application is to verify that the support-defined pieces are two-sided ideals and that the local comparison kernels on the graded pieces are nilpotent.

Proposition 6 (Semismall verification criterion). *Assume a semismall or support-decomposition datum for $f: X \rightarrow Y$ has the following features:*

1. *Finitely many relevant support strata.*
2. *Finitely many chosen generators on each stratum, together with chosen ordinary lifts whose generated comparison subring is finite-dimensional.*
3. *After choosing a total order refining the closure order, finite filtrations by two-sided ideals:*

$$0 = F^{\leq -1}R \subseteq F^{\leq 0}R \subseteq \cdots \subseteq F^{\leq N}R = R,$$

$$0 = F^{\leq -1}E_{\Gamma}(M) \subseteq \cdots \subseteq F^{\leq N}E_{\Gamma}(M) = E_{\Gamma}(M).$$

4. *The ideal filtrations are compatible with support composition, so that*

$$F^{\leq a}R \cdot F^{\leq b}R \subseteq F^{\leq \max\{a,b\}}R.$$

5. *On each associated graded piece, after the chosen identification with the corresponding local stratum algebra, the graded defect ideal is the local comparison kernel, and this kernel is nilpotent.*

Then, the associated projector datum determines a support-filtered representability datum.

Proof. The stated filtrations are precisely the source and target filtrations required in Definition 15, and the chosen ordinary comparison algebra is finite-dimensional. The support-composition condition gives the required multiplicativity, while the last assumption gives nilpotence of the graded defect ideals. Thus, the stated geometric data form a support-filtered representability datum. $\square$

Consequently, if the local defect ideal on each relevant stratum is nilpotent after passage to the corresponding local comparison algebra, Theorem 5 gives ordinary representatives for the compatible motivated idempotents lying in the finite support-decomposition comparison algebra generated by the semismall data.

Proposition 7 (Worked semismall verification criterion). *Suppose $f: X \rightarrow Y$ is semismall and that the finite datum on $M(X)$ is generated by finitely many local projectors indexed by the relevant strata. Choose a total order*

$$Y_0, Y_1, \dots, Y_N$$

refining the closure order, and put

$$Z_{\leq a} = \bigcup_{b \leq a} \overline{Y_b}.$$

Assume that the chosen finite comparison algebra admits support ideals: the elements of the finite-dimensional comparison subring R represented by correspondences supported over $Z_{\leq a}$ form a two-sided ideal $F^{\leq a}R$, and the same is true for the finite comparison algebra $E_{\Gamma}(M)$. Assume also that products of elements supported over $Z_{\leq a}$ and $Z_{\leq b}$ are supported over $Z_{\leq \max\{a,b\}}$, that the comparison homomorphism maps $F^{\leq a}R$ onto $F^{\leq a}E_{\Gamma}(M)$, and that the associated graded piece for Y_a is identified with the corresponding local stratum algebra. If the resulting local kernels are nilpotent, then every compatible motivated idempotent lying in the resulting finite support-decomposition comparison algebra is ordinarily representable.

Proof. The ideals $F^{\leq a}R$ form an increasing finite filtration because the order on the relevant strata is total. Since elements of $F^{\leq a}R$ and $F^{\leq b}R$ are represented by correspondences supported over $Z_{\leq a}$ and $Z_{\leq b}$, the support-composition estimate of Lemma 7, together with the assumed compatibility of the chosen support order, gives

$$F^{\leq a}R \cdot F^{\leq b}R \subseteq F^{\leq \max\{a,b\}}R.$$

The target algebra carries the analogous filtration, and the comparison map is filtration-preserving and surjective on each level by assumption. Passing to the quotient by $F^{\leq a-1}$ removes lower-support terms, so the graded kernel is the local comparison kernel under the chosen local identification. Nilpotence of these local kernels gives a support-filtered representability datum by Proposition 6, and Theorem 5 gives the asserted global representability. $\square$

Remark 10. *In this way, a stratumwise nilpotence calculation feeds directly into the filtered lifting theorem.*

7.4. Relative CK/MCK Relation Defects

Relative Chow–Künneth and multiplicative Chow–Künneth structures lead to finite projector algebras stable under multiplication and duality; compare [9–14]. For multiplicative relations one must also record maps such as the small diagonal, whose natural source is a tensor power of the motive. We therefore place the finite list of objects which occur in the relations inside a single finite tensor envelope and work with block endomorphisms of that envelope. The idempotents to be represented remain the idempotents in the original M -corner.

Definition 16 (Relative CK/MCK representability datum). *A relative CK/MCK representability datum for $M \in \mathcal{A}$ consists of the following:*

1. *A finite tensor envelope*

$$M^{\boxplus} = T_0 \oplus \cdots \oplus T_s,$$

where $T_0 = M$ and the remaining summands are the finitely many tensor expressions in M that occur as sources or targets of the chosen CK/MCK operations.

