

Article

On Milne–Mercer-Type Inequalities via Proportional Caputo–Hybrid–Tempered–Conformable Fractional Integrals

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Abstract

In this paper, we construct a new parameterized integral kernel based on a proportional Caputo–hybrid–tempered–conformable kernel architecture. The proposed kernel combines proportional Caputo–hybrid weighting, conformable geometric scaling, and exponential tempering within a single kernel design. This formulation provides a flexible setting for fractional integral inequalities and induces an integral operator whose fundamental properties, including linearity, compatibility, and well-definedness, are established. We first derive a fundamental Milne-type identity corresponding to the new kernel. Using this identity, new upper bounds are obtained under the assumption that $|f'|$ and $|f''|$, as well as $|f'|^q$ and $|f''|^q$, are convex. The resulting inequalities are formulated through the tempered–conformable kernel and provide quantitative estimates for the deviation between fractional integral averages and Milne–Mercer-type expressions. Several limiting cases are also examined. Several existing inequalities involving proportional Caputo–hybrid, tempered, and conformable kernels arise naturally as limiting parameter configurations of the present construction. Consequently, the present results extend existing Milne–Mercer inequalities through a new parameterized kernel design, while preserving several important limiting cases.

Keywords: Milne–Mercer inequality; proportional Caputo–hybrid fractional integral; conformable fractional integral; tempered fractional inequalities

MSC: 26A51; 26D10; 26D15

1. Introduction

The classical Jensen–Mercer inequality introduced by Mercer [1] has become an important analytical tool in the development of Mercer-type integral inequalities. Based on this framework, various Milne-type and Milne–Mercer-type inequalities have been established for different classes of convex functions. For instance, Djenaoui and Meftah [2] obtained Milne-type inequalities for differentiable s -convex functions, while Benaissa and Sarikaya [3] recently developed Milne–Mercer-type inequalities for h -convex functions. These results reveal the effectiveness of Milne-type estimates in approximation theory and integral analysis.

Subsequently, several authors extended Milne-type inequalities to broader convexity settings. Munir et al. [4] investigated Milne-type inequalities for various convex function classes and provided several applications. Moreover, Farid [5] employed convexity techniques in the study of fractional operators and demonstrated their usefulness in deriving new integral estimates.



Academic Editor: Duarte Valério

Received: 18 June 2026

Revised: 15 July 2026

Accepted: 19 July 2026

Published: 21 July 2026

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The study of fractional integral inequalities has attracted considerable attention following the pioneering work of Sarikaya et al. [6], who established Hermite–Hadamard-type inequalities for fractional integrals. Since then, numerous fractional versions of classical inequalities have been proposed. Budak et al. [7] derived new Milne-type inequalities involving fractional integrals, while Mahajan and Nagar [8] established fractional Newton-type integral inequalities for Caputo fractional operators.

Fractional operators with nonsingular kernels have also received increasing attention. The classical Caputo derivative introduced by Caputo [9] has played a central role in fractional modeling, whereas Caputo and Fabrizio [10] proposed a new nonsingular fractional derivative with improved analytical properties. Based on Caputo–Fabrizio operators, Sahoo et al. [11] obtained new midpoint-type Hermite–Hadamard–Mercer inequalities and demonstrated their applicability in fractional analysis.

2. Preliminaries

Definition 1. A function $f : [a, b] \rightarrow \mathbb{R}$ is called convex on $[a, b]$ for all $x, y \in [a, b]$ and $t \in [0, 1]$ if it satisfies the following inequality:

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

In 2003, Mercer [1] introduced a refined version of the classical Jensen inequality, now known as the Jensen–Mercer inequality, stated as follows.

For a convex mapping $f : [a, b] \rightarrow \mathbb{R}$, the following inequality holds for each $x_j \in [a, b]$:

$$f\left(a + b - \sum_{j=1}^n u_j x_j\right) \leq f(a) + f(b) - \sum_{j=1}^n u_j f(x_j),$$

where $u_j \in [0, 1]$ and $\sum_{j=1}^n u_j = 1$.

Milne’s open-type formula is parallel to Simpson’s closed-type formula, since both are valid under comparable smoothness assumptions.

Theorem 1. Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is a four times continuously differentiable mapping on (a, b) and let $\|f^{(4)}\|_{\infty} = \sup_{t \in (a, b)} |f^{(4)}(t)| < \infty$. Then, the following inequality holds:

$$\left| \frac{1}{3} \left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b) \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{7(b-a)^4}{23040} \|f^{(4)}\|.$$

The proportional Caputo–hybrid operator has recently been proposed as a nonlocal singular operator. Its definition incorporates both derivative and integral components, and it can be expressed through a linear combination of Riemann–Liouville integral and Caputo derivative terms.

Definition 2 ([12]). Let $f : I \subset \mathbb{R}^+ \rightarrow \mathbb{R}$ be a differentiable function on I^o and $f, f' \in L^1(I)$. Then, the proportional Caputo–hybrid operator may be defined as

$${}^{\text{PC}}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t [K_1(\alpha, \tau) f(\tau) + K_0(\alpha, \tau) f'(\tau)] (t - \tau)^{-\alpha} d\tau$$

where $\alpha \in [0, 1]$ and K_0 and K_1 are functions satisfying

$$\lim_{\alpha \rightarrow 0^+} K_0(\alpha, \tau) = 0; \quad \lim_{\alpha \rightarrow 1} K_0(\alpha, \tau) = 1; \quad K_0(\alpha, \tau) \neq 0, \quad \alpha \in (0, 1];$$

$$\lim_{\alpha \rightarrow 0} K_1(\alpha, \tau) = 0; \lim_{\alpha \rightarrow 1^-} K_1(\alpha, \tau) = 1; K_1(\alpha, \tau) \neq 0, \alpha \in [0, 1).$$

Afterwards, Sarikaya presented a related definition by employing distinct K_1 and K_0 functions as follows.

Definition 3 ([13]). Let $f : I \subset \mathbb{R}^+ \rightarrow \mathbb{R}$ be a differentiable function on I^0 and $f, f' \in L^1(I)$. The left-sided and right-sided proportional Caputo–hybrid operators of order α are defined as follows:

$${}^{PC}D_b^\alpha f(b) = \frac{1}{\Gamma(1-\alpha)} \int_a^b [K_1(\alpha, b-\tau)f(\tau) + K_0(\alpha, b-\tau)f'(\tau)](b-\tau)^{-\alpha} d\tau$$

and

$${}^{PC}D_a^\alpha f(a) = \frac{1}{\Gamma(1-\alpha)} \int_a^b [K_1(\alpha, \tau-a)f(\tau) + K_0(\alpha, \tau-a)f'(\tau)](\tau-a)^{-\alpha} d\tau$$

where $\alpha \in [0, 1]$, $K_0(\alpha, t) = (1-\alpha)^2 t^{1-\alpha}$, and $K_1(\alpha, t) = \alpha^2 t^\alpha$.

Sarikaya established Hermite–Hadamard inequalities for convex functions via the proportional Caputo–hybrid operator.

Theorem 2 ([13]). Let $f : I \subset \mathbb{R}^+ \rightarrow \mathbb{R}$ be a differentiable function on I^0 , the interior of the interval I , where $a, b \in I^0$ with $a < b$, and f, f' be convex functions on I . Then, the following inequalities hold:

$$\begin{aligned} & \alpha^2(b-a)^\alpha f\left(\frac{a+b}{2}\right) + \frac{1}{2}(1-\alpha)(b-a)^{1-\alpha} f'\left(\frac{a+b}{2}\right) \\ & \leq \frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}} \left[{}^{PC}D_b^\alpha f(b) + {}^{PC}D_a^\alpha f(a) \right] \\ & \leq \alpha^2(b-a)^\alpha \left[\frac{f(a) + f(b)}{2} \right] + (1-\alpha)(b-a)^{1-\alpha} \left[\frac{f'(a) + f'(b)}{4} \right]. \end{aligned}$$

Another important direction is conformable fractional calculus. Jarad et al. [14] introduced a new conformable fractional integral framework preserving many desirable properties of classical calculus. Using this setting, Çelik et al. [15] established generalized Milne-type inequalities, whereas Ying et al. [16] developed several conformable fractional Milne-type inequalities and investigated their analytical properties.

In [14], Jarad et al. introduced conformable fractional integrals.

Definition 4. The conformable fractional integral operators $I_{a^+}^\beta f(x)$ and $I_{b^-}^\beta f(x)$ of order $\beta \in \mathbb{C}$, with $\text{Re}(\beta) > 0$ and $\alpha \in (0, 1]$, are given by

$$\left({}^{Conf}I_{a^+}^{\beta, \alpha} f \right)(t) = \frac{1}{\Gamma(\beta)} \int_a^t \frac{1}{(s-a)^{1-\alpha}} \left[\frac{(t-a)^\alpha - (s-a)^\alpha}{\alpha} \right]^{\beta-1} f(s) ds,$$

and

$$\left({}^{Conf}I_{b^-}^{\beta, \alpha} f \right)(t) = \frac{1}{\Gamma(\beta)} \int_t^b \frac{1}{(b-s)^{1-\alpha}} \left[\frac{(b-t)^\alpha - (b-s)^\alpha}{\alpha} \right]^{\beta-1} f(s) ds.$$

For $\alpha = 1$, the previous operators are reduced to Riemann–Liouville fractional operators with order $\beta > 0$ as follows:

$$(J_{a+}^{\beta} f)(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s) ds, \quad t > a,$$

and

$$(J_{b-}^{\beta} f)(t) = \frac{1}{\Gamma(\beta)} \int_t^b (s-t)^{\beta-1} f(s) ds, \quad t < b.$$

The beta-Euler function $\beta(\cdot, \cdot)$ is defined for any $p, q > 0$ as follows:

$$\beta(p, q) = \int_0^1 (1-y)^{q-1} y^{p-1} dy.$$

On the other hand, tempered fractional operators were introduced to incorporate exponential attenuation effects into classical fractional kernels. Their mathematical properties and numerical aspects were investigated by Li et al. [17]. Mohammed et al. [18] established generalized Hermite–Hadamard inequalities via tempered fractional integrals, while Haider et al. [19] presented a comprehensive study of Milne-type inequalities involving tempered fractional integrals.

More recently, tempered fractional inequalities have been extended to other quadrature-type inequalities. Lakhdari et al. [20] investigated tempered fractional Bullen-type inequalities under various functional settings. These studies show that tempered kernels provide useful tools for refining quadrature-type integral estimates.

Definition 5 ([17]). For the real number $\alpha > 0$ and $x, \lambda \geq 0$, the λ -incomplete gamma function is defined as follows:

$$\gamma_{\lambda}(\alpha, x) = \int_0^x t^{\alpha-1} e^{-\lambda t} dt.$$

If $\lambda = 1$, it is equivalent to the incomplete gamma function:

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt.$$

Here, $0 < \alpha < \infty$ and $\lambda \geq 0$.

Remark 1. For the real number $\alpha > 0$ and $x, \lambda \geq 0$, we attain

1. $\gamma_{\lambda(k-l)}(\alpha, 1) = \int_0^1 t^{\alpha-1} e^{-(k-l)t} dt = \frac{1}{(k-l)^{\alpha}} \gamma_{\lambda}(\alpha, k-l).$
2. $\int_0^1 \gamma_{\lambda(k-l)}(\alpha, x) dx = \frac{\gamma_{\lambda}(\alpha, k-l)}{(k-l)^{\alpha}} - \frac{\gamma_{\lambda}(\alpha+1, k-l)}{(k-l)^{\alpha+1}}.$

Definition 6 ([17]). Tempered fractional integral operators are defined as follows:

$$J_{a+}^{(\alpha, \lambda)} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} e^{-\lambda(x-t)} f(t) dt, \quad x \in [a, b],$$

and

$$J_{b-}^{(\alpha, \lambda)} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} e^{-\lambda(t-x)} f(t) dt, \quad x \in [a, b]$$

for $\alpha > 0$ and $\lambda \geq 0$.

A significant recent development is the proportional Caputo–hybrid framework. Sarikaya [13] first established Hermite–Hadamard-type inequalities via proportional Caputo–hybrid operators. Subsequently, Demir et al. [21] developed new Hermite–Hadamard-type inequalities based on this operator, while Butt et al. [22] derived Hermite–Hadamard inequalities for harmonically convex functions. Furthermore, Demir [23] introduced a novel method for deriving Milne-type inequalities via proportional Caputo–hybrid operators, highlighting the flexibility of this framework in integral inequality theory. In related published work, the author established Hermite–Hadamard–Mercer-type and Milne–Mercer-type inequalities via Atangana–Baleanu–conformable fractional integral operators [24,25]. These studies indicate that generalized kernel constructions can yield broader classes of inequalities and richer recovery properties.

Recently, Munir et al. [26] investigated integral inequalities under generalized convexity assumptions. Their work focused primarily on generalized convex functions associated with existing fractional operators. By contrast, the present study introduces a proportional Caputo–hybrid–tempered–conformable kernel formulation, derives a new Milne-type identity, and establishes several Milne–Mercer-type inequalities. These results broaden the class of kernel-based integral estimates beyond those considered in the above work.

We introduce a proportional Caputo–hybrid–tempered–conformable kernel together with its corresponding integral operator.

Definition 7. For $\lambda \geq 0$, define the auxiliary kernel function δ_λ by

$$\delta_\lambda^{\alpha,\beta,\eta}(t) = \int_0^t (1-s^\beta)^{1-2\alpha\eta} \frac{1}{s^{1-\beta}} e^{\lambda s} ds.$$

Let $f : I \subset \mathbb{R}^+ \rightarrow \mathbb{R}$ be a differentiable function on I^0 and $f, f' \in L^1(I)$. For $0 < \alpha \leq 1$, $\eta > 0$, $0 < \alpha\eta < 1$, $\beta \in (0, 1]$, and $\lambda \geq 0$, the associated integral operators of order α are defined as follows:

$$\begin{aligned} & {}_{a^+}^{PCH-Conf} D^{\alpha,\beta,\eta,\lambda} f(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_a^t [K_1(\alpha\eta, t, \tau) f(\tau) + K_0(\alpha\eta, t, \tau) f'(\tau)] \\ & \quad \times \frac{1}{(\tau-a)^{1-\beta}} \left[\frac{(t-a)^\beta - (\tau-a)^\beta}{\beta} \right]^{-\alpha\eta} e^{-\lambda(t-\tau)} d\tau \end{aligned}$$

where

$$K_0(\alpha\eta, t, \tau) = (1-\alpha\eta)^2 e^{\lambda(t-\tau)} \left[\frac{(t-a)^\beta - (\tau-a)^\beta}{\beta} \right]^{1-\alpha\eta}$$

and

$$K_1(\alpha\eta, t, \tau) = (\alpha\eta)^2 (\tau-a)^{1-\beta} \left[\frac{(t-a)^\beta - (\tau-a)^\beta}{\beta} \right]^{\alpha\eta}$$

and

$$\begin{aligned} & {}_{b^-}^{PCH-Conf} D^{\alpha,\beta,\eta,\lambda} f(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_t^b [L_1(\alpha\eta, t, \tau) f(\tau) + L_0(\alpha\eta, t, \tau) f'(\tau)] \\ & \quad \times \frac{1}{(b-\tau)^{1-\beta}} \left[\frac{(b-t)^\beta - (b-\tau)^\beta}{\beta} \right]^{-\alpha\eta} e^{-\lambda(\tau-t)} d\tau \end{aligned}$$

where

$$L_0(\alpha\eta, t, \tau) = (1-\alpha\eta)^2 e^{\lambda(\tau-t)} \left[\frac{(b-t)^\beta - (b-\tau)^\beta}{\beta} \right]^{1-\alpha\eta}$$

and

$$L_1(\alpha\eta, t, \tau) = (\alpha\eta)^2(b - \tau)^{1-\beta} \left[\frac{(b-t)^\beta - (b-\tau)^\beta}{\beta} \right]^{\alpha\eta}$$

and $\Gamma(\cdot)$ is the gamma function.

2.1. Analytical Motivation of the Proposed Kernel

The proposed PCHTC construction is not obtained by directly combining existing operators in an algebraic manner. Instead, it is derived from the design of a four-parameter kernel with a distinct mathematical structure,

$$K_{\text{PCHTC}}(t, \tau; \alpha, \beta, \eta, \lambda),$$

whose structure simultaneously integrates three independent mechanisms:

$$K_{\text{PCHTC}} = K_{\text{PCH}} \cdot K_{\text{Conf}} \cdot K_{\text{Temp}}.$$

Here, K_{PCH} controls the proportional Caputo–hybrid weighting, K_{Conf} introduces conformable geometric scaling, and K_{Temp} provides exponential localization through the tempering parameter. Rather than acting independently, these components interact within a single integral kernel.

As a consequence, the resulting kernel admits a four-dimensional parameter space. Previously studied models appear only under particular parameter selections. The mathematical motivation of the present work is to construct a parameterized kernel that provides greater analytical flexibility for Milne–Mercer-type analysis.

Proposition 1 (Linearity). Let $f, g \in L^1(I)$. For arbitrary constants $c_1, c_2 \in \mathbb{R}$,

$${}^{PCH-Conf}D^{\alpha, \beta, \eta, \lambda}_{a^+}[c_1 f + c_2 g] = c_1 \cdot {}^{PCH-Conf}D^{\alpha, \beta, \eta, \lambda}_{a^+}[f] + c_2 \cdot {}^{PCH-Conf}D^{\alpha, \beta, \eta, \lambda}_{a^+}[g]$$

and similarly,

$${}^{PCH-Conf}D^{\alpha, \beta, \eta, \lambda}_{b^-}[c_1 f + c_2 g] = c_1 \cdot {}^{PCH-Conf}D^{\alpha, \beta, \eta, \lambda}_{b^-}[f] + c_2 \cdot {}^{PCH-Conf}D^{\alpha, \beta, \eta, \lambda}_{b^-}[g].$$

Proof. Since the coefficient functions K_0, K_1, L_0 , and L_1 are independent of f and g , the assertion follows directly from the linearity of integration.

Therefore,

$${}^{PCH-Conf}D^{\alpha, \beta, \eta, \lambda}[af + bg] = a \cdot {}^{PCH-Conf}D^{\alpha, \beta, \eta, \lambda}[f] + b \cdot {}^{PCH-Conf}D^{\alpha, \beta, \eta, \lambda}[g].$$

Hence, the desired result follows. $\square$

Proposition 2 (Compatibility).

- (1) If $\lambda = 0$, then the ${}^{PCH-Conf}D^{\alpha, \beta, \eta, \lambda}_{a^+} f(t)$ operator reduces the proportional Caputo–hybrid conformable operator.
- (2) If $\beta = \eta = 1$, then the ${}^{PCH-Conf}D^{\alpha, \beta, \eta, \lambda}_{a^+} f(t)$ operator reduces the proportional Caputo–hybrid–tempered operator.
- (3) If $\lambda = 0$ and $\beta = \eta = 1$, then the ${}^{PCH-Conf}D^{\alpha, \beta, \eta, \lambda}_{a^+} f(t)$ operator reduces the classical proportional Caputo–hybrid operator.

Hence, the kernel formulation is compatible with several existing operators reported in the literature.

Proposition 3 (Well-definedness). *Let $0 < \alpha\eta < 1$, $\beta \in (0, 1]$, and $\lambda \geq 0$, and let $f, f' \in L^1[a, b]$. Then, the left-sided and right-sided proportional Caputo–hybrid–tempered–conformable operators introduced in Definition 7 are well defined for $t \in (a, b]$ and $t \in [a, b)$, respectively.*

Proof. We prove the assertion for the left-sided operator; the proof for the right-sided operator is analogous.

For $a < \tau < t$, the kernel appearing in the left-sided operator is of the form

$$\frac{1}{(\tau - a)^{1-\beta}} \left[\frac{(t - a)^\beta - (\tau - a)^\beta}{\beta} \right]^{-\alpha\eta} e^{-\lambda(t-\tau)}.$$

We examine the possible singularities at the endpoints $\tau = a$ and $\tau = t$.

First, as $\tau \rightarrow a^+$, we have $(t - a)^\beta - (\tau - a)^\beta \rightarrow (t - a)^\beta > 0$.

Hence, $\left[\frac{(t-a)^\beta - (\tau-a)^\beta}{\beta} \right]^{-\alpha\eta}$ remains bounded near $\tau = a$. Therefore, the only possible singular factor near $\tau = a$ is $(\tau - a)^{\beta-1}$.

Since $\beta \in (0, 1]$, we have $\beta - 1 \geq -1$ only at the endpoint $\beta = 0$, which is excluded; hence, $\beta - 1 > -1$. Consequently, $\int_a^{a+\varepsilon} (\tau - a)^{\beta-1} d\tau < \infty$ for sufficiently small $\varepsilon > 0$. Thus, the kernel is integrable near $\tau = a$.

Next, as $\tau \rightarrow t^-$, the factor $(\tau - a)^{\beta-1}$ is bounded, because $t > a$. By the mean value theorem, there exists ξ between $\tau - a$ and $t - a$ such that $(t - a)^\beta - (\tau - a)^\beta = \beta \xi^{\beta-1} (t - \tau)$.

Since ξ stays in a compact subinterval of $(0, \infty)$ when τ is close to t , there exists a constant $C > 0$ such that $(t - a)^\beta - (\tau - a)^\beta \geq C(t - \tau)$.

Hence, near $\tau = t$,

$$\left[\frac{(t - a)^\beta - (\tau - a)^\beta}{\beta} \right]^{-\alpha\eta} \leq C_1 (t - \tau)^{-\alpha\eta}$$

for some constant $C_1 > 0$. Since $0 < \alpha\eta < 1$, we have $\int_{t-\varepsilon}^t (t - \tau)^{-\alpha\eta} d\tau < \infty$.

Therefore, the kernel is integrable near $\tau = t$.

Finally, since $\lambda \geq 0$ and $a \leq \tau \leq t$, the exponential tempering factor satisfies $0 < e^{-\lambda(t-\tau)} \leq 1$.

Thus, the tempering term does not introduce any additional singularity.

Combining the above observations, the kernel is integrable on $[a, t]$. Since $f, f' \in L^1[a, b]$ and the coefficient functions K_0, K_1, L_0, L_1 in Definition 7 are finite under the stated parameter conditions, the corresponding integral exists. Hence, the integral operator generated by the proposed kernel is well defined.

The right-sided case follows by the same argument, with $b - \tau$ and $\tau - t$ replacing $\tau - a$ and $t - \tau$, respectively. $\square$

Remark 2. *The proposed proportional Caputo–hybrid–tempered–conformable framework is generated from a new parameterized kernel rather than from a direct algebraic combination of several existing operators. The mathematical novelty lies in the analytical architecture of the kernel.*

The principal objective of the present work is not to develop a complete theory of fractional calculus. The proposed kernel is designed specifically for the derivation of Milne–Mercer-type inequalities under a flexible parameter setting.

Moreover, the present paper does not claim that the integral operator induced by the proposed kernel satisfies every theoretical property of classical fractional operators, such as the semigroup composition law. The investigation of these operator-theoretic properties will be considered in future work.

Therefore, the proposed kernel should not be interpreted merely as an integer-order low-pass or high-pass filter, since it modifies the integral kernel through three mathematically independent analytical mechanisms rather than applying a conventional filtering process.

Remark 3. *The terminology “fractional integral operator” is retained only where it is used in the existing literature on proportional Caputo–hybrid, conformable, and tempered operators. In the present manuscript, the emphasis is placed on the generalized kernel formulation and the associated integral inequalities, rather than on the construction of a complete operator theory.*

The above propositions establish the fundamental analytical properties required for the subsequent development of the proposed Milne–Mercer-type inequalities. We now proceed to investigate the corresponding integral identities and inequality estimates.

Although significant progress has been achieved in Milne-type and Milne–Mercer-type inequalities, existing studies have mainly focused on either tempered fractional kernels [18,19], conformable fractional kernels [15,16], or proportional Caputo–hybrid kernels [11–13,21–23] separately. To the best of our knowledge, no study has investigated Milne–Mercer-type inequalities for a parameterized kernel that simultaneously incorporates proportional Caputo–hybrid weighting, conformable geometric scaling, and exponential tempering. Consequently, the development of such a kernel formulation and its associated inequalities remains an open problem.

Motivated by the above developments, we introduce a proportional Caputo–hybrid–tempered–conformable kernel formulation. The associated integral operator serves as an analytical tool for deriving Milne–Mercer-type inequalities, while the principal novelty lies in the four-parameter kernel family from which several existing operators arise as boundary cases.

The principal contributions of the present paper are summarized below.

- (i) A new parameterized proportional Caputo–hybrid–tempered–conformable kernel is introduced by simultaneously incorporating proportional weighting, conformable scaling, and exponential localization.
- (ii) The proposed construction introduces a parameterized kernel with greater analytical flexibility than existing PCH, tempered, or conformable kernels.
- (iii) A new Milne-type identity corresponding to the proposed generalized kernel is established.
- (iv) New Milne–Mercer-type inequalities are derived under convexity assumptions on $|f''|$ and $|f''|^q$.
- (v) The proposed formulation preserves compatibility with several existing kernel models through suitable parameter choices.
- (vi) The integral operator induced by the proposed kernel is shown to satisfy the basic analytical properties required in this work, including linearity, compatibility, and well-definedness.
- (vii) The proposed kernel provides a systematic approach to establishing new Milne–Mercer-type inequalities.

The objective of the present paper is not merely to introduce another fractional integral construction. Rather, the novelty lies in the analytical design of the proposed kernel and its associated Milne-type identity whose analytical architecture simultaneously incorporates proportional Caputo–hybrid weighting, conformable scaling, and exponential tempering. This kernel provides greater analytical flexibility while remaining compatible with several existing kernel models through suitable parameter selections.

Compared with the existing literature, the novelty of the present work may be summarized as follows.

Table 1 demonstrates that the novelty of the present work is not merely the recovery of previously known operators. Instead, the proposed PCHTC kernel architecture provides additional analytical flexibility by simultaneously incorporating proportional Caputo–hybrid weighting, conformable scaling, and exponential tempering. Existing operators arise as limiting or boundary cases of the resulting parameter space, while the proposed kernel formulation allows the derivation of Milne–Mercer-type inequalities that are not available within the individual frameworks alone.

Table 1. Comparison of the present work with existing studies.

Study	Kernel Structure	Identity	Milne	Mercer	Boundary Cases
Sarikaya [13]	PCH	×	✓	×	×
Celik et al. [15] and Ying et al. [16]	Conf	✓	✓	×	RL-type
Haider et al. [19]	Tempered	✓	✓	×	Classical tempered
Present paper	PCHTC	✓	✓	✓	PCH + Tempered + Conf

2.2. Mathematical Gain of the Proposed Kernel Framework

Although the proposed formulation possesses a more complicated kernel than previously introduced frameworks, the additional analytical complexity is not introduced merely for formal generalization. Instead, each parameter controls an independent mathematical mechanism. The proportional Caputo–hybrid structure regulates the weighting of local and nonlocal information, the conformable parameter governs the geometric scaling of the kernel, and the exponential tempering parameter controls kernel localization.

Consequently, the proposed kernel permits independent adjustment of three analytical mechanisms within one formulation, thereby providing greater flexibility in deriving Milne–Mercer-type inequalities. Therefore, the mathematical contribution of the present paper lies in enlarging the admissible analytical kernel class rather than merely increasing the algebraic complexity of the associated integral operator.

Unlike existing proportional Caputo–hybrid, tempered, or conformable kernels, the proposed parameterized kernel allows three independent analytical mechanisms to be adjusted simultaneously. Consequently, the associated Milne-type identity depends on four independent parameters, rather than one or two parameters as in previous studies. This enlarged parameter space produces integral estimates whose parameter sensitivity cannot be reproduced by any existing single-mechanism kernel. Therefore, the present contribution is not merely a unification of previous operators but the construction of a genuinely broader analytical kernel family supporting new classes of Milne–Mercer-type inequalities.

Therefore, the additional analytical complexity is justified because each parameter contributes an independent analytical degree of freedom, allowing the resulting Milne–Mercer-type inequalities to capture parameter-dependent analytical behavior that is inaccessible within previously available single-mechanism kernel frameworks.

2.3. Relation to Existing Methodology

The proofs in this paper rely on standard techniques, including integration by parts, the Jensen–Mercer inequality, Hölder’s inequality, and convexity arguments. The novelty of the present work therefore lies not in the proof methodology but in the construction of the proposed kernel. By combining proportional Caputo–hybrid weighting, conformable scaling, and exponential tempering within a unified kernel, the resulting integral identities and Milne–Mercer-type inequalities extend beyond those derived from existing single-component kernels.

3. Main Results

Lemma 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable on $[a, b]$, and suppose that $f'' \in L^1[a, b]$. Let $0 \leq a < m < n \leq b$, $0 < \alpha \leq 1$, $0 < \alpha\eta < 1$, $\beta \in (0, 1]$, and $\lambda \geq 0$. Then, the following identity is valid for the associated proportional Caputo–hybrid–tempered–conformable integral operators.

$$\begin{aligned}
 & \frac{(1 - \alpha\eta)^2 (n - m)^{2\beta - 2\alpha\beta\eta} e^{-\lambda(\frac{n-m}{2})}}{2^{2\beta - 2\alpha\beta\eta + 2}} \\
 & \times \left\{ \int_0^1 \left[\frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(t) \right] f'' \left(a + b - \left[\frac{(1+t)}{2} m + \frac{(1-t)}{2} n \right] \right) dt \right. \\
 & - \int_0^1 \left[\frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(1-t) \right] f'' \left(a + b - \left[\left(\frac{1-t}{2} \right) m + \left(\frac{1+t}{2} \right) n \right] \right) dt \Big\} \\
 & + \frac{(\alpha\eta)^2 (n - m) \beta^{2\alpha\eta - 1}}{8} \left\{ \int_0^1 \left[\frac{1}{3} + t \right] f' \left(a + b - \left[\left(\frac{1+t}{2} \right) m + \left(\frac{1-t}{2} \right) n \right] \right) dt \right. \\
 & - \int_0^1 \left[\frac{4}{3} - (1-t) \right] f' \left(a + b - \left[\left(\frac{1-t}{2} \right) m + \left(\frac{1+t}{2} \right) n \right] \right) dt \Big\} \\
 & = \frac{(1 - \alpha\eta)^2 (n - m)^{2\beta - 2\alpha\beta\eta - 1} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(1) e^{-\lambda(\frac{n-m}{2})}}{3} \\
 & \times \left[2f'(a + b - m) - f' \left(a + b - \frac{m+n}{2} \right) + 2f'(a + b - n) \right] \\
 & + \frac{(\alpha\eta)^2 \beta^{2\alpha\eta - 1}}{6} \left[2f(a + b - m) - f \left(a + b - \frac{m+n}{2} \right) + 2f(a + b - n) \right] \\
 & - \frac{\Gamma(1 - \alpha) (\beta^{2\alpha\eta - 1})}{2(n - m)} \left[{}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^+} D^{\alpha, \beta, \eta, \lambda} f(a + b - m) + {}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^-} D^{\alpha, \beta, \eta, \lambda} f(a + b - n) \right].
 \end{aligned}$$

where δ_λ is the incomplete gamma function for $\lambda \geq 0$,

$$\delta_\lambda^{\alpha, \beta, \eta}(t) = \int_0^t (1 - s^\beta)^{1 - 2\alpha\eta} \frac{1}{s^{1-\beta}} e^{\lambda s} ds.$$

and $\Gamma(\cdot)$ is the gamma function.

Proof. For convenience, set

$$A = \frac{(1 - \alpha\eta)^2 (n - m)^{2\beta - 2\alpha\beta\eta} e^{-\lambda(\frac{n-m}{2})}}{2^{2\beta - 2\alpha\beta\eta + 2}}$$

and

$$B = \frac{(\alpha\eta)^2 (n - m) \beta^{2\alpha\eta - 1}}{8}.$$

Step 1. By integration by parts, we get

$$\begin{aligned} I_1 &= \int_0^1 \left[\frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(t) \right] f'' \left(a + b - \left[\frac{(1+t)}{2} m + \frac{(1-t)}{2} n \right] \right) dt \\ &= \frac{8}{3(n-m)} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(1) f'(a+b-m) - \frac{2}{3(n-m)} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(1) f' \left(a + b - \frac{m+n}{2} \right) \\ &\quad - \frac{2}{(n-m)} \int_0^1 f' \left(a + b - \left[\frac{(1+t)}{2} m + \frac{(1-t)}{2} n \right] \right) \frac{(t^\beta - 1)^{1-2\alpha\eta}}{t^{1-\beta}} e^{\lambda(\frac{n-m}{2})t} dt. \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_0^1 \left[\frac{1}{3} + t \right] f' \left(a + b - \left[\left(\frac{1+t}{2} \right) m + \left(\frac{1-t}{2} \right) n \right] \right) dt \\ &= \frac{8f(a+b-m)}{3(n-m)} - \frac{2f(a+b-\frac{m+n}{2})}{3(n-m)} \\ &\quad - \frac{2}{n-m} \int_0^1 f \left(a + b - \left[\left(\frac{1+t}{2} \right) m + \left(\frac{1-t}{2} \right) n \right] \right) dt. \end{aligned}$$

Step 2. Using the substitution $x = a + b - \left[\frac{(1+t)}{2} m + \frac{(1-t)}{2} n \right]$, we have

$$\begin{aligned} I_1 &= \frac{8}{3(n-m)} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(1) f'(a+b-m) - \frac{2}{3(n-m)} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(1) f' \left(a + b - \frac{m+n}{2} \right) \\ &\quad - \frac{4}{(n-m)^2} \int_{a+b-\frac{m+n}{2}}^{a+b-m} f'(x) \frac{\left(1 - \left(\frac{2[x-(a+b-\frac{m+n}{2})]}{n-m} \right)^\beta \right)^{1-2\alpha\eta}}{\left(\frac{2[x-(a+b-\frac{m+n}{2})]}{n-m} \right)^{1-\beta}} e^{\lambda[x-(a+b-\frac{m+n}{2})]} dx \\ &= \frac{8}{3(n-m)} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(1) f'(a+b-m) - \frac{2}{3(n-m)} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(1) f' \left(a + b - \frac{m+n}{2} \right) \\ &\quad - \frac{4}{(n-m)^2} \left(\frac{2}{n-m} \right)^{2\beta-2\alpha\beta\eta-1} \beta^{2\alpha\eta-1} \int_{a+b-\frac{m+n}{2}}^{a+b-m} f'(x) e^{\lambda[x-(a+b-\frac{m+n}{2})]} \\ &\quad \times \frac{1}{[x-(a+b-\frac{m+n}{2})]^{1-\beta}} \\ &\quad \times \left(\frac{[(a+b-m)-(a+b-\frac{m+n}{2})]^\beta - [x-(a+b-\frac{m+n}{2})]^\beta}{\beta} \right)^{1-2\alpha\eta} dx \\ &= \frac{8}{3(n-m)} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(1) f'(a+b-m) - \frac{2}{3(n-m)} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta}(1) f' \left(a + b - \frac{m+n}{2} \right) \\ &\quad - \frac{\beta^{2\alpha\eta-1} 2^{2\beta-2\alpha\beta\eta+1}}{(n-m)^{2\beta-2\alpha\beta\eta+1}} \int_{a+b-\frac{m+n}{2}}^{a+b-m} f'(x) e^{\lambda[x-(a+b-\frac{m+n}{2})]} \\ &\quad \times \frac{1}{[x-(a+b-\frac{m+n}{2})]^{1-\beta}} \\ &\quad \times \left(\frac{[(a+b-m)-(a+b-\frac{m+n}{2})]^\beta - [x-(a+b-\frac{m+n}{2})]^\beta}{\beta} \right)^{1-2\alpha\eta} dx \end{aligned}$$

and

$$I_2 = \frac{8f(a+b-m)}{3(n-m)} - \frac{2f(a+b-\frac{m+n}{2})}{3(n-m)} - \frac{4}{(n-m)^2} \int_{a+b-\frac{m+n}{2}}^{a+b-m} f(x) dx.$$

Step 3. By multiplying the results by A and B , then we have

$$\begin{aligned} J_1 &= AI_1 + BI_2 \\ &= \frac{(1-\alpha\eta)^2(n-m)^{2\beta-2\alpha\beta\eta-1}\delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1)e^{-\lambda(\frac{n-m}{2})}}{3(2^{2\beta-2\alpha\beta\eta})} \left[2f'(a+b-n) - \frac{1}{2}f'\left(a+b-\frac{m+n}{2}\right) \right] \\ &\quad - \frac{(1-\alpha\eta)^2(\beta^{2\alpha\eta-1})e^{-\lambda(\frac{n-m}{2})}}{2(n-m)\delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1)} \int_{a+b-\frac{m+n}{2}}^{a+b-m} f'(x)e^{\lambda[x-(a+b-\frac{m+n}{2})]} \\ &\quad \times \frac{1}{[x-(a+b-\frac{m+n}{2})]^{1-\beta}} \\ &\quad \times \left(\frac{[(a+b-m)-(a+b-\frac{m+n}{2})]^\beta - [x-(a+b-\frac{m+n}{2})]^\beta}{\beta} \right)^{1-\alpha\eta} dx \\ &\quad + \frac{(\alpha\eta)^2\beta^{2\alpha\eta-1}}{6} \left[2f(a+b-m) - \frac{1}{2}f\left(a+b-\frac{m+n}{2}\right) \right] \\ &\quad - \frac{(\alpha\eta)^2\beta^{2\alpha\eta-1}}{2(n-m)} \int_{a+b-\frac{m+n}{2}}^{a+b-m} f(x) dx. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} I_3 &= \int_0^1 \left[\frac{4}{3}\delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1-t) \right] f''\left(a+b - \left[\left(\frac{1-t}{2}\right)m + \left(\frac{1+t}{2}\right)n\right]\right) dt \\ &= \frac{-8}{3(n-m)}\delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1)f'(a+b-n) + \frac{2}{3(n-m)}\delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1)f'\left(a+b-\frac{m+n}{2}\right) \\ &\quad + \frac{2}{(n-m)} \int_0^1 f'\left(a+b - \left[\left(\frac{1-t}{2}\right)m + \left(\frac{1+t}{2}\right)n\right]\right) \frac{((1-t)^\beta - 1)^{1-2\alpha\eta}}{(1-t)^{1-\beta}} e^{\lambda(\frac{n-m}{2})(1-t)} dt \end{aligned}$$

and

$$\begin{aligned} I_4 &= \int_0^1 \left[\frac{4}{3} - (1-t) \right] f'\left(a+b - \left[\left(\frac{1-t}{2}\right)m + \left(\frac{1+t}{2}\right)n\right]\right) dt \\ &= \frac{-8f(a+b-n)}{3(n-m)} + \frac{2f(a+b-\frac{m+n}{2})}{3(n-m)} \\ &\quad + \frac{2}{(n-m)} \int_0^1 f\left(a+b - \left[\left(\frac{1-t}{2}\right)m + \left(\frac{1+t}{2}\right)n\right]\right) dt. \end{aligned}$$

By changing variable $x = a+b - \left[\left(\frac{1-t}{2}\right)m + \left(\frac{1+t}{2}\right)n\right]$, we have

$$\begin{aligned}
I_3 &= \frac{-8}{3(n-m)} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta} (1) f'(a+b-n) + \frac{2}{3(n-m)} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta} (1) f' \left(a+b - \frac{m+n}{2} \right) \\
&\quad + \frac{4}{(n-m)^2} \int_{a+b-n}^{a+b-\frac{m+n}{2}} f'(x) \frac{\left(1 - \left(\frac{2[(a+b-\frac{m+n}{2})-x]}{n-m} \right)^\beta \right)^{1-2\alpha\eta}}{\left(\frac{2[(a+b-\frac{m+n}{2})-x]}{n-m} \right)^{1-\beta}} e^{\lambda[(a+b-\frac{m+n}{2})-x]} dx \\
&= \frac{-8}{3(n-m)} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta} (1) f'(a+b-n) + \frac{2}{3(n-m)} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta} (1) f' \left(a+b - \frac{m+n}{2} \right) \\
&\quad + \frac{\beta^{2\alpha\eta-1} 2^{2\beta-2\alpha\beta\eta+1}}{(n-m)^{2\beta-2\alpha\beta\eta+1}} \int_{a+b-n}^{a+b-\frac{m+n}{2}} f'(x) e^{\lambda[(a+b-\frac{m+n}{2})-x]} \\
&\quad \times \frac{1}{[(a+b-\frac{m+n}{2})-x]^{1-\beta}} \\
&\quad \times \left(\frac{[(a+b-\frac{m+n}{2})-(a+b-n)]^\beta - [(a+b-\frac{m+n}{2})-x]^\beta}{\beta} \right)^{1-2\alpha\eta} dx
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= \frac{-8f(a+b-n)}{3(n-m)} + \frac{2f(a+b-\frac{m+n}{2})}{3(n-m)} \\
&\quad + \frac{4}{(n-m)^2} \int_{a+b-n}^{a+b-\frac{m+n}{2}} f(x) dx.
\end{aligned}$$

By multiplying the results by A and B , then we have

$$\begin{aligned}
J_2 &= AI_3 + BI_4 \\
&= \frac{(1-\alpha\eta)^2 (n-m)^{2\beta-2\alpha\beta\eta-1} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta} (1) e^{-\lambda(\frac{n-m}{2})}}{3(2^{2\beta-2\alpha\beta\eta})} \left[-2f'(a+b-n) + \frac{1}{2} f' \left(a+b - \frac{m+n}{2} \right) \right] \\
&\quad + \frac{(1-\alpha\eta)(\beta^{2\alpha\eta-1})}{2(n-m)} \int_{a+b-n}^{a+b-\frac{m+n}{2}} f'(x) e^{\lambda[(a+b-\frac{m+n}{2})-x]} \\
&\quad \times \frac{1}{[(a+b-\frac{m+n}{2})-x]^{1-\beta}} \\
&\quad \times \left(\frac{[(a+b-\frac{m+n}{2})-(a+b-n)]^\beta - [(a+b-\frac{m+n}{2})-x]^\beta}{\beta} \right)^{1-2\alpha\eta} dx \\
&\quad + \frac{(\alpha\eta)^2 \beta^{2\alpha\eta-1}}{6} \left[-2f(a+b-n) + \frac{1}{2} f \left(a+b - \frac{m+n}{2} \right) \right] \\
&\quad + \frac{(\alpha\eta)^2 \beta^{2\alpha\eta-1}}{2(n-m)} \int_{a+b-n}^{a+b-\frac{m+n}{2}} f(x) dx.
\end{aligned}$$

Subtracting J_2 from J_1 , and using the definitions of the left-sided and right-sided PCHTC integral operators, we obtain

$$\begin{aligned}
& J_1 - J_2 \\
&= \frac{(1 - \alpha\eta)^2(n - m)^{2\beta - 2\alpha\beta\eta - 1} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta} (1)e^{-\lambda(\frac{n-m}{2})}}{3} \left[2f'(a + b - m) - f'\left(a + b - \frac{m+n}{2}\right) + 2f'(a + b - n) \right] \\
&+ \frac{(\alpha\eta)^2 \beta^{2\alpha\eta - 1}}{6} \left[2f(a + b - m) - f\left(a + b - \frac{m+n}{2}\right) + 2f(a + b - n) \right] \\
&- \frac{(\beta^{2\alpha\eta - 1})}{2(n - m)} \int_{a+b-\frac{m+n}{2}}^{a+b-m} (1 - \alpha\eta)^2 \frac{1}{[x - (a + b - \frac{m+n}{2})]^{1-\beta}} \\
&\times \left(\frac{[(a + b - m) - (a + b - \frac{m+n}{2})]^\beta - [x - (a + b - \frac{m+n}{2})]^\beta}{\beta} \right)^{1-\alpha\eta} f'(x) e^{-\lambda[(a+b-m)-x]} dx \\
&- \frac{\beta^{2\alpha\eta - 1}}{2(n - m)} \int_{a+b-\frac{m+n}{2}}^{a+b-m} (\alpha\eta)^2 f(x) dx \\
&- \frac{(\beta^{2\alpha\eta - 1})}{2(n - m)} \int_{a+b-\frac{m+n}{2}}^{a+b-n} (1 - \alpha\eta)^2 \frac{1}{[(a + b - \frac{m+n}{2}) - x]^{1-\beta}} \\
&\times \left(\frac{[(a + b - \frac{m+n}{2}) - (a + b - n)]^\beta - [(a + b - \frac{m+n}{2}) - x]^\beta}{\beta} \right)^{1-\alpha\eta} f'(x) e^{\lambda[x - (a+b-n)]} dx \\
&- \frac{\beta^{2\alpha\eta - 1}}{2(n - m)} \int_{a+b-n}^{a+b-\frac{m+n}{2}} (\alpha\eta)^2 f(x) dx \\
&= \frac{(1 - \alpha\eta)^2(n - m)^{2\beta - 2\alpha\beta\eta - 1} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta} (1)e^{-\lambda(\frac{n-m}{2})}}{3} \left[2f'(a + b - m) - f'\left(a + b - \frac{m+n}{2}\right) + 2f'(a + b - n) \right] \\
&+ \frac{(\alpha\eta)^2 \beta^{2\alpha\eta - 1}}{6} \left[2f(a + b - m) - f\left(a + b - \frac{m+n}{2}\right) + 2f(a + b - n) \right] \\
&- \frac{\Gamma(1 - \alpha)(\beta^{2\alpha\eta - 1})}{2(n - m)} \left[{}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^+} D^{\alpha, \beta, \eta, \lambda} f(a + b - m) + {}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^-} D^{\alpha, \beta, \eta, \lambda} f(a + b - n) \right].
\end{aligned}$$

We now establish the first Milne–Mercer-type inequality associated with the proposed generalized kernel under the convexity assumption of $|f'|$. Compared with previous Milne-type inequalities, the estimate explicitly reflects the combined influence of proportional Caputo–hybrid weighting, conformable scaling, and exponential tempering. $\square$

Theorem 3. Let $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable on $[a, b]$ such that $f, f' \in L^1[a, b]$, with $0 \leq a < m < n < b$, $\alpha \in (0, 1]$, $\eta > 0$, $0 < \alpha\eta < 1$, $\beta \in (0, 1]$, and $\lambda \geq 0$. If $|f'|$ and $|f''|$ are convex, then the following Milne-type inequality for proportional Caputo–hybrid–tempered–conformable fractional integral holds:

$$\begin{aligned}
& \left| \frac{(1 - \alpha\eta)^2(n - m)^{2\beta - 2\alpha\beta\eta - 1} \delta_{\lambda(\frac{n-m}{2})}^{\alpha, \beta, \eta} (1)e^{-\lambda(\frac{n-m}{2})}}{3} \right. \\
& \times \left[2f'(a + b - m) - f'\left(a + b - \frac{m+n}{2}\right) + 2f'(a + b - n) \right] \\
& + \frac{(\alpha\eta)^2 \beta^{2\alpha\eta - 1}}{6} \left[2f(a + b - m) - f\left(a + b - \frac{m+n}{2}\right) + 2f(a + b - n) \right] \\
& \left. - \frac{\Gamma(1 - \alpha)(\beta^{2\alpha\eta - 1})}{2(n - m)} \left[{}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^+} D^{\alpha, \beta, \eta, \lambda} f(a + b - m) + {}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^-} D^{\alpha, \beta, \eta, \lambda} f(a + b - n) \right] \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(1-\alpha\eta)^2(n-m)^{2\beta-2\alpha\beta\eta}e^{-\lambda(\frac{n-m}{2})}}{2^{2\beta-2\alpha\beta\eta+2}} \\
&\quad \times \left\{ f''(a+b-m) \int_0^1 t \left| \frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(t) \right| dt \right. \\
&\quad + f''(a+b-n) \int_0^1 t \left| \frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1-t) \right| dt \\
&\quad + f''\left(a+b-\frac{m+n}{2}\right) \int_0^1 (1-t) \left[\left| \frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(t) \right| + \left| \frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1-t) \right| \right] dt \Big\} \\
&\quad + \frac{(\alpha\eta)^2\beta^{2\alpha\eta-1}}{4} \left[\frac{f'(a+b-m) + f'(a+b-n)}{2} + \frac{2}{3} f'\left(a+b-\frac{m+n}{2}\right) \right]
\end{aligned}$$

where δ_λ is the incomplete gamma function for $\lambda \geq 0$:

$$\delta_\lambda^{\alpha,\beta,\eta}(t) = \int_0^t (1-s^\beta)^{1-2\alpha\eta} \frac{1}{s^{1-\beta}} e^{\lambda s} ds.$$

Proof. By Lemma 1, the Jensen–Mercer inequality, and the convexity of $|f'|$ and $|f''|$ on $[a, b]$, we have

$$\begin{aligned}
&\left| \frac{(1-\alpha\eta)^2(n-m)^{2\beta-2\alpha\beta\eta-1} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) e^{-\lambda(\frac{n-m}{2})}}{3} \right. \\
&\quad \times \left[2f'(a+b-m) - f'\left(a+b-\frac{m+n}{2}\right) + 2f'(a+b-n) \right] \\
&\quad + \frac{(\alpha\eta)^2\beta^{2\alpha\eta-1}}{6} \left[2f(a+b-m) - f\left(a+b-\frac{m+n}{2}\right) + 2f(a+b-n) \right] \\
&\quad \left. - \frac{\Gamma(1-\alpha)(\beta^{2\alpha\eta-1})}{2(n-m)} \left[{}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^+} D^{\alpha,\beta,\eta,\lambda} f(a+b-m) + {}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^-} D^{\alpha,\beta,\eta,\lambda} f(a+b-n) \right] \right| \\
&\leq \frac{(1-\alpha\eta)^2(n-m)^{2\beta-2\alpha\beta\eta}e^{-\lambda(\frac{n-m}{2})}}{2^{2\beta-2\alpha\beta\eta+2}} \\
&\quad \times \left\{ \int_0^1 \left| \frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(t) \right| \left| f''\left(a+b-\left[\frac{(1+t)}{2}m + \frac{(1-t)}{2}n\right]\right) \right| dt \right. \\
&\quad + \int_0^1 \left| \frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1-t) \right| \left| f''\left(a+b-\left[\left(\frac{1-t}{2}\right)m + \left(\frac{1+t}{2}\right)n\right]\right) \right| dt \Big\} \\
&\quad + \frac{(\alpha\eta)^2(n-m)\beta^{2\alpha\eta-1}}{8} \left\{ \int_0^1 \left| \frac{1}{3} + t \right| \left| f'\left(a+b-\left[\left(\frac{1+t}{2}\right)m + \left(\frac{1-t}{2}\right)n\right]\right) \right| dt \right. \\
&\quad \left. - \int_0^1 \left| \frac{1}{3} + t \right| \left| f'\left(a+b-\left[\left(\frac{1-t}{2}\right)m + \left(\frac{1+t}{2}\right)n\right]\right) \right| dt \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(1-\alpha\eta)^2(n-m)^{2\beta-2\alpha\beta\eta}e^{-\lambda(\frac{n-m}{2})}}{2^{2\beta-2\alpha\beta\eta+2}} \\
&\quad \times \left\{ \int_0^1 \left| \frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(t) \right| \left| t f''(a+b-m) + (1-t) f''\left(a+b-\frac{m+n}{2}\right) \right| dt \right. \\
&\quad + \int_0^1 \left| \frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1-t) \right| \left| t f''(a+b-n) + (1-t) f''\left(a+b-\frac{m+n}{2}\right) \right| dt \Big\} \\
&\quad + \frac{(\alpha\eta)^2(n-m)\beta^{2\alpha\eta-1}}{8} \left\{ \int_0^1 \left| \frac{1}{3} + t \right| \left| t f'(a+b-m) + (1-t) f'\left(a+b-\frac{m+n}{2}\right) \right| dt \right. \\
&\quad + \int_0^1 \left| \frac{1}{3} + t \right| \left| t f'(a+b-m) + (1-t) f'\left(a+b-\frac{m+n}{2}\right) \right| dt \Big\} \\
&= \frac{(1-\alpha\eta)^2(n-m)^{2\beta-2\alpha\beta\eta}e^{-\lambda(\frac{n-m}{2})}}{2^{2\beta-2\alpha\beta\eta+2}} \\
&\quad \times \left\{ |f''(a+b-m)| \int_0^1 t \left| \frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(t) \right| dt \right. \\
&\quad + |f''(a+b-n)| \int_0^1 t \left| \frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1-t) \right| dt \\
&\quad + \left| f''\left(a+b-\frac{m+n}{2}\right) \right| \int_0^1 (1-t) \left[\left| \frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(t) \right| + \left| \frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1-t) \right| \right] dt \Big\} \\
&\quad + \frac{(\alpha\eta)^2(n-m)\beta^{2\alpha\eta-1}}{8} \\
&\quad \times \left\{ |f'(a+b-m)| \int_0^1 t \left(\frac{1}{3} + t \right) dt + |f'(a+b-n)| \int_0^1 t \left(\frac{1}{3} + t \right) dt \right. \\
&\quad + \left| f'\left(a+b-\frac{m+n}{2}\right) \right| \int_0^1 2(1-t) \left(\frac{1}{3} + t \right) dt \Big\} \\
&= \frac{(1-\alpha\eta)^2(n-m)^{2\beta-2\alpha\beta\eta}e^{-\lambda(\frac{n-m}{2})}}{2^{2\beta-2\alpha\beta\eta+2}} \\
&\quad \times \left\{ |f''(a+b-m)| \int_0^1 t \left| \frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(t) \right| dt \right. \\
&\quad + |f''(a+b-n)| \int_0^1 t \left| \frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1-t) \right| dt \\
&\quad + \left| f''\left(a+b-\frac{m+n}{2}\right) \right| \int_0^1 (1-t) \left[\left| \frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(t) \right| + \left| \frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1-t) \right| \right] dt \Big\} \\
&\quad + \frac{(\alpha\eta)^2(n-m)\beta^{2\alpha\eta-1}}{8} \left[\frac{|f'(a+b-m)| + |f'(a+b-n)|}{2} + \frac{2}{3} \left| f'\left(a+b-\frac{m+n}{2}\right) \right| \right].
\end{aligned}$$

□

Remark 4. Theorem 3 cannot be obtained directly from the proportional Caputo–hybrid framework, the tempered framework, or the conformable framework separately, because the kernel appearing in Lemma 1 simultaneously depends on the four independent parameters $(\alpha, \beta, \eta, \lambda)$. Consequently, the resulting Milne–Mercer-type estimate represents a genuinely new inequality rather than a direct specialization of previously known results.

Remark 5. When the tempering parameter $\lambda = 0$, the associated operator reduces to the proportional Caputo–hybrid–conformable fractional integral.

$$\begin{aligned}
& \left| \frac{(1-\alpha\eta)^2(n-m)^{2\beta-2\alpha\beta\eta-1}\delta_0^{\alpha,\beta,\eta}(1)}{3} \left[2f'(a+b-m) - f'\left(a+b-\frac{m+n}{2}\right) + 2f'(a+b-n) \right] \right. \\
& + \frac{(\alpha\eta)^2\beta^{2\alpha\eta-1}}{6} \left[2f(a+b-m) - f\left(a+b-\frac{m+n}{2}\right) + 2f(a+b-n) \right] \\
& \left. - \frac{\Gamma(1-\alpha)(\beta^{2\alpha\eta-1})}{2(n-m)} \left[{}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^+} D^{\alpha,\beta,\eta} f(a+b-m) + {}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^-} D^{\alpha,\beta,\eta} f(a+b-n) \right] \right| \\
& \leq \frac{(1-\alpha\eta)^2(n-m)^{2\beta-2\alpha\beta\eta}}{2^{2\beta-2\alpha\beta\eta+2}} \\
& \times \left\{ |f''(a+b-m)| \int_0^1 t \left| \frac{1}{3} \delta_0^{\alpha,\beta,\eta}(1) + \delta_0^{\alpha,\beta,\eta}(t) \right| dt \right. \\
& + |f''(a+b-n)| \int_0^1 t \left| \frac{4}{3} \delta_0^{\alpha,\beta,\eta}(1) - \delta_0^{\alpha,\beta,\eta}(1-t) \right| dt \\
& + \left| f''\left(a+b-\frac{m+n}{2}\right) \right| \int_0^1 (1-t) \left[\left| \frac{1}{3} \delta_0^{\alpha,\beta,\eta}(1) + \delta_0^{\alpha,\beta,\eta}(t) \right| + \left| \frac{4}{3} \delta_0^{\alpha,\beta,\eta}(1) - \delta_0^{\alpha,\beta,\eta}(1-t) \right| \right] dt \Big\} \\
& + \frac{(\alpha\eta)^2(n-m)\beta^{2\alpha\eta-1}}{8} \left[\frac{|f'(a+b-m)| + |f'(a+b-n)|}{2} + \frac{2}{3} \left| f'\left(a+b-\frac{m+n}{2}\right) \right| \right].
\end{aligned}$$

where

$$\delta_0^{\alpha,\beta,\eta}(t) = \int_0^t (1-s^\beta)^{1-2\alpha\eta} \frac{1}{s^{1-\beta}} ds.$$

Remark 6. When the conformable parameter $\beta = 1$, the associated operator reduces to the proportional Caputo–hybrid–tempered fractional integral.

$$\begin{aligned}
& \left| \frac{(1-\alpha\eta)^2(n-m)^{1-2\alpha\eta}\delta_{\lambda(\frac{n-m}{2})}^{\alpha,\eta}(1)e^{-\lambda(\frac{n-m}{2})}}{3} \left[2f'(a+b-m) - f'\left(a+b-\frac{m+n}{2}\right) + 2f'(a+b-n) \right] \right. \\
& + \frac{(\alpha\eta)^2}{6} \left[2f(a+b-m) - f\left(a+b-\frac{m+n}{2}\right) + 2f(a+b-n) \right] \\
& \left. - \frac{\Gamma(1-\alpha)}{2(n-m)} \left[{}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^+} D^{\alpha,\eta} f(a+b-m) + {}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^-} D^{\alpha,\eta} f(a+b-n) \right] \right| \\
& \leq \frac{(1-\alpha\eta)^2(n-m)^{2-2\alpha\eta}e^{-\lambda(\frac{n-m}{2})}}{2^{2\beta-2\alpha\beta\eta+2}} \\
& \times \left\{ |f''(a+b-m)| \int_0^1 t \left| \frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\eta}(t) \right| dt \right. \\
& + |f''(a+b-n)| \int_0^1 t \left| \frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\eta}(1-t) \right| dt \\
& + \left| f''\left(a+b-\frac{m+n}{2}\right) \right| \int_0^1 (1-t) \left[\left| \frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\eta}(t) \right| + \left| \frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\eta}(1-t) \right| \right] dt \Big\} \\
& + \frac{(\alpha\eta)^2(n-m)}{8} \left[\frac{|f'(a+b-m)| + |f'(a+b-n)|}{2} + \frac{2}{3} \left| f'\left(a+b-\frac{m+n}{2}\right) \right| \right]
\end{aligned}$$

where

$$\delta_{\lambda}^{\alpha,\eta}(t) = \int_0^t (1-s)^{1-2\alpha\eta} e^{\lambda s} ds.$$

Remark 7. When the tempering parameter $\lambda = 0$ and the conformable parameter $\beta = \eta = 1$, the associated operator reduces to the proportional Caputo–hybrid fractional integral.

$$\begin{aligned} & \left| \frac{(1-\alpha)^2(n-m)^{1-2\alpha}\delta_0^\alpha(1)}{3} \right. \\ & \times \left[2f'(a+b-m) - f'\left(a+b-\frac{m+n}{2}\right) + 2f'(a+b-n) \right] \\ & + \frac{\alpha^2}{6} \left[2f(a+b-m) - f\left(a+b-\frac{m+n}{2}\right) + 2f(a+b-n) \right] \\ & \left. - \frac{\Gamma(1-\alpha)}{2(n-m)} \left[{}^{PCH}_{(a+b-\frac{m+n}{2})^+} D^\alpha f(a+b-m) + {}^{PCH}_{(a+b-\frac{m+n}{2})^-} D^\alpha f(a+b-n) \right] \right| \\ & \leq \frac{(1-\alpha)^2(n-m)^{2-2\alpha}}{2^{4-2\alpha}} \\ & \times \left\{ |f''(a+b-m)| \int_0^1 t \left| \frac{1}{3}\delta_0^\alpha(1) + \delta_0^\alpha(t) \right| dt \right. \\ & + |f''(a+b-n)| \int_0^1 t \left| \frac{4}{3}\delta_0^\alpha(1) - \delta_0^\alpha(1-t) \right| dt \\ & + \left| f''\left(a+b-\frac{m+n}{2}\right) \right| \int_0^1 (1-t) \left[\left| \frac{1}{3}\delta_0^\alpha(1) + \delta_0^\alpha(t) \right| + \left| \frac{4}{3}\delta_0^\alpha(1) - \delta_0^\alpha(1-t) \right| \right] dt \Big\} \\ & + \frac{\alpha^2(n-m)}{8} \left[\frac{|f'(a+b-m)| + |f'(a+b-n)|}{2} + \frac{2}{3} \left| f'\left(a+b-\frac{m+n}{2}\right) \right| \right]. \end{aligned}$$

where

$$\delta_0^\alpha(t) = \int_0^t (1-s)^{1-2\alpha} ds = \frac{1}{2-2\alpha} [1 - (1-t)^{2-2\alpha}].$$

The following inequality extends Theorem 3 by employing Hölder's inequality, leading to sharper estimates for functions satisfying the convexity of $|f'|^q$.

Theorem 4. Assume that the conditions of Lemma 1 hold. If $|f'|^q$ and $|f''|^q$ are convex with $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following Milne-type inequality for the proportional Caputo–hybrid–tempered–conformable fractional integral holds.

Proof. By Lemma 1, and using Hölder's inequality together with the Jensen–Mercer inequality and the convexity of $|f'|^q$ and $|f''|^q$, we have

$$\begin{aligned} & \left| \frac{(1-\alpha\eta)^2(n-m)^{2\beta-2\alpha\beta\eta-1}\delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1)e^{-\lambda(\frac{n-m}{2})}}{3} \right. \\ & \times \left[2f'(a+b-m) - f'\left(a+b-\frac{m+n}{2}\right) + 2f'(a+b-n) \right] \\ & + \frac{(\alpha\eta)^2\beta^{2\alpha\eta-1}}{6} \left[2f(a+b-m) - f\left(a+b-\frac{m+n}{2}\right) + 2f(a+b-n) \right] \\ & \left. - \frac{\Gamma(1-\alpha)(\beta^{2\alpha\eta-1})}{2(n-m)} \left[{}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^+} D^{\alpha,\beta,\eta} f(a+b-m) + {}^{PCH-Conf}_{(a+b-\frac{m+n}{2})^-} D^{\alpha,\beta,\eta} f(a+b-n) \right] \right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(1-\alpha\eta)^2(n-m)^{2\beta-2\alpha\beta\eta}e^{-\lambda(\frac{n-m}{2})}}{2^{2\beta-2\alpha\beta\eta+2}} \\
&\quad \times \left\{ \left(\int_0^1 \left| \frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(t) \right|^p dt \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(\int_0^1 \left| f'' \left(a+b - \left[\frac{(1+t)}{2}m + \frac{(1-t)}{2}n \right] \right) \right|^q dt \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left| \frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1-t) \right|^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^1 \left| f'' \left(a+b - \left[\left(\frac{1-t}{2} \right)m + \left(\frac{1+t}{2} \right)n \right] \right) \right|^q dt \right)^{\frac{1}{q}} \Bigg\} \\
&\quad + \frac{(\alpha\eta)^2(n-m)\beta^{2\alpha\eta-1}}{8} \left\{ \int_0^1 \left| \frac{1}{3} + t \right| dt \right. \\
&\quad \times \left(\int_0^1 \left| f' \left(a+b - \left[\left(\frac{1+t}{2} \right)m + \left(\frac{1-t}{2} \right)n \right] \right) \right| dt \right) \\
&\quad + \int_0^1 \left| \frac{1}{3} + t \right| dt \\
&\quad \left. - \int_0^1 \left| f' \left(a+b - \left[\left(\frac{1-t}{2} \right)m + \left(\frac{1+t}{2} \right)n \right] \right) \right| dt \right\} \\
\\
&\leq \frac{(1-\alpha\eta)^2(n-m)^{2\beta-2\alpha\beta\eta}e^{-\lambda(\frac{n-m}{2})}}{2^{2\beta-2\alpha\beta\eta+2}} \\
&\quad \times \left\{ \left(\int_0^1 \left| \frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(t) \right|^p dt \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(\int_0^1 t \left| f''(a+b-m) \right|^q + (1-t) \left| f'' \left(a+b - \frac{m+n}{2} \right) \right|^q dt \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left| \frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1-t) \right|^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^1 t \left| f''(a+b-n) \right|^q + (1-t) \left| f'' \left(a+b - \frac{m+n}{2} \right) \right|^q dt \right)^{\frac{1}{q}} \Bigg\} \\
&\quad + \frac{(\alpha\eta)^2(n-m)\beta^{2\alpha\eta-1}}{8} \left\{ \left(\int_0^1 \left| \frac{1}{3} + t \right|^p dt \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(\int_0^1 t \left| f'(a+b-m) \right|^q + (1-t) \left| f' \left(a+b - \frac{m+n}{2} \right) \right|^q dt \right) \\
&\quad + \left(\int_0^1 \left| \frac{1}{3} + t \right|^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left. \int_0^1 t \left| f'(a+b-n) \right|^q + (1-t) \left| f' \left(a+b - \frac{m+n}{2} \right) \right|^q dt \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(1-\alpha\eta)^2(n-m)^{2\beta-2\alpha\beta\eta}e^{-\lambda(\frac{n-m}{2})}}{2^{2\beta-2\alpha\beta\eta+2}} \\
&\times \left\{ \frac{[f''(a+b-m) + |f''(a+b-\frac{m+n}{2})|]}{2} \left(\int_0^1 \left| \frac{1}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) + \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(t) \right|^p dt \right)^{\frac{1}{p}} \right. \\
&+ \frac{[f''(a+b-n) + |f''(a+b-\frac{m+n}{2})|]}{2} \left(\int_0^1 \left| \frac{4}{3} \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1) - \delta_{\lambda(\frac{n-m}{2})}^{\alpha,\beta,\eta}(1-t) \right|^p dt \right)^{\frac{1}{p}} \Bigg\} \\
&+ \frac{(\alpha\eta)^2(n-m)\beta^{2\alpha\eta-1}}{8} \left\{ \left(\frac{\left(\frac{4}{3}\right)^{p+1} - \left(\frac{1}{3}\right)^{p+1}}{p+1} \right)^{\frac{1}{p}} \right. \\
&\times \frac{[|f'(a+b-m)| + 2|f'(a+b-\frac{m+n}{2})| + |f'(a+b-n)|]}{2} \Bigg\}.
\end{aligned}$$

□

Remark 8. Theorem 3 provides a direct convexity-based estimate involving weighted L^1 -type kernel integrals. In contrast, Theorem 4 applies Hölder's inequality and yields an $L^p - L^q$ -type estimate. Therefore, neither estimate is universally sharper for all admissible functions and parameters. Theorem 3 is more directly computable when pointwise bounds for $|f'|$ and $|f''|$ are available, whereas Theorem 4 is more suitable when q -power convexity or L^q -type control is known.

4. Numerical Examples and Parameter Sensitivity for Theorem 3

In this section, we present four numerical examples to verify Theorem 3 and to examine the influence of the parameters α , β , η , and λ on the upper bound of the proposed Milne–Mercer-type inequality. Throughout this section, unless otherwise stated, we take

$$a = 0, \quad b = 1, \quad m = 0.2, \quad n = 0.8, \quad \alpha = 0.5, \quad \beta = 0.8, \quad \eta = 0.8, \quad \lambda = 0.5.$$

The term LHS denotes the absolute deviation term appearing on the left-hand side of Theorem 3, while RHS denotes the corresponding upper bound. The gap is defined by

$$\text{Gap} = \text{RHS} - \text{LHS}.$$

Thus, the positivity of the gap numerically confirms the validity of Theorem 3 for the selected data.

This example illustrates the applicability of Theorem 3 to a polynomial function.

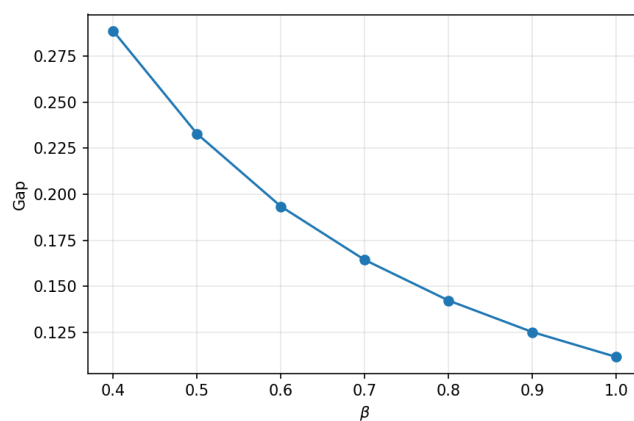
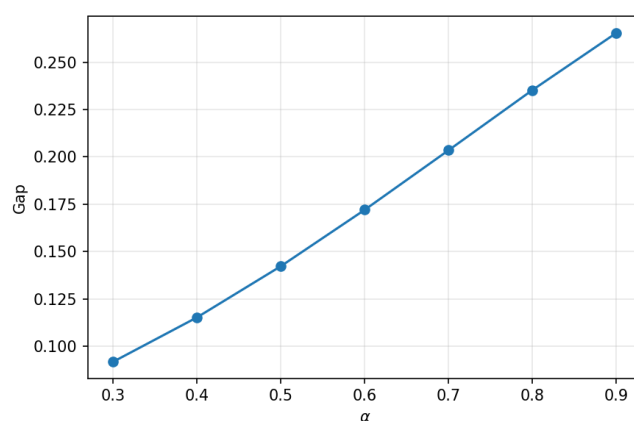
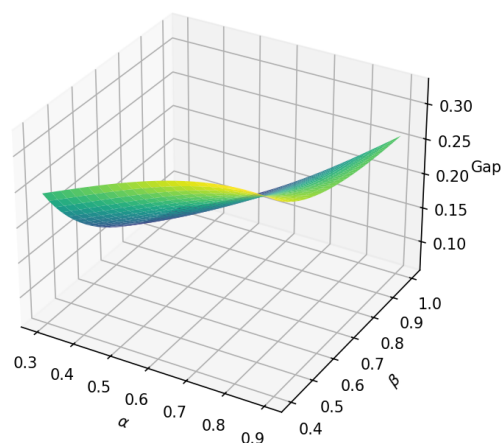
Example 1 (Verification and sensitivity for $f(x) = x^2$). Let $f(x) = x^2$ on $[0, 1]$. Since $|f'|$ and $|f''|$ are convex on $[0, 1]$, the assumptions of Theorem 3 are satisfied. The following Tables 2 and 3 and Figures 1–3 verify the inequality and describe how the gap changes when the parameters vary.

Table 2. Verification of Theorem 3 for $f(x) = x^2$ under different values of β .

β	LHS	RHS	Gap	Relative Margin (%)
0.40	0.102969	0.391722	0.288753	73.71
0.50	0.061763	0.294624	0.232861	79.04
0.60	0.038807	0.232214	0.193407	83.29
0.70	0.025417	0.189768	0.164351	86.61
0.80	0.017369	0.159654	0.142286	89.12
0.90	0.012443	0.137597	0.125154	90.96
1.00	0.009405	0.121033	0.111628	92.23

Table 3. Verification of Theorem 3 for $f(x) = x^2$ under different values of α .

α	LHS	RHS	Gap	Relative Margin (%)
0.30	0.018091	0.109909	0.091818	83.54
0.40	0.017936	0.133191	0.115255	86.53
0.50	0.017369	0.159654	0.142286	89.12
0.60	0.016262	0.188319	0.172057	91.36
0.70	0.014581	0.218103	0.203522	93.31
0.80	0.012477	0.247780	0.235303	94.96
0.90	0.010465	0.275903	0.265438	96.21

**Figure 1.** Gap variation with respect to β for $f(x) = x^2$.**Figure 2.** Gap variation with respect to α for $f(x) = x^2$.**Figure 3.** Three-dimensional gap surface with respect to α and β for $f(x) = x^2$; the color scale represents the magnitude of the gap.

Discussion for Example 1. The numerical results show that all gaps are positive. Hence, the computed LHS is bounded above by the RHS of Theorem 3 for every tested value of α and β . The gap decreases monotonically as the conformable parameter β increases, which means that larger values of β give a sharper estimate in this polynomial case. When α increases, the RHS and the gap increase, while the LHS changes only moderately. The surface plot confirms that the smallest gaps occur near large β and small α .

The following example examines the performance of the proposed inequality for an exponential function.

Example 2 (Verification and sensitivity for $f(x) = e^x$). Let $f(x) = e^x$ on $[0, 1]$. Since $|f|$, $|f'|$, and $|f''|$ are convex on $[0, 1]$, Theorem 3 is applicable. The following Tables 4 and 5 and Figures 4–6 verify the inequality and examine the effects of β and λ on the gap.

Table 4. Verification of Theorem 3 for $f(x) = e^x$ under different values of β .

β	LHS	RHS	Gap	Relative Margin (%)
0.40	0.130218	0.410980	0.280762	68.32
0.50	0.083033	0.320866	0.237833	74.12
0.60	0.055542	0.262946	0.207404	78.88
0.70	0.038652	0.223502	0.184851	82.71
0.80	0.027863	0.195445	0.167582	85.74
0.90	0.020768	0.174809	0.154040	88.12
1.00	0.016000	0.159224	0.143224	89.95

Table 5. Verification of Theorem 3 for $f(x) = e^x$ under different values of λ .

λ	LHS	RHS	Gap	Relative Margin (%)
0.00	0.031060	0.202874	0.171813	84.69
0.25	0.029410	0.199056	0.169646	85.23
0.50	0.027863	0.195445	0.167582	85.74
0.75	0.026411	0.192028	0.165616	86.25
1.00	0.025050	0.188793	0.163743	86.73
1.25	0.023774	0.185731	0.161957	87.20
1.50	0.022576	0.182831	0.160254	87.65

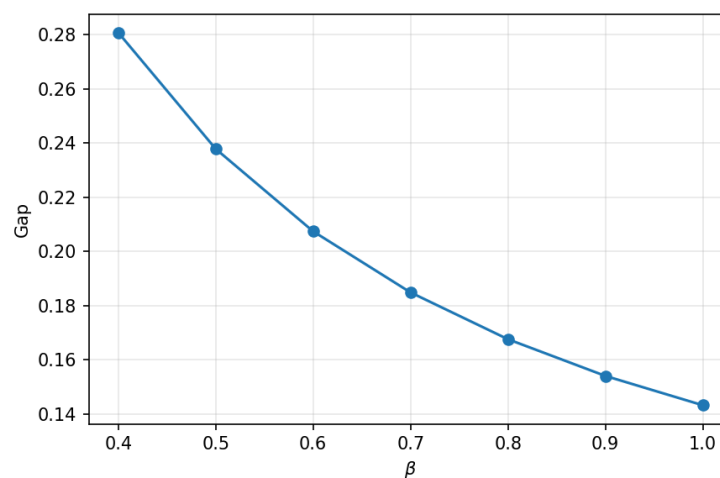


Figure 4. Gap variation with respect to β for $f(x) = e^x$.

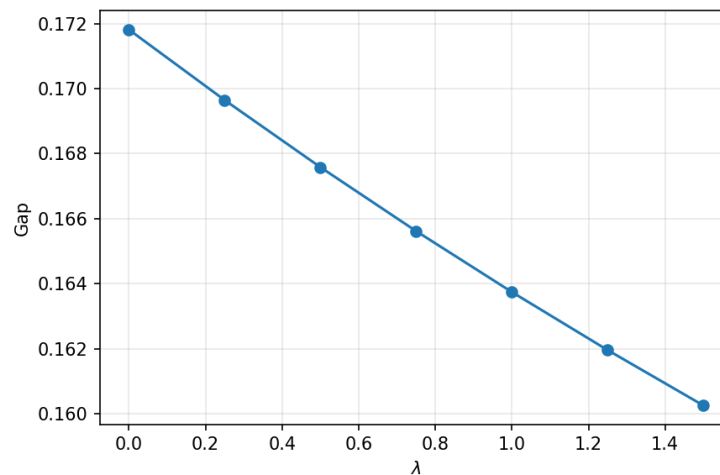


Figure 5. Gap variation with respect to λ for $f(x) = e^x$.

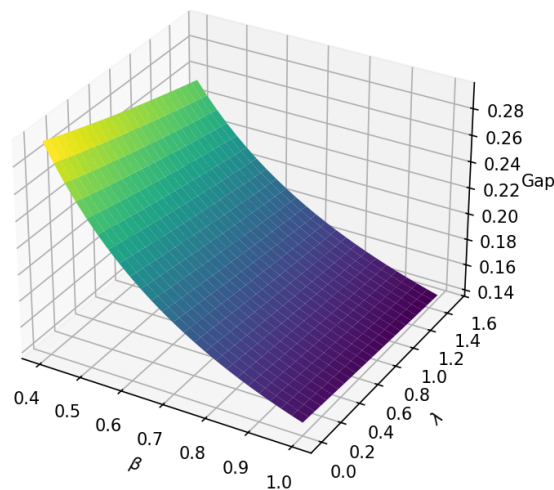


Figure 6. Three-dimensional gap surface with respect to β and λ for $f(x) = e^x$; the color scale represents the magnitude of the gap.

Discussion for Example 2. All reported gaps are again positive, so Theorem 3 is numerically verified for the exponential test function. Increasing β reduces the LHS, RHS, and gap, indicating that β has a substantial effect on the sharpness of the bound. Increasing λ also decreases the gap, but the change is milder. This agrees with the role of the tempering factor, which attenuates the kernel without changing the structure as strongly as the conformable parameter.

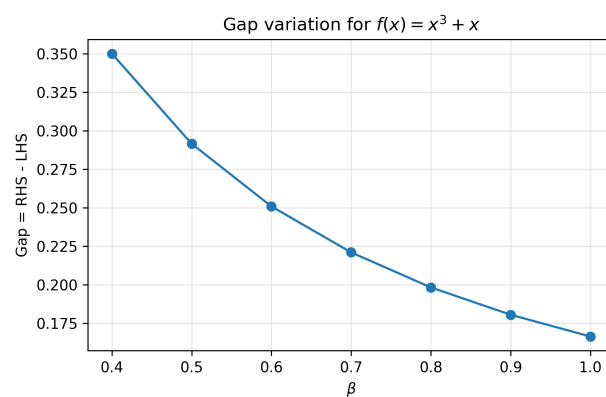
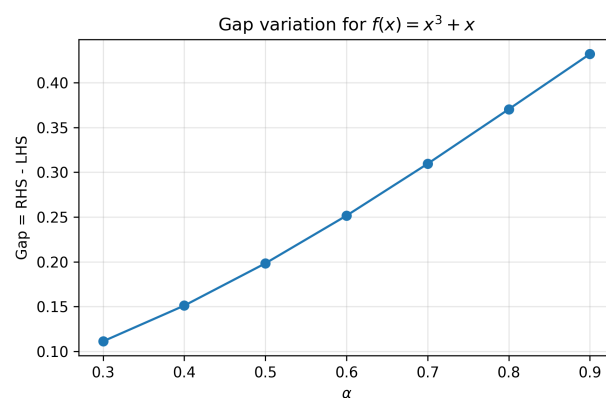
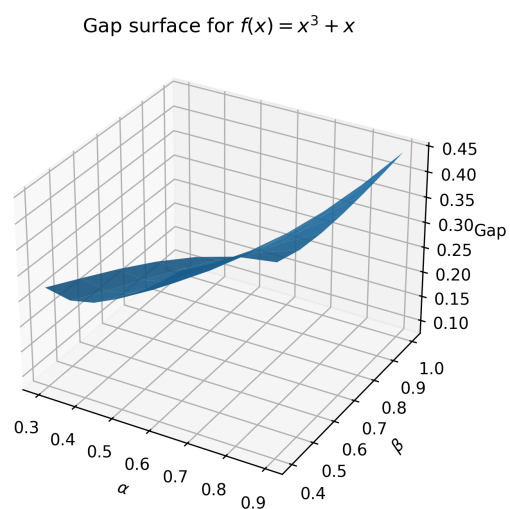
Example 3 (Verification and sensitivity for $f(x) = x^3 + x$). Let $f(x) = x^3 + x$ on $[0, 1]$. This example provides a higher-order polynomial verification of Theorem 3. The following Tables 6 and 7 and Figures 7–9 examine the effects of β and α on the gap under the same baseline setting.

Table 6. Verification of Theorem 3 for $f(x) = x^3 + x$ under different values of β .

β	LHS	RHS	Gap	Relative Margin (%)
0.40	0.315072	0.665219	0.350147	52.64
0.50	0.206274	0.497905	0.291631	58.57
0.60	0.141737	0.392671	0.250935	63.90
0.70	0.101068	0.322115	0.221048	68.62
0.80	0.074332	0.272631	0.198299	72.74
0.90	0.056196	0.236738	0.180542	76.26
1.00	0.043591	0.210009	0.166417	79.24

Table 7. Verification of Theorem 3 for $f(x) = x^3 + x$ under different values of α .

α	LHS	RHS	Gap	Relative Margin (%)
0.30	0.071538	0.182800	0.111262	60.87
0.40	0.073538	0.224661	0.151123	67.27
0.50	0.074332	0.272631	0.198299	72.74
0.60	0.073446	0.324997	0.251550	77.40
0.70	0.070465	0.379947	0.309482	81.45
0.80	0.065155	0.435552	0.370397	85.04
0.90	0.057712	0.489751	0.432039	88.22

**Figure 7.** Gap variation with respect to β for $f(x) = x^3 + x$.**Figure 8.** Gap variation with respect to α for $f(x) = x^3 + x$.**Figure 9.** Three-dimensional gap surface with respect to α and β for $f(x) = x^3 + x$.

Discussion for Example 3. The higher-order polynomial example gives positive gaps for all tested values. As β increases from 0.40 to 1.00, both the RHS and the gap decrease, so the bound becomes sharper. By contrast, increasing α enlarges the RHS and the gap even though the LHS remains comparatively stable. The surface plot therefore shows that the smallest gaps occur near larger β and smaller α .

Example 4 (Verification and sensitivity for $f(x) = \ln(1+x)$). Let $f(x) = \ln(1+x)$ on $[0, 1]$. This logarithmic test provides an additional non-polynomial benchmark for the numerical behavior of Theorem 3. The following Tables 8 and 9 and Figures 10–12 examine the effects of β and λ on the gap under the same baseline setting.

Table 8. Verification of Theorem 3 for $f(x) = \ln(1+x)$ under different values of β .

β	LHS	RHS	Gap	Relative Margin (%)
0.40	0.007298	0.124788	0.117490	94.15
0.50	0.002506	0.101853	0.099347	97.54
0.60	0.000298	0.086820	0.086523	99.66
0.70	0.000669	0.076329	0.075660	99.12
0.80	0.001011	0.068680	0.067669	98.53
0.90	0.001036	0.062918	0.061882	98.35
1.00	0.000904	0.058466	0.057562	98.45

Table 9. Verification of Theorem 3 for $f(x) = \ln(1+x)$ under different values of λ .

λ	LHS	RHS	Gap	Relative Margin (%)
0.00	0.000931	0.070581	0.069650	98.68
0.25	0.000974	0.069605	0.068630	98.60
0.50	0.001011	0.068680	0.067669	98.53
0.75	0.001041	0.067803	0.066761	98.46
1.00	0.001066	0.066971	0.065905	98.41
1.25	0.001086	0.066183	0.065097	98.36
1.50	0.001100	0.065435	0.064334	98.32

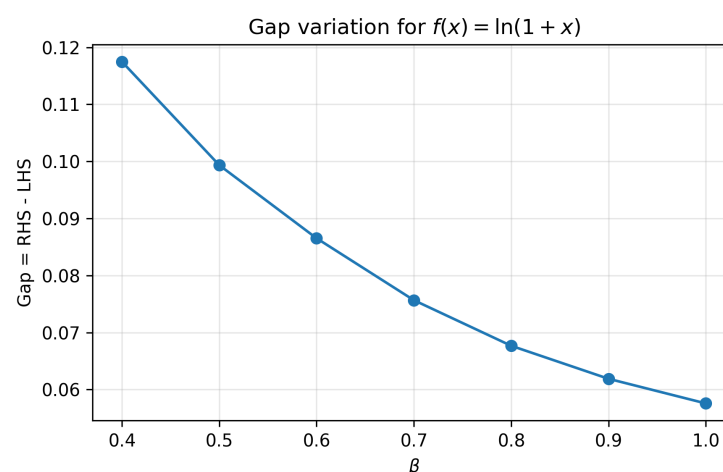


Figure 10. Gap variation with respect to β for $f(x) = \ln(1+x)$.

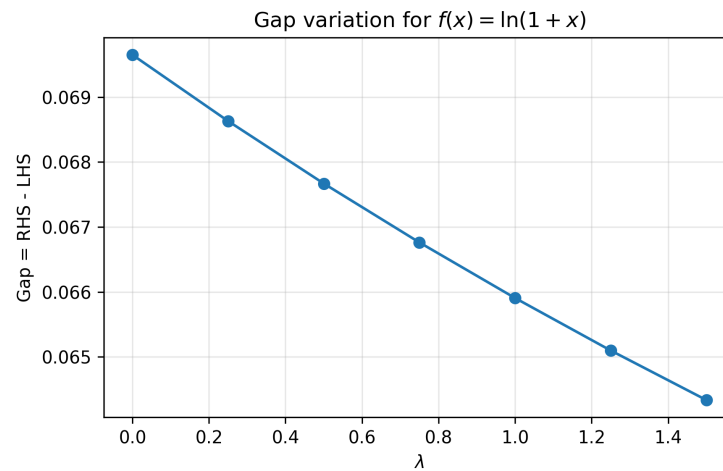


Figure 11. Gap variation with respect to λ for $f(x) = \ln(1 + x)$.

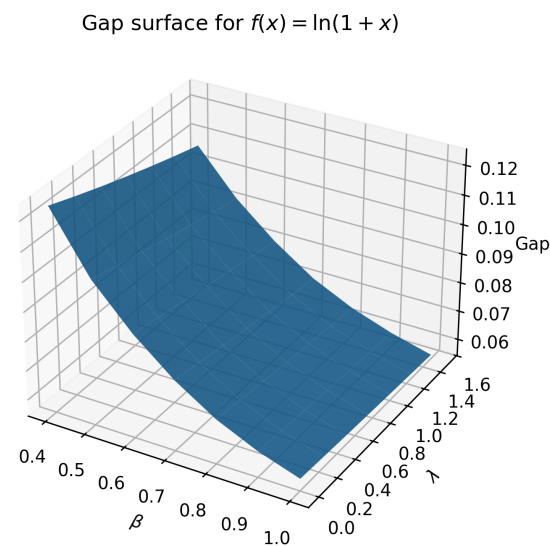


Figure 12. Three-dimensional gap surface with respect to β and λ for $f(x) = \ln(1 + x)$.

Discussion for Example 4. The logarithmic example also produces positive gaps throughout the tested parameter ranges. The values of LHS are much smaller than the corresponding RHS values, and the relative margin remains above 94%. Increasing β decreases the RHS and the gap. Increasing λ also gradually decreases the gap, but its effect is milder. The surface plot confirms that the β -direction has a stronger impact than the λ -direction for this example.

Comprehensive Sensitivity with Respect to α , β , λ , and η

After verifying Theorem 3 for the four test functions, we compare the effects of all four parameters in a uniform way. One parameter is varied at a time while the remaining parameters are fixed at the baseline values:

$$\alpha = 0.5, \quad \beta = 0.8, \quad \eta = 0.8, \quad \lambda = 0.5.$$

The parameters are varied over the following grids:

$$\begin{aligned} \alpha &\in \{0.3, 0.4, \dots, 0.9\}, & \beta &\in \{0.4, 0.5, \dots, 1.0\}, \\ \eta &\in \{0.4, 0.5, \dots, 0.9, 0.95\}, & \lambda &\in \{0, 0.25, 0.5, 0.75, 1.0, 1.25, 1.5\}. \end{aligned}$$

For each parameter, the relative gap variation is computed by

$$\frac{\max(\text{Gap}) - \min(\text{Gap})}{\text{mean}(\text{Gap})} \times 100\%.$$

The resulting gap and RHS sensitivities are summarized in Table 10. Figures 13 and 14 compare the corresponding relative gap and RHS variations, respectively, across the four examples.

Table 10. Comprehensive gap and RHS sensitivity for the four numerical examples.

Function	Parameter	Gap Min	Gap Max	Gap Range	Relative Gap Variation (%)	Relative RHS Change (%)
x^2	α	0.091819	0.265481	0.173662	99.18	87.19
x^2	β	0.111628	0.289765	0.178137	98.99	124.61
x^2	λ	0.131748	0.148369	0.016622	11.90	14.58
x^2	η	0.081682	0.170135	0.088453	70.10	60.37
e^x	α	0.090371	0.377774	0.287403	128.55	112.47
e^x	β	0.143224	0.281471	0.138248	70.30	101.23
e^x	λ	0.160255	0.171814	0.011559	6.97	10.42
e^x	η	0.075548	0.211953	0.136404	95.20	79.94
$x^3 + x$	α	0.111262	0.432039	0.320778	123.10	93.00
$x^3 + x$	β	0.166417	0.350147	0.183729	77.52	122.68
$x^3 + x$	λ	0.188394	0.204017	0.015623	7.98	13.83
$x^3 + x$	η	0.094441	0.248070	0.153628	89.92	64.36
$\ln(1+x)$	α	0.037100	0.151447	0.114346	127.18	127.08
$\ln(1+x)$	β	0.057562	0.117490	0.059929	74.10	80.06
$\ln(1+x)$	λ	0.064334	0.069650	0.005316	7.95	7.58
$\ln(1+x)$	η	0.031248	0.085205	0.053957	92.96	93.41

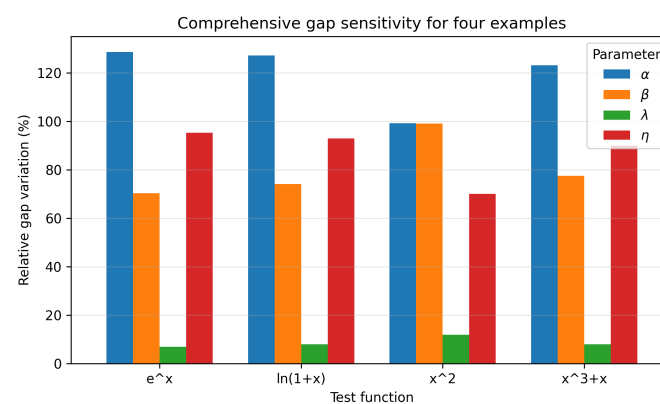


Figure 13. Relative gap variation generated by α , β , λ , and η for all four examples.

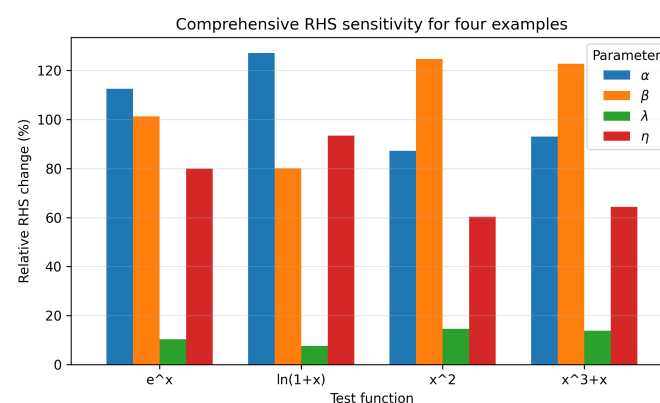


Figure 14. Relative RHS variation generated by α , β , λ , and η for all four examples.

Discussion of the comprehensive parameter sensitivity. Across all four numerical examples, the proposed inequality remains valid throughout the investigated parameter ranges because all the computed gaps are strictly positive. The magnitude of the gap depends on the growth behavior of the test function, but the same kernel parameters consistently control the sharpness of the estimate.

The fractional-order parameter α has a strong effect on the gap for $f(x) = e^x$, $f(x) = x^3 + x$, and $f(x) = \ln(1 + x)$, and its influence is also substantial for $f(x) = x^2$. This indicates that the order component of the kernel directly affects the size of the fractional contribution in Theorem 3. The conformable parameter β remains one of the most influential parameters for the RHS upper bound in all four examples. In particular, increasing β consistently reduces the gap in the one-parameter scans, which shows that β plays a central role in improving the numerical sharpness of the estimate.

The hybrid parameter η produces a secondary but still visible effect. Its impact is especially clear for the exponential, cubic polynomial, and logarithmic examples, where the derivative terms interact more strongly with the factor $\alpha\eta$. In contrast, the tempering parameter λ gives the smallest relative variation in every tested function. This confirms that λ mainly acts as a mild attenuation parameter in the present numerical range, whereas α and β govern the dominant structural behavior of the proposed kernel.

Overall, the four examples support the following practical sensitivity ranking for the proposed inequality:

$$\alpha \text{ and } \beta > \eta > \lambda.$$

The relative dominance of α or β may vary with the growth rate of the test function; however, both parameters are consistently more influential than η and λ in the numerical verification of Theorem 3.

5. Conclusions

This paper established new Milne–Mercer-type inequalities associated with the proposed PCHTC kernel. A corresponding Milne-type identity was first derived, and several integral estimates were obtained under convexity assumptions on $|f''|$ and $|f''|^q$.

Numerical examples demonstrated the validity of the theoretical estimates and illustrated the influence of the governing parameters. The numerical investigation showed that different combinations of the proportional, conformable, and tempering parameters affect the resulting bounds in distinct ways, confirming the flexibility of the proposed kernel.

Future research will investigate additional mathematical properties of the proposed kernel, including composition formulas, inverse operators, and applications to generalized fractional differential equations. Extensions to other classes of convex functions and related quadrature-type inequalities will also be considered.

Funding: This research received no external funding. The APC was funded by the author.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Mercer, A.M. A variant of Jensen's inequality. *J. Inequal. Pure Appl. Math.* **2003**, *4*, 73.
2. Djenaoui, M.; Meftah, B. Milne type inequalities for differentiable s -convex functions. *Honam Math. J.* **2022**, *44*, 325–338.

3. Benaissa, B.; Sarikaya, M.Z. Milne–Mercer-type inequalities for h -convex functions. *Filomat* **2026**, *40*, 2393–2408. <https://doi.org/10.2298/FIL2607393B>.
4. Munir, A.; Qayyum, A.; Budak, H.; Faiz, I.; Kara, H.; Supadi, S.S. A study of Milne-type inequalities for several convex function classes with applications. *Filomat* **2024**, *38*, 11539–11551. <https://doi.org/10.2298/FIL2432539M>.
5. Farid, G. On Caputo fractional derivatives via convexity. *Kragujev. J. Math.* **2020**, *44*, 393–399. <https://doi.org/10.46793/KgJMat2003.393F>.
6. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Basak, N. Hermite–Hadamard’s inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model* **2013**, *57*, 2403–2407.
7. Budak, H.; Kösem, P.; Kara, H. On new Milne-type inequalities for fractional integrals. *J. Ineq. Appl.* **2023**, *2023*, 10. <https://doi.org/10.1186/s13660-023-02921-5>.
8. Mahajan, Y.; Nagar, H. Fractional Newton-type integral inequalities for the Caputo fractional operator. *Math. Meth. Appl. Sci.* **2025**, *48*, 5244–5254. <http://doi.org/10.1002/mma.10600>.
9. Caputo, M. Linear model of dissipation whose Q is almost frequency independent-II. *Geophys. J. Int.* **1967**, *13*, 529–539. <http://doi.org/10.1111/j.1365-246X.1967.tb02303.x>.
10. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85.
11. Sahoo, S.K.; Hamed, Y.S.; Mohammed, P.O.; Kodamasingh, B.; Nonlaopon, K. New midpoint type Hermite–Hadamard–Mercer inequalities pertaining to Caputo–Fabrizio fractional operators. *Alex. Eng. J.* **2023**, *65*, 689–698. <http://doi.org/10.1016/j.aej.2022.10.019>.
12. Gürbüz, M.; Akdemir, A.O.; Dokuyucu, M.A. Novel approaches for differentiable convex functions via the proportional Caputo–hybrid operators. *Fractal Fract.* **2022**, *6*, 258.
13. Sarikaya, M.Z. On Hermite–Hadamard-type inequalities for proportional Caputo–Hybrid operator. *Konuralp J. Math.* **2023**, *11*, 31–39.
14. Jarad, F.; Ugurlu, E.; Abdeljawad, T.; Baleanu, D. On a new class of fractional operators. *Adv. Diff. Equ.* **2017**, *2017*, 247. <http://doi.org/10.1186/s13662-017-1306-z>.
15. Çelik, B.; Budak, H.; Set, E. On generalized Milne type inequalities for new conformable fractional integrals. *Filomat* **2024**, *38*, 1807–1823. <http://doi.org/10.2298/FIL2405807C>.
16. Ying, R.; Lakhdari, A.; Xu, H.Y.; Saleh, W.; Meftah, B. On conformable fractional Milne-type inequalities. *Symmetry* **2024**, *16*, 196. <http://doi.org/10.3390/sym16020196>.
17. Li, C.; Deng, W.H.; Zhao, L.J. Well-posedness and numerical algorithm for the tempered fractional differential equations. *Discret. Contin. Dyn. Syst. B* **2019**, *24*, 1989–2015. <https://doi.org/10.3934/dcdsb.2019026>.
18. Mohammed, P.O.; Sarikaya, M.Z.; Baleanu, D. On the generalized Hermite–Hadamard inequalities via the tempered fractional integrals. *Symmetry* **2020**, *12*, 595. <https://doi.org/10.3390/sym12040595>.
19. Haider, W.; Budak, H.; Shehzadi, A.; Hezenci, F.; Chen, H. A comprehensive study on Milne-type inequalities with tempered fractional integrals. *Bound. Value Probl.* **2024**, *2024*, 53. <https://doi.org/10.1186/s13661-024-01855-1>.
20. Lakhdari, A.; Sarikaya, M.Z.; Budak, H.; Meftah, B. On tempered fractional Bullen-type inequalities: Analysis in different function settings. *Kuwait J. Sci.* **2026**, *53*, 100515. <https://doi.org/10.1016/j.kjs.2025.100515>.
21. Demir, İ.; Tunç, T.; Sarikaya, M.Z. A new approach to Hermite–Hadamard-type inequality with proportional Caputo–hybrid operator. *J. Nonlinear Model. Anal.* **2025**, *7*, 2137–2156. <http://doi.org/10.12150/jnma.2025.2137>.
22. Butt, S.I.; Umar, M.; Khan, D.; Seol, Y.S.; Tipurić-Spužević, S. Hermite–Hadamard-type inequalities for harmonically convex functions via proportional Caputo–hybrid operators with applications. *Fractal Fract.* **2025**, *9*, 77. <http://doi.org/10.3390/fractalfract9020077>.
23. Demir, İ. A novel method for Milne-type inequalities using proportional Caputo–hybrid operator. *Miskolc Math. Notes* **2025**, *26*, 701–725. <http://doi.org/10.18514/MMN.2025.4903>.
24. Lo, J.C. Some Hermite–Hadamard–Mercer type inequalities via Atangana–Baleanu–Conformable fractional operator. *AIMS Math.* **2026**, *11*, 17937–17950. <https://doi.org/10.3934/math.2026731>.
25. Lo, J.C. On Some Milne–Mercer-Type Inequalities via Atangana–Baleanu–Conformable Fractional Integrals for h -convex functions. *Mathematics* **2026**, *14*, 2278. <https://doi.org/10.3390/math14132278>.
26. Munir, A.; Li, S.; Kashur, A. Analysis on Milne-type inequalities through generalized convexity with applications. *Filomat* **2025**, *39*, 11473–11490. <https://doi.org/10.2298/FIL2532473M>.

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