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Exact Asymptotics of the Ruin Probability in the Sparre Andersen Model for Non-Life Insurance with Investments in a Lévy Process

Platon Promyslov 

Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, Moscow 119991, Russia;
platon.promyslov@gmail.com

Abstract

We establish the exact power-law asymptotics of the ruin probability, as a function of the initial capital, in the Sparre Andersen model for non-life insurance with investments in an arbitrary Lévy process. The main advance over previous work, which only established two-sided estimates, is a proof of the existence of an exact limiting equality with a positive finite constant. The method combines a reduction to discrete time, the one-dimensional Kesten–Goldie theorem for the stationary measure of the associated affine stochastic recursion, and Goldie’s result on the asymptotics of the supremum of a perpetuity. The limiting constant is bounded below by the integral Goldie constant of the stationary measure. This bound need not be tight, and an explicit expression for the limiting constant is currently available only in the Cramér–Lundberg submodel. The exponent coincides with the positive Cramér root of the Laplace exponent of the Lévy process given by the logarithm of the risky-asset price; the distribution of the inter-jump times of the business process affects only the constant, not the exponent. In the final section, the constant is compared with an explicit formula in terms of double confluent Heun functions, obtained for the Cramér–Lundberg submodel with proportional investment in a geometric Brownian motion; the agreement of the two approaches is illustrated numerically.

Keywords: ruin probability; Sparre Andersen model; non-life insurance; implicit renewal theory; Kesten–Goldie theorem; perpetuity equation

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1. Introduction

The study of the ruin probability of an insurance company in the presence of risky investments goes back to the work of Paulsen [1], where the dramatic change in the nature of the asymptotics of the ruin probability $\Psi(u)$ was first observed—from exponential, in models without investments, to power-law, in models with investments in a risky asset. In parallel, Kalashnikov and Norberg [2] obtained power-law asymptotics in the Cramér–Lundberg model with risky investments by applying the Kesten–Goldie theory to the associated affine stochastic equation. For the Cramér–Lundberg model with exponentially distributed claims and the capital fully invested in a geometric Brownian motion, the exact power-law asymptotics of the ruin probability was established by Frolova, Kabanov, and Pergamenschikov [3]. The theory has developed along two lines: the analytic approach, via integro-differential equations [4–10], and the probabilistic approach, via

distributional equations and implicit renewal theory [11–14]. A modern exposition of the latter is contained in the monograph [11], and the general theory of ruin in [15].

Two-sided estimates for the order of the ruin probability in the Cramér–Lundberg model (with a random bounded premium rate) with investments in a geometric Brownian motion were obtained by Pergamenshchikov and Zeitouni [16]. For the annuity case with exponentially distributed payments, an exact asymptotics was established by Kabanov and Pergamenshchikov [17]. Eberlein, Kabanov, and Schmidt [12] considered the non-life insurance case in the Sparre Andersen model with investments in an arbitrary Lévy process and established, under a non-degeneracy alternative reproduced in assumption (A4) of Section 2 below, the two-sided estimate for the ruin probability

$$0 < \liminf_{u \rightarrow \infty} u^\beta \Psi(u) \leq \limsup_{u \rightarrow \infty} u^\beta \Psi(u) < \infty, \quad (1)$$

where $\beta > 0$ is the unique positive root of the cumulant $H(q) := \ln \mathbb{E}[e^{-qV_{T_1}}]$, and V is the logarithm of the risky-asset price. In [14] the conditions of these estimates were relaxed, and analogous two-sided estimates were obtained in the mixed model using Proposition 2.5.4 of [11]. The two-sided estimate (1) establishes only the order of the asymptotics: it does not exclude oscillation of $u^\beta \Psi(u)$ between two distinct constants $0 < C_- < C_+ < \infty$, or other non-convergent behavior. The existence of an exact limit $\lim_{u \rightarrow \infty} u^\beta \Psi(u) = C^* \in (0, \infty)$ remained an open question in the Sparre Andersen model for the non-life case with an arbitrary Lévy process. The present work resolves this question: the limit exists, and the limiting constant C^* is characterized as $C^* = \lim_{u \rightarrow \infty} u^\beta \mathbb{P}(M^* > u)$, where $M^* = \sup_n Y_n$ is the supremum of the associated perpetuity.

In parallel, in [8], by reducing the problem to an integro-differential equation, an exact limit $\Psi(u) \sim Cu^{-\beta}$ with an explicit computable constant C was obtained for a particular case—the Cramér–Lundberg model for the life insurance case with proportional investment in a geometric Brownian motion and a constant fraction $\kappa \in (0, 1]$ invested in the risky asset. In [7], the existence of a classical C^2 -smooth solution to the integro-differential equation for an analogous model, but for the non-life insurance case, was established under the moment condition $\mathbb{E}[|\xi|^{\gamma-1}] < \infty$. This condition was relaxed in [9], where C^2 -smoothness was proved under the minimal moment condition $\mathbb{E}[|\xi|^\varepsilon] < \infty$ for some $\varepsilon > 0$, which improves the classical result of Grandits [5], that yielded $W_{\text{loc}}^{2,1}$ -regularity under the same moment condition. A subsequent analysis via special functions in [10] gave a representation in terms of double confluent Heun functions. The extension to the general model—the Sparre Andersen model with an arbitrary Lévy process—requires an essentially different apparatus and is the content of this paper.

The main result of this paper is the following: in the Sparre Andersen model for non-life insurance with investments in a general Lévy process, the two-sided estimate (1) turns into the exact equality

$$\lim_{u \rightarrow \infty} u^\beta \Psi(u) = C^* \in (0, \infty), \quad (2)$$

where $C^* \geq C_+^Y$, and C_+^Y is the integral Goldie constant of the stationary measure of the associated affine recursion, having the explicit representation (9); this is a lower bound for C^* , not the constant of the ruin probability itself.

Compared with the two-sided estimate (1) of [12], the main advance is the upgrade from the order of the asymptotics to the exact limit. Paulsen [4] obtained an exact Cramér-like asymptotics $\Psi(u) \sim Cu^{-\beta}$ for a model in which the surplus-generating process and the return on investments are independent Lévy processes (covering the Cramér–Lundberg model with risky investments), under additional integrability conditions (see also [1]).

The present result establishes the exact limit in the Sparre Andersen model, where the business process is a renewal process and need not be Lévy, with an arbitrary Lévy price process, under assumptions (A1)–(A4) and a mild non-arithmeticity condition (N) on the law of V_{T_1} , stated in Section 2.

The proof relies on three steps. First, the reduction to a discrete-time model from [12]: in the non-life case, ruin occurs only at the jump times T_n of the business process, which reduces the problem to the analysis of the tail of the supremum of the discrete chain $\sup_n Y_n \geq u$, where $Y_n := Y_{T_n}$. Second, the Kesten–Goldie theorem in its one-dimensional form ([11] (Theorem 2.4.4), going back to the works of Kesten [18] and Goldie [13]) gives the power-law asymptotics of the stationary measure $\mathbb{P}(Y_\infty > u) \sim C_+^Y u^{-\beta}$. Third, Goldie’s result [13] on the fixed-point equation $M \stackrel{d}{=} (AM + B)_+$ provides the passage from the asymptotics of the stationary measure to the asymptotics of the supremum, $\mathbb{P}(\sup_n Y_n > u) \sim C^* u^{-\beta}$. Finally, the passage from $\mathbb{P}(\sup_n Y_n > u)$ to $\Psi(u) = \mathbb{P}(\sup_n Y_n \geq u)$ is carried out by means of the two-sided bound $\mathbb{P}(\sup_n Y_n > u) \leq \Psi(u) \leq \mathbb{P}(\sup_n Y_n > u - \varepsilon)$ and a subsequent limiting argument. We note that this two-sided bound and the limiting argument require no additional conditions.

The key technical obstacle that prevented this combination from being achieved earlier is the verification of the conditions of Goldie’s Theorem 6.2 for the supremum of the perpetuity in the Sparre Andersen model: the identification $M \stackrel{d}{=} \sup_n Y_n$, the moment estimates for (M_1, Q_1) , and the verification of the non-lattice condition (N). These are resolved in Lemmas 2, 3 and 5–7. As a by-product, we correct the non-degeneracy alternative of [12] (Theorem 2.3): its exceptional case must also include Lévy measures with $\Pi(|h|) = 0$; see assumption (A4) and the example following it.

Conceptually, the exponent β admits a twofold interpretation: as the Cramér root of the cumulant H of the random walk V_{T_n} (the formulation of [12,14]) or, equivalently, as the Cramér root of the cumulant ψ_V of the Lévy process V itself, since

$$H(\beta) = 0 \iff \mathbb{E}[e^{T_1 \psi_V(\beta)}] = 1 \iff \psi_V(\beta) = 0,$$

by the strict monotonicity of the function $s \mapsto \mathbb{E}[e^{sT_1}]$ (a consequence of $T_1 > 0$ a.s.). This duality gives a transparent interpretation: the exponent is determined by the investment structure V itself, while the distribution of the inter-jump times affects only the constant C^* in (2).

The transfer of the result to the annuity and mixed models is a natural direction for further research. The key obstacle is that both extensions require abandoning the discrete-time reduction, which is the cornerstone of the present approach. In the annuity model, the trajectories of Y have a positive continuous drift between jumps, so that $\sup_t Y_t \neq \sup_n Y_{T_n}$ and the discrete reduction fails. In the mixed model, F_ζ charges both half-axes, and the left tail $\mathbb{P}(Y_\infty < -u)$ requires a separate analysis in the spirit of Goldie’s theorem applied to the transform $-Y_\infty$.

The paper is organized as follows. Section 2 describes the model, states the assumptions and the non-lattice condition, gives the reduction to a discrete-time model in the spirit of [12], and establishes the convergence of the associated perpetuity. Section 3 states and proves the main result (2). Section 4 establishes the connection with the exact analytic solution [10] for the Cramér–Lundberg submodel with proportional investment and gives a numerical verification. Section 5 contains a discussion and directions for further research.

2. The Model and Assumptions

All random elements are defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions.

Let $R = (R_t)_{t \geq 0}$ be a Lévy process with triplet (a, σ^2, Π) , $\Pi((-\infty, -1]) = 0$. The stochastic exponential $\mathcal{E}(R) = e^V$ is interpreted as the price of the risky asset. We introduce the log-price

$$V_t = at - \frac{1}{2}\sigma^2 t + \sigma W_t + h * (\mu - \nu)_t + (\ln(1+x) - h) * \mu_t,$$

where $h(x) = x\mathbf{1}_{\{|x| \leq 1\}}$, μ is the jump measure of R , and ν is its compensator. Then, V is again a Lévy process. By $\psi_V(q) := \ln \mathbb{E}[e^{-qV_1}]$ we denote its Laplace exponent.

The business process P is independent of R and has the form $P_t = ct + \sum_{i=1}^{N_t} \xi_i$, where $N = (N_t)_{t \geq 0}$ is a renewal counting process with jump times $0 < T_1 < T_2 < \dots$; the inter-jump times $U_i := T_i - T_{i-1}$, $i \geq 1$, form a sequence of independent identically distributed random variables with law F , where $F(\{0\}) = 0$, and the jumps $\xi_i := \Delta P_{T_i}$ are independent identically distributed random variables with law F_ξ , $F_\xi(\{0\}) = 0$. The sequences (U_i) and (ξ_i) are mutually independent. We set $T_0 := 0$.

In this paper we consider the non-life insurance case: $c > 0$ and $\xi_i < 0$ almost surely. The assumption $\xi_i < 0$ a.s. corresponds to the pure non-life case; extensions to models where F_ξ charges both half-axes (the mixed model) are discussed in Section 5. The process is observed starting from the moment $T_0 := 0$, which corresponds to the standard setting of [12].

The capital process X^u satisfies the linear stochastic equation

$$X_t^u = u + \int_0^t X_{s-}^u dR_s + P_t, \quad X_0^u = u > 0.$$

We define the ruin time $\tau^u := \inf\{t \geq 0 : X_t^u \leq 0\}$ and the ruin probability $\Psi(u) := \mathbb{P}(\tau^u < \infty)$.

By the stochastic Cauchy formula,

$$X_t^u = e^{V_t}(u - Y_t), \quad Y_t := - \int_{(0,t]} e^{-V_{s-}} dP_s.$$

The strict positivity of e^{V_t} yields $\tau^u = \inf\{t \geq 0 : Y_t \geq u\}$.

Lemma 1. *In the non-life insurance case, the trajectories of Y decrease continuously on each interval (T_{n-1}, T_n) and have positive jumps at the moments T_n . Consequently,*

$$\Psi(u) = \mathbb{P}\left(\sup_{n \geq 0} Y_{T_n} \geq u\right). \quad (3)$$

Identity (3) holds for every $u > 0$ that is not an atom of $\sup_{n \geq 0} Y_{T_n}$; for all $u > 0$ one has $\mathbb{P}(\sup_{n \geq 0} Y_{T_n} > u) \leq \Psi(u) \leq \mathbb{P}(\sup_{n \geq 0} Y_{T_n} \geq u)$.

Proof. Between jumps $dP_s = c ds$, hence $dY_s = -e^{-V_{s-}} c ds < 0$. At the moment T_n the jump equals $\Delta Y_{T_n} = -e^{-V_{T_n-}} \xi_n > 0$ because $\xi_n < 0$. Since Y decreases between jumps, the supremum over $[T_n, T_{n+1})$ is attained at T_n . Hence, in view of $\tau^u = \inf\{t : Y_t \geq u\}$, ruin occurs if and only if $Y_{T_n} \geq u$ for some $n \geq 0$. The event $\{Y_{T_n} \geq u \text{ for some } n \geq 0\}$ contains $\{\sup_n Y_{T_n} > u\}$ and is contained in $\{\sup_n Y_{T_n} \geq u\}$; the enclosing events differ only when the supremum equals u without being attained, which yields the two-sided bound for all $u > 0$ and identity (3) outside the atoms of the supremum. $\square$

Lemma 1 is a particular case of the reduction to a discrete-time model stated explicitly in [12]: in the non-life insurance model it suffices to track the reserve process along the sequence of jump times T_n . Set $Y_n := Y_{T_n}$ and $M^* := \sup_{n \geq 0} Y_n$. By (3), $\mathbb{P}(M^* > u) \leq \Psi(u) \leq \mathbb{P}(M^* \geq u)$, which is all that is used in the proof of Theorem 1 below.

Put

$$M_m := e^{-(V_{T_m} - V_{T_{m-1}})}, \quad Q_m := - \int_{T_{m-1}}^{T_m} e^{-(V_s - V_{T_{m-1}})} dP_s, \quad m \geq 1.$$

By the independence of the increments of the Lévy process V , the independence of P and V , and the independence and identical distribution of the inter-jump times U_i , the pairs $(M_m, Q_m)_{m \geq 1}$ form a sequence of independent identically distributed random pairs. A direct computation gives the explicit representation of the chain (Y_n) :

$$Y_n = \sum_{k=1}^n \left(\prod_{i=1}^{k-1} M_i \right) Q_k, \quad Y_0 = 0. \quad (4)$$

We emphasize that (4) is not the affine recursion $Y_n = M_n Y_{n-1} + Q_n$. The latter holds for the backward (time-reversed) sequence

$$\tilde{Y}_0 := 0, \quad \tilde{Y}_n := M_n \tilde{Y}_{n-1} + Q_n, \quad n \geq 1, \quad (5)$$

for which, for each fixed n , one has $\tilde{Y}_n \stackrel{d}{=} Y_n$. Indeed, $\tilde{Y}_n = \sum_{k=1}^n (\prod_{i=k+1}^n M_i) Q_k$ is obtained from (4) by reversing the order of the pairs (M_i, Q_i) , and since these pairs are independent and identically distributed, reversing the order does not change the joint law. Associated with the recursion (5) is the distributional fixed-point equation

$$Y_\infty \stackrel{d}{=} M_1 Y_\infty + Q_1 \quad (Y_\infty \text{ independent of } (M_1, Q_1)), \quad (6)$$

the convergence to whose unique solution is established below, after the assumptions are stated.

We retain the standard assumptions of [14], complemented by the non-degeneracy alternative of [12] (Theorem 2.3):

- (A1) The process R is non-degenerate: $\sigma^2 > 0$ or $\Pi \neq 0$.
- (A2) The cumulant $H(q) := \ln \mathbb{E}[e^{-qV_{T_1}}]$ has a unique root $\beta > 0$ lying strictly inside $\text{int}(\text{dom } H)$.
- (A3) $\mathbb{E}[|\xi_1|^\beta] < \infty$ and $\mathbb{E}[e^{\varepsilon T_1}] < \infty$ for some $\varepsilon > 0$.
- (A4) If $\sigma^2 = 0$, the law of $|\xi_1|$ is bounded, and the jumps of R are one-sided and of finite variation ($\Pi((-1, 0)) \Pi((0, \infty)) = 0$ and $\Pi(|h|) < \infty$), then $\mathbb{P}(T_1 \leq t) > 0$ for every $t > 0$.

Assumption (A4) is a corrected and slightly strengthened form of the non-degeneracy alternative of [12] (Theorem 2.3), whose exceptional configuration is stated there under the condition $0 < \Pi(|h|) < \infty$; the example below shows that the case $\Pi(|h|) = 0$ must be included as well. Outside the exceptional configuration listed in (A4) no additional requirement is imposed. Its role is to guarantee that the support of the stationary measure $\mathcal{L}(Y_\infty)$ constructed below is unbounded from above [12] (Section 5), which yields the positivity of the limiting constants in the main result. An assumption of this kind cannot be dispensed with. Indeed, let $V_t = -t + 2N_t$, where N is a Poisson process with unit intensity, let T_1 be uniformly distributed on $[1, 2]$, and let $\xi_1 \equiv -K$ with $0 < K < c(1 - e^{-1})$. Assumptions (A1)–(A3) and (N) hold (with $\beta \approx 0.797$), yet, since the downward motion of V consists only of the unit drift, $\int_0^{T_1} e^{V_{T_1} - V_s} ds \geq 1 - e^{-T_1} \geq 1 - e^{-1}$, whence $Q_1 = e^{-V_{T_1}} (|\xi_1| - c \int_0^{T_1} e^{V_{T_1} - V_s} ds) < 0$ almost surely for the pair (M_1, Q_1) defined above. Then, the sequence (Y_n) of (4) is strictly decreasing, $\Psi(u) = 0$ for all $u > 0$, and no asymptotics with a positive constant can hold. In this example $\mathbb{P}(T_1 \leq t) = 0$ for $t < 1$, so (A4) fails. We note that here the jumps of R are of size $e^2 - 1 > 1$, so that $\Pi(|h|) = 0$: the

example lies outside the exceptional case as formulated in [12] (Theorem 2.3), and it shows that the lower bound in (1) may fail for $\Pi(|h|) = 0$ unless $\mathbb{P}(T_1 \leq t) > 0$ for all $t > 0$; this is why (A4) is stated with the condition $\Pi(|h|) < \infty$. The proofs of [12] (Lemmas 5.3 and 5.4) remain valid in this case: the only property of the jump part of V used there is that it stays bounded on a fixed time interval with positive probability, and when all jumps are large this holds on the event that no jumps occur at all.

For the application of the Kesten–Goldie theorem (Theorem 4.1 of [13]) an additional non-lattice condition is required:

(N) The law of the random variable V_{T_1} is non-arithmetic: there exist no $\delta > 0$ and $a \in \mathbb{R}$ such that $\mathbb{P}(V_{T_1} \in a + \delta\mathbb{Z}) = 1$; equivalently, $|\mathbb{E} e^{itV_{T_1}}| < 1$ for all $t \neq 0$.