2. *Comparison-side block idempotents $\epsilon_0, \dots, \epsilon_s$ corresponding to the projectors onto the summands $T_0, \dots, T_s$, with $\epsilon_0 = \epsilon_M$, so that*

$$\epsilon_i \epsilon_j = \delta_{ij} \epsilon_i, \quad \sum_{j=0}^s \epsilon_j = 1.$$

3. *A finite family $q_1, \dots, q_m$ of block endomorphisms of $M^{\boxplus}$ encoding the relative projectors, the block maps induced by tensor products of these projectors which occur in the relations, the small diagonal or multiplication maps, and any auxiliary correspondences used in the relations, such that these elements together with the block idempotents $\epsilon_0, \dots, \epsilon_s$ generate a finite comparison algebra*

$$E_{\Gamma}(M^{\boxplus}).$$

4. *A finite-dimensional ordinary comparison subring*

$$R \subseteq \text{End}_{\text{Kar}(\mathcal{A})}(M^{\boxplus})$$

generated by chosen lifts $\tilde{q}_1, \dots, \tilde{q}_m$ of these block endomorphisms, together with the actual ordinary block projectors $\tilde{\epsilon}_0, \dots, \tilde{\epsilon}_s$ onto the summands $T_0, \dots, T_s$. Thus,

$$\tilde{\epsilon}_i \tilde{\epsilon}_j = \delta_{ij} \tilde{\epsilon}_i, \quad \sum_{j=0}^s \tilde{\epsilon}_j = 1,$$

and $\tilde{\epsilon}_0$ is the ordinary projector onto the summand $T_0 = M$; multiplicative composites are taken inside R .

5. A surjective comparison homomorphism

$$\rho: R \twoheadrightarrow E_{\Gamma}(M^{\boxplus})$$

sending each $\tilde{q}_i$ to q_i and each ordinary block idempotent $\tilde{\epsilon}_j$ to ϵ_j .

6. A finite presentation

$$E_{\Gamma}(M^{\boxplus}) \cong \mathbb{Q}\langle x_1, \dots, x_m, e_0, \dots, e_s \rangle / (r_1, \dots, r_u)$$

in which e_j represents the block idempotent ϵ_j . The listed relations $r_1, \dots, r_u$ generate the full kernel of the presentation map. In particular, they include the block-idempotent relations

$$e_i e_j - \delta_{ij} e_i, \quad \sum_{j=0}^s e_j - 1,$$

the block-source and block-target conditions for the operations used in the datum, and the CK/MCK relations under consideration.

7. The corresponding lifted relation defects

$$D_{\ell} := r_{\ell}(\tilde{q}_1, \dots, \tilde{q}_m, \tilde{\epsilon}_0, \dots, \tilde{\epsilon}_s) \in R, \quad 1 \leq \ell \leq u,$$

and nilpotence of the two-sided ideal $I = (D_1, \dots, D_u) \subseteq R$.

A motivated idempotent generated by the datum is said to lie in the M -corner if it belongs to

$$\epsilon_0 E_{\Gamma}(M^{\boxplus}) \epsilon_0.$$

Here, $\epsilon_0 = \epsilon_M$ denotes the block idempotent on the comparison side, while $\tilde{\epsilon}_0 = \tilde{\epsilon}_M$ denotes the actual ordinary projector onto the summand $M \subset M^{\boxplus}$.

Remark 11 (Why finite relation data suffice). The finite presentation fixes the relations whose ordinary defects have to be controlled. The listed relations are not merely a convenient subcollection of CK/MCK identities; they generate the full presentation kernel of the chosen finite algebra. The generator list is understood to include the tensor-induced block endomorphisms and all block idempotents that appear in those relations. Proposition 8 says that these defects give exactly the comparison kernel in the finite tensor envelope; passing to the M -corner then gives the kernel relevant to the original motive.

Example 3 (Typical CK/MCK relation defects). For a Chow–Künneth datum with motivated projectors $q_0, \dots, q_{2d}$, the finite presentation usually contains the orthogonal idempotent relations

$$q_i q_j - \delta_{ij} q_i = 0, \quad \sum_i q_i - 1 = 0.$$

Given chosen ordinary lifts $\tilde{q}_i$, the corresponding ordinary defects are

$$\tilde{q}_i \tilde{q}_j - \delta_{ij} \tilde{q}_i, \quad \sum_i \tilde{q}_i - 1.$$

For a multiplicative Chow–Künneth datum, one also includes the finite multiplicativity relations governed by the small diagonal. The small diagonal is a map from a tensor square to M , and in the tensor envelope

$$M^{\boxplus} = M \oplus (M \otimes M)$$

it is read as an off-diagonal block endomorphism. The finite generator list is enlarged, if necessary, to include the block endomorphisms induced by tensor products, such as $q_i \otimes q_j$ on $M \otimes M$. In this block notation the usual relation has the form

$$q_k \circ \Delta_{\text{sm}} \circ (q_i \otimes q_j) = 0 \quad \text{whenever } k \neq i + j,$$

and the ordinary defect is the block endomorphism

$$\tilde{q}_k \circ \tilde{\Delta}_{\text{sm}} \circ (\tilde{q}_i \otimes \tilde{q}_j) \quad (k \neq i + j).$$

If the small diagonal is treated as a fixed ordinary correspondence, the block endomorphism is built from that fixed lift; otherwise it is included among the chosen generators. Once the displayed CK/MCK relations, together with the block-idempotent and block-source/target relations, are known to generate the full finite presentation ideal, nilpotence of the two-sided ideal generated by their ordinary defects gives simultaneous lifting of the CK/MCK idempotents in the M -corner.

Proposition 8 (Finite relation-generation lemma). *In the setting of Definition 16, let $I \subseteq R$ be the two-sided ideal generated by the listed relation defects. Then, the comparison homomorphism ρ factors through an isomorphism*

$$\bar{\rho}: R/I \xrightarrow{\sim} E_{\Gamma}(M^{\boxplus}).$$

In particular, $I = \ker(\rho)$.

