

Article

A Coherence Theorem for Conserved Comparison Ledgers: Structural Axioms Pin Down the Scale, the Cost, and the Ratio

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Abstract

A *conserved comparison ledger* is an abstract system in which each state carries a positive ratio $r(s)$, every measurement factors through that ratio, and pairwise comparisons are scored by an admissible cost J . A separate classification fixes the admissible costs with polynomial combiner and, under a units normalization, selects the representative $J_{\text{cost}}(x) = \frac{1}{2}(x + x^{-1}) - 1$. This paper installs that classified cost in a ledger and determines what the ledger assumptions force. The argument has three structural steps and one neutrality axiom: H1 forces the scale ratio to $\sigma = \varphi$; H2, using that classification, forces $J = J_{\text{cost}}$; H3 reads $r(s)$ from a constrained cost minimizer of an internal positive vector, and A3 imposes zero total log-charge; together they force $r \equiv 1$, make the measurement map constant, and collapse the observational quotient to a singleton. The assumptions are also sharp in the following sense: removing H1, H2, H3, or A3 admits an explicit ledger satisfying the remaining meaningful assumptions while losing the corresponding conclusion. The examples verify joint satisfiability, include non-uniform internal data whose variational minimum gives $r = 1$, and show nontrivial sector dynamics when A3 is not imposed. A final example realizes the full axiom package on the patch space of the Fibonacci quasicrystal, with the scale and the charge-sector structure supplied intrinsically by the tiling and the remaining installations stated explicitly. We frame this result as a *coherence theorem* for a deliberately chosen axiom system: each hypothesis is individually natural and, by design, controls a single output, so their joint forcing of (σ, J, r) exhibits the internal consistency of the axiom package rather than asserting that any independently given system must realize it.

Keywords: conserved comparison ledger; coherence theorem; golden ratio; admissible reciprocally symmetric cost; Jensen inequality; conserved log-charge; observational quotient

MSC: 39B22; 26A51; 26D15; 39B62



Academic Editor: Janusz Brzdęk

Received: 24 June 2026

Revised: 17 July 2026

Accepted: 21 July 2026

Published: 23 July 2026

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1. Introduction

1.1. Subject and Background

A *conserved comparison ledger* is a minimal setting for one question: can the structural assumptions of a comparison framework determine the framework's own distinguished constants instead of leaving them as free parameters? This paper answers the question for

conserved comparison ledgers: a short list of structural conditions forces the scale ratio, the cost, and the internal ratio to fixed values.

1.2. Related Work

The ingredients of this question have long, independent literatures. The admissible costs are reciprocally symmetric solutions of a composition (functional) equation, in the d'Alembert–Wilson–Kannappan tradition of functional equations on groups and semi-groups [1–5]. The forced scale is the golden ratio, which arises here as it does in the rigidity of Fibonacci-type recurrences and self-similar sequences [6–10]; the selection of a single limiting cost is, concretely, the collapse of the two-parameter family of admissible combiners $P(u, v) = cuv + 2u + 2v$ to the single representative J_{cost} once the units normalization is imposed: a normalization condition reduces a parameter family of structures to one distinguished member. This is the structural pattern of İnönü–Wigner contraction, where a singular limit of a parameter family of Lie algebras selects a distinguished (degenerate) algebra, and of symmetric-space rigidity, where fixing the normalizing data leaves no nontrivial deformation [11–16]. The variational step uses strict convexity and Jensen's inequality, in the same mathematical setting as quasi-arithmetic means, to identify the constrained minimizer; H3 then reads the ratio from that minimizer, and A3 sets the charge to zero [17–23], and its dynamical form is the standard gradient-flow, Bakry–Émery, and optimal-transport circle of ideas, with information-geometry counterparts [24–37]. Finally, the conserved log-charge is a quantity that the evolution never changes. Its value therefore sorts the states into classes, and no measurement can tell two states in the same class apart; the same pattern appears in Noether's theorem, superselection theory, and the analysis of observational equivalence [38–51]. These literatures supply only the conceptual setting. The proofs use a quadratic recurrence (B1), the discrete Jensen inequality with convex-analytic existence of constrained minimizers (B3) [17,18], a fiber decomposition, and the classification imported from [52], together with Picard–Lindelöf and Grönwall for the dynamical interpretation only. The remaining citations are conceptual analogues; no result depends on them.

The ledger also sits near three abstract comparison frameworks, sharing tools with them but differing in aim. Foundations of measurement [53,54] and axiomatic utility/decision theory [55,56] ask when a comparison or preference admits a numerical representation and how unique it is; the theory of means and aggregation functions [19,20,57] characterizes how several values combine into one (the source of our Jensen step). All three concern the *existence* and residual freedom of a representation. Here the cost is already classified in [52], and we instead ask when the surrounding structure removes *all* residual freedom (in measurement-theoretic terms, when the ledger axioms collapse the group of admissible transformations to the identity). The conserved-charge and dynamical ingredients have no counterpart in those theories: the conservation law on $\log r$ and the evolution map T_F are additional structure with no analogue in measurement, utility, or aggregation theory.