Lemma 2. *If $\sigma^2 > 0$, then (N) holds.*

Proof. If $\sigma^2 > 0$, then, conditionally on $T_1 = t$, the variable V_{T_1} contains the Gaussian component $\sigma W_t - (\sigma^2/2)t$, which has an absolutely continuous density. The unconditional law of V_{T_1} , being a mixture, has an absolutely continuous component, which rules out concentration on any coset $a + \delta\mathbb{Z}$ (a Lebesgue-null set); hence (N) holds. $\square$

Lemma 3. *If the Lévy measure Π_V has an absolutely continuous component charging an open interval in $\mathbb{R}$, then (N) holds.*

Proof. Let $I \subset \mathbb{R}$ be an open interval on which Π_V has an absolutely continuous component. The restriction $\Pi_V|_I$ is finite (since Π_V is integrable outside any neighborhood of zero by the definition of a Lévy measure; if $0 \in I$, one may pass to a subinterval not containing zero). Consider the decomposition of the Lévy process V into the sum of two independent processes: a pure-jump process J_t^I with Lévy measure $\Pi_V|_I$ (a compound Poisson process with finite intensity, the law of whose individual jump has an absolutely continuous component) and the remainder $V_t^c := V_t - J_t^I$. The law of J_t^I for $t > 0$ is a mixture over the number of jumps,

$$\mathbb{P}(J_t^I \in \cdot) = \sum_{n \geq 0} \frac{(\Pi_V(I) \cdot t)^n}{n!} e^{-\Pi_V(I) \cdot t} \nu_I^{*n},$$

where $\nu_I := \Pi_V|_I / \Pi_V(I)$ is a probability measure having an absolutely continuous component. Then, already $\nu_I^{*1} = \nu_I$ has an absolutely continuous component. Consequently, the mixture $\mathbb{P}(J_t^I \in \cdot)$ for $t > 0$ has an absolutely continuous component. The independent convolution with the law of V_t^c preserves the absolutely continuous component. Hence, V_t has an absolutely continuous component for every $t > 0$. Averaging over the random moment T_1 with a law not concentrated at zero (that is, $F(\{0\}) = 0$) preserves the presence of an absolutely continuous component in V_{T_1} , which implies (N). $\square$

Remark 1. *Note that the law F of the inter-jump times does not automatically imply that V_{T_1} is non-arithmetic. For example, if V is a standard Poisson process, then $V_t \in \mathbb{Z}_+$ almost surely for every t , and hence $V_{T_1} \in \mathbb{Z}_+$ for any law of T_1 , which is lattice with span 1. Sufficient conditions for (N) must be imposed on the structure of V itself. This example illustrates only the lattice mechanism and does not satisfy (A2): for $V_{T_1} \geq 0$ the cumulant H has no positive root. However, there exist models satisfying (A2) for which (N) fails: for instance, let V be a compound Poisson process whose jumps take the values $+\delta$ and $-\delta$ ($\delta > 0$) with probabilities p and $1 - p$, where $p > 1/2$. Then, V_{T_1} takes values in the lattice $\delta\mathbb{Z}$, so (N) fails, while $\mathbb{E}[V_{T_1}] > 0$ and the cumulant H has a unique positive root β (so that (A2) holds whenever $\mathbb{E}[e^{\epsilon T_1}] < \infty$). In such models the Kesten–Goldie theorem is inapplicable and the asymptotics may exhibit logarithmic oscillations; see Remark 2.*

Remark 2. If condition (N) fails, that is, V_{T_1} is lattice with span $\delta > 0$, the Kesten–Goldie theorem in its direct form is inapplicable. In this case, classical renewal theory (Feller [19] (Chapter XI), [11] (Section 2.4)) yields an asymptotics of the form $u^\beta \Psi(u) = L(u) + o(1)$, where L is a function periodic in $\log u$ with period δ ; then, $u^\beta \Psi(u)$ oscillates and has no limit, so a relation with $\sim$ is not correct. The lattice case is not considered in this paper.

Let us verify the conditions for the convergence of the perpetuity. From the convexity of H , the equality $H(0) = 0$, and the existence of a root $\beta > 0$ (assumption (A2)) it follows that $\mathbb{E}[\log M_1] = -\mathbb{E}[V_{T_1}] = H'(0) < 0$. From (A3) we obtain $\mathbb{E}[\log_+ |Q_1|] < \infty$, since $\mathbb{E}[|Q_1|^\beta] < \infty$ by [12] (Lemma 2.1, Corollary 2.2).

Lemma 4. Under assumptions (A1)–(A3) the forward sum (Y_n) converges almost surely to a finite random variable Y_∞^{fwd} , while the backward recursion $\{\tilde{Y}_n\}$ from (5) converges in distribution to a random variable Y_∞ satisfying Equation (6). The law $\mathcal{L}(Y_\infty)$ is the unique stationary measure of (6), and $Y_\infty^{\text{fwd}} \stackrel{d}{=} Y_\infty$. Below Y_∞ denotes a random variable with law $\mathcal{L}(Y_\infty)$.

Proof. The almost sure convergence $Y_n \rightarrow Y_\infty^{\text{fwd}}$ is established in [12] (Lemma 4.1). It can also be seen directly: the terms of the series $\sum_k (\prod_{i < k} M_i) Q_k$ decay geometrically, since $\mathbb{E}[\log M_1] < 0$. The convergence in distribution of the recursion $\tilde{Y}_n$ to a solution of (6) and the uniqueness of the stationary measure, under the conditions $\mathbb{E}[\log |M_1|] < 0$, $\mathbb{E}[\log_+ |Q_1|] < \infty$ and non-degeneracy, are provided by the Goldie–Maller theorem [11] (Theorem 2.1.1, Corollary 2.1.2) and [20]. The recursion $\tilde{Y}_n$ itself does not converge almost surely, since the difference $\tilde{Y}_n - M_n \tilde{Y}_{n-1} = Q_n$ does not tend to zero; it is precisely the forward sum Y_n that converges almost surely. Finally, the equality of laws $Y_\infty^{\text{fwd}} \stackrel{d}{=} Y_\infty$ follows from the equality $Y_n \stackrel{d}{=} \tilde{Y}_n$ established above for each n and the passage to the limit. $\square$

All asymptotic statements in the work depend only on the law $\mathcal{L}(Y_\infty)$, so below we work with Y_∞ as a typical representative of this law.

3. Main Results

The main result of this paper is the following theorem.

Theorem 1. Under assumptions (A1), (A2), (A3), (A4), and (N),

$$\lim_{u \rightarrow \infty} u^\beta \Psi(u) = C^* \in (0, \infty), \quad (7)$$

where the constant C^* is given by the limit $C^* = \lim_{u \rightarrow \infty} u^\beta \mathbb{P}(M^* > u)$ and satisfies $C^* \geq C_+^Y$, where C_+^Y is the integral Goldie constant (9) of the stationary measure $\mathcal{L}(Y_\infty)$. This is a lower bound for C^* , not the constant of the ruin probability itself.

Let us explain the logic of the proof. The reduction (Lemma 1) reduces the ruin probability to the tail of the supremum of the chain (Y_n) ; the Kesten–Goldie theorem yields the power-law tail of the stationary measure $\mathcal{L}(Y_\infty)$; Goldie’s result for the maximum equation transfers the power-law tail to the supremum M^* . Finally, the two-sided bound $\mathbb{P}(M^* > u) \leq \Psi(u) \leq \mathbb{P}(M^* \geq u)$ of Lemma 1 and a limiting argument give the exact equality for $\Psi(u)$. This passage does not require continuity of the distribution of M^* ; the bound holds regardless of whether M^* has atoms. Technically, the proof consists of the following lemmas:

- Lemma 5: moment identities for the pair (M_1, Q_1) ;

- Lemma 6: the asymptotics of the stationary measure $\mathbb{P}(Y_\infty > u) \sim C_+^\gamma u^{-\beta}$ by the Kesten–Goldie theorem [11] (Theorem 2.4.4);
- Lemma 7: the asymptotics of the supremum $\mathbb{P}(M^* > u) \sim C^* u^{-\beta}$ by [13] (Theorem 6.2);
- the proof of Theorem 1 itself, with the passage from $\mathbb{P}(M^* > u)$ to $\Psi(u)$ via the two-sided bound and a limiting argument.

As a meaningful comment on the exponent β , we note the following observation, which gives it a transparent Lévy interpretation.

Remark 3. Condition (A2), $H(\beta) = 0$, has an equivalent form in terms of the Laplace exponent ψ_V of the Lévy process V itself. Since V is a Lévy process and T_1 is independent of V ,

$$\mathbb{E}[e^{-\beta V_{T_1}}] = \mathbb{E}[\mathbb{E}[e^{-\beta V_t}]_{t=T_1}] = \mathbb{E}[e^{T_1 \psi_V(\beta)}].$$

The identity $\mathbb{E}[e^{T_1 \psi_V(\beta)}] = 1$, together with $T_1 > 0$ a.s. and the strict monotonicity of the function $s \mapsto \mathbb{E}[e^{sT_1}]$, implies $\psi_V(\beta) = 0$. In substance: the exponent β is the Cramér root of the cumulant ψ_V of the process V itself, rather than of the random walk V_{T_n} . The distribution of the inter-jump times affects only the value of the constant C^* in (7), not the exponent.

Lemma 5. Under assumptions (A1)–(A3) the following hold:

- (i) $\mathbb{E}[M_1^\beta] = 1$;
- (ii) $\mathbb{E}[M_1^\beta | \log M_1] < \infty$;
- (iii) $m := \mathbb{E}[M_1^\beta \log M_1] = H'(\beta) > 0$;
- (iv) $\mathbb{E}[|Q_1|^\beta] < \infty$.

Proof. (i) is a reformulation of the definition of β :

$$\mathbb{E}[M_1^\beta] = \mathbb{E}[e^{-\beta V_{T_1}}] = e^{H(\beta)} = 1.$$

(ii) follows from $\beta \in \text{int}(\text{dom } H)$ (assumption (A2)): there exists $\delta > 0$ such that $\beta \pm \delta \in \text{dom } H$. Then the elementary inequality

$$|x| e^{-\beta x} \leq \frac{1}{\delta} (e^{-(\beta-\delta)x} + e^{-(\beta+\delta)x}), \quad x \in \mathbb{R}, \quad (8)$$

holds, being equivalent (after division by $e^{-\beta x}$) to $|\delta x|/2 \leq \cosh(\delta x)$. Applying (8) to V_{T_1} and taking expectations,

$$\mathbb{E}[|V_{T_1}| e^{-\beta V_{T_1}}] \leq \frac{1}{\delta} (\mathbb{E}[e^{-(\beta-\delta)V_{T_1}}] + \mathbb{E}[e^{-(\beta+\delta)V_{T_1}}]) = \frac{1}{\delta} (e^{H(\beta-\delta)} + e^{H(\beta+\delta)}) < \infty.$$

Since $|\log M_1| = |V_{T_1}|$ and $M_1^\beta = e^{-\beta V_{T_1}}$, this gives $\mathbb{E}[M_1^\beta | \log M_1] < \infty$.

(iii) is a direct computation (differentiation under the integral sign is justified by (ii)):

$$m = \mathbb{E}[M_1^\beta \log M_1] = \mathbb{E}[e^{-\beta V_{T_1}} \cdot (-V_{T_1})] = -\mathbb{E}[V_{T_1} e^{-\beta V_{T_1}}] = H'(\beta).$$

For the positivity $H'(\beta) > 0$: H is strictly convex on the interior of its domain (strict convexity follows from the non-degeneracy of V_{T_1} , ensured by (A1)), and $H(0) = H(\beta) = 0$; by Rolle's theorem there exists $\zeta \in (0, \beta)$ with $H'(\zeta) = 0$, and from the strict increase of H' we obtain $H'(\beta) > H'(\zeta) = 0$.

(iv) is the content of Lemma 2.1 and Corollary 2.2 of [12]. $\square$

Lemma 6. Under assumptions (A1)–(A4) and (N) there exists $C_+^Y \in (0, \infty)$ such that

$$\mathbb{P}(Y_\infty > u) \sim C_+^Y u^{-\beta} \quad \text{as } u \rightarrow \infty,$$

and

$$C_+^Y = \frac{1}{\beta H'(\beta)} \mathbb{E}[(M_1 Y_\infty + Q_1)_+^\beta - ((M_1 Y_\infty)_+^\beta)], \quad (9)$$

where Y_∞ on the right-hand side of (9) is independent of (M_1, Q_1) .

Proof. We apply the Kesten–Goldie theorem in its one-dimensional form [11] (Theorem 2.4.4) (going back to Theorem 5 of [18] and Theorem 4.1 of [13]) to the stationary measure $\mathcal{L}(Y_\infty)$ of Equation (6). The conditions of [11] (Theorem 2.4.4) are:

- $M_1 > 0$ almost surely ($M_1 = e^{-V_{T_1}}$);
- $\log M_1 = -V_{T_1}$ is non-arithmetic—condition (N);
- $\mathbb{E}[M_1^\beta] = 1$, $\mathbb{E}[|Q_1|^\beta] < \infty$, $\mathbb{E}[M_1^\beta \log_+ M_1] < \infty$ —Lemma 5;
- $\mathbb{P}(M_1 x + Q_1 = x) < 1$ for every $x \in \mathbb{R}$. Indeed, if $\mathbb{P}(M_1 x + Q_1 = x) = 1$ for some x , then δ_x solves (6) and, by the uniqueness of the stationary measure (Lemma 4), $\mathcal{L}(Y_\infty) = \delta_x$, contradicting the unboundedness from above of the support of $\mathcal{L}(Y_\infty)$, which holds under (A4) (see the end of the proof). When ξ_1 is non-degenerate, one can also argue directly: conditionally on the pair $(T_1, (V_s)_{s \leq T_1})$, the variable $Q_1 = -c \int_0^{T_1} e^{-V_s} ds - e^{-V_{T_1}} \xi_1$ is a non-constant affine transformation of ξ_1 , whence $\mathbb{P}(Q_1 = x(1 - M_1)) < 1$.

By [11] (Theorem 2.4.4), $\mathbb{P}(Y_\infty > u) \sim C_+^Y u^{-\beta}$ with the explicit Formula (9). The positivity $C_+^Y > 0$ is ensured by the unboundedness of the support of $\mathcal{L}(Y_\infty)$ from above. By [11] (Theorems 2.5.5 and 2.4.6), the support of the stationary measure is a half-line or the whole real line, and if it is unbounded from above, then $C_+^Y > 0$. The unboundedness from above of the support holds under assumption (A4): by [12] (Section 5), if $\sigma^2 > 0$ or the law of $|\xi_1|$ is unbounded, then Y_∞ is unbounded from above without further conditions, and in the remaining exceptional configuration of (A4) the same conclusion holds under the requirement $\mathbb{P}(T_1 \leq t) > 0$ for all $t > 0$ (Lemmas 5.3 and 5.4, [12]); see also [14] (Theorem 3.1) for the mixed model. As the example after (A4) shows, this assumption cannot be dropped: without it the support of $\mathcal{L}(Y_\infty)$ may be contained in $(-\infty, 0]$ and Ψ may vanish identically. $\square$

Lemma 7. Under assumptions (A1)–(A4) and (N) there exists $C^* \in [C_+^Y, \infty)$ such that

$$\mathbb{P}(M^* > u) \sim C^* u^{-\beta} \quad \text{as } u \rightarrow \infty.$$

Proof. We apply Goldie’s result [13] (Theorem 6.2) on the solution of the distributional equation $M \stackrel{d}{=} (M_1 M + Q_1)_+$. This is a particular case of Letac’s model from [13] with $L = -Q_1/M_1$. The pair (M_1, Q_1) satisfies the same Kesten–Goldie conditions as in Lemmas 5 and 6. The additional moment condition $\mathbb{E}[(M_1 L^+)^beta] = \mathbb{E}[(Q_1^-)^beta] \leq \mathbb{E}[|Q_1|^\beta] < \infty$ holds automatically. Under these assumptions there exists a unique solution M of this equation, and

$$\mathbb{P}(M > u) \sim C^* u^{-\beta} \quad \text{as } u \rightarrow \infty.$$

The identification $M \stackrel{d}{=} M^* = \sup_n Y_n$ is described in [11] (discussion after Equation (5.6.32)). The iterated sequence $Z_n := (M_n Z_{n-1} + Q_n)_+$ with $Z_0 := 0$ satisfies $Z_n \stackrel{d}{=} \max_{0 \leq j \leq n} Y_j$, since the pairs (M_i, Q_i) are independent and identically distributed.

The monotone maximum $\max_{0 \leq j \leq n} Y_j$ increases to $\sup_n Y_n = M^*$ almost surely, so Z_n converges in distribution to M^* . By Lemma 4, we have $Y_\infty^{\text{fwd}} = \lim Y_n \leq \sup_n Y_n = M^*$ almost surely, whence $\mathbb{P}(M^* > u) \geq \mathbb{P}(Y_\infty > u)$ for all u , and therefore $C^* \geq C_+^Y$. $\square$

Proof of Theorem 1. By Lemma 1, in the non-life insurance case $\mathbb{P}(M^* > u) \leq \Psi(u) \leq \mathbb{P}(M^* \geq u)$. By Lemma 7, $\lim_{u \rightarrow \infty} u^\beta \mathbb{P}(M^* > u) = C^*$. We establish $\lim_{u \rightarrow \infty} u^\beta \Psi(u) = C^*$ via a two-sided bound and a limiting argument. For any $\varepsilon > 0$,

$$\mathbb{P}(M^* > u) \leq \Psi(u) \leq \mathbb{P}(M^* \geq u) \leq \mathbb{P}(M^* > u - \varepsilon).$$

Multiplying by u^β and passing to the limit as $u \rightarrow \infty$,

$$\begin{aligned} C^* = \lim u^\beta \mathbb{P}(M^* > u) &\leq \liminf u^\beta \Psi(u) \leq \limsup u^\beta \Psi(u) \leq \\ &\leq \lim u^\beta \mathbb{P}(M^* > u - \varepsilon) = C^*, \end{aligned}$$

where the last equality uses $u^\beta \sim (u - \varepsilon)^\beta$ as $u \rightarrow \infty$. Hence $u^\beta \Psi(u) \rightarrow C^*$, which proves (7). $\square$

4. Connection with Exact Analytic Solutions in the Cramér–Lundberg Model with Investments

We compare the established result with the exact analytic solution for the classical submodel—the Cramér–Lundberg model with proportional investment in a geometric Brownian motion, studied in [7,10]. The exact power-law asymptotics in this submodel goes back to [3]. In this submodel the inter-jump times are exponentially distributed with intensity $\lambda > 0$, the price of the risky asset is a geometric Brownian motion, and a fixed fraction $\kappa \in (0, 1]$ of the capital is invested in the risky asset, the rest in a riskless account with rate $\tilde{r} \geq 0$. The effective log-value of the portfolio is

$$V_t = (\tilde{a}_\kappa - \frac{1}{2}\tilde{\sigma}_\kappa^2)t + \tilde{\sigma}_\kappa W_t, \quad \tilde{a}_\kappa := \tilde{r} + \kappa(\tilde{a} - \tilde{r}), \quad \tilde{\sigma}_\kappa := \kappa\tilde{\sigma}.$$

This is a purely diffusive Lévy process. The jumps ξ_i are strictly negative when $c > 0$. The Laplace exponent of V is

$$\psi_V(q) = -(\tilde{a}_\kappa - \frac{1}{2}\tilde{\sigma}_\kappa^2)q + \frac{\tilde{\sigma}_\kappa^2}{2}q^2.$$

The equation $\psi_V(\beta) = 0$ with $\beta > 0$ gives

$$\beta = \frac{2\tilde{a}_\kappa}{\tilde{\sigma}_\kappa^2} - 1 = \frac{2(\tilde{r} + \kappa(\tilde{a} - \tilde{r}))}{(\kappa\tilde{\sigma})^2} - 1 = \gamma - 1, \quad (10)$$

where $\gamma := 2\tilde{a}_\kappa/\tilde{\sigma}_\kappa^2$ is the standard notation of [7,10]. The notation β originates from the works of Pergamenschikov, Kabanov, Eberlein, and Schmidt and is defined as the positive root of the cumulant-generating function H . The notation γ is used by Grandits and Gaier–Grandits in the context of heavy-tailed claim models. The two parameters are related by $\gamma = \beta + 1$, which connects the two lines of research. By Theorem 1, $\Psi(u) \sim C^* u^{-\beta} = C^* u^{-\gamma+1}$, which agrees with the known result of [7]; there, this asymptotics is established by analytic methods through reduction to a Volterra integral equation, and the constant is obtained explicitly via the solution of a special integral equation. In the more general Sparre Andersen model with an arbitrary distribution of the inter-jump times, the analytic approach via integro-differential equations loses its applicability, since there is no closed second-order equation for Ψ , whereas the method considered here,

based on Goldie's theorem and the asymptotics of the supremum of the perpetuity, works without changes.

Model parameters and the exact representation. For definiteness we fix the control values from [10]: $\lambda = 1$, $\mu = 1$ (the parameter of the exponential claim distribution), $\tilde{a} = 0.15$, $\tilde{\sigma} = 0.4$, $c = 0.5$, $\tilde{r} = 0.05$, $\kappa = 0.9$. From (10),

$$\beta = \frac{2 \cdot 0.14}{0.1296} - 1 \approx 1.1605.$$

By the exact-solution theorem of [10], the survival probability $\Phi(u) = 1 - \Psi(u)$ for this model is expressed through an integral of the regular branch of the double confluent Heun function HeunD [21], with parameters determined by the set $(\tilde{a}, \tilde{\sigma}, c, \kappa, \tilde{r}, \lambda, \mu)$.

Agreement of the asymptotics. From the properties of the analytic continuation of the Heun function near the irregular singular point at infinity (the variable of the canonical Heun form is $\zeta = -\mu u$), studied in [10] (Proposition 1), it follows that the ruin probability has the power-law asymptotics

$$\Psi(u) \sim \mathcal{K} u^{-\gamma+1} \quad \text{as } u \rightarrow \infty,$$

where $\gamma = 2((\tilde{a} - \tilde{r})\kappa + \tilde{r})/(\kappa\tilde{\sigma})^2$. In our notation $\gamma - 1$ coincides exactly with the exponent β , and the constant $\mathcal{K}$ of the asymptotics $\Psi(u) \sim \mathcal{K} u^{-(\gamma-1)}$ obtained in [10] from the representation via Heun functions coincides with C^* , both being the limit of $u^{\gamma-1}\Psi(u)$ for the same model. Thus, Theorem 1 agrees with the asymptotics obtained in [10] by the analytic theory of differential equations. The coincidence of the two methodologies—Goldie's implicit renewal theory and the analysis of special functions—on their common intersection serves as a nontrivial cross-check of both methods. We note that the work does not recompute the constant itself by the second method: the exponent and the internal consistency of the tail are confirmed numerically, while the equality $\mathcal{K} = C^*$ holds by the uniqueness of the limit.

Numerical illustration of the agreement. For the parameters above, the survival probability $\Phi(u) = 1 - \Psi(u)$ is found by numerically integrating the corresponding ODE in canonical form

$$u^2 g'' + [\mu u^2 + (\gamma + 2)u + \theta]g' + [\mu\gamma u + (\mu\theta + \gamma - \nu)]g = 0$$

from [10] (where, for convenience, we have renamed the ODE coefficients $\theta := 2c/(\kappa\tilde{\sigma})^2$ and $\nu := 2\lambda/(\kappa\tilde{\sigma})^2$, to avoid confusion with the Cramér exponent $\beta = \gamma - 1$); this is neither Monte Carlo simulation nor an approximation, but a direct solution of the Cauchy problem, equivalent to evaluating the Heun function. The ODE is solved using the LSODA method (via `scipy.integrate.solve_ivp`) with adaptive step control, relative tolerance 10^{-10} and absolute tolerance 10^{-13} . The normalization integral $\int_0^\infty g(s) ds$ is propagated as an additional state variable of the same solver, and the solution was cross-checked with the implicit Radau method and by a Monte Carlo simulation of the reserve process. The limiting constant C^* is extracted by extrapolating the tail coefficient $u^\beta \Psi(u)$ as $u \rightarrow \infty$ (Richardson extrapolation), which gives $C^* \approx 10.269$.

Table 1 lists the values of $\Psi(u)$ and $u^\beta \Psi(u)$; one observes a stable convergence $u^\beta \Psi(u) \rightarrow C^*$, with the remainder $u^\beta \Psi(u) - C^*$ decaying roughly like u^{-1} (as u doubles, the relative error decreases roughly by half)—in qualitative agreement with the known rate-of-convergence estimates for perpetuities [13] (Theorem 3.2).

A rigorous transfer of such estimates to $\Psi(u)$ is a separate question, see Section 5. Figure 1 shows the graph of $\Psi(u)$ on a double logarithmic scale: for large u the curve

approaches a straight line of slope $-\beta \approx -1.1605$ and coincides with the asymptote $C^*u^{-\beta}$, which constitutes the geometric content of Theorem 1.

Table 1. Numerical verification of the asymptotics $\Psi(u) \sim C^*u^{-\beta}$ for the Cramér–Lundberg model with proportional investment ($\kappa = 0.9$, $\beta \approx 1.1605$). Parameters: $\mu = \lambda = 1$, $\tilde{a} = 0.15$, $\tilde{\sigma} = 0.4$, $c = 0.5$, $\tilde{r} = 0.05$. Limiting constant $C^* \approx 10.269$.

u	$\Psi(u)$	$u^\beta \Psi(u)$	$(u^\beta \Psi(u) - C^*)/C^*$
10	4.59×10^{-1}	6.646	-3.53×10^{-1}
50	1.01×10^{-1}	9.438	-8.08×10^{-2}
100	4.70×10^{-2}	9.848	-4.10×10^{-2}
200	2.15×10^{-2}	10.057	-2.06×10^{-2}
500	7.51×10^{-3}	10.184	-8.28×10^{-3}
1000	3.37×10^{-3}	10.226	-4.14×10^{-3}
2000	1.51×10^{-3}	10.247	-2.08×10^{-3}

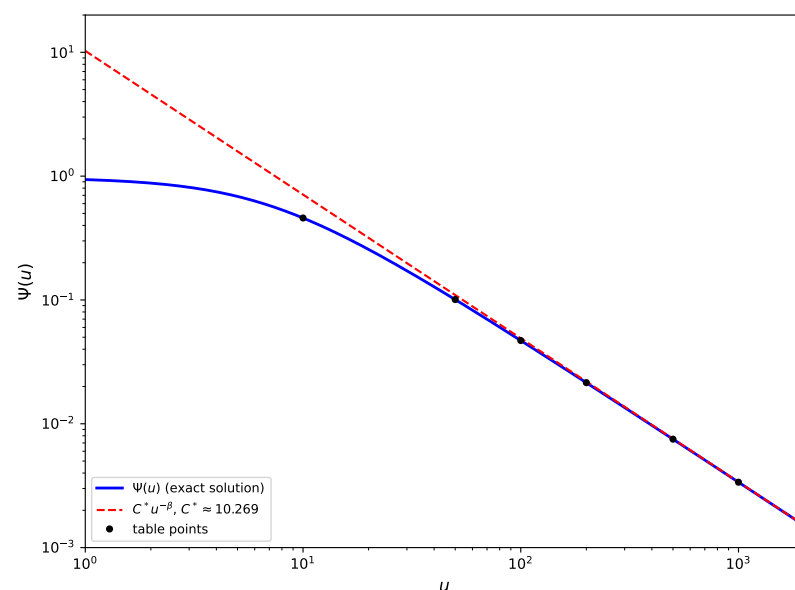


Figure 1. The ruin probability $\Psi(u)$ (blue solid) and its asymptote $C^*u^{-\beta}$ (red dashed) on a double logarithmic scale ($\kappa = 0.9$, $C^* \approx 10.269$). Black dots correspond to the values in Table 1.

5. Discussion

The established result—the exact equality $\lim u^\beta \Psi(u) = C^*$ —gives a complete description of the leading term of the asymptotics of the ruin probability in the Sparre Andersen model for non-life insurance with investments in an arbitrary Lévy process. The exponent β is the Cramér root of the Laplace exponent ψ_V of the Lévy process V , independent of the distribution of the inter-jump times. The distribution F affects only the value of the constant, not the exponent. At the same time, no explicit formula for C^* itself is available in the general model (see below); only the lower bound $C^* \geq C_+^Y$ is known, where C_+^Y has the integral representation (9). This inequality may be strict: the forward sum Y_n is not monotonically increasing in general, because the premium term in Q_k enters with a negative sign, so the supremum M^* may exceed the limit Y_∞ . Compared with the two-sided estimate (1) of [12,14], the new content is precisely the existence of an exact limit rather than merely the order of the asymptotics.

The natural directions for continuing the research are the following.

Annuity models. For annuity payments ($c < 0$, $\xi_i > 0$ a.s.) the trajectories of Y have a positive continuous drift between jumps and negative jumps at the moments T_n ,

so that $\sup_{t \geq 0} Y_t \neq \sup_{n \geq 0} Y_{T_n}$ in general. The discrete reduction of Lemma 1 ceases to work directly: ruin may occur continuously between jumps, and for the Markov property one has to consider the pair (X_t, D_t) with the age process D_t . An analytic study of the annuity model via an integro-differential equation was carried out in [8]. The transfer of the probabilistic method to the annuity case requires a substantial adaptation, since the continuous drift makes the discrete-time reduction inapplicable.

The mixed model. When F_{ζ} charges both half-axes, Y_{∞} takes values of both signs, and the question of the two-sided tail of $\mathcal{L}(Y_{\infty})$ arises: the left tail $\mathbb{P}(Y_{\infty} < -u)$ has the same power-law form $\sim C^- u^{-\beta}$ under appropriate conditions on the support of $\mathcal{L}(Y_{\infty})$ (a two-sided version of Proposition 2.5.4 of [11]). This topic requires a separate analysis, since the sign change of Q_k breaks the monotonicity argument used in the non-life case.

Quantitative rate-of-convergence estimates. The limiting equality obtained does not provide a quantitative estimate of the rate of convergence $|u^{\beta}\Psi(u) - C^*| \leq Ku^{-\delta}$. Such estimates in perpetuity asymptotics are known [13] (Theorem 3.2) under additional conditions on the moment characteristics of the pair (M_1, Q_1) . Their transfer to $\Psi(u)$ through the links we have built is an open problem, as it requires control of the error introduced by the discrete-time reduction and the passage from M^* to $\Psi(u)$.

An explicit formula for C^* . Obtaining an explicit (computable) formula for the constant C^* in the general model is an open question. In the narrow Cramér–Lundberg model with geometric Brownian motion, an exact formula is established by other methods (see Section 4); the agreement of the two approaches on their common intersection gives a nontrivial cross-check of both programs.

Multi-dimensional extension. A natural generalization is the multi-dimensional case, i.e. insurance portfolios with multiple lines of business investing in a vector Lévy process, which fits the multivariate Kesten–Goldie framework (see, e.g., the works of Basrak, Davis, and Mikosch on multivariate regular variation). A full discussion is beyond the scope of this paper.

Another meaningful direction is the relaxation of the moment condition on the claim sizes $\mathbb{E}[|\xi|^{\beta}] < \infty$, which excludes heavy-tailed claims with regularly varying tails. Within the implicit renewal theory, regularly varying claim sizes are handled by a different mechanism, in which the claim tail rather than the Cramér root drives the asymptotics (the Grincevicius–Grey theorem cited below); an adaptation is thus possible, but it yields an asymptotics of a different nature. This direction already has some results. In [22], a model with investments in general semimartingales and heavy-tailed claims is considered. In [6], concrete asymptotic estimates are obtained for a model with investments at a constant interest rate under regularly varying claim tails. The theory of regularly varying functions and their probabilistic applications is presented in [23], and the classical theory of extremal events in insurance and finance in [24]. The transfer of the apparatus of this paper to the heavy-tailed case is not straightforward. When the condition $\mathbb{E}[|\xi|^{\beta}] < \infty$ is dropped, Goldie’s theorem ceases to give a purely power-law asymptotics with exponent β , and the asymptotic behavior is determined by the competition between the claim tail and the Cramér tail of the perpetuity. When the tail of Q_1 varies regularly with exponent less than β , the tail of Y_{∞} inherits the same regular variation. This result was first established by Grincevicius [25]. An improved proof and the converse direction were given by Grey [26].

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