Proof. Choose the finite list of block endomorphism generators $q_1, \dots, q_m$ and the block idempotents $\epsilon_0, \dots, \epsilon_s$ from Definition 16. Put

$$F = \mathbb{Q}\langle x_1, \dots, x_m, e_0, \dots, e_s \rangle.$$

There is a surjective evaluation map

$$\pi_{\text{mot}}: F \twoheadrightarrow E_{\Gamma}(M^{\boxplus}), \quad x_i \longmapsto q_i, \quad e_j \longmapsto \epsilon_j.$$

Let $J_{\text{mot}} := \ker(\pi_{\text{mot}})$. By the finite presentation condition, the ideal J_{mot} is generated by the listed relations $r_1, \dots, r_u$.

Similarly, the ordinary lifts give a surjective evaluation map:

$$\pi_R: F \twoheadrightarrow R, \quad x_i \longmapsto \tilde{q}_i, \quad e_j \longmapsto \tilde{\epsilon}_j.$$

The chosen comparison map is compatible with these evaluations:

$$\rho \circ \pi_R = \pi_{\text{mot}}.$$

The image of r_{ℓ} under π_R is the defect D_{ℓ} , and I is the two-sided ideal generated by these defects. Since the r_{ℓ} generate J_{mot} as a two-sided ideal, $\pi_R(J_{\text{mot}}) \subseteq I$. Hence, π_R descends to a surjection:

$$\bar{\pi}_R: F/J_{\text{mot}} \twoheadrightarrow R/I.$$

Using the chosen presentation $F/J_{\text{mot}} \cong E_{\Gamma}(M^{\boxplus})$, we obtain a surjection:

$$\psi: E_{\Gamma}(M^{\boxplus}) \twoheadrightarrow R/I.$$

Conversely, the comparison map $\rho: R \rightarrow E_\Gamma(M^\boxplus)$ kills every D_ℓ and therefore kills I . It follows that ρ factors through a surjective map:

$$\bar{\rho}: R/I \rightarrow E_\Gamma(M^\boxplus).$$

The composite $\bar{\rho} \circ \psi$ fixes the generators q_i and the block idempotents ϵ_j , hence is the identity on $E_\Gamma(M^\boxplus)$. The composite $\psi \circ \bar{\rho}$ fixes the images of $\tilde{q}_i$ and $\tilde{\epsilon}_j$ in R/I , hence is the identity on R/I . Thus, $\bar{\rho}$ is an isomorphism, and $I = \ker(\rho)$. $\square$

Proposition 9 (CK/MCK relation defects). *In the setting of Definition 16, the M -corner comparison datum satisfies the hypotheses of Proposition 1.*

Proof. By Proposition 8, the two-sided ideal I generated by the complete list of lifted presentation defects is exactly the kernel of the comparison map from R to $E_\Gamma(M^\boxplus)$. This ideal is nilpotent by the final condition in the CK/MCK datum. Put

$$R_M := \tilde{\epsilon}_0 R \tilde{\epsilon}_0, \quad E_M := \epsilon_0 E_\Gamma(M^\boxplus) \epsilon_0,$$

and let $\rho_M: R_M \rightarrow E_M$ be the restriction of ρ . Since $\rho(\tilde{\epsilon}_0) = \epsilon_0$, this restriction is well defined, and the corner comparison sits in the commutative square below:

$$\begin{array}{ccc} R_M & \xrightarrow{\rho_M} & E_M \\ \downarrow & & \downarrow \\ R & \xrightarrow{\rho} & E_\Gamma(M^\boxplus) \end{array}$$

The upper horizontal map is surjective: if $y = \epsilon_0 a \epsilon_0 \in E_M$ and $r \in R$ satisfies $\rho(r) = a$, then

$$\rho_M(\tilde{\epsilon}_0 r \tilde{\epsilon}_0) = y.$$

Its kernel is $R_M \cap I = \tilde{\epsilon}_0 I \tilde{\epsilon}_0$. Indeed, an element of R_M killed by ρ_M lies in I , and every element of $R_M \cap I$ is equal to $\tilde{\epsilon}_0 x \tilde{\epsilon}_0$ for some $x \in I$. The corner kernel is nilpotent, since

$$(\tilde{\epsilon}_0 I \tilde{\epsilon}_0)^n \subseteq \tilde{\epsilon}_0 I^n \tilde{\epsilon}_0$$

for all n . Under the usual identification,

$$\tilde{\epsilon}_0 \operatorname{End}(M^\boxplus) \tilde{\epsilon}_0 \cong \operatorname{End}(M),$$

the ring R_M is a finite-dimensional comparison subring for the original motive M , and the restricted comparison map extends the word comparison map inherited from the CK/MCK datum. Hence, Proposition 1 applies to the M -corner. $\square$

Theorem 6 (Relative CK/MCK representability theorem). *Let $M \in \mathcal{A}$ admit a relative CK/MCK representability datum. Then, every compatible motivated idempotent lying in*

$$\epsilon_0 E_\Gamma(M^\boxplus) \epsilon_0$$

and generated by the chosen finite CK/MCK datum is represented by an ordinary idempotent in $\tilde{\epsilon}_0 R \tilde{\epsilon}_0$, viewed through the standard identification

$$\tilde{\epsilon}_0 \operatorname{End}(M^\boxplus) \tilde{\epsilon}_0 \cong \operatorname{End}(M).$$

Proof. Proposition 9 identifies the finite-dimensional comparison subring $\tilde{e}_0 R \tilde{e}_0$ in the M -corner, with nilpotent defect ideal. Equivalently, the corner map $\tilde{e}_0 R \tilde{e}_0 \rightarrow e_0 E_\Gamma(M^\boxplus) e_0$ constructed there is a finite comparison map to which Theorem 2 applies. The main lifting theorem therefore gives an idempotent in $\tilde{e}_0 R \tilde{e}_0$ representing any compatible motivated idempotent in that corner. Under $\tilde{e}_0 \operatorname{End}(M^\boxplus) \tilde{e}_0 \cong \operatorname{End}(M)$, this is an ordinary projector on the original motive M . $\square$

Remark 12. Thus, multiplicativity is recorded in the finite tensor envelope, while the represented idempotents return to the original M -corner.

7.5. Triangular Decompositions from Projective Bundles and Blow-Ups

Blow-up steps, projective-bundle steps, and finite iterated combinations of such moves come with standard block decompositions. In the applications below, finite block-stable upper-triangular comparison algebras compatible with these splittings are also chosen. The diagonal blocks are the lower-complexity pieces, while the permitted off-diagonal blocks record transition correspondences between them. The following triangular form is the geometric ingredient used in the applications below: it gives an explicit nilpotence bound for the whole defect ideal in terms of the diagonal ones.

For the formulas below, we use the notation $M(-)$ for the chosen motivic object. In Chow motives this is the usual $h(-)$ notation and the cohomological shifts are omitted; in triangulated motives the same decompositions are written with the indicated Tate twists and shifts.

Definition 17 (Triangular reduction datum). A triangular reduction datum for $M \in \mathcal{A}$ consists of the following:

1. A finite-dimensional ordinary comparison subring $R \subseteq \mathcal{E}_{\mathcal{A}}(M)$ equipped with a Peirce decomposition indexed by a finite ordered set:

$$1_R = e_1 + \cdots + e_m, \quad R_{ij} := e_i R e_j,$$

where the e_i are pairwise orthogonal idempotents and $R_{ij} = 0$ for $i > j$.

2. A target comparison algebra S equipped with a Peirce decomposition indexed by a finite ordered set:

$$1_S = \bar{e}_1 + \cdots + \bar{e}_m, \quad S_{ij} := \bar{e}_i S \bar{e}_j,$$

where the $\bar{e}_i$ are pairwise orthogonal idempotents and $S_{ij} = 0$ for $i > j$.

3. A surjective comparison homomorphism

$$\rho: R \twoheadrightarrow S$$

such that $\rho(e_i) = \bar{e}_i$ and hence $\rho(R_{ij}) \subseteq S_{ij}$ for all i, j .

4. Diagonal blocks corresponding to lower-complexity geometric pieces produced by the reduction step.
5. Nilpotence of each diagonal comparison kernel $\ker(\rho_{ii}: R_{ii} \rightarrow S_{ii})$.

Remark 13 (Typical origin of the block decomposition). In a blow-up step the diagonal blocks come from the ambient piece and the center piece. In a projective-bundle step they are Tate-shifted copies of the base comparison algebra. The upper-triangular condition is a property of the chosen finite comparison algebra, not a consequence of the direct-sum splitting alone. Once such an algebra has been chosen, the off-diagonal transition blocks may remain present; they form a finite strictly upper-triangular ideal.

Theorem 7 (Split triangular stability). *Let*

$$M = \bigoplus_{i=1}^m M_i$$

be an ordered finite direct-sum decomposition in an ordinary pseudo-abelian motivic category. For each i , let

$$\rho_i: R_i \twoheadrightarrow S_i$$

be a surjective comparison homomorphism between finite-dimensional $\mathbb{Q}$ -algebras, and put $K_i = \ker(\rho_i)$. Assume that $K_i^{a_i} = 0$ for some positive integers a_i . Let $R \subseteq \text{End}(M)$ be a finite-dimensional block-stable upper-triangular algebra of the form

$$R = \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1m} \\ 0 & R_2 & \cdots & R_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_m \end{pmatrix},$$

where multiplication is Peirce block multiplication. Let

$$S := \prod_{i=1}^m S_i$$

and let

$$\rho: R \longrightarrow S$$

be the homomorphism which applies ρ_i on the i -th diagonal block and kills the strictly upper-triangular blocks. Then, ρ is surjective and $K = \ker(\rho)$ is nilpotent. More precisely,

$$K^{a_1 + \cdots + a_m} = 0.$$

In particular, if $K_i^{a_i} = 0$ for all i , then $K^{am} = 0$.

Proof. Surjectivity follows from the surjectivity of the diagonal maps ρ_i . The kernel K consists of upper-triangular block matrices whose i -th diagonal block lies in K_i ; all strictly upper-triangular blocks lie in K because ρ kills them.

Consider a product $x_1 \cdots x_q$ with $x_\nu \in K$. Its (i_0, i_q) -block is a sum of terms indexed by nondecreasing chains:

$$i_0 \leq i_1 \leq \cdots \leq i_q,$$

where the ν -th factor contributes a block from $R_{i_{\nu-1}, i_\nu}$. If $i_{\nu-1} = i_\nu = i$, that factor is a diagonal block belonging to K_i ; if $i_{\nu-1} < i_\nu$, a strictly upper-triangular block is used. Along such a chain there can be at most $m - 1$ strict increases. Put $A = a_1 + \cdots + a_m$ and take $q = A$. If, for every i , the chain contained at most $a_i - 1$ diagonal factors at level i , then the total number of diagonal factors would be at most $A - m$, and the total number of factors would be at most

$$(A - m) + (m - 1) = A - 1,$$

a contradiction. Hence, some level i contributes at least a_i diagonal factors from K_i . Since the indices are nondecreasing, these factors occur in one consecutive diagonal stretch at level i , and their product lies in $K_i^{a_i} = 0$. The corresponding block term is therefore zero. All block terms in every product of A elements of K vanish, so $K^A = 0$. $\square$

The special case in which all diagonal maps are identities is the closed triangular model used repeatedly below.

Setup 1 (Closed split triangular model). *Let*

$$M = \bigoplus_{i=1}^m M_i$$

be an ordered finite direct-sum decomposition. For each i , let $A_i \subseteq \text{End}(M_i)$ be a finite-dimensional ordinary comparison algebra and let $R \subseteq \text{End}(M)$ be a finite-dimensional block-stable upper-triangular algebra:

$$R = \bigoplus_{1 \leq i \leq j \leq m} R_{ij}, \quad R_{ij} = e_i R e_j, \quad R_{ii} = A_i.$$

Let $S = \prod_i A_i$ and let $\rho: R \rightarrow S$ be diagonal projection.

Corollary 1 (Closed triangular nilpotence). *In Setup 1, the kernel*

$$N = \ker(\rho) = \bigoplus_{1 \leq i < j \leq m} R_{ij}$$

satisfies $N^m = 0$.

Proof. This is Theorem 7 with $S_i = R_i = A_i$ and $a_i = 1$ for every diagonal block. Equivalently, a product of m strictly upper-triangular blocks would require $m + 1$ strictly increasing indices among $\{1, \dots, m\}$, which is impossible. $\square$

With respect to the ordered splitting, the closed model can be displayed as

$$R = \begin{pmatrix} A_1 & R_{12} & \cdots & R_{1m} \\ 0 & A_2 & \cdots & R_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{pmatrix}, \quad \ker(\rho) = \begin{pmatrix} 0 & R_{12} & \cdots & R_{1m} \\ 0 & 0 & \cdots & R_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus, every idempotent of S lifts to R by Lemma 4, and two lifts of the same idempotent are conjugate by a unit of $1 + \ker(\rho)$ by Proposition 3.

Example 4 (The two-block triangular calculation). *Let*

$$R = \begin{pmatrix} R_1 & B \\ 0 & R_2 \end{pmatrix}, \quad S = S_1 \times S_2,$$

and let $\rho: R \rightarrow S$ be induced by surjective maps $\rho_i: R_i \twoheadrightarrow S_i$, killing the off-diagonal block B . If $K_i = \ker(\rho_i)$ and $K_i^{a_i} = 0$, then

$$K := \ker(\rho) = \begin{pmatrix} K_1 & B \\ 0 & K_2 \end{pmatrix}.$$

We claim directly that $K^{a_1+a_2} = 0$. Indeed, a non-zero block term in a product of elements of K either stays on the first diagonal, uses one off-diagonal block and then stays on the second diagonal, or stays on the second diagonal. In the first and third cases, one obtains a product of at least a_1 elements of K_1 or at least a_2 elements of K_2 , respectively, once the length is $a_1 + a_2$. In the mixed case there is only one possible transition from the first block to the second; hence the remaining

$a_1 + a_2 - 1$ diagonal factors cannot all avoid both thresholds a_1 and a_2 . Thus, each block term contains either a factor in $K_1^{a_1}$ or a factor in $K_2^{a_2}$, and so it vanishes. This is the first nontrivial instance of Theorem 7.

Corollary 2 (Projective-bundle stability). *Let B be smooth projective and let $p: P = \mathbb{P}_B(E) \rightarrow B$ be the projective bundle associated with a vector bundle E of rank r . Suppose a finite comparison map for the base datum*

$$\rho_B: R_B \rightarrow S_B$$

has defect ideal K_B with $K_B^N = 0$. Fix the standard projective-bundle splitting, in the form used for Chow motives by Manin and in intersection-theoretic form by Fulton [19,20], and write it in triangulated notation as

$$M(P) \simeq \bigoplus_{a=0}^{r-1} M(B)(a)[2a]$$

(and as $h(P) \simeq \bigoplus_{a=0}^{r-1} h(B)(a)$ in Chow motives). Let R_P be a finite-dimensional block-stable upper-triangular comparison subring for the induced finite comparison datum on P . We assume that R_P contains the corresponding word algebra and is equipped with a surjective induced comparison homomorphism onto the product of the diagonal targets, extending the word comparison map; its diagonal maps are the Tate-shifted copies of ρ_B . Denote the corresponding diagonal target copies by $S_{B,a}$, $0 \leq a \leq r-1$; each is identified with S_B after the indicated Tate twist, and after the cohomological shift in triangulated notation. Thus, the induced defect ideal is

$$K_P := \ker \left(R_P \rightarrow \prod_{a=0}^{r-1} S_{B,a} \right).$$

Then, K_P is nilpotent and satisfies

$$K_P^{rN} = 0.$$

Consequently, every compatible idempotent in this finite projective-bundle comparison algebra induced by the diagonal targets is ordinarily representable. Whenever two representatives of the same compatible idempotent are compared inside a common finite comparison satisfying the same nilpotence hypothesis, they are unipotently conjugate there.

Proof. The projective-bundle formula gives the displayed direct-sum decomposition. In Chow-motivic notation the same statement is obtained by deleting the shifts. Thus, the map from R_P to $\prod_{a=0}^{r-1} S_{B,a}$ is a diagonal-target upper-triangular comparison map of the form required in Theorem 7. The diagonal target copies $S_{B,a}$ have comparison kernels identified with the base kernel K_B , so the diagonal exponents are $N, \dots, N$ over the r summands. Applying the theorem with $m = r$ gives the exponent $N + \dots + N = rN$, hence $K_P^{rN} = 0$. Since the comparison homomorphism on R_P extends the word comparison map for the induced datum, the representability and rigidity conclusions follow from Theorems 2 and 3. $\square$

Example 5 (The projective-bundle exponent). *In Corollary 2 the diagonal target has exactly r copies of the base target algebra, one for each summand $M(B)(a)[2a]$, $0 \leq a \leq r-1$. If the base defect ideal satisfies $K_B^N = 0$, then each diagonal copy has the same nilpotence exponent N . The triangular theorem therefore gives*

$$K_P^{N+\dots+N} = K_P^{rN} = 0.$$

For a $\mathbb{P}^1$ -bundle, $r = 2$, this reads $K_P^{2N} = 0$. If the base comparison is already exact, so that $K_B = 0$, then each diagonal exponent may be taken to be 1; in this case $K_P^r = 0$, reflecting the fact that the only remaining defect is strictly upper-triangular with respect to the r -term projective-bundle splitting.

Corollary 3 (Smooth blow-up stability). *Let X be smooth projective and let $Z \subset X$ be a smooth center of codimension $c \geq 2$. Put $\tilde{X} = \text{Bl}_Z X$. Suppose finite comparison maps*

$$\rho_X: R_X \twoheadrightarrow S_X, \quad \rho_Z: R_Z \twoheadrightarrow S_Z$$

have defect ideals K_X and K_Z , with

$$K_X^{N_X} = 0, \quad K_Z^{N_Z} = 0.$$

Fix the standard blow-up splitting, again in the standard motivic/intersection-theoretic form [19,20], and write it in triangulated notation as

$$M(\tilde{X}) \simeq M(X) \oplus \bigoplus_{a=1}^{c-1} M(Z)(a)[2a]$$

(and as $h(\tilde{X}) \simeq h(X) \oplus \bigoplus_{a=1}^{c-1} h(Z)(a)$ in Chow motives). Let $R_{\tilde{X}}$ be a finite-dimensional block-stable upper-triangular comparison subring for the induced finite comparison datum on $\tilde{X}$. We assume that $R_{\tilde{X}}$ contains the corresponding word algebra and is equipped with a surjective induced comparison homomorphism onto the product of the diagonal targets, extending the word comparison map; its first diagonal map is ρ_X and its remaining diagonal maps are Tate-shifted copies of ρ_Z . Denote the corresponding diagonal target copies by $S_{Z,a}$, $1 \leq a \leq c-1$, each identified with S_Z after the indicated Tate twist and, in triangulated notation, shift. Thus, the induced defect ideal is

$$K_{\tilde{X}} := \ker \left(R_{\tilde{X}} \longrightarrow S_X \times \prod_{a=1}^{c-1} S_{Z,a} \right).$$

Then, $K_{\tilde{X}}$ is nilpotent and satisfies

$$K_{\tilde{X}}^{N_X + (c-1)N_Z} = 0.$$

In particular,

$$K_{\tilde{X}}^{c \max\{N_X, N_Z\}} = 0.$$

The compatible idempotents in this finite blow-up comparison algebra induced by the diagonal target are therefore ordinarily representable. Whenever two representatives of the same compatible idempotent are compared inside a common finite comparison satisfying the same nilpotence hypothesis, they are unipotently conjugate there.

Proof. The blow-up formula gives the displayed ordered direct-sum decomposition, with the Chow-motivic version obtained by omitting the cohomological shifts. Thus, the map from $R_{\tilde{X}}$ to $S_X \times \prod_{a=1}^{c-1} S_{Z,a}$ is a diagonal-target upper-triangular comparison map of the form required in Theorem 7. The diagonal exponents are therefore N_X for the X -block and $N_Z, \dots, N_Z$ for the $c-1$ copies of the Z -block. The theorem gives the exponent $N_X + (c-1)N_Z$. The weaker bound $c \max\{N_X, N_Z\}$ is immediate. Since the comparison homomorphism on $R_{\tilde{X}}$ extends the word comparison map for the induced datum, the final sentence follows from the main representability and rigidity theorems. $\square$

Corollary 4 (Finite towers of projective bundles and blow-ups). *Let a finite construction be obtained by a sequence of projective-bundle steps and smooth blow-ups along smooth centers. Assume that every step is equipped with its standard motivic splitting and with a chosen finite,*

block-stable, upper-triangular comparison algebra for that splitting, whose comparison target is the product of the diagonal targets. At each stage, this algebra is assumed to contain the word algebra of the induced finite datum and to carry a surjective induced comparison homomorphism onto the product of the diagonal targets, extending the word comparison map. Assume moreover that every initial variety and every smooth center used as an input to a blow-up step carries a finite comparison datum with nilpotent defect ideal. Then, every defect ideal produced along the tower is nilpotent. Hence, every compatible idempotent in the resulting diagonal-target finite comparison algebra is ordinarily representable. If two finite presentations admit a common later block-stable triangular stage with nilpotent defect ideal, the resulting summands become isomorphic at that stage.

Proof. Argue by induction on the length of the construction. At each step, the chosen comparison algebra is finite-dimensional by hypothesis, and a finite sequence of such choices therefore keeps the construction inside finite comparison data. At a projective-bundle step, all diagonal pieces are copies, up to Tate twist and, in triangulated notation, shift, of the previous base piece, so Corollary 2 transfers nilpotence to the new defect ideal. At a blow-up step, the diagonal pieces are the ambient piece and copies of the chosen smooth center. The ambient piece has a nilpotent defect ideal by the induction hypothesis, and the center has one by the standing assumption on all smooth centers used in blow-up steps. Corollary 3 therefore transfers nilpotence from the ambient and center pieces to the blow-up. Since the sequence has finite length and nilpotence is assumed on all initial inputs and blow-up centers, every stage has a nilpotent defect ideal. The representability conclusion follows from Theorem 2. The final assertion follows from Theorem 3 at the common triangular stage. $\square$

Remark 14 (Scope of the triangular stability statements). *The projective-bundle and blow-up statements are formulated for fixed standard splittings and chosen finite block-stable upper-triangular comparison algebras equipped with their induced comparison maps extending the corresponding word comparison maps. The triangular hypothesis is therefore part of the finite comparison datum. In this setting defect nilpotence is stable under the standard geometric operations, with explicit nilpotence bounds.*

8. Concluding Remarks

The paper gives a finite algebraic route from nilpotent defect control to ordinary motivic projector representability and refinement rigidity. The required inputs are part of the finite datum: a finite-dimensional comparison algebra, an ordinary comparison subring mapping onto it, and a nilpotent defect ideal. Once these inputs are available, the lifting and conjugacy conclusions are formal consequences of standard idempotent lifting modulo nilpotent ideals.

The non-formal burden in applications is therefore the verification of the finite datum. The paper isolates several checks that can be made in practice: support-defined pieces must be two-sided ideals in the selected comparison algebra, CK/MCK relations must generate the finite presentation kernel, and triangular geometric decompositions must be implemented by block-stable upper-triangular comparison algebras. When the defect is non-zero, one must also supply an external nilpotence input, for example, by showing that the selected finite kernel lies inside a nilpotent numerical, homological, or base-change kernel. Under these hypotheses, defect nilpotence is stable, with explicit exponents, for the projective-bundle, smooth blow-up, and finite-tower constructions considered above.

Accordingly, the results are finite reduction statements rather than unconditional representability theorems for arbitrary motivated projectors. Geometric or motivic input supplies nilpotent defect control, and the present arguments then produce ordinary idempotent representatives and common-refinement uniqueness of the resulting summands.

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Appendix A. Ring-Theoretic Details

This appendix gives the elementary algebraic verifications used in the body of the paper. It fixes the points in the argument where nilpotence is passed through filtrations, quotients, and triangular decompositions. The nilpotence statements for subideals, quotients, and extensions do not use the existence of a unit, and hence also apply to the non-unital graded pieces which occur in the filtered argument.

Appendix A.1. Idempotent Lifting over Nilpotent Ideals

We recall the standard calculation behind Lemma 3. First suppose that $I^2 = 0$ and choose any lift $x \in R$ of an idempotent $\bar{e} \in R/I$. Put

$$\delta = x^2 - x \in I.$$

The element δ commutes with x , since

$$x\delta = x(x^2 - x) = x^3 - x^2 = (x^2 - x)x = \delta x.$$

Moreover, $\delta^2 = 0$. Define

$$e = x - (2x - 1)\delta.$$

Then, e still maps to $\bar{e}$. We check the idempotent equation explicitly. Since $\delta^2 = 0$ and δ commutes with x , the correction term $(2x - 1)\delta$ also commutes with x , and

$$\begin{aligned} e^2 &= (x - (2x - 1)\delta)^2 \\ &= x^2 - 2x(2x - 1)\delta \\ &= x + \delta - (4x^2 - 2x)\delta. \end{aligned}$$

Using $x^2 = x + \delta$ and $\delta^2 = 0$, we have $x^2\delta = x\delta$. Hence,

$$e^2 = x + \delta - (4x - 2x)\delta = x + \delta - 2x\delta = x - (2x - 1)\delta = e.$$

No commutativity of R is used.

For a general nilpotent ideal I with $I^N = 0$, one applies the square-zero case successively. Assume that an idempotent has been lifted modulo I^r . To lift it modulo I^{r+1} , apply the square-zero case to the ideal I^r/I^{r+1} in R/I^{r+1} ; its square is zero because $I^{2r} \subseteq I^{r+1}$ for $r \geq 1$. Starting with the given idempotent modulo I and increasing r from 1 to $N - 1$ gives an idempotent in R .

Appendix A.2. Nilpotent Subideals, Quotients, and Extensions

Lemma A1 (Subideals and quotients of nilpotent ideals). *Let I be a nilpotent two-sided ideal in a ring R .*

1. *Every two-sided ideal $J \triangleleft R$ with $J \subseteq I$ is nilpotent.*
2. *For every two-sided ideal $J \triangleleft R$ with $J \subseteq I$, the quotient I/J is nilpotent as an ideal of R/J .*

Proof. If $I^N = 0$ and $J \subseteq I$, then $J^N \subseteq I^N = 0$, proving the first assertion. For the quotient assertion, one has

$$(I/J)^N = (I^N + J)/J = J/J = 0$$

in R/J . $\square$

Lemma A2 (Nilpotent extensions). *Let $J \subseteq I \subseteq R$ be two-sided ideals. Assume J is nilpotent and I/J is nilpotent as an ideal of R/J . Then, I is nilpotent.*

Proof. Choose integers $a, b \geq 1$ with $J^a = 0$ and $(I/J)^b = 0$. The second condition means $I^b \subseteq J$. Therefore, $I^{ab} \subseteq J^a = 0$, proving nilpotence of I . $\square$

Appendix A.3. Filtered Nilpotence as a Finite Induction Device

Proposition A1 (Expanded proof of the filtered nilpotence criterion). *Let $\varphi: R \rightarrow S$ be a surjective ring homomorphism equipped with a filtered defect-nilpotence datum:*

$$0 = F^{-1}R \subseteq F^0R \subseteq \cdots \subseteq F^NR = R, \quad 0 = F^{-1}S \subseteq F^0S \subseteq \cdots \subseteq F^NS = S,$$

where each F^iR and F^iS is a two-sided ideal, $\varphi(F^iR) = F^iS$, and each graded kernel

$$K_i := \ker(\mathrm{gr}_F^i \varphi: \mathrm{gr}_F^i R \rightarrow \mathrm{gr}_F^i S)$$

is nilpotent. Then, $\ker(\varphi)$ is nilpotent.

Proof. Write $K = \ker(\varphi)$ and $K^{(i)} := K \cap F^iR$. Since F^iR is a two-sided ideal of R , each $K^{(i)}$ is a two-sided ideal of R contained in K . We prove by induction on i that $K^{(i)}$ is nilpotent.

For $i = 0$, the map $K^{(0)} \rightarrow \mathrm{gr}_F^0 R = F^0R$ is injective because $F^{-1}R = 0$. If $x \in K^{(0)}$, then $\varphi(x) = 0$, so the class of x lies in $K_0 = \ker(\mathrm{gr}_F^0 \varphi)$. Thus, $K^{(0)}$ is a two-sided subideal of K_0 , and is nilpotent by Lemma A1.

Assume that $K^{(i-1)}$ is nilpotent. The natural map

$$K^{(i)} / K^{(i-1)} \longrightarrow F^iR / F^{i-1}R = \mathrm{gr}_F^i R$$

is injective. Its image is a two-sided subideal of K_i . Indeed, for $x \in K^{(i)}$ one has $\varphi(x) = 0$, hence the graded class of x maps to zero under $\mathrm{gr}_F^i \varphi$. If $r \in F^iR$, then rx and xr lie in F^iR because F^iR is a two-sided ideal; after passing to $F^iR / F^{i-1}R$ this is precisely left and right multiplication by the class of r . Thus, the image of $K^{(i)} / K^{(i-1)}$ is stable under multiplication by the graded ring $\mathrm{gr}_F^i R$. Since K_i is nilpotent, Lemma A1 shows that this image is nilpotent. Equivalently, for some b , every product of b elements of $K^{(i)}$ has zero class in $F^iR / F^{i-1}R$; because all factors still lie in K , that product belongs to $K \cap F^{i-1}R = K^{(i-1)}$. Hence, $K^{(i)} / K^{(i-1)}$ is a nilpotent quotient ideal. Lemma A2 applied to $K^{(i-1)} \subseteq K^{(i)}$ then gives nilpotence of $K^{(i)}$. After finitely many steps, $K = K^{(N)}$ is nilpotent. $\square$

Appendix A.4. Triangular Decompositions

Proposition A2 (Expanded proof of the Peirce triangular criterion). *Let $\rho: R \rightarrow S$ be a surjective homomorphism preserving the Peirce decompositions indexed by the same finite ordered set*

$$R_{ij} = e_i R e_j, \quad S_{ij} = \bar{e}_i S \bar{e}_j, \quad \rho(e_i) = \bar{e}_i,$$

with $R_{ij} = S_{ij} = 0$ for $i > j$. Assume that each diagonal kernel $\ker(\rho_{ii}: R_{ii} \rightarrow S_{ii})$ is nilpotent. Then, $\ker(\rho)$ is nilpotent.

Proof. The Peirce identities give $R_{ij}R_{\ell k} = 0$ when $j \neq \ell$ and $R_{ij}R_{jk} \subseteq R_{ik}$, and similarly for S . Hence, $N_R = \bigoplus_{i < j} R_{ij}$ and $N_S = \bigoplus_{i < j} S_{ij}$ are two-sided strictly upper-triangular ideals. A non-zero product of strictly upper-triangular blocks would have to follow a strictly increasing chain of indices $i_0 < i_1 < \dots < i_q$. There is no such chain of length $q = m$ inside $\{1, \dots, m\}$. Therefore, any product of at least m elements of N_R or N_S vanishes, and $N_R^m = 0$ and $N_S^m = 0$.

Passing to the quotients by these nilpotent ideals identifies the map induced by ρ with the product of the diagonal maps:

$$\bar{\rho} = \prod_{i=1}^m \rho_{ii} : \prod_{i=1}^m R_{ii} \longrightarrow \prod_{i=1}^m S_{ii}.$$

Its kernel is

$$\ker(\bar{\rho}) = \prod_{i=1}^m \ker(\rho_{ii}),$$

which is nilpotent because it is a finite product of nilpotent ideals. Now, $\ker(\rho) / (\ker(\rho) \cap N_R)$ identifies with a subideal of $\ker(\bar{\rho})$, hence is nilpotent by Lemma A1. Since $\ker(\rho) \cap N_R$ is a subideal of the nilpotent ideal N_R , it is nilpotent as well. A final application of Lemma A2 yields nilpotence of $\ker(\rho)$. $\square$

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