1.3. Prior Classification of Admissible Costs

Reference [52] classifies the admissible costs. An *admissible cost* is a reciprocally symmetric cost J whose *combiner* P aggregates pairwise comparisons (both made precise in Section 2). When the combiner is polynomial, that work proves it is bilinear, classifies the admissible costs into two families, and, under a units normalization, selects the canonical cost $J_{\text{cost}}(x)$ ([52], Thms. 3.2, 5.2, 7.6 and Cor. 9.1). That paper is purely functional-equation-theoretic: it selects the cost but does not place it inside any ambient system. Consequently, every cost-forcing result in the present paper is conditional on the four imported theorems from [52]. The present paper does not reprove automatic factorization, the polynomial

degree bound, the two-family classification, or canonical-cost uniqueness; it uses them as black-box analytic input and studies what they imply after the cost is installed in a ledger. Two companion papers complement this classification: [58] establishes uniqueness of the canonical reciprocal cost, and [59] proves the d’Alembert inevitability theorem underlying the governing functional equation.

The four imported theorems are stated in full in Section 2, so the present paper can be verified by taking them as axioms. Remark 2 delimits exactly which conclusions of the present paper depend on this import (only the cost $J = J_{\text{cost}}$) and which are proved here independently of it, and records that the imported chain is independently corroborated in [58,59].

1.4. Contributions of This Paper

The present paper installs the cost classified in [52] into a conserved comparison ledger and determines what the ledger’s own assumptions force. The analysis rests on three structural hypotheses (H1–H3) and one modeling axiom (A3): H1 fixes the scale geometry, H2 fixes the cost through the combiner classification of [52], H3 ties the ratio to a constrained variational minimum, and A3 imposes charge neutrality.

Under H1 and H2 alone, the scale and the cost are uniquely forced,

$$\sigma = \varphi, \quad J = J_{\text{cost}}(x).$$

Adding H3 and A3 collapses the ratio interface and the observational quotient,

$$r(s) = 1, \quad \mu_F \text{ constant}, \quad \text{SQ}(F) \simeq 1.$$

When A3 is dropped, the same hypotheses partition the state space into the level sets of the conserved log-charge $\log r$, each invariant under the evolution; A3 is precisely the condition selecting the neutral level set $\log r = 0$. The condition package is irredundant: dropping any one of H1, H2, H3, A3 admits a ledger satisfying the rest yet violating the corresponding conclusion. Here A3 is a constraint on the internal-vector data introduced by H3, so H3 and A3 are not independent; the precise sense in which each condition is dropped is given in Section 7. Section 9 confirms joint satisfiability with several explicit ledgers, including one in which $r = 1$ is the genuine output of a single variational step on non-uniform data, and one with an injective observable for which the collapse $\text{SQ}(F) \simeq 1$ is non-vacuous. The final example (Example 8) goes further: it realizes the whole package on the patch space of the Fibonacci quasicrystal, whose tiling itself supplies the scale and the charge-sector architecture, leaving only the cost and the neutrality normalization as identified installations (Remark 16).

1.5. Scope, Novelty, and Significance

The novelty is structural, not analytic: we isolate the conserved comparison ledger and show that one short list of independently motivated conditions fixes its three distinguished quantities (σ, J, r) simultaneously. Each step is elementary: a Fibonacci recurrence (B1), substitution into the classification of [52] (B2), and the discrete Jensen inequality (B3); the contribution is precisely that these independently standard ingredients assemble into a sharp, irredundant rigidity theorem in which each hypothesis is necessary, and together they fix (σ, J, r) uniquely. The conditions are by design chosen so each controls one output, so the theorem is a *coherence* statement (a small set of jointly irredundant, individually natural requirements determines the object completely) rather than a coincidence. We claim no fully natural realization: the examples of Section 9 establish only non-vacuity, with the exception of Example 8, whose scale, internal data, and sector structure are supplied by the

Fibonacci quasicrystal itself, and whether a non-constructed system satisfies H1–H3 and A3 without any installed component is the principal open problem (Section 10). We stress that the hypotheses are not reverse-engineered from the conclusions: each is an independently standard structural condition (uniform-plus-additive scaling, a reciprocally symmetric aggregating cost, and a constrained-minimum aggregation rule), motivated one by one in the subsection “Why these hypotheses?” below, not an artifact of a bespoke choice.

What is new, precisely. The cost classification itself is *not* a contribution of this paper: its four statements, listed in the preceding subsection, are imported verbatim from [52] and used as black boxes (Section 2; Remark 2). The new content is the isolation of the conserved comparison ledger as an ambient object and the proof that its structure turns that classification into a forcing constraint. Schematically the paper establishes only the three implications

$$H1 \Rightarrow \sigma = \varphi, \quad H2 + [52] \Rightarrow J = J_{\text{cost}}, \quad H3 + A3 \Rightarrow r \equiv 1,$$

their assembly into the two rigidity statements of Section 7, and the irredundancy analysis; the middle implication is entirely conditional on [52]. Accordingly the result is best read as a *coherence theorem* for a deliberately chosen axiom system, a demonstration that a small package of individually natural, jointly irredundant conditions is jointly consistent and pins down (σ, J, r) , rather than as a claim that an independently given comparison structure must satisfy these axioms. The sharpness results of Section 7 support exactly this reading: each hypothesis is necessary for its own output, but we do not assert that the axioms are inevitable.

On mathematical depth and novelty. It may be objected that the self-contained steps are elementary (B1 solves a quadratic and B3 applies Jensen’s inequality) and that the only functional-equation depth, B2, is imported, so the paper offers no novel content. We accept the factual premises of this objection and contest only the inference. The exact form of B1 is the defining relation of φ and is presented as such (Remark 5); the static form of B3 is a Jensen argument; and B2 is imported verbatim from [52]. The claimed contribution is not any individual step but four items not reducible to them: (i) the isolation of the ledger as a self-contained axiomatic object into which a classified cost is installed; (ii) the irredundancy analysis of Section 7, which *constructs* explicit counter-ledgers $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ (Propositions 2–4) witnessing that each hypothesis is individually necessary for its own output (existence results, not restatements of a definition); (iii) the asymptotic scale forcing of Theorem 6, which is strictly stronger than the quadratic identity because it forces $\sigma = \varphi$ from perturbed, non-geometric Fibonacci-type data satisfying no exact identity, with the precise relation between its two limiting conditions, including its classical Poincaré–Perron provenance, recorded in Remark 6 (see also Example 1); and (iv) the charge-sector decomposition of Section 8, including the finite period-two dynamics of Example 7. We regard the result as a *coherence theorem* in precisely the sense recorded in the preceding paragraph. Coherence results of this shape (a small set of natural constraints forces a rigid normal form) are a recognized genre in mathematics, the paradigm being Mac Lane’s coherence theorem [60,61], whose value lies not in the difficulty of any single diagram but in the guarantee that the entire package of constraints is mutually consistent. A second precedent, closer in form to the present paper, is the axiomatic characterization of Shannon entropy: each of the Shannon–Khinchin axioms is individually elementary and the uniqueness argument is short, yet the theorem is foundational precisely because it certifies that one natural package of requirements admits exactly one solution [62,63]. The present paper is offered in that spirit, and two concrete steps meet the depth objection directly: Proposition 1 shows that the H3 aggregation rule, far from being engineered, is canonically forced up to a single residual modeling choice, whose elimination is posed as a self-contained functional-

equation problem in Section 10; and Example 8 realizes the full axiom package on the Fibonacci quasicrystal's patch space, in the semi-natural sense already noted above and made precise in Remark 16.

1.6. Why These Hypotheses?

Each condition in the package is independently motivated; none is chosen retroactively to match a desired conclusion.

H1 (scale geometry). *Uniform geometric growth and additive self-similarity are structurally independent properties of a scale sequence. Each alone is compatible with any ratio $\sigma > 1$; jointly, they are compatible only when $\sigma^2 = \sigma + 1$, whose unique positive root greater than 1 is φ . The golden ratio is not imposed; it is the only value for which both structural properties of (σ, ℓ) can coexist. The sharpness result confirms that dropping H1 immediately admits $\sigma \neq \varphi$.*

H2 (polynomial compositionality and calibration). *Polynomial compositionality is a structural statement rather than a regularity assumption: it says the cost's comparative information is aggregated by a finite algebraic rule, which allows the classification of [52] to enter as a black box (Section 5). Its normalization and calibration are unit conventions that pin that classification to the canonical representative J_{cost} . Each sub-condition is load-bearing: dropping any one leaves a free parameter; relaxing the normalization alone, for instance, permits $J(x) = (\ln x)^2$.*

H3 (variational origin of the ratio). *H3 distinguishes an equilibrium ratio, one that minimizes cost subject to a fixed log-charge constraint, from an arbitrary assignment. Without this mechanism, r carries no variational content and cannot be controlled by A3; a ledger satisfying H1 and H2 may then carry $r \equiv 2$ simply because r was assigned by hand. As motivation (not a theorem): once J governs all comparisons, a natural permutation-symmetric, cost-based way to summarize a multi-channel state $x(s) \in \mathbb{R}_{>0}^{N_s}$ by one scalar is to replace it by the least-cost configuration at fixed log-charge, the constrained minimizer $T_{N_s}(x(s))$. Other summaries are possible; the point is that this one is permutation-symmetric and cost-based, and it turns out to be the uniform configuration. Aggregation-function theory [20,57] suggests one could go further and derive H3 from such aggregation axioms (symmetry, cost-monotonicity, and a consistency condition under refinement); we take the constrained-minimizer form as a hypothesis here and record the axiomatic derivation as a natural strengthening (Section 10).*

A3 (charge neutrality). *The log-sum-zero condition $\Lambda(x(s)) = 0$ is the constraint used here; it places the Jensen equilibrium at the uniform configuration $(1, \dots, 1)$ and hence forces $r(s) = 1$. Any nonzero log-sum Q places the equilibrium at $(e^{Q/N}, \dots, e^{Q/N})$ with $r(s) = e^{Q/N} \neq 1$. Section 8 describes the resulting level-set structure in full.*

1.7. Notation

We fix the following notation, used throughout the paper. We write $\mathbb{R}$ for the reals, $\mathbb{C}$ for the complex numbers, $\mathbb{R}_{>0} := (0, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, and $\mathbb{N} := \{0, 1, 2, \dots\}$ (including 0). The golden ratio is $\varphi := \frac{1+\sqrt{5}}{2}$, the unique positive real root greater than 1 of $\sigma^2 = \sigma + 1$. A conserved comparison ledger $F = (S_F, T_F, O_F, \mu_F, r, \bar{\mu}_F, J)$ consists of a countable state space S_F , an evolution map $T_F: S_F \rightarrow S_F$, an observable set O_F , a measurement map $\mu_F: S_F \rightarrow O_F$ factoring as $\mu_F = \bar{\mu}_F \circ r$ through the ratio interface $r: S_F \rightarrow \mathbb{R}_{>0}$, and an admissible cost $J: \mathbb{R}_{>0} \rightarrow \mathbb{R}$; $\text{LogCharge}(s) := \log r(s)$ is the state-level log-charge, conserved by T_F . For a vector $c = (c_1, \dots, c_N) \in \mathbb{R}_{>0}^N$ we write $\Lambda(c) := \sum_{i=1}^N \log c_i$ for its vector-level log-charge; under hypothesis H3 the two are related by $\text{LogCharge}(s) = \Lambda(x(s))/N_s$. The log-substitution of the cost is $G(t) := J(e^t)$, with evenness $G(t) = G(-t)$ from reciprocal symmetry; $P: \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}$ is the unique continuous combiner of J delivered by [52], and $J_{\text{cost}}(x) := \frac{1}{2}(x + x^{-1}) - 1$ is its distinguished representative. A hierarchical structure is (σ, ℓ) ; under H1, $\sigma = \varphi$. The words “charge”, “conservation”, and “log-charge” are used throughout purely as bookkeeping labels

for the additive quantity $\log r$ and its invariance under T_F ; no physical content is asserted, and every such term could be replaced by the neutral phrase “additive T_F -invariant” without altering any definition, statement, or proof. We retain the shorter labels only for brevity.

1.8. Method and Outline

It is convenient to keep the three forcing steps separate, since each draws on different structural input. *Step B1* fixes the scale from H1 alone and is purely algebraic (Section 4). *Step B2* fixes the cost from H2 together with the classification imported from [52], and it is the only step that depends on it (Section 5). *Step B3* fixes the ratio interface from H3 and A3 through a single constrained-minimization (Jensen) argument (Section 6). The dependence is one-directional: B3 uses the cost fixed in B2, whereas B1 and B2 neither use nor are affected by H3 or A3. This layering is what lets the scale-and-cost rigidity and the full collapse stand as separate theorems, and it is what makes the A3-free analysis of Section 8 a clean restriction of the same argument rather than a different one.

The paper is laid out along the proof sequence introduced above: inheritance from [52], ledger setup, the three rigidity layers B1/B2/B3, the main rigidity theorems, the no-A3 alternative, and the examples. Section 2 records the precise analytic input from [52], together with a basic non-negativity lemma that justifies the domain of the combiner. Section 3 defines frameworks, ledgers, the observational quotient, and the foundational axioms. Section 4 establishes scale forcing under H1 in both exact and asymptotic forms. Section 5 applies the imported classification to force the canonical cost under H2. Section 6 develops the Jensen minimizer and the collapse $r \equiv 1$; the Lyapunov and gradient-flow refinements are deferred to Appendix A. Section 7 assembles the A3-free and full rigidity theorems and proves the condition package irredundant (dropping any one condition breaks a specific conclusion). Section 8 develops the level-set structure of $\log r$ when A3 is dropped and identifies A3 as the condition picking the level set $\log r = 0$. Section 9 gives several examples: a canonical static ledger, an N -channel ledger with non-uniform internal data, an explicit $N = 2$ cost-drop calculation, and a ledger with an injective observable exhibiting both observational collapse under A3 and a nontrivial multi-sector quotient when A3 is dropped; a further example exhibits non-trivial conservative dynamics that exercise the conservation law and evolution map. A final finite-sector example shows that nontrivial invariant level sets can already occur with two states and period-two evolution. Section 10 recapitulates the proof sequence layer-by-layer, restates the three forced quantities and their two corollaries, and situates the present paper relative to the classification of [52].

2. Imported Analytic Input from Prior Work

This section records the analytic input taken from [52]. It states, without proof, the four results from [52] that the present paper uses as black-box inputs; they form a logical chain: automatic factorization produces the combiner, the polynomial degree bound forces it to a bilinear form, the two-family classification identifies all admissible costs compatible with that bilinear combiner, and canonical-cost uniqueness selects a single representative once normalization and calibration are imposed. The four theorems below are imported from [52]; Lemma 1 is an elementary consequence proved here. Later sections also use standard external facts, cited locally, such as existence of constrained convex minimizers, Jensen’s inequality, and Grönwall’s estimate.

Definition 1 (Admissible cost, ([52], Def. 2.1)). *A function $J: \mathbb{R}_{>0} \rightarrow \mathbb{R}$ is an admissible cost if it satisfies the following three conditions:*

1. *Reciprocal symmetry:* $J(x) = J(x^{-1})$ for all $x > 0$.
2. *Unit normalization:* $J(1) = 0$.

3. *Strict log-convexity: the log-substitution $G(t) := J(e^t)$ is strictly convex on $\mathbb{R}$ (equivalently, J is strictly convex as a function of the logarithmic coordinate $t = \ln x$ on $\mathbb{R}_{>0}$).*

Remark 1 (Continuity is automatic). *Continuity of J on $\mathbb{R}_{>0}$ is automatic and is therefore not listed as an axiom: convexity of G implies continuity of G on $\mathbb{R}$ ([18], Thm. 1.3.3); ([52], Remark following Def. 2.1)), hence $J(x) = G(\ln x)$ is continuous on $\mathbb{R}_{>0}$.*

Before introducing the combiner, we record a basic consequence of admissibility that underpins the rest of the section: every admissible cost takes non-negative values. This fact justifies the domain $[0, \infty)^2$ of the combiner P in the automatic factorization theorem below, and it will be used again in Section 6.

Lemma 1 (Non-negativity of admissible costs). *If J is an admissible cost and $G := J \circ \exp$, then G is even and*

$$G(t) \geq 0 \text{ for all } t \in \mathbb{R}, \quad G(t) > 0 \text{ for } t \neq 0.$$

Equivalently,

$$J(x) \geq 0 \text{ for all } x > 0, \quad J(x) = 0 \iff x = 1.$$

Proof. Evenness follows from the symmetry axiom: $G(-t) = J(e^{-t}) = J((e^t)^{-1}) = J(e^t) = G(t)$. Also, $G(0) = J(1) = 0$. For $t \neq 0$, strict convexity applied to the distinct points t and $-t$ gives

$$G(0) = G\left(\frac{t + (-t)}{2}\right) < \frac{G(t) + G(-t)}{2} = G(t),$$

so $G(t) > 0$ for $t \neq 0$ and hence $G(t) \geq 0$ for all t . Since $J(x) = G(\log x)$, the stated properties of J follow. $\square$

Theorem 1 (Automatic factorization, ([52], Automatic Factorization Theorem, Thm. 3.2)). *For every admissible cost J there is a unique continuous function $P: [0, \infty)^2 \rightarrow \mathbb{R}$ such that*

$$J(xy) + J(x/y) = P(J(x), J(y)) \quad (x, y > 0).$$

The combiner P is symmetric and satisfies $P(u, 0) = 2u$ and $P(0, v) = 2v$.

Theorem 2 (Polynomial combiner classification, ([52], Continuous degree bound, Thm. 5.2; bilinear-form corollary, Cor. 5.5)). *If an admissible cost has a polynomial combiner P then*

$$P(u, v) = cuv + 2u + 2v$$

for some $c \geq 0$.

Theorem 3 (Two-family classification, ([52], Full classification, Thm. 7.6)). *Let J be an admissible cost with polynomial combiner. Then exactly one of the following holds:*

- (i) *Hyperbolic family:*

$$J(x) = \frac{1}{c}(x^\lambda + x^{-\lambda}) - \frac{2}{c} \quad (c, \lambda > 0).$$

- (ii) *Degenerate quadratic family:*

$$J(x) = a(\ln x)^2 \quad (a > 0).$$

Theorem 4 (Canonical cost uniqueness, ([52], Canonical cost, Cor. 9.1)). *If an admissible cost satisfies $P(u, v) = 2uv + 2u + 2v$ and the calibration convention $(J \circ \exp)''(0) = 1$ then*

$$J(x) = J_{\text{cost}}(x) := \frac{1}{2}(x + x^{-1}) - 1.$$

Remark 2 (Scope of the dependence on [52]). *The four imported statements above are used in exactly one place. Only the cost-forcing layer B2 (Section 5) invokes Theorems 1–4; the scale-forcing layer B1 (Section 4) and the variational layer B3 (Section 6) are self-contained and use nothing from [52]. Consequently every forced conclusion except $J = J_{\text{cost}}$, namely $\sigma = \varphi$ (under H1), $r \equiv 1$ with the constant measurement map and the singleton observational quotient (under H3 and A3), and the entire no-A3 level-set analysis of Section 8, is proved here in full from the ledger axioms and its own structural hypothesis, with no input from [52]; the single conclusion $J = J_{\text{cost}}$ is conditional on precisely the four statements reproduced here, which the reader may adopt as axioms. These four are moreover independently corroborated rather than taken on faith: canonical-cost selection is reproved in [58], the bilinear reduction of the combiner is obtained directly from the governing functional equation in [59], and both reduce, under continuity, or under measurability with local boundedness (Remark 10), to the classical d’Alembert–Wilson–Kannappan theory [1,2,4]. The black-box import is therefore an expository economy, not an unverifiable dependence.*

3. Frameworks, Ledgers, and Axioms

Having recorded the analytic input from [52], we now establish the abstract objects in which that input will be installed. We begin with the general notion of a framework, then we specialize to the conserved comparison ledger, which adds four further requirements: countability of the state space, a ratio interface r through which all observables are read, an admissible cost governing comparisons, and a conservation law on the log-charge. The foundational axioms (A1)–(A2) assert that the state set is non-empty and that the object satisfies the ledger definition. The structural hypotheses H1–H3 and the modeling axiom A3 are introduced separately, in the sections where they are used. Keeping them out of this section, where the definitions fix only the ambient object, prevents the main theorem from being built into the definition of a ledger.

Definition 2 (Framework). *A framework is a tuple $F = (S_F, T_F, O_F, \mu_F)$ where S_F is a set of states, $T_F: S_F \rightarrow S_F$ is an evolution map, O_F is a set of observables, and $\mu_F: S_F \rightarrow O_F$ is a measurement map.*

Two states are observationally indistinguishable when the measurement map assigns them the same value. The observational quotient encodes this identification globally.

Definition 3 (Observational quotient). *For a framework F , define $s_1 \sim_F s_2$ if $\mu_F(s_1) = \mu_F(s_2)$. The observational quotient is*

$$\text{SQ}(F) := S_F / \sim_F.$$

We say the quotient collapses if $\text{SQ}(F)$ is a singleton.

Lemma 2 (Constant measurement gives a singleton quotient). *If $S_F \neq \emptyset$ and μ_F is constant on S_F then $\text{SQ}(F) \simeq \mathbf{1}$ (i.e., $\text{SQ}(F)$ contains exactly one element).*

Proof. A constant measurement map places every pair of states in the same equivalence class under $\sim_F$, so $S_F / \sim_F$ is a singleton. $\square$

Remark 3 (Notation $\text{SQ}(F) \simeq \mathbf{1}$). *Throughout the paper, $\mathbf{1}$ denotes a fixed one-element set, and $\simeq$ denotes bijection.*

The ledger specializes the framework by requiring that all observables be read through a single positive ratio variable, that a cost function govern comparisons, and that the log-charge of the ratio variable be preserved by the dynamics.

Definition 4 (Conserved comparison ledger). A conserved comparison ledger is a tuple $F = (S_F, T_F, O_F, \mu_F, r, \bar{\mu}_F, J)$ in which (S_F, T_F, O_F, μ_F) is a framework and the additional data satisfy:

1. S_F is countable;
2. $r: S_F \rightarrow \mathbb{R}_{>0}$ is a distinguished ratio interface and $\bar{\mu}_F: \mathbb{R}_{>0} \rightarrow O_F$ is a distinguished reduced measurement map, with $\mu_F = \bar{\mu}_F \circ r$;
3. $J: \mathbb{R}_{>0} \rightarrow \mathbb{R}$ is a chosen admissible cost;
4. for every $s \in S_F$,

$$\text{LogCharge}(T_F(s)) = \text{LogCharge}(s), \quad \text{LogCharge}(s) := \log r(s).$$

The ratio interface r and the reduced measurement map $\bar{\mu}_F$ are part of the ledger's data, not merely asserted to exist; all subsequent ledger-level statements refer to this distinguished pair.

Condition (1) (countability of S_F) is a modeling convention, not an analytic ingredient: the proofs in Section 6 operate on the finite-dimensional internal vector $x(s) \in \mathbb{R}_{>0}^{N_s}$ state by state and never sum or integrate over S_F . Concretely, the proofs are pointwise in s and use only the fiber decomposition of $s \mapsto \log r(s)$; the sole place in which cardinality enters a conclusion is the countably infinite quotient of Example 6, a property of that example. The invariant $\text{LogCharge}(s) = \log r(s)$ fixed by condition (4) is called the *log-charge* of s .

Definition 5 (Foundational axioms).

(A1) F is inhabited: $S_F \neq \emptyset$.

(A2) F is a conserved comparison ledger (i.e., it satisfies all four conditions of Definition 4).

Note that (A2) is not circular: Definition 2 defines frameworks, which are not ledgers; (A2) asserts that F is specifically a ledger in the sense of Definition 4.

Remark 4 (Hypotheses act on disjoint data). The three structural hypotheses act on disjoint pieces of the ledger data. H1 constrains only the hierarchical scale pair (σ, ℓ) . H2 constrains only the admissible cost J through its combiner. H3 augments the ledger with an internal vector $x(s)$ (defined precisely in Definition 11) from which the ratio interface is recovered by one Jensen step; A3 (Definition 12) then imposes the log-sum-zero condition $\Lambda(x(s)) = 0$ on that data. Each step can therefore be proved independently, and the three conclusions are combined in Section 7 without further argument.

4. Step B1: Hierarchical Forcing

The first structural hypothesis concerns the scale geometry of the ledger. If the ledger carries a sequence of scales that is simultaneously uniformly geometric (each term a fixed multiple of the previous) and additively self-similar (the third term equal to the sum of the first two) then the two requirements are jointly compatible only at the golden ratio. This section proves that conclusion in exact and limiting-ratio form.

Definition 6 (Hierarchical structure on a ledger). Let F be a conserved comparison ledger. A hierarchical structure on F is a pair (σ, ℓ) , designated as additional data of F , with $\sigma > 1$ and $\ell: \mathbb{N} \rightarrow \mathbb{R}_{>0}$ satisfying:

1. Uniform scaling: $\ell(k+1) = \sigma \ell(k)$ for all $k \geq 0$;
2. Additive composition: $\ell(2) = \ell(1) + \ell(0)$.

We say F carries a hierarchical structure if such a pair is specified as part of F 's data.

Theorem 5 (Scale fixed by exact hierarchy). *If F carries a hierarchical structure (σ, ℓ) then $\sigma = \varphi := \frac{1+\sqrt{5}}{2}$.*

Proof. Uniform scaling gives $\ell(1) = \sigma\ell(0)$ and $\ell(2) = \sigma^2\ell(0)$. Additive composition gives $\ell(2) = \ell(1) + \ell(0) = (\sigma + 1)\ell(0)$. Since $\ell(0) > 0$, we obtain $\sigma^2 = \sigma + 1$, i.e., $\sigma^2 - \sigma - 1 = 0$. The two roots are $(1 \pm \sqrt{5})/2$; since $\sigma > 1$, the root $(1 - \sqrt{5})/2 < 0$ is excluded, leaving $\sigma = (1 + \sqrt{5})/2 =: \varphi$. $\square$

Although condition (2) of Definition 6 constrains only the triple $(0, 1, 2)$, uniform scaling propagates it: $\ell(k) = \sigma^k\ell(0)$ with $\sigma^2 = \sigma + 1$ gives $\ell(k+2) = \ell(k+1) + \ell(k)$ for all $k \geq 0$, the full Fibonacci recurrence.

Remark 5 (The golden-ratio characterization is classical). *We claim no originality for Theorem 5: that $\sigma > 1$ with $\sigma^2 = \sigma + 1$ is the golden ratio is a classical fact about Fibonacci-type recurrences [6–8], recalled only to fix the form of H1. The one mildly novel point is the observation that ratio convergence does not imply asymptotic additivity; the converse implication is classical (Poincaré–Perron), as recorded in Remark 6. The exact case simply restates the defining relation of φ , and we record it as a normalization rather than claim it as a result. The substance of the layer lies in the asymptotic case: Theorem 6 concludes $\sigma = \varphi$ for sequences satisfying no exact geometric or Fibonacci identity (Example 1 is genuinely non-geometric, its ratios $1, 2, \frac{3}{2}, \frac{5}{3}, \dots$ never constant), so layer B1 is not exhausted by the quadratic $\sigma^2 = \sigma + 1$.*

The same equation $\sigma^2 = \sigma + 1$ is recovered when the hierarchical identities hold only in the limit, as the next definition and theorem make precise.

Definition 7 (Asymptotic hierarchical structure on a ledger). *Let F be a conserved comparison ledger. An asymptotic hierarchical structure on F is a pair (σ, ℓ) , designated as additional data of F , with $\sigma > 1$ and $\ell: \mathbb{N} \rightarrow \mathbb{R}_{>0}$ such that*

1. $\lim_{k \rightarrow \infty} \frac{\ell(k+1)}{\ell(k)} = \sigma;$
2. $\lim_{k \rightarrow \infty} \frac{\ell(k+2)}{\ell(k+1) + \ell(k)} = 1.$

Remark 6 (Strength of the asymptotic additivity assumption). *Condition (2) in Definition 7 does not follow from the existence of the limiting ratio in condition (1). A sequence may satisfy $\ell(k+1)/\ell(k) \rightarrow \sigma > 1$ without satisfying $\ell(k+2)/(\ell(k+1) + \ell(k)) \rightarrow 1$. Thus the asymptotic version of H1 is not intended to follow from ratio convergence alone; it is a strong asymptotic additivity requirement, useful for allowing perturbations of the exact Fibonacci-type relation while preserving the same limiting forcing equation. The converse implication, by contrast, does hold, and we record its classical provenance: writing condition (2) as $\ell(k+2) = a_k(\ell(k+1) + \ell(k))$ with $a_k \rightarrow 1$ exhibits ℓ as a solution of a linear difference equation of Poincaré type with limiting characteristic equation $\lambda^2 = \lambda + 1$, whose roots φ and $1 - \varphi$ have distinct moduli; by the Poincaré–Perron theorem [64,65] the ratio $\ell(k+1)/\ell(k)$ of a nonvanishing solution converges to one of the roots, and positivity of ℓ excludes the negative root $1 - \varphi$. For positive scales, condition (2) therefore already forces $\ell(k+1)/\ell(k) \rightarrow \varphi$; the role of condition (1) is to designate the limit σ as structural data of the ledger, on which Theorem 6 (whose two-line proof we keep self-contained) then enforces $\sigma = \varphi$.*

Theorem 6 (Asymptotic hierarchical forcing). *If F carries an asymptotic hierarchical structure (σ, ℓ) with limiting ratio $\sigma > 1$ then $\sigma = \varphi$.*

Proof. Define $q_k := \ell(k+1)/\ell(k)$ and $a_k := \ell(k+2)/(\ell(k+1) + \ell(k))$. By assumption, $q_k \rightarrow \sigma$ and $a_k \rightarrow 1$. Dividing the identity $\ell(k+2) = a_k(\ell(k+1) + \ell(k))$ by $\ell(k+1)$ gives

$$q_{k+1} = a_k(1 + q_k^{-1}).$$

Since $\{q_{k+1}\}$ is the sequence $\{q_k\}$ shifted by one index, $q_{k+1} \rightarrow \sigma$ as $k \rightarrow \infty$. Taking limits on both sides and using $a_k \rightarrow 1$, $q_k^{-1} \rightarrow \sigma^{-1}$ yields $\sigma = 1 \cdot (1 + \sigma^{-1})$, hence $\sigma^2 = \sigma + 1$. Since $\sigma > 1$, the unique solution is φ . $\square$

Example 1 (A genuinely asymptotic hierarchical scale). The Fibonacci scale $\ell(k) = F_{k+1}$ (with $F_1 = F_2 = 1$, so $\ell(0), \ell(1), \ell(2), \dots = 1, 1, 2, 3, 5, 8, \dots$) is not exactly geometric: the ratios $\ell(k+1)/\ell(k) = 1, 2, \frac{3}{2}, \frac{5}{3}, \dots$ are never constant, so no (exact) hierarchical structure of Definition 6 describes it. It does, however, carry an asymptotic hierarchical structure with $\sigma = \varphi$: condition (2) of Definition 7 holds exactly, since $\ell(k+2) = \ell(k+1) + \ell(k)$, and condition (1) holds in the limit, since $\ell(k+1)/\ell(k) \rightarrow \varphi$. Thus Theorem 6 forces $\sigma = \varphi$ for a scale beyond the reach of the exact theorem, which is why the asymptotic disjunct is retained in H1.

Remark 7 (Hypothesis H1 and its scope). In the rest of the paper, hypothesis (H1) means the following disjunction: F carries either a hierarchical structure (Definition 6) or an asymptotic hierarchical structure (Definition 7); Example 1 exhibits a scale of the second kind. In either form, Theorems 5 and 6 force $\sigma = \varphi$. Because (σ, ℓ) is designated as part of F 's data, the forced value $\sigma = \varphi$ is a property of F itself, not of an extraneous auxiliary object. H1 is a structural hypothesis imposed on the ledger; the paper does not assert that every ledger must carry such a structure.

5. Step B2: Cost Forcing from the Imported Classification

The second structural hypothesis constrains the admissible cost installed in the ledger. Reference [52] has classified all admissible costs with polynomial combiner; the task here is to identify the unique cost selected by the specific normalization and calibration imposed by H2. The “uniqueness” asserted below is uniqueness after fixing units: H2(1) forces only the hyperbolic family, while the normalization H2(2) and calibration H2(3) are unit conventions selecting the representative J_{cost} within it (Remark 9).

Definition 8 (Polynomial compositionality). An admissible cost satisfies polynomial compositionality if its combiner P is a polynomial.

Definition 9 (Hypothesis H2). A ledger satisfies (H2) if its cost satisfies the following three conditions:

1. polynomial compositionality;
2. the canonical normalization $P(1, 1) = 6$;
3. the log-substitution $G := J \circ \exp$ is twice differentiable in a neighborhood of 0, and the calibration convention $(J \circ \exp)''(0) = 1$.

The three conditions act in sequence: condition (1) yields a polynomial (hence, by Theorem 2, bilinear) combiner, condition (2) fixes its single free parameter, and condition (3) selects the canonical member of the resulting family. The arithmetic is carried out once, in the proof of Theorem 7 below.

The polynomial-combiner condition H2(1) is a hypothesis of the present theorem, not a conclusion of the ledger axioms. It is a convenient sufficient condition: it lets the classification of [52] be imported as a single black box (Theorems 2 and 3), replacing a continuous bootstrap by a one-line algebraic reduction. The same canonical cost J_{cost} is reached without it: under continuity, reciprocal symmetry, unit normalization, and the same curvature calibration, the classical continuous d'Alembert–Wilson–Kannappan theory [1,2,4]

isolates the same hyperbolic family, so the polynomial form is an algebraic convenience for a self-contained import.

Remark 8 (Condition H2(3) is well-posed). *The expression $(J \circ \exp)''(0)$ in condition (3) presupposes twice-differentiability, which admissibility alone does not provide (Definition 1 requires only strict convexity of $G = J \circ \exp$). It is, however, automatic once condition (1) holds: by Theorem 2 the combiner is bilinear, so Theorem 3 places J in either the hyperbolic family $\frac{1}{c}(x^\lambda + x^{-\lambda}) - \frac{2}{c}$ or the degenerate family $a(\ln x)^2$, both of which are C^∞ on $\mathbb{R}_{>0}$. Hence, G is smooth and $G''(0)$ exists, so condition (3) is well-posed.*

Theorem 7 (Cost forcing under H2). *Let J be an admissible cost satisfying (H2) (Definition 9). Then, $J = J_{\text{cost}}$.*

Proof. By Theorem 2, the polynomial combiner takes the form $P(u, v) = cuv + 2u + 2v$ for some $c \geq 0$. Evaluating at $(u, v) = (1, 1)$ gives $P(1, 1) = c + 4$, and the normalization $P(1, 1) = 6$ forces $c = 2$. Since $c = 2 > 0$, the degenerate quadratic family of Theorem 3 (for which $c = 0$ and $P(1, 1) = 4$) is excluded: J belongs to the hyperbolic family. Applying Theorem 4 with $c = 2$ and the calibration $(J \circ \exp)''(0) = 1$ then yields $J = J_{\text{cost}}$. $\square$

Remark 9 (What the normalization $P(1, 1) = 6$ does and does not do). *We do not claim 6 is singled out by a first principle. By Theorem 2 the combiner is $P(u, v) = cuv + 2u + 2v$ with $c = P(1, 1) - 4$, so fixing $P(1, 1)$ only fixes c . The normalization-independent content of B2 is the dichotomy of Theorem 3: hyperbolic ($c > 0$, $P(1, 1) > 4$) versus degenerate ($c = 0$, $P(1, 1) = 4$). Any $P(1, 1) > 4$ gives the hyperbolic family, and after the calibration $(J \circ \exp)''(0) = 1$ gives a cost of $\frac{1}{2}(x^\lambda + x^{-\lambda})$ -type; the value 6 ($c = 2$) is the units convention normalizing the representative to J_{cost} . A reader objecting to 6 may replace H2(2) by “ $P(1, 1) > 4$ ” and read the conclusion as “ J is a calibration of J_{cost} .”*

Remark 10 (The polynomial-combiner hypothesis is mild). *Hypothesis H2(1) is far weaker than a finite-degree truncation adopted merely for convenience, for two reasons. First, no regularity is smuggled in. For any admissible cost the combiner delivered by Theorem 1 is already continuous, and in the classical d’Alembert–Wilson theory measurable solutions of the underlying composition equation are automatically continuous [2,4,5]; indeed, J  rai’s regularity theory upgrades mere measurability (or the Baire property) to C^∞ smoothness for wide classes of non-composite functional equations, d’Alembert’s among them [66]. Hence H2(1) cannot be circumvented by passing to a merely measurable “wild” combiner while staying inside the admissible class. Second, the degree is essentially forced, not assumed. The companion d’Alembert inevitability theorem [59] establishes that a symmetric polynomial combiner of degree ≥ 3 with non-canceling leading term admits no nonconstant continuous solution; therefore, within the admissible class, every polynomial combiner is already the bilinear form $P(u, v) = cuv + 2u + 2v$ of Theorem 2. Thus H2(1) does not discard a rich higher-degree family of admissible costs, there is none, and the only genuinely unaddressed case is a non-polynomial (e.g., transcendental) combiner, which we isolate as an open problem in Section 10.*

With $\sigma = \varphi$ forced by B1 and $J = J_{\text{cost}}$ forced by B2, only the ratio interface remains free. The next section imposes H3 and A3 to determine it.

6. Step B3: Constrained Minimizer and Its Dynamics

The third structural hypothesis augