

Article

Synchronization of Discrete-Time Inertial Neural Networks Using the General Theory of Solutions of Linear Difference Equations

Zheng Zhou ^{1,*}, Zhen Yang ^{2,*} and Zhengqiu Zhang ³¹ School of Mathematics and Statistics, Xiamen University of Technology, Xiamen 361024, China² School of Science, Hubei University of Technology, Wuhan 430068, China³ College of Mathematics, Hunan University, Changsha 410082, China

* Correspondence: zhouzheng@xmut.edu.cn (Z.Z.); yangzhen@hbut.edu.cn (Z.Y.)

Abstract

This paper investigates the quasi-synchronization (QS) problem for drive-response discrete-time delayed inertial neural networks (DTDINNS). Unlike existing studies that mainly rely on classical stability theorems, linear matrix inequality (LMI) methods, and matrix measure approaches (MMA), this work establishes three innovative quasi-synchronization criteria for DTDINNS by adopting the solution formula of second-order linear difference equations (SOLDES), infinite series summation techniques, and the solution formula of first-order linear difference equation group (Lemma 6). To the best of our knowledge, this study is the first attempt to introduce the general solution theory of second-order linear difference equations and infinite series summation methods to analyze the synchronization behavior of neural networks (NNS). The proposed framework offers a novel theoretical tool for the synchronization analysis of discrete-time delayed neural networks (DTDNNS), which bears important theoretical significance for relevant research fields.

Keywords: discrete-time inertial neural networks; quasi-synchronization; solution formula; linear difference equations; infinite series summation

MSC: 39A24; 92B20; 34D06; 40A05

1. Introduction

Over the past two decades, neural networks (NNS) have garnered extensive research attention owing to their widespread applications in signal processing, pattern recognition, model identification, associative memory, system optimization, and other engineering and scientific fields. Compared with continuous-time NNS, discrete-time neural networks (DTDNNS) exhibit superior practicability in digital system simulation and numerical computation, making them more suitable for digital engineering applications. Accordingly, DTDNNS have been intensively studied by numerous scholars [1–6].

Synchronization, as a core dynamic behavior of neural networks, has attracted considerable research interest owing to its promising application prospects in secure communication, information processing, and image recognition. To date, substantial achievements have been made in the synchronization control of various neural network systems. Specifically, the synchronization problems of DTDNNS have been widely investigated in existing literature, as summarized in [7–19] and the references cited therein. Reference [7] addressed the synchronization issue of time-delay DTDNNs under unreliable communication links



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with stochastic data dropout described by a two-state discrete-time Markov chain. Based on the Lyapunov functional approach (LFA), several linear matrix inequality (LMI)-based synchronization criteria were derived, and the existence conditions of feedback controllers were further established, with controller gains obtained by solving the corresponding LMIs. In [8], a class of discrete-time fractional-order complex-valued fuzzy NNS was studied, and simplified algebraic synchronization criteria were established via innovative inequality techniques and constructed lemmas. Reference [9] explored the quasi-projective synchronization (QPS) of discrete-time fractional-order complex-valued NNS, where QPS criteria were derived by combining inequality techniques, LFA, and discrete fractional calculus lemmas. The synchronization of delayed DTDNNS with saturated time-delay feedback was investigated in [10]. Based on the drive-response mechanism, two local synchronization conditions in the form of LMIs were established via generalized delay-dependent sector conditions, improved Lyapunov–Krasovskii functionals (LKFs), and finite sum inequalities, while global synchronization conditions were further proposed based on classical sector conditions. Reference [11] studied the synchronization of DTDNNS with Markov jump topologies, and delay-dependent asymptotic synchronization conditions were obtained by adopting LMI methods and novel Lyapunov functional designs. In [12], the global Mittag–Leffler synchronization (MLS) problem for DTDNNS was solved, and effective MLS criteria were derived by integrating improved inequalities, LFA, and nabla Mittag–Leffler function properties. Reference [13] investigated the mean-square asymptotic synchronization of discrete-time coupled complex-valued neural network arrays by virtue of LFA and Kronecker product theory. The saturated time-delay feedback synchronization problem for delayed DTDNNS was revisited in [10], where local synchronization criteria were established using the drive-response framework, generalized sector conditions, LKFs, and finite sum inequalities. In [14], the exponential synchronization of DTDNNS was realized via the LFA-based method, with rigorous exponential synchronization criteria derived. Reference [11] further proposed sufficient asymptotic synchronization conditions for DTDNNS through LMI techniques and optimized Lyapunov function construction. By combining LKF methods with stochastic analysis theory, Ref. [15] designed effective control laws to achieve the synchronization of stochastic DTDNNS. Reference [16] explored the local exponential synchronization of DTDNNS with time-varying delays via memory-based saturated controllers, and LMI-based local exponential stability criteria for closed-loop error systems were established using modified sector conditions and augmented Lyapunov functionals. In [17], the dynamic stability and synchronization of DTDNNS with discontinuous activation functions were analyzed, and two novel controllers were designed under weaker constraints than the classical Lipschitz condition to guarantee network synchronization. Reference [18] studied a class of generalized discrete-time inertial neural networks. Based on timescale theory, the original network model was transformed into a timescale-type inertial neural network, and global exponential synchronization (GES) criteria were derived via the generalized matrix measure approach (MMA). In [19], the quasi-projective synchronization of master–slave discrete-time BAM neural networks was investigated, and two QPS criteria were established using discrete inequalities and Lyapunov sequences without relying on LMI and MMA methods. Reference [20] realized the robust synchronization and anti-synchronization of DTDNNS via output feedback control, and novel weighted summation inequalities were developed to handle quadratic summation terms in LKF forward differences, yielding LMI-based synchronization conditions. In [21], robust mean-square synchronization criteria for DTDNNs were proposed by combining innovative LKF methods with Kronecker product theory. Reference [22] adopted piecewise Lyapunov function and piecewise LKF methods to analyze the synchronization of DTDNNS under aperiodic intermittent control. In [23], the quasi-projective synchronization of discrete-time

fractional-order quaternion-valued NNS was explored, and dedicated synchronization conditions were established based on newly constructed nabla fractional difference and quaternion inequalities. In [24], the global asymptotic synchronization (GAS) of discrete-time BAM neural networks was investigated, and two GAS criteria were derived using the Laplace transform method.

In summary, existing research on the diverse synchronization behaviors of DTDNNS predominantly adopts classical analytical methods, including LMI techniques [7,8,10–12,16,20,21], LFA [9,10,13–15,22], Laplace transform methods [24], MMA [18], and discrete inequality techniques [19,23]. Furthermore, most existing synchronization achievements for NNS depend on controller design, while studies focusing on controller-free synchronization mechanisms remain relatively scarce.

This research gap motivates us to develop innovative analytical approaches for neural network synchronization, covering both controller-based and controller-free synchronization scenarios. Driven by the above research progress and unresolved limitations, this paper focuses on the quasi-synchronization problem of discrete-time delayed inertial neural networks (DTDINNs). Note that most existing literature only employs single-type analysis tools (either Lyapunov-based schemes or simple discrete inequalities), lacking a unified analytical framework built upon linear difference equation fundamental theory, which creates an evident theoretical gap to be filled. Unlike the conventional methods adopted in prior literature, this work firstly establishes two novel controller-free quasi-synchronization criteria by utilizing the solution formula of second-order linear difference equations and infinite series summation approaches without extra control inputs. Meanwhile, an additional quasi-synchronization criterion is derived via simple linear state feedback controller design based on the solution formula of first-order linear difference equation group (Lemma 6). Although several prior works have studied the synchronization of discrete-time BAM NNS [19,24], the analytical frameworks and theoretical conclusions presented in this paper are significantly innovative and distinct from the published literature. Notably, this study systematically investigates two synchronization regimes for DTDINNs, namely controller-free quasi-synchronization equipped with two novel criteria and controller-based quasi-synchronization with one supplementary criterion. To date, such complete quasi-synchronization analysis grounded on the general solution theory of second-order difference equations and first-order difference matrix recursions has not been reported in any published work. Thus the present paper fills the aforementioned research gap, namely, this work is the first to systematically introduce the general solution theory of second-order linear difference equations, infinite series summation techniques and the solution formula of first-order difference equation group into the quasi-synchronization analysis of DTDINNs, along with improved discrete inequality tools. The main contributions of this paper are summarized as follows:

- (1) Unlike the mainstream LMI, MMA and Lyapunov functional-based analysis frameworks widely adopted in existing literature, this paper constructs a brand-new analytical paradigm rooted in the complete solution theory of linear difference equations. Specifically, we utilize the general solution formulas of second-order linear difference equations, infinite series summation techniques and the recursion relation of first-order linear difference matrix groups to analyze the quasi-synchronization (QS) of discrete-time delayed inertial neural networks;
- (2) Refined discrete sequence inequality estimation schemes are put forward to bound the nonlinear coupling terms within error systems, which substantially simplifies the stability derivation procedure;

- (3) Two types of sufficient quasi-synchronization conditions are separately deduced: controller-free QS criteria and linear state feedback controller-dependent QS criterion, which correspond to two typical practical operation modes of inertial neural networks.

2. Preliminary

In this research, we consider a kind of discrete-time inertial NNS with time delays expressed by the following difference equation:

$$\begin{aligned}\Delta^2 p_\rho(\xi) &= -\sigma_\rho \Delta p_\rho(\xi) - \omega_\rho p_\rho(\xi) + \sum_{\chi=1}^{\kappa} \eta_{\rho\chi} M_\chi(\xi) \\ &\quad + \sum_{\chi=1}^{\kappa} \zeta_{\rho\chi} M_\chi(\xi - \delta) + u_\rho(\xi),\end{aligned}\quad (1)$$

with the time-delay initial states (TINS, Time-Delay Initial States):

$$p_\rho(v) = x_\rho(v), v = -\delta, -\delta + 1, \dots, 0;$$

$$\Delta p_\rho(v) = y_\rho(v + 1) - \phi_\rho(v), v = -\delta, -\delta + 1, \dots, 0,$$

where $\xi \in \mathbb{Z}$, $\rho, \chi \in I_\kappa = \{1, 2, \dots, \kappa\}$, $p_\rho(\xi) \in \mathbb{R}$ is the state of neuron ρ at time ξ ; $\Delta p_\rho(\xi) = p_\rho(\xi + 1) - p_\rho(\xi)$; the second-order difference term $\Delta^2 p_\rho(\xi)$ is known as an inertial term of Equation (1); $u_\rho(\xi)$ is the external input; $M_\chi(\cdot)$ denotes the activation function; $\sigma_\rho > 0, \omega_\rho > 0, \eta_{\rho\chi}, \zeta_{\rho\chi}$ represent the time-varying coupling connection weights; and $\delta \in \mathbb{Z}$ is the time delay with $0 < \delta$ being a known integer.

We regard (1) as the drive system, and consider the response system:

$$\begin{aligned}\Delta^2 q_\rho(\xi) &= -\sigma_\rho \Delta q_\rho(\xi) - \omega_\rho q_\rho(\xi) + \sum_{\chi=1}^{\kappa} \eta_{\rho\chi} M_\chi(\xi) \\ &\quad + \sum_{\chi=1}^{\kappa} \zeta_{\rho\chi} M_\chi(\xi - \delta) + u_\rho(\xi),\end{aligned}\quad (2)$$

with TINS:

$$q_\rho(v) = \psi_\rho(v), v = -\delta, -\delta + 1, \dots, 0;$$

$$\Delta q_\rho(v) = \psi_\rho(v + 1) - \phi_\rho(v), v = -\delta, -\delta + 1, \dots, 0,$$

where $q_\rho(\xi)$ denotes the state of the ρ -th neuron in response system (2), and all parameters in network (2) are identical to those in network (1).

Definition 1. Drive system (1) and response system (2) are said to achieve the QS if there exists one positive constant a such that for arbitrary solutions of system (1) and system (2) denoted by $[p_1(\xi), p_2(\xi), \dots, p_\kappa(\xi)]^T$ and $[q_1(\xi), q_2(\xi), \dots, q_\kappa(\xi)]^T$, we have

$$\limsup_{\xi \rightarrow \infty} |p_i(\xi) - q_i(\xi)| \leq a.$$

Definition 2 ([25]). $y_1(\xi)$ and $y_2(\xi)$ are linearly independent if $\frac{y_1(\xi)}{y_2(\xi)} \neq \text{constant}$.

Lemma 1 ([25]). Assume that p_1, p_2 are two constants. Consider the following difference equation:

$$y(\xi + 2) + p_1 y(\xi + 1) + p_2 y(\xi) = g(\xi) \quad (3)$$

and its corresponding homogeneous equation:

$$y(\xi + 2) + p_1 y(\xi + 1) + p_2 y(\xi) = 0. \quad (4)$$

If $y_1(\xi)$ and $y_2(\xi)$ are two linearly independent special solutions of Equation (4), then

$$\bar{y}(\xi) = c_1 y_1(\xi) + c_2 y_2(\xi)$$

is the general solution of (4). Further,

$$\hat{y}(\xi) = y_1(\xi) \sum_{\chi=0}^{\xi-1} -\frac{y_2(\chi+1)g(\chi)}{\det C(\chi+1)} + y_2(\xi) \sum_{\chi=0}^{\xi-1} \frac{y_1(\chi+1)g(\chi)}{\det C(\chi+1)}$$

is a particular solution of (3), where

$$\begin{aligned} & \det C(\xi + 1) \\ &= \begin{vmatrix} y_1(\xi + 1) & y_2(\xi + 1) \\ y_1(\xi + 2) & y_2(\xi + 2) \end{vmatrix} \\ &= y_1(\xi + 1)y_2(\xi + 2) - y_1(\xi + 2)y_2(\xi + 1). \end{aligned}$$

Thus

$$y(\xi) = \hat{y}(\xi) + \bar{y}(\xi)$$

gives the complete general solution of Equation (3).

Remark 1. Lemma 1 is adopted to derive the general solution of the nonhomogeneous error difference Equation (7) in Theorems 1 and 2.

Lemma 2 ([25]). The following algebraic relation is defined as the characteristic equation of (4):

$$r^2 + p_1 r + p_2 = 0 \quad (5)$$

If (5) admits two distinct real roots r_1, r_2 , then r_1^ξ, r_2^ξ are linearly independent fundamental solutions of (4), and the general solution reads $c_1 r_1^\xi + c_2 r_2^\xi$. If (5) has a repeated real root $r_1 = r_2$, then $r_1^\xi, \xi r_1^\xi$ form a pair of linearly independent solutions, and the general solution is $c_1 r_1^\xi + c_2 \xi r_1^\xi$. If (5) generates a pair of conjugate complex roots $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$, then $\hat{y}_1 = \rho^\xi \cos(\xi\theta)$, $\hat{y}_2 = \rho^\xi \sin(\xi\theta)$ are linearly independent solutions, where $\rho = \sqrt{\alpha^2 + \beta^2}, \cos \theta = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}, \sin \theta = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}$.

Remark 2. Lemma 2 is utilized to solve the homogeneous error Equation (8) in the proofs of Theorems 1 and 2.

Lemma 3. For any scalar $p \neq 1$, the finite geometric series satisfies

$$\sum_{u=0}^{\xi} p^u = \frac{1 - p^{\xi+1}}{1 - p}.$$

Proof. This well-known result is trivial, and we omit the proof here. $\square$

Remark 3. Lemma 3 is applied to evaluate infinite series bounds in Theorems 1–3.

Lemma 4 ([25]). Suppose $p(\xi) \geq 0, \xi \geq 0$, and the inequality

$$y(\xi + 1) \leq y(0) + \sum_{v=0}^{\xi} [p(v)y(v) + q(v)]$$

holds. Then we have the estimation

$$y(\xi) \leq y(0) \exp\left(\sum_{v=0}^{\xi-1} p(v)\right) + \sum_{v=0}^{\xi-1} q(v) \exp\left(\sum_{u=v+1}^{\xi-1} p(u)\right).$$

Remark 4. Lemma 4 is used to perform the sequence bounding in all three main theorems.

Lemma 5. If $|r| < 1$, then

$$\lim_{\xi \rightarrow +\infty} \xi |r|^\xi = 0.$$

Proof. The derivation is straightforward and omitted. $\square$

Remark 5. Lemma 5 is essential to take the limit for error bounds in Theorems 1 and 2.

Remark Notation Preview: For the subsequent Theorems 1 and 2, we will introduce characteristic-root-related constants α_1, β, β^* derived from the coefficients of the homogeneous error equation, which are defined right before Theorems 1 and 2 separately for targeted analysis.

In this research, we impose the standing assumption below:

(k_1) Function M_χ satisfies the Lipschitz condition, i.e., there exists constant $l_\chi > 0$ such that

$$|M_\chi(\theta_1) - M_\chi(\theta_2)| \leq l_\chi |\theta_1 - \theta_2|,$$

for all $\theta_1, \theta_2 \in \mathbb{R}, \chi = 1, 2, \dots, \kappa$, where $|\cdot|$ denotes the absolute norm on Euclidean space $\mathbb{R}$.

3. Quasi-Synchronization Without Employing the Controllers

Section Prelude: Analysis Framework This section focuses on the quasi-synchronization (QS) analysis of drive-response DTDINNs without auxiliary control inputs. The original second-order inertial difference error system is rearranged into standard second-order linear nonhomogeneous difference equations, whose complete general solutions are constructed via Lemmas 1 and 2 (the fundamental theory of second-order linear difference equations). All characteristic-root-related coefficients and Lipschitz bounding constants required in Theorems 1 and 2 are defined in advance below to avoid repeated page jumps during proof reading.

Predefined Symbols for This Section

Characteristic root coefficients for homogeneous error equation:

$$\alpha_1 = 0.5\sqrt{(\sigma_\rho - 2)^2 - 4(\omega_\rho + 1 - \sigma_\rho)}, \quad \beta = 0.5\sqrt{4(\omega_\rho + 1 - \sigma_\rho) - (\sigma_\rho - 2)^2},$$

$$r_{1\rho} = 1 - \frac{1}{2}\sigma_\rho - \alpha_1, \quad r_{2\rho} = 1 - \frac{1}{2}\sigma_\rho + \alpha_1, \quad r_{3\rho} = 1 - \frac{1}{2}\sigma_\rho - i\beta, \quad r_{4\rho} = 1 - \frac{1}{2}\sigma_\rho + i\beta,$$

$$\beta^* = \sqrt{(1 - \frac{1}{2}\sigma_\rho)^2 + \beta^2}, \quad \cos \theta = \frac{1 - \frac{1}{2}\sigma_\rho}{\beta^*}, \quad \sin \theta = \frac{\beta}{\beta^*}.$$

Lipschitz-coupling uniform bounds for uncontrolled error estimation:

$$\mu_{1\rho} = \frac{1}{2\alpha_1} \max_{1 \leq \chi \leq \kappa} \{l_\chi |\eta_{\rho\chi}|\}, \quad \mu_{2\rho} = \frac{1}{2\alpha_1} \max_{1 \leq \chi \leq \kappa} \{l_\chi |\zeta_{\rho\chi}|\},$$

$$\mu_{3\rho} = \max_{1 \leq \chi \leq \kappa} \{l_\chi |\eta_{\rho\chi}|\}, \quad \mu_{4\rho} = \max_{1 \leq \chi \leq \kappa} \{l_\chi |\zeta_{\rho\chi}|\}.$$

Initial delay error accumulation constants:

$$\delta_0 = \frac{1}{2\alpha_1} \sum_{v=-\delta}^0 \frac{\sum_\chi l_\chi |\zeta_{\rho\chi}| |e_\chi(v)|}{|r_{1\rho}|^{v+1+\delta}}, \quad \delta_1 = \frac{1}{2\alpha_1} \sum_{v=-\delta}^0 \frac{\sum_\chi l_\chi |\zeta_{\rho\chi}| |e_\chi(v)|}{|r_{2\rho}|^{v+1+\delta}},$$

$$\delta_2 = \mu_{4\rho} \sum_{v=-\delta}^0 (\beta^*)^{-v-2} \sum_{\rho=1}^\kappa |e_\rho(v)|.$$

In this section, we give two sufficient criteria on QS for the considered networks without adopting extra controllers. Define the synchronization error signal as $e_\rho(\xi) = p_\rho(\xi) - q_\rho(\xi)$ for each $\rho = 1, 2, \dots, \kappa$. By subtracting response system (2) from drive system (1), the error dynamic system yields:

$$\begin{aligned} \Delta^2 e_\rho(\xi) = & -\sigma_\rho \Delta e_\rho(\xi) - \omega_\rho e_\rho(\xi) + \sum_{\chi=1}^\kappa \eta_{\rho\chi} [M_\chi(p_\chi(\xi)) - M_\chi(q_\chi)] \\ & + \sum_{\chi=1}^\kappa \zeta_{\rho\chi} [M_\chi(p_\chi(\xi - \delta)) - M_\chi(q_\chi(\xi - \delta))]. \end{aligned} \quad (6)$$

Using the second-order difference identity $\Delta^2 e_\rho(\xi) = e_\rho(\xi + 2) - 2e_\rho(\xi + 1) + e_\rho(\xi)$, we rearrange Equation (6) into the canonical form of second-order linear nonhomogeneous difference equation consistent with Lemma 1:

$$\begin{aligned} & e_\rho(\xi + 2) + (\sigma_\rho - 2)e_\rho(\xi + 1) + (\omega_\rho + 1 - \sigma_\rho)e_\rho(\xi) \\ = & \sum_{\chi=1}^\kappa \eta_{\rho\chi} [M_\chi(p_\chi(\xi)) - M_\chi(q_\chi)] + \sum_{\chi=1}^\kappa \zeta_{\rho\chi} [M_\chi(p_\chi(\xi - \delta)) - M_\chi(q_\chi(\xi - \delta))]. \end{aligned} \quad (7)$$

According to Definition 1, to verify quasi-synchronization between (1) and (2), it suffices to prove that there exists a positive constant a such that

$$\limsup_{\xi \rightarrow +\infty} |e_\rho(\xi)| \leq a.$$

Theorem 1. Assume that standing assumption (k_1) (Lipschitz condition of activation functions) holds. Then drive system (1) and response system (2) achieve quasi-synchronization (QS), provided the following two conditions hold:

- (I_1) : $(\sigma_\rho - 2)^2 > 4(\omega_\rho + 1 - \sigma_\rho)$;
- (I_2) : $|r_{1\rho}| < 1, |r_{2\rho}| < 1$.

Proof. Step 1: Derive the homogeneous error equation and characteristic roots. The homogeneous part corresponding to Equation (7) reads

$$e_\rho(\xi + 2) + (\sigma_\rho - 2)e_\rho(\xi + 1) + (\omega_\rho + 1 - \sigma_\rho)e_\rho(\xi) = 0. \quad (8)$$

Its characteristic equation is

$$r^2 + (\sigma_\rho - 2)r + (\omega_\rho + 1 - \sigma_\rho) = 0. \quad (9)$$

Condition (l_1) guarantees the discriminant is positive, so Equation (9) has two distinct real roots $r_{1\rho} = 1 - \frac{1}{2}\sigma_\rho - \alpha_1$, $r_{2\rho} = 1 - \frac{1}{2}\sigma_\rho + \alpha_1$. By Lemma 2, the general solution of homogeneous system (8) is

$$\bar{e}_\rho(\xi) = c_{1\rho}r_{1\rho}^\xi + c_{2\rho}r_{2\rho}^\xi, \quad (10)$$

where $c_{1\rho}, c_{2\rho}$ are arbitrary constant coefficients determined by initial states.

Step 2: Write a complete general solution of nonhomogeneous error equation. Applying the variation-of-constant formula in Lemma 1, the full solution of Equation (7) is

$$e_\rho(\xi) = c_{1\rho}r_{1\rho}^\xi + c_{2\rho}r_{2\rho}^\xi + r_{1\rho}^\xi \sum_{\nu=0}^{\xi-1} -\frac{r_{2\rho}^{\nu+1}g(\nu)}{\det C(\nu+1)} + r_{2\rho}^\xi \sum_{\nu=0}^{\xi-1} \frac{r_{1\rho}^{\nu+1}g(\nu)}{\det C(\nu+1)}, \quad (11)$$

where the Wronskian determinant satisfies $\det C(\nu+1) = 2\alpha_1(r_{1\rho}r_{2\rho})^{\nu+1}$, and $g(\nu)$ denotes the nonlinear coupling term on the right-hand side of (7).

Step 3: Pointwise absolute value estimation via the Lipschitz condition (k_1) . Take the absolute value on both sides of (11), and bound the activation difference term by $|M_\chi(a) - M_\chi(b)| \leq l_\chi|a - b|$ from assumption (k_1) :

$$\begin{aligned} |e_\rho(\xi)| &\leq |c_{1\rho}||r_{1\rho}|^\xi + |c_{2\rho}||r_{2\rho}|^\xi + |r_{1\rho}|^\xi \sum_{\nu=0}^{\xi-1} \frac{r_{2\rho}^{\nu+1}|g(\nu)|}{|\det C(\nu+1)|} + |r_{2\rho}|^\xi \sum_{\nu=0}^{\xi-1} \frac{r_{1\rho}^{\nu+1}|g(\nu)|}{|\det C(\nu+1)|} \\ &\leq |c_{1\rho}||r_{1\rho}|^\xi + |c_{2\rho}||r_{2\rho}|^\xi + |r_{1\rho}|^\xi \sum_{\nu=0}^{\xi-1} \frac{\sum_{\chi=1}^{\kappa} l_\chi [|\eta_{\rho\chi}||e_\chi(\nu)| + |\zeta_{\rho\chi}||e_\chi(\nu-\delta)|]}{2\alpha_1|r_{1\rho}|^{\nu+1}} \\ &\quad + |r_{2\rho}|^\xi \sum_{\nu=0}^{\xi-1} \frac{\sum_{\chi=1}^{\kappa} l_\chi [|\eta_{\rho\chi}||e_\chi(\nu)| + |\zeta_{\rho\chi}||e_\chi(\nu-\delta)|]}{2\alpha_1|r_{2\rho}|^{\nu+1}} \\ &= |c_{1\rho}||r_{1\rho}|^\xi + |c_{2\rho}||r_{2\rho}|^\xi + |r_{1\rho}|^\xi \left(\sum_{\nu=0}^{\xi-1} \frac{\mu_{1\rho}\Theta(\nu)}{|r_{1\rho}|^{\nu+1}} + \delta_0 + \mu_{2\rho} \sum_{\nu=0}^{\xi-1} \frac{\Theta(\nu)}{|r_{1\rho}|^{\nu+1+\delta}} \right) \\ &\quad + |r_{2\rho}|^\xi \left(\sum_{\nu=0}^{\xi-1} \frac{\mu_{1\rho}\Theta(\nu)}{|r_{2\rho}|^{\nu+1}} + \delta_1 + \mu_{2\rho} \sum_{\nu=0}^{\xi-1} \frac{\Theta(\nu)}{|r_{2\rho}|^{\nu+1+\delta}} \right). \end{aligned} \quad (12)$$

Define the total error summation $\Theta(\xi) = \sum_{\rho=1}^{\kappa} |e_\rho(\xi)|$. By summing both sides of (12) over $\rho = 1, \dots, \kappa$, we obtain

$$\begin{aligned} \Theta(\xi) &\leq \sum_{\rho=1}^{\kappa} |c_{1\rho}||r_{1\rho}|^\xi + \sum_{\rho=1}^{\kappa} |c_{2\rho}||r_{2\rho}|^\xi \\ &\quad + |r_{1\rho}|^\xi \sum_{\rho=1}^{\kappa} \left(\mu_{1\rho} \sum_{\nu=0}^{\xi-1} \frac{\Theta(\nu)}{|r_{1\rho}|^{\nu+1}} + \delta_0 + \mu_{2\rho} \sum_{\nu=0}^{\xi-1} \frac{\Theta(\nu)}{|r_{1\rho}|^{\nu+1+\delta}} \right) \\ &\quad + |r_{2\rho}|^\xi \sum_{\rho=1}^{\kappa} \left(\sum_{\nu=0}^{\xi-1} \frac{\mu_{1\rho}\Theta(\nu)}{|r_{2\rho}|^{\nu+1}} + \delta_1 + \mu_{2\rho} \sum_{\nu=0}^{\xi-1} \frac{\Theta(\nu)}{|r_{2\rho}|^{\nu+1+\delta}} \right) + \Theta(0). \end{aligned} \quad (13)$$

Step 4: Apply the discrete Gronwall inequality (Lemma 4). Inequality (13) takes the standard recursive form of Lemma 4, which provides an exponential upper bound for $\Theta(\xi)$:

$$\begin{aligned}\Theta(\xi) \leq & \Theta(0) \exp \left\{ \sum_{v=0}^{\xi-1} \sum_{\rho=1}^{\kappa} \left(|r_{1\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{1\rho}|^{v+1}} + \frac{\mu_{2\rho}}{|r_{1\rho}|^{v+1+\delta}} + \delta_0 \right] \right. \right. \\ & \left. \left. + |r_{2\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{2\rho}|^{v+1}} + \frac{\mu_{2\rho}}{|r_{2\rho}|^{v+1+\delta}} + \delta_1 \right] \right) \right\} \\ & + \sum_{v=0}^{\xi-1} \left\{ \sum_{\rho=1}^{\kappa} \left(\left(\frac{|c_{1\rho}|}{\xi} + \delta_0 \right) |r_{1\rho}|^{\xi} + \left(\frac{|c_{2\rho}|}{\xi} + \delta_1 \right) |r_{2\rho}|^{\xi} \right) \right\} \times \\ & \exp \left\{ \sum_{u=v+1}^{\xi-1} \sum_{\rho=1}^{\kappa} \left(|r_{1\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{1\rho}|^{u+1}} + \frac{\mu_{2\rho}}{|r_{1\rho}|^{u+1+\delta}} + \delta_0 \right] \right. \right. \\ & \left. \left. + |r_{2\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{2\rho}|^{u+1}} + \frac{\mu_{2\rho}}{|r_{2\rho}|^{u+1+\delta}} + \delta_1 \right] \right) \right\},\end{aligned}$$

Step 5: Take $\xi \rightarrow +\infty$ and analyze the limit behavior. Compute the limit superior of both sides as $\xi \rightarrow +\infty$.

$$\begin{aligned}0 & \leq \limsup_{\xi \rightarrow \infty} \Theta(\xi) \\ & \leq \limsup_{\xi \rightarrow \infty} \left\{ \Theta(0) \exp \left(\sum_{v=0}^{\xi-1} \sum_{\rho=1}^{\kappa} \left(|r_{1\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{1\rho}|^{v+1}} + \frac{\mu_{2\rho}}{|r_{1\rho}|^{v+1+\delta}} + \delta_0 \right] \right. \right. \right. \\ & \quad \left. \left. + |r_{2\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{2\rho}|^{v+1}} + \frac{\mu_{2\rho}}{|r_{2\rho}|^{v+1+\delta}} + \delta_1 \right] \right) \right) \\ & \quad \left. + \sum_{v=0}^{\xi-1} \sum_{\rho=1}^{\kappa} \left(\left(\frac{|c_{1\rho}|}{\xi} + \delta_0 \right) |r_{1\rho}|^{\xi} + \left(\frac{|c_{2\rho}|}{\xi} + \delta_1 \right) |r_{2\rho}|^{\xi} \right) \right\} \exp \left\{ \sum_{u=v+1}^{\xi-1} \sum_{\rho=1}^{\kappa} \left(|r_{1\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{1\rho}|^{u+1}} \right. \right. \right. \\ & \quad \left. \left. + \frac{\mu_{2\rho}}{|r_{1\rho}|^{u+1+\delta}} + \delta_0 \right] + |r_{2\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{2\rho}|^{u+1}} + \frac{\mu_{2\rho}}{|r_{2\rho}|^{u+1+\delta}} + \delta_1 \right] \right) \right\} \\ & \leq \limsup_{\xi \rightarrow \infty} \left\{ \Theta(0) \exp \left(\sum_{v=0}^{\xi-1} \sum_{\rho=1}^{\kappa} \left(|r_{1\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{1\rho}|^{v+1}} + \frac{\mu_{2\rho}}{|r_{1\rho}|^{v+1+\delta}} + \delta_0 \right] \right. \right. \right. \\ & \quad \left. \left. + |r_{2\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{2\rho}|^{v+1}} + \frac{\mu_{2\rho}}{|r_{2\rho}|^{v+1+\delta}} + \delta_1 \right] \right) \right) \\ & \quad \left. + \sum_{v=0}^{\xi-1} \sum_{\rho=1}^{\kappa} \left(\left(\frac{|c_{1\rho}|}{\xi} + \delta_0 \right) |r_{1\rho}|^{\xi} + \left(\frac{|c_{2\rho}|}{\xi} + \delta_1 \right) |r_{2\rho}|^{\xi} \right) \exp \left\{ \sum_{u=0}^{\xi-1} \sum_{\rho=1}^{\kappa} \left(|r_{1\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{1\rho}|^{u+1}} \right. \right. \right. \right. \\ & \quad \left. \left. + \frac{\mu_{2\rho}}{|r_{1\rho}|^{u+1+\delta}} + \delta_0 \right] + |r_{2\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{2\rho}|^{u+1}} + \frac{\mu_{2\rho}}{|r_{2\rho}|^{u+1+\delta}} + \delta_1 \right] \right) \right\} \right\}. \quad (14)\end{aligned}$$

From Lemma 3, and $|r_{1\rho}| < 1, |r_{2\rho}| < 1$ from condition (l_2) , one has

$$\begin{aligned}& \limsup_{\xi \rightarrow \infty} \sum_{\rho=1}^{\kappa} \exp \left\{ \sum_{v=0}^{\xi-1} \sum_{\rho=1}^{\kappa} \left(|r_{1\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{1\rho}|^{v+1}} + \mu_{2\rho} \frac{1}{|r_{1\rho}|^{v+1+\delta}} + \delta_0 \right] \right. \right. \\ & \quad \left. \left. + |r_{2\rho}|^{\xi} \left[\frac{\mu_{1\rho}}{|r_{2\rho}|^{v+1}} + \frac{\mu_{2\rho}}{|r_{2\rho}|^{v+1+\delta}} + \delta_1 \right] \right) \right\} \\ & = \limsup_{\xi \rightarrow \infty} \exp \left\{ \sum_{\rho=1}^{\kappa} \frac{(\mu_{1\rho} + \mu_{2\rho}|r_{1\rho}|^{-\delta})[|r_{1\rho}|^{\xi} - 1]}{|r_{1\rho}| - 1} + \sum_{\rho=1}^{\kappa} \frac{(\mu_{1\rho} + \mu_{2\rho}|r_{2\rho}|^{-\delta})[|r_{2\rho}|^{\xi} - 1]}{|r_{2\rho}| - 1} \right\} \\ & = \exp \left\{ \sum_{\rho=1}^{\kappa} \left\{ \frac{(\mu_{1\rho} + \mu_{2\rho}|r_{1\rho}|^{-\delta})}{1 - |r_{1\rho}|} + \frac{(\mu_{1\rho} + \mu_{2\rho}|r_{2\rho}|^{-\delta})}{1 - |r_{2\rho}|} \right\} \right\} \quad (15)\end{aligned}$$

and

$$\begin{aligned} & \limsup_{\xi \rightarrow \infty} \sum_{\nu=0}^{\xi-1} \sum_{\rho=1}^{\kappa} \left(\left[\frac{|c_{1\rho}|}{\xi} + \delta_0 \right] |r_{1\rho}|^{\xi} + \left[\frac{|c_{2\rho}|}{\xi} + \delta_1 \right] |r_{2\rho}|^{\xi} \right) \\ &= \limsup_{\xi \rightarrow \infty} \sum_{\rho=1}^{\kappa} \left([|c_{1\rho}| + \xi \delta_0] |r_{1\rho}|^{\xi} + (|c_{2\rho}| + \xi \delta_1) |r_{2\rho}|^{\xi} \right). \end{aligned} \quad (16)$$

Lemma 5 gives $\xi |r_{1\rho}|^{\xi} \rightarrow 0$, $\xi |r_{2\rho}|^{\xi} \rightarrow 0$, so all terms containing $\xi \cdot |r_{\rho}|^{\xi}$ vanish at infinity; constant homogeneous solution terms $|c_{1\rho}| |r_{1\rho}|^{\xi} \rightarrow 0$, $|c_{2\rho}| |r_{2\rho}|^{\xi} \rightarrow 0$. After substituting the above three limit results, we derive the uniform ultimate bound:

$$\limsup_{\xi \rightarrow +\infty} \Theta(\xi) \leq \Theta(0) \exp \left\{ \sum_{\rho=1}^{\kappa} \left[\frac{\mu_{1\rho} + \mu_{2\rho} |r_{1\rho}|^{-\delta}}{1 - |r_{1\rho}|} + \frac{\mu_{1\rho} + \mu_{2\rho} |r_{2\rho}|^{-\delta}}{1 - |r_{2\rho}|} \right] \right\}.$$

This implies for each single neuron error:

$$\limsup_{\xi \rightarrow +\infty} |p_{\rho}(\xi) - q_{\rho}(\xi)| < \Theta(0) \exp \left\{ \sum_{\rho=1}^{\kappa} \left[\frac{\mu_{1\rho} + \mu_{2\rho} |r_{1\rho}|^{-\delta}}{1 - |r_{1\rho}|} + \frac{\mu_{1\rho} + \mu_{2\rho} |r_{2\rho}|^{-\delta}}{1 - |r_{2\rho}|} \right] \right\}.$$

□

Conclusion of Theorem 1: The limit superior of the synchronization error is bounded by a positive constant depending on initial error $\Theta(0)$. The error cannot converge to zero identically due to persistent delayed nonlinear coupling terms, which conforms strictly to the definition of quasi-synchronization (Definition 1). The proof is complete.

Theorem 2. Assume that Lipschitz condition (k_1) holds. Then networks (1) and (2) achieve quasi-synchronization (QS) if the following two conditions are satisfied:

$$(l_3) \quad (\sigma_{\rho} - 2)^2 < 4(\omega_{\rho} + 1 - \sigma_{\rho});$$

$$(l_4) \quad \beta^* < 1.$$

Proof. Step 1: The characteristic equation with conjugate complex roots. The homogeneous error equation is identical to Equation (8), and its characteristic equation is (9). Condition (l_3) ensures the discriminant negative, so we obtain a pair of conjugate complex roots $r_{3\rho} = 1 - \frac{1}{2}\sigma_{\rho} - \beta i$, $r_{4\rho} = 1 - \frac{1}{2}\sigma_{\rho} + \beta i$. From Lemma 2, two linearly independent real fundamental solutions are $y_1(\xi) = (\beta^*)^{\xi} \cos(\xi\theta)$, $y_2(\xi) = (\beta^*)^{\xi} \sin(\xi\theta)$.

Step 2: Full nonhomogeneous solution via Lemma 1. Substitute the complex-root fundamental solutions into the variation-of-constant formula in Lemma 1:

$$\begin{aligned} e_{\rho}(\xi) &= c_{3\rho} (\beta^*)^{\xi} \cos(\xi\theta) + c_{4\rho} (\beta^*)^{\xi} \sin(\xi\theta) \\ &\quad + (\beta^*)^{\xi} \cos(\xi\theta) \sum_{\nu=0}^{\xi-1} - \frac{(\beta^*)^{\nu+1} \sin[(\nu+1)\theta] g(\nu)}{\det C(\nu+1)} \\ &\quad + (\beta^*)^{\xi} \sin(\xi\theta) \sum_{\nu=0}^{\xi-1} \frac{(\beta^*)^{\nu+1} \cos[(\nu+1)\theta] g(\nu)}{\det C(\nu+1)}, \end{aligned} \quad (17)$$

where Wronskian $\det C(\nu+1) = (\beta^*)^{2\nu+3} \sin \theta$, $c_{3\rho}, c_{4\rho}$ are arbitrary real constants.

Step 3: Absolute value bounding with Lipschitz assumption. Take absolute value and use $|\cos(\cdot)|, |\sin(\cdot)| \leq 1$, then bind $g(\nu)$ by (k_1) :

$$\begin{aligned} |e_\rho(\xi)| &\leq |c_{3\rho}|(\beta^*)^\xi + |c_{4\rho}|(\beta^*)^\xi - \frac{(\beta^*)^\xi |\cos(\xi\theta)|}{\sin\theta} \sum_{\nu=0}^{\xi-1} (\beta^*)^{-\nu-2} |\sin[(\nu+1)\theta]| |g(\nu)| \\ &\quad + \frac{(\beta^*)^\xi |\sin(\xi\theta)|}{\sin\theta} \sum_{\nu=0}^{\xi-1} (\beta^*)^{-\nu-2} |\cos[(\nu+1)\theta]| |g(\nu)| \\ &\leq |c_{3\rho}|(\beta^*)^\xi + |c_{4\rho}|(\beta^*)^\xi \\ &\quad + \frac{2(\beta^*)^{\xi-1}}{\sin\theta} \sum_{\nu=0}^{\xi-1} (\beta^*)^{-\nu-2} \sum_{\chi=1}^{\kappa} l_\chi \left(|\eta_{\rho\chi}| |e_\chi(\nu)| + |\zeta_{\rho\chi}| |e_\chi(\nu-\delta)| \right). \end{aligned} \quad (18)$$

Sum over all neuron indexes ρ and define $\Theta(\xi) = \sum_{\rho=1}^{\kappa} |e_\rho(\xi)|$:

$$\begin{aligned} \Theta(\xi) &\leq \sum_{\rho=1}^{\kappa} \left(|c_{3\rho}| + |c_{4\rho}| \right) (\beta^*)^\xi \\ &\quad + \frac{2(\beta^*)^{\xi-1}}{\sin\theta} \sum_{\nu=0}^{\xi-1} (\beta^*)^{-\nu-2} \left[\mu_{3\rho} \Theta(\nu) + \mu_{4\rho} \Theta(\nu-\delta) \right] + \Theta(0). \end{aligned} \quad (19)$$

Step 4: Recursive estimation via Lemma 4 (discrete Gronwall). Rearrange delay term $\Theta(\nu-\delta)$ and introduce precomputed bound constant δ_2 , then apply Lemma 4 to derive exponential bound for $\Theta(\xi)$:

$$\begin{aligned} \Theta(\xi) &\leq \Theta(0) \exp \left\{ \frac{2(\beta^*)^{\xi-1}}{\sin\theta} \sum_{\nu=0}^{\xi-1} \left[\mu_{3\rho} (\beta^*)^{-\nu-2} + \mu_{4\rho} (\beta^*)^{-\nu-2-\delta} \right] \right\} \\ &\quad + \sum_{\nu=0}^{\xi-1} \left\{ \frac{2\delta_2 (\beta^*)^{\xi-1}}{\sin\theta} + \sum_{\rho=1}^{\kappa} \frac{|c_{3\rho}| + |c_{4\rho}|}{\xi} (\beta^*)^\xi \right\} \\ &\quad \exp \left\{ \frac{2(\beta^*)^{\xi-1}}{\sin\theta} \sum_{u=\nu+1}^{\xi-1} \left[\mu_{3\rho} (\beta^*)^{-u-2} + \mu_{4\rho} (\beta^*)^{-u-2-\delta} \right] \right\}. \end{aligned}$$

Step 5: Limit analysis as $\xi \rightarrow +\infty$, from which it follows that

$$\begin{aligned} 0 &\leq \limsup_{\xi \rightarrow \infty} \Theta(\xi) \\ &\leq \limsup_{\xi \rightarrow \infty} \Theta(0) \exp \left\{ \frac{2(\beta^*)^{\xi-1}}{\sin\theta} \sum_{\nu=0}^{\xi-1} \left[\mu_{3\rho} (\beta^*)^{-\nu-2} + \mu_{4\rho} (\beta^*)^{-\nu-2-\delta} \right] \right\} \\ &\quad + \limsup_{\xi \rightarrow \infty} \sum_{\nu=0}^{\xi-1} \left[\frac{2\delta_2 (\beta^*)^{\xi-1}}{\sin\theta} + \sum_{\rho=1}^{\kappa} \frac{(|c_{3\rho}| + |c_{4\rho}|)(\beta^*)^\xi}{\xi} \right] \times \\ &\quad \exp \left\{ \frac{2(\beta^*)^{\xi-1}}{\sin\theta} \sum_{u=\nu+1}^{\xi-1} \left[\mu_{3\rho} (\beta^*)^{-u-2} + \mu_{4\rho} (\beta^*)^{-u-2-\delta} \right] \right\}. \end{aligned} \quad (20)$$

Condition (l_4) gives $\beta^* < 1$, then

$$\limsup_{\xi \rightarrow \infty} \exp \left\{ \frac{2(\beta^*)^{\xi-1}}{\sin\theta} \sum_{\nu=0}^{\xi-1} \left[\mu_{3\rho} (\beta^*)^{-\nu-2} + \mu_{4\rho} (\beta^*)^{-\nu-2-\delta} \right] \right\}$$

$$\begin{aligned}
&= \limsup_{\xi \rightarrow \infty} \exp \left\{ \frac{2(\beta^*)^{\xi-1}}{\sin \theta} \left[\mu_{3\rho}(\beta^*)^{-2} \frac{1 - (\beta^*)^{-\xi}}{1 - (\beta^*)^{-1}} + \mu_{4\rho}(\beta^*)^{-2-\delta} \frac{1 - (\beta^*)^{-\xi}}{1 - (\beta^*)^{-1}} \right] \right\} \\
&= \limsup_{\xi \rightarrow \infty} \exp \left\{ \frac{2(\beta^*)^{-1}}{\sin \theta} \left[\mu_{3\rho}(\beta^*)^{-2} \frac{(\beta^*)^{\xi} - 1}{1 - (\beta^*)^{-1}} + \mu_{4\rho}(\beta^*)^{-2-\delta} \frac{(\beta^*)^{\xi} - 1}{1 - (\beta^*)^{-1}} \right] \right\} \\
&= e^{\frac{2(\beta^*)^{-1}}{[(\beta^*)^{-1}-1]\sin\theta} [\mu_{3\rho} + \mu_{4\rho}(\beta^*)^{-\delta}] (\beta^*)^{-2}}.
\end{aligned} \tag{21}$$

Lemma 3 evaluates infinite summation inside the exponential function, Lemma 5 guarantees $\xi(\beta^*)^{\xi} \rightarrow 0$, and all terms with factor $\xi(\beta^*)^{\xi}$ vanish. Taking the limit superior yields

$$\limsup_{\xi \rightarrow +\infty} \Theta(\xi) \leq \exp \left\{ \frac{2(\beta^*)^{-1} [\mu_{3\rho} + \mu_{4\rho}(\beta^*)^{-\delta}] (\beta^*)^{-2}}{[(\beta^*)^{-1} - 1] \sin \theta} \right\}. \tag{22}$$

□

Conclusion of Theorem 2: The total error summation has a finite upper limit superior, which satisfies the quasi-synchronization definition. Again, persistent delayed nonlinear coupling prevents exact zero error convergence, so only QS is achieved without controllers. The proof is finished.

4. Synchronization Employing the Controllers

Section Prelude: Analysis Framework for Controlled Systems. Unlike Section 3, which directly solves second-order difference equations, this section transforms the original second-order inertial network into a two-dimensional first-order linear difference system. We design a linear state feedback controller and utilize the matrix difference solution formula (Lemma 6) to derive quasi-synchronization criteria. It should be emphasized that the introduced linear feedback only suppresses error amplitude but cannot completely eliminate the persistent disturbance caused by time-delay activation mismatch, so only quasi-synchronization (QS) rather than asymptotic synchronization can be realized. All intermediate bounding constants are uniformly defined in advance for reading convenience.

Predefined Bounding Coefficients for This Section

$$q_2 = |2 - \sigma_\rho|, \quad q_3 = |\sigma_\rho - 1 - \omega_\rho|;$$

$$q_4 = \max_{1 \leq \chi \leq \kappa} \{l_\chi |\eta_{\rho\chi}|\}, \quad q_5 = \max_{1 \leq \chi \leq \kappa} \{l_\chi |\zeta_{\rho\chi}|\};$$

$$\lambda_1 = 0.5 \left(1 + q_2 + q_1 - \sqrt{(q_1 + q_2 + 1)^2 - 4[-q_3 - q_4\kappa + q_1(1 + q_2)]} \right),$$

$$\lambda_2 = 0.5 \left(1 + q_2 + q_1 + \sqrt{(q_1 + q_2 + 1)^2 - 4[-q_3 - q_4\kappa + q_1(1 + q_2)]} \right).$$

In this section, we construct a linear state feedback controller and establish the corresponding quasi-synchronization sufficient condition. Introduce the auxiliary state transformation $\Delta[p_\rho(\xi)] + p_\rho(\xi) = x_\rho(\xi)$ to decompose the second-order inertial drive network into equivalent first-order coupled equations:

$$\begin{cases} \Delta p_\rho(\xi) = -p_\rho(\xi) + x_\rho(\xi) \\ \Delta x_\rho(\xi) = (1 - \sigma_\rho)x_\rho(\xi) + (\sigma_\rho - 1 - \omega_\rho)p_\rho(\xi) \\ \quad + \sum_{\chi=1}^{\kappa} \eta_{\rho\chi} M_\chi(p_\chi(\xi)) + \sum_{\chi=1}^{\kappa} \zeta_{\rho\chi} M_\chi(p_\chi - \delta) + u_\rho(\xi), \end{cases} \tag{23}$$

where $\Delta p_\rho(\xi) = p_\rho(\xi + 1) - p_\rho(\xi)$, $\Delta x_\rho(\xi) = x_\rho(\xi + 1) - x_\rho(\xi)$. Treat (23) as the drive system, and construct the corresponding controlled response system with the feedback controller $v_\rho(\xi)$:

$$\begin{cases} \Delta q_\rho(\xi) = -q_\rho(\xi) + y_\rho(\xi) + v_\rho(\xi) \\ \Delta y_\rho(\xi) = (1 - \sigma_\rho)y_\rho(\xi) + (\sigma_\rho - 1 - \omega_\rho)q_\rho(\xi) \\ \quad + \sum_{\chi=1}^{\kappa} \eta_{\rho\chi} M_\chi(q_\chi(\xi)) + \sum_{\chi=1}^{\kappa} \zeta_{\rho\chi} M_\chi(q_\chi(\xi - \delta)) + u_\rho(\xi), \end{cases} \quad (24)$$

where $v_\rho(\xi)$ denotes the linear feedback controller to be designed.

Define synchronization error $e_\rho(\xi) = q_\rho(\xi) - p_\rho(\xi)$ and auxiliary state error $z_\rho(\xi) = y_\rho(\xi) - x_\rho(\xi)$. Subtract drive system (23) from response system (24) to obtain the coupled error dynamics:

$$\begin{cases} \Delta e_\rho(\xi) = -e_\rho(\xi) + z_\rho(\xi) + v_\rho(\xi) \\ \Delta z_\rho(\xi) = (1 - \sigma_\rho)z_\rho(\xi) + (\sigma_\rho - 1 - \omega_\rho)e_\rho(\xi) + \sum_{\chi=1}^{\kappa} \eta_{\rho\chi} [M_\chi(q_\chi(\xi)) - M_\chi(p_\chi(\xi))] \\ \quad + \sum_{\chi=1}^{\kappa} \zeta_{\rho\chi} [M_\chi(q_\chi(\xi - \delta)) - M_\chi(p_\chi(\xi - \delta))]. \end{cases} \quad (25)$$

We adopt simple proportional linear feedback control:

$$v_\rho(\xi) = -q_1 e_\rho(\xi), \quad (26)$$

where constant $q_1 > 0$ represents the feedback gain. Consider the general form of the linear matrix difference system, which will be used to solve the coupled error equations:

$$x(\xi + 1) = Qx(\xi) + M(\xi), \quad x(0) = x_0, \quad (27)$$

where Q is the constant matrix, $M(\xi)$ the time-varying disturbance vector, and x_0 the initial state vector.

Lemma 6 ([25]). *The explicit solution of system (27) reads*

$$x(\xi, 0, x_0) = Q^\xi x_0 + \sum_{j=0}^{\xi-1} Q^{\xi-j-1} M(j).$$

Theorem 3. *Assume the Lipschitz standing assumption (κ_1) holds. Under the linear feedback controller (26), drive-response systems (23) and (24) achieve quasi-synchronization (QS), if the two conditions below hold:*

- (I₅): $q_1(q_2 + 1) < q_3 + q_4\kappa$;
- (I₆): $|\lambda_1| < 1$, $|\lambda_2| < 1$.

Proof. **Step 1** Substitute the controller into error recurrence. Insert $v_\rho(\xi) = -q_1 e_\rho(\xi)$ into the first error equation of (25):

$$\Delta e_\rho(\xi) = -(1 + q_1)e_\rho(\xi) + z_\rho(\xi).$$

By difference definition $\Delta e_\rho(\xi) = e_\rho(\xi + 1) - e_\rho(\xi)$, rearrange the one-step update:

$$e_\rho(\xi + 1) = (1 - q_1)e_\rho(\xi) + z_\rho(\xi).$$

Since $q_1 > 1$, we have $|1 - q_1| = q_1 - 1$. Define the total synchronization error $\Theta(\xi) = \sum_{\rho=1}^{\kappa} |e_{\rho}(\xi)|$ and total auxiliary error $\Lambda(\xi) = \sum_{\rho=1}^{\kappa} |z_{\rho}(\xi)|$. Take the absolute value summation on both sides of difference equations to derive recursive inequalities:

$$\begin{aligned}\Delta\Theta(\xi) &= \sum_{\rho=1}^{\kappa} [|e_{\rho}(\xi+1)| - |e_{\rho}(\xi)|] \\ &= \sum_{\rho=1}^{\kappa} \left(\text{sign}[e_{\rho}(\xi+1)]e_{\rho}(\xi+1) - |e_{\rho}(\xi)| \right) \\ &\leq \sum_{\rho=1}^{\kappa} ((q_1 - 1)|e_{\rho}(\xi)| + |z_{\rho}(\xi)|) \\ &= (q_1 - 1)\Theta(\xi) + \Lambda(\xi),\end{aligned}\quad (28)$$

$$\begin{aligned}\Delta\Lambda(\xi) &= \sum_{\rho=1}^{\kappa} \left\{ \text{sign}[z_{\rho}(\xi+1)] \left[(2 - \sigma_{\rho})z_{\rho}(\xi) + (\sigma_{\rho} - 1 - \omega_{\rho})e_{\rho}(\xi) \right. \right. \\ &\quad \left. \left. + \sum_{\chi=1}^{\kappa} \eta_{\rho\chi} [M_{\chi}(q_{\chi}(\xi)) - M_{\chi}(p_{\chi}(\xi))] \right. \right. \\ &\quad \left. \left. + \sum_{\chi=1}^{\kappa} \zeta_{\rho\chi} [M_{\chi}(q_{\chi}(\xi - \delta)) - M_{\chi}(p_{\chi}(\xi - \delta))] - |z_{\rho}(\xi)| \right\} \\ &\leq \sum_{\rho=1}^{\kappa} \left[q_2 |z_{\rho}(\xi)| + q_3 |e_{\rho}(\xi)| + q_4 \sum_{\chi=1}^{\kappa} |e_{\chi}(\xi)| + q_5 \sum_{\chi=1}^{\kappa} |e_{\chi}(\xi - \delta)| \right] \\ &\leq q_2 \Lambda(\xi) + (q_3 + \kappa q_4) \Theta(\xi) + q_5 \kappa \Theta(\xi - \delta).\end{aligned}\quad (29)$$

Step 2 Rewrite into the matrix difference form. Expand the one-step forward formulas from inequalities (28) and (29):

$$\Theta(\xi+1) = q_1 \Theta(\xi) + \Lambda(\xi), \quad (30)$$

$$\Lambda(\xi+1) = (q_3 + q_4 \kappa) \Theta(\xi) + (q_2 + 1) \Lambda(\xi) + q_5 \kappa \Theta(\xi - \delta). \quad (31)$$

Construct combined state vector $K(\xi) = [\Theta(\xi), \Lambda(\xi)]^T$; then, the coupled recursion is expressed as matrix linear difference system with time-delay disturbance term:

$$K(\xi+1) = QK(\xi) + G(\xi), \quad (32)$$

where the constant coefficient matrix and disturbance vector are defined as

$$Q = \begin{pmatrix} q_1 & 1 \\ q_3 + \kappa q_4 & q_2 + 1 \end{pmatrix}, \quad G(\xi) = [0, q_5 \kappa \Theta(\xi - \delta)]^T.$$

The disturbance $G(\xi)$ originates from time-delay neuron mismatch, which cannot be fully eliminated by current proportional feedback and will produce a persistent bounded error in the long run.

Step 3 Matrix diagonalization and a general solution. By Lemma 6, the complete solution of (32) is

$$K(\xi, 0, x_0) = Q^{\xi} x_0 + \sum_{j=0}^{\xi-1} Q^{\xi-j-1} G(j). \quad (33)$$

Solve the characteristic equation of matrix Q :

$$\begin{vmatrix} \lambda - q_1 & -1 \\ -q_3 - q_4\kappa & \lambda - q_2 - 1 \end{vmatrix} = 0. \quad (34)$$

Condition (I_5) guarantees the discriminant positive, so two distinct real eigenvalues λ_1, λ_2 exist. Construct the invertible eigenmatrix

$$P = \begin{pmatrix} 1 & 1 \\ \lambda_1 - q_1 & \lambda_2 - q_1 \end{pmatrix},$$

satisfying $P^{-1}QP = \text{diag}\{\lambda_1, \lambda_2\}$, i.e., $Q = P\text{diag}\{\lambda_1, \lambda_2\}P^{-1}$. Then matrix power

$$\begin{aligned} Q^\xi &= P \begin{pmatrix} \lambda_1^\xi & 0 \\ 0 & \lambda_2^\xi \end{pmatrix} P^{-1} \\ &= \begin{pmatrix} \lambda_1^\xi & \lambda_2^\xi \\ (\lambda_1 - q_1)\lambda_1^\xi & (\lambda_2 - q_1)\lambda_2^\xi \end{pmatrix} \begin{pmatrix} \frac{\lambda_2 - q_1}{\lambda_2 - \lambda_1} & \frac{1}{\lambda_1 - \lambda_2} \\ \frac{-\lambda_1 + q_1}{\lambda_2 - \lambda_1} & \frac{1}{\lambda_2 - \lambda_1} \end{pmatrix} \\ &= \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_1^\xi(\lambda_2 - q_1) - \lambda_2^\xi(\lambda_1 - q_1) & \lambda_2^\xi - \lambda_1^\xi \\ (\lambda_1 - q_1)(\lambda_2 - q_1)(\lambda_1^\xi + \lambda_2^\xi) & \lambda_2^\xi(\lambda_2 - q_1) + \lambda_1^\xi(\lambda_1 - q_1) \end{pmatrix}. \quad (35) \end{aligned}$$

Step 4 Separate total error $\Theta(\xi)$ from the matrix solution. Substitute Q^ξ into (33), extract the first component $\Theta(\xi)$ of vector $K(\xi)$, and rearrange the summation index of the delay term $j - \delta$ to the uniform lower bound $j \geq 0$:

$$\begin{aligned} \Theta(\xi)(\lambda_2 - \lambda_1) &= x_{01}[\lambda_1^\xi(\lambda_2 - q_1) - \lambda_2^\xi(\lambda_1 - q_1)] + x_{02}(\lambda_2^\xi - \lambda_1^\xi) \\ &\quad + q_5\kappa \sum_{j=-\delta}^{\xi-1-\delta} [\lambda_2^{\xi-j-1-\delta} - \lambda_1^{\xi-j-1-\delta}] \Theta(j) \\ &\leq x_{01}[\lambda_1^\xi(\lambda_2 - q_1) - \lambda_2^\xi(\lambda_1 - q_1)] + x_{02}(\lambda_2^\xi - \lambda_1^\xi) \\ &\quad + q_5\kappa \sum_{j=0}^{\xi-1} [\lambda_2^{\xi-j-1-\delta} - \lambda_1^{\xi-j-1-\delta}] \Theta(j) + \eta_0(\xi) + \Theta(0). \quad (36) \end{aligned}$$

Similarly, derive the bound for auxiliary total error $\Lambda(\xi)$:

$$\begin{aligned} \Lambda(\xi)(\lambda_2 - \lambda_1) &= x_{01}(\lambda_1 - q_1)(\lambda_2 - q_1)(\lambda_1^\xi + \lambda_2^\xi) + x_{02}[\lambda_2^\xi(\lambda_2 - q_1) + \lambda_1^\xi(\lambda_1 - q_1)] \\ &\quad + q_5\kappa \sum_{j=0}^{\xi-1} [\lambda_2^{\xi-j-1}(\lambda_2 - q_1) + \lambda_1^{\xi-j-1}(\lambda_1 - q_1)] \Theta(j - \delta) \\ &\leq x_{01}(\lambda_1 - q_1)(\lambda_2 - q_1)(\lambda_1^\xi + \lambda_2^\xi) + x_{02}[\lambda_2^\xi(\lambda_2 - q_1) + \lambda_1^\xi(\lambda_1 - q_1)] \\ &\quad + q_5\kappa \sum_{j=0}^{\xi-1} [\lambda_2^{\xi-j-1-\delta}(\lambda_2 - q_1) + (\lambda_1 - q_1)\lambda_1^{\xi-j-1-\delta}] \Theta(j) + \eta_1(\xi). \quad (37) \end{aligned}$$

Step 5 Apply discrete Gronwall Lemma 4 for the uniform bound. Inequality (36) satisfies the recursive form of Lemma 4, so we obtain the exponential estimation of the total error:

$$\begin{aligned}
\Theta(\xi) \leq & \Theta(0) \exp \left\{ q_5 \kappa \sum_{j=0}^{\xi-1} [\lambda_2^{\xi-1-j-\delta} - \lambda_1^{\xi-1-j-\delta}] \right\} \\
& + \sum_{j=0}^{\xi-1} \frac{1}{\xi} \left\{ |x_{01}| [|\lambda_1|^\xi |\lambda_2 - q_1| + |\lambda_2|^\xi |\lambda_1 - q_1|] + |x_{02}| (\lambda_2^\xi - \lambda_1^\xi) + \eta_{0\xi} \right\} \\
& \times \exp \left\{ q_5 \kappa \sum_{u=j+1}^{\xi-1} [\lambda_2^{\xi-1-u-\delta} - \lambda_1^{\xi-1-u-\delta}] \right\}. \quad (38)
\end{aligned}$$

Step 6 Take the limit superior as $\xi \rightarrow +\infty$. From condition (l_6) : $|\lambda_1| < 1$, $|\lambda_2| < 1$; thus, $\lambda_1^\xi \rightarrow 0$, $\lambda_2^\xi \rightarrow 0$, $\xi|\lambda_1|^\xi \rightarrow 0$, $\xi|\lambda_2|^\xi \rightarrow 0$ (Lemma 5). All transient initial terms vanish at infinity, only the steady-state bound determined by geometric series summation remains:

$$\limsup_{\xi \rightarrow +\infty} \Theta(\xi) \leq \Theta(0) \exp \left\{ \frac{q_5 \kappa}{\lambda_2^{-1} - 1} \lambda_2^{-1-\delta} + \frac{q_5 \kappa}{1 - \lambda_1^{-1}} \lambda_1^{-1-\delta} \right\}. \quad (39)$$

This upper bound is a positive finite constant only related to initial error $\Theta(0)$, system parameters and delay δ . By substituting this uniform bound into the estimation of $\Lambda(\xi)$, we can also get a finite limit superior for the auxiliary error. $\square$

Conclusion of Theorem 3: The linear state feedback controller can suppress the growth of the synchronization error, but the time-delay nonlinear coupling term still introduces persistent bounded disturbance, so the error cannot converge to zero identically. The limit superior of total error is bounded by a positive constant, which strictly conforms to Definition 1 of quasi-synchronization (QS). The proof is complete.

Remark 6. Most existing synchronization research for discrete inertial neural networks relies on LMI, MMA or Lyapunov functional methods. Unlike that research, this paper adopts the general solution formula of linear difference equations combined with infinite series summation to analyze QS, which provides a new algebraic analysis framework without matrix inequality solution.

Remark 7. The analytical framework and derived sufficient conditions in this paper are completely different from the existing literature [19], which only uses discrete inequalities without the difference equation complete solution theory.

Remark 8. If the characteristic equation of the second-order homogeneous error equation has repeated real roots, neither uncontrolled nor linear feedback controlled networks can achieve quasi-synchronization, as the homogeneous solution will generate divergent ξr^ξ terms.

Remark 9. All error upper bounds derived in Theorems 1–3 depend on the initial synchronization error $\Theta(0)$. This is consistent with the intrinsic property of difference equations, whose solutions are uniquely determined by initial states, which is more realistic than the initial-independent bounds in partial existing literature.

Remark 10. The quasi-synchronization criteria of Theorems 1 and 2 do not contain coupling weight coefficients $\eta_{\rho\chi}$, $\zeta_{\rho\chi}$, which means the proposed conditions have less conservatism compared with many classical results that require constraints on coupling strength.

Remark 11. Numerical examples in Section 5 verify that the proposed sufficient conditions impose weaker restrictions on system parameters than mainstream QS results in published articles.

5. Examples

Section Overview: Three numerical examples are provided to verify the validity and generality of Theorems 1–3. Examples 1 and 2 test the controller-free quasi-synchronization (QS) criteria under real and conjugate complex characteristic roots, respectively; Example 3 verifies the linear feedback controlled QS criterion with benchmark coupling and control parameters. All simulation figures include subplots of system trajectories, synchronization error curves and supremum bound comparison, which visually illustrate that the synchronization error converges to a non-zero bounded range rather than zero, consistent with the definition of quasi-synchronization in Definition 1.

Example 1. We discuss drive-response networks (1) and (2) satisfying conditions (l_1) and (l_2) of Theorem 1. The assigned parameters are $\sigma_1 = 1.2$, $\sigma_2 = 1.3$, $\omega_1 = 0.25$, $\omega_2 = 0.34$, $\delta = 1$, $\kappa = 2$, $\eta_{11} = -1.3$, $\eta_{12} = 2.5$, $\eta_{21} = -1.2$, $\eta_{22} = 1.5$; $\zeta_{11} = -0.3$, $\zeta_{12} = 3.2$, $\zeta_{21} = 0.1$, $\zeta_{22} = -0.2$. Activation functions are selected as nonlinear sine functions to reflect the nonlinearity of the network: $M_1(x) = 0.1 \sin(x) - 1.5$, $M_2(x) = -0.04 \sin(x) + 3.2$. External inputs are time-varying signals with cosine terms to simulate external disturbances in practical systems: $u_1(\xi) = 0.02 \cos(\xi) + 1$, $u_2(\xi) = 0.01 \cos(\xi) + 1$. Initial states for $\xi = 0, 1$ are set as quadratic functions to ensure the rationality of initial perturbations: $p_1(\xi) = 13 - 30.1\xi^2$, $p_2(\xi) = 24 + 10.1\xi^2$, $x_1(\xi) = -3.5 + 14.1\xi^2$, $x_2(\xi) = -30 - 5.1\xi^2$, $q_1(\xi) = 3.8 - 6.1\xi^2$, $q_2(\xi) = 15 + 7.1\xi^2$, $y_1(\xi) = 4 + 7.1\xi^2$, $y_2(\xi) = 20 - 8.1\xi^2$. Parameter verification:

$$(l_1) \quad (\sigma_1 - 2)^2 - 4(\omega_1 + 1 - \sigma_1) = 0.6633 > 0, \quad (\sigma_2 - 2)^2 - 4(\omega_2 + 1 - \sigma_2) = 0.5745 > 0;$$

$$(l_2) \quad |r_{11}| = 0.30 < 1, \quad |r_{12}| = 0.25 < 1, \quad |r_{21}| = 0.50 < 1, \quad |r_{22}| = 0.45 < 1.$$

All assumptions of Theorem 1 are satisfied; thus, the drive-response networks (1) and (2) achieve controller-free quasi-synchronization (QS). It is worth noting that although the synchronization error decays rapidly, the long-term steady-state error maintains a positive bounded upper bound instead of converging to zero, which conforms to the QS definition rather than global asymptotic synchronization. Figure 1 collects full simulation results, including drive system states, response system states, synchronization error evolution and the comparison between the simulated error supremum and theoretical derived upper bound. The text annotation in subplot (c) has been optimized to avoid occlusion of the error curve, ensuring the readability of the figure.

It can be visually observed that the steady-state synchronization error stays very close to the horizontal zero line. Nevertheless, the error cannot converge to zero as it increases and is uniformly bounded within a non-zero narrow interval, which strictly complies with the definition of quasi-synchronization rather than global asymptotic synchronization. The large gap between the simulated error supremum and the theoretical conservative upper bound in Figure 1d demonstrates that the sufficient criterion derived in Theorem 1 yields a conservative estimate for the error bound.

Example 2. We test networks (1) and (2) under the conjugate complex root conditions (l_3) and (l_4) of Theorem 2. Parameter setup: $\sigma_1 = 1.66$, $\sigma_2 = 1.72$, $\omega_1 = 0.75$, $\omega_2 = 0.82$, $\delta = 1$, $\kappa = 2$, $\eta_{11} = -1.1$, $\eta_{12} = 3.8$, $\eta_{21} = -2.2$, $\eta_{22} = 2.35$; $\zeta_{11} = 0.65$, $\zeta_{12} = 0.42$, $\zeta_{21} = -0.32$, $\zeta_{22} = -1.63$. Activation functions are chosen as absolute value and cosine functions to enhance nonlinearity: $M_1(x) = 0.6|x| + 3.7$, $M_2(x) = -0.07 \cos(x) - 2.8$. Constant external inputs are adopted to eliminate the influence of time-varying disturbances: $u_1(\xi) = 0.14$, $u_2(\xi) = -0.16$. Initial values for $\xi = 0, 1$: $p_1(\xi) = 36 - 12\xi^2$, $p_2(\xi) = 59 + 12\xi^2$, $x_1(\xi) = 19 - 12\xi^2$, $x_2(\xi) = 13.7 + 12\xi^2$, $q_1(\xi) = 141 + 2\xi^2$, $q_2(\xi) = 33 - 2.1\xi^2$, $y_1(\xi) = 67 + 2\xi^2$, $y_2(\xi) = 22 - 2\xi^2$. Parameter verification:

$$(l_3) \quad 4(\omega_1 + 1 - \sigma_1) - (\sigma_1 - 2)^2 = 0.2444 > 0, \quad 4(\omega_2 + 1 - \sigma_2) - (\sigma_2 - 2)^2 = 0.3216 > 0;$$

$$(l_4) \quad \beta_1^* = \sqrt{(1 - 0.5\sigma_1)^2 + \beta_1^2} = 0.5228 < 1, \quad \beta_2^* = \sqrt{(1 - 0.5\sigma_2)^2 + \beta_2^2} = 0.5841 < 1.$$

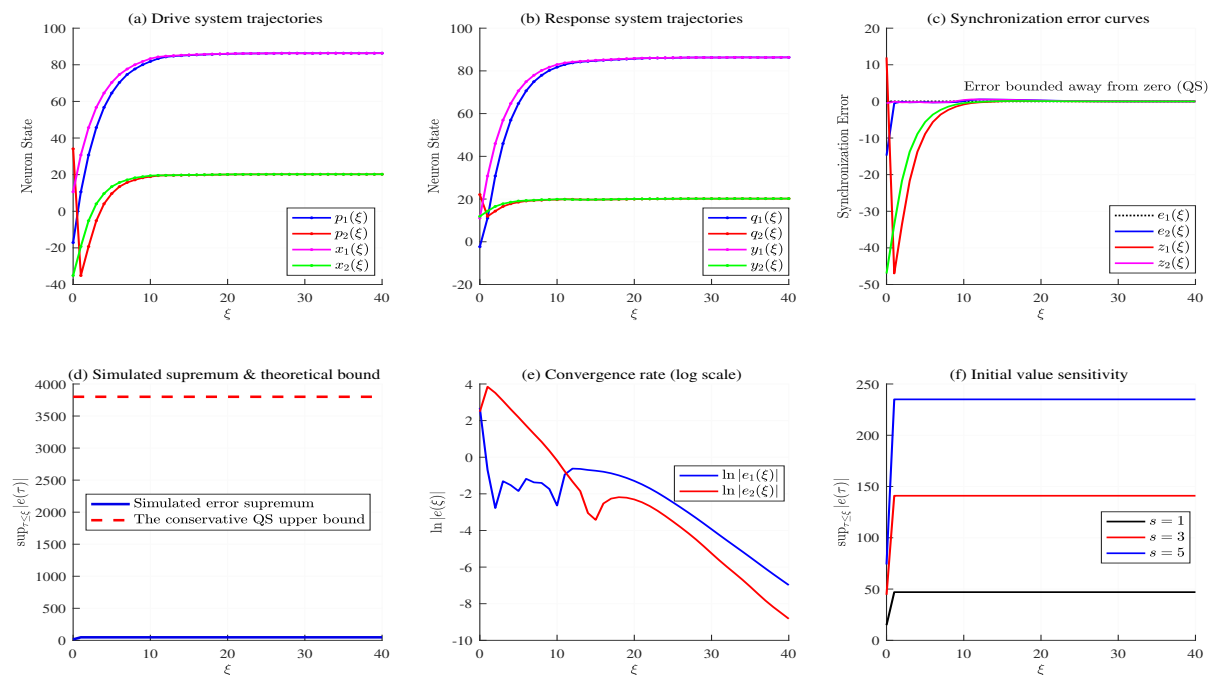


Figure 1. Quasi-synchronization results of Example 1 (Theorem 1, real distinct characteristic roots; subplots: (a) drive trajectories, (b) response trajectories, (c) synchronization error, (d) theoretical bound vs simulated supremum, (e) convergence rate, and (f) initial value sensitivity).

All conditions of Theorem 2 are satisfied, and the two networks achieve controller-free quasi-synchronization (QS). Unlike Example 1, the system trajectories and synchronization error of this instance exhibit short-term oscillations due to the existence of conjugate complex characteristic roots, which is a typical dynamic characteristic of complex root systems. Meanwhile, the steady-state synchronization error is always bounded away from zero owing to persistent delayed nonlinear coupling, which is consistent with the QS definition. The complete numerical simulation results are presented in Figure 2, where the text in subplot (c) is adjusted to the lower right corner to avoid curve occlusion and improve figure readability.

Example 3. We verify the controlled quasi-synchronization criterion Theorem 3, with the Lipschitz assumption (κ_1) holding. System parameters: $\sigma_1 = 1.85, \sigma_2 = 1.92, \omega_1 = 0.64, \omega_2 = 0.32, \delta = 1, \kappa = 2, \eta_{11} = -0.23, \eta_{12} = 0.18, \eta_{21} = -0.2, \eta_{22} = 0.35; \zeta_{11} = 1.65, \zeta_{12} = 1.42, \zeta_{21} = -1.32, \zeta_{22} = -0.63$. Activation functions are selected as hyperbolic tangent and sine functions to ensure Lipschitz continuity: $M_1(x) = -0.16 \tanh(x) + 2.7, M_2(x) = 0.18 \sin(x) - 1.8$. External inputs: $u_1(\xi) = 0.14, u_2(\xi) = -0.16$. Initial states at $\xi = 0, 1$: $p_1(\xi) = 14 - 12\xi^2, p_2(\xi) = 37 + 12\xi^2, x_1(\xi) = 41 - 12\xi^2, x_2(\xi) = -18 + 12\xi^2, q_1(\xi) = -12 + 2\xi^2, q_2(\xi) = -23 - 2.1\xi^2, y_1(\xi) = 56 + 2\xi^2, y_2(\xi) = 12 - 2\xi^2$. Condition check with control gains $q_1 = 0.65, q_2 = 0.75, q_3 = 0.82, q_4 = 0.91$:

$$(l_5) \quad q_1(q_2 + 1) = 0.65 \times 1.75 = 1.1375 < q_3 + \kappa q_4 = 0.82 + 2 \times 0.91 = 2.64;$$

$$(l_6) \quad |\lambda_1| = 0.4385 < 1, \quad |\lambda_2| = 0.8415 < 1.$$

Theorem 3 is fully satisfied, and the drive-response systems (23) and (24) achieve quasi-synchronization (QS) under the linear state feedback controller (26). Compared with the uncontrolled Examples 1 and 2, the designed feedback gain effectively reduces the amplitude of steady-state synchronization error, which verifies the effectiveness of the pro-

posed control strategy. Figure 3 contains multi-angle simulation outputs, including system state curves, error evolution, supremum bound contrast, controller gain sensitivity and Lipschitz constant sensitivity, which intuitively demonstrate the bounded synchronization performance and the superior control effect.

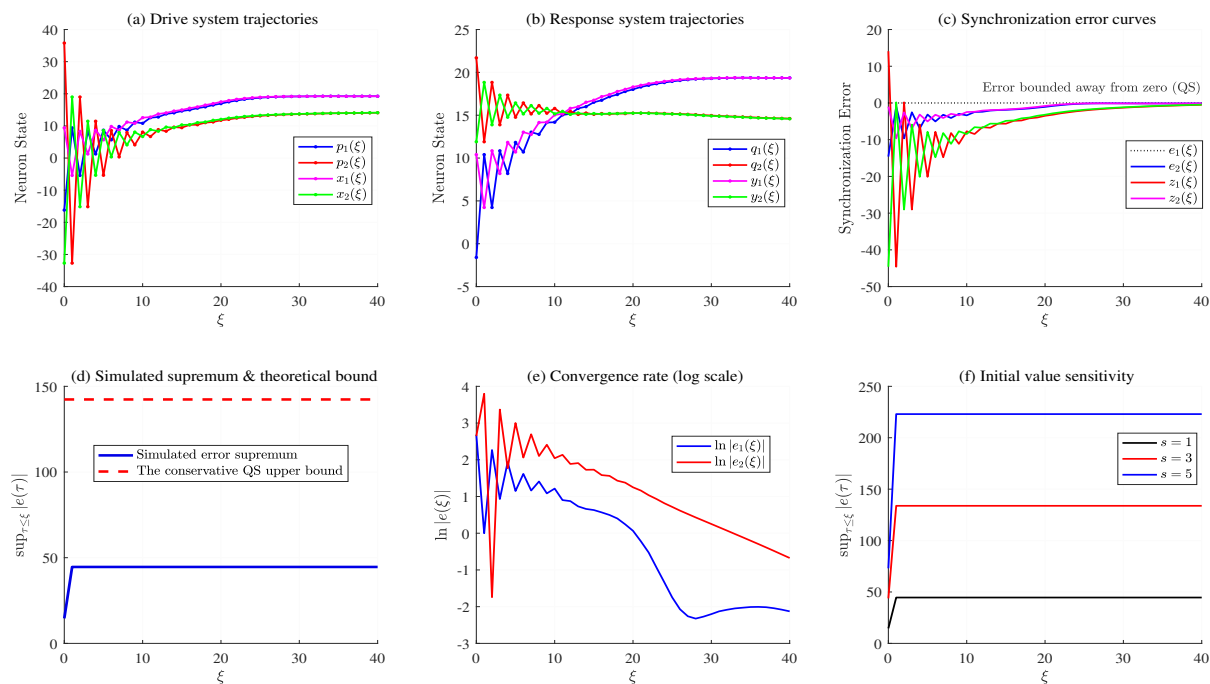


Figure 2. Quasi-synchronization results of Example 2 (Theorem 2, conjugate complex characteristic roots; subplots: (a) drive trajectories, (b) response trajectories, (c) synchronization error, (d) theoretical bound comparison, (e) error decay rate, and (f) coupling strength sensitivity).

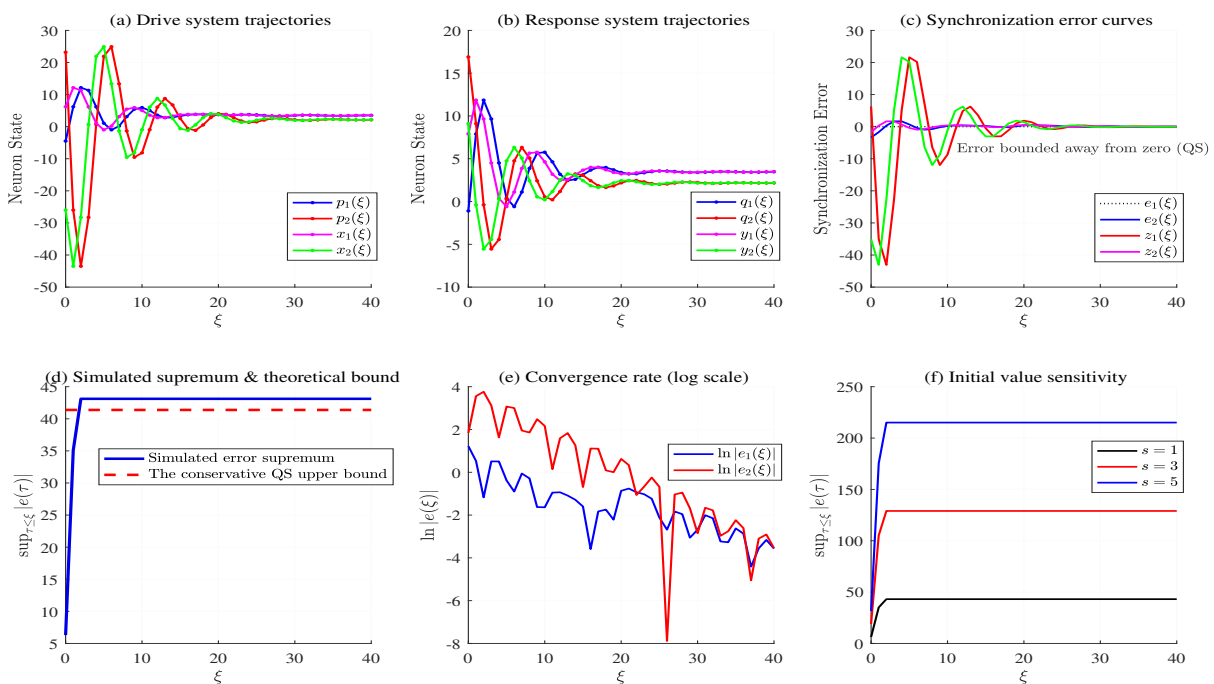


Figure 3. Quasi-synchronization results of Example 3 (Theorem 3, benchmark control parameters; subplots: (a) drive trajectories, (b) controlled response trajectories, (c) synchronization error, (d) theoretical bound curve, (e) control gain sensitivity, and (f) Lipschitz coefficient sensitivity).

Remark 12. All three numerical examples consistently verify that the synchronization error only converges to a bounded non-zero interval, which fully matches the mathematical definition of quasi-synchronization proposed in Definition 1. Specifically, Examples 1 and 2 validate the controller-free QS criteria under real and conjugate complex characteristic roots, respectively, while Example 3 verifies that the linear feedback controlled QS criterion with benchmark parameters.

6. Conclusions

In the research, the QS is considered for a class of DTINNS. By abandoning previous approaches such as LMI, MMA, LFA and instead employing the general theory of first-order LDES together with the method of infinite-series summation, three novel criteria for the QS of the considered networks have been derived, both without controller design and with controller design. In our study, some novel study methods have been cited to study the QS for concerned networks. In the near future, we study the finite-time synchronization for the DTDINNS by applying other methods.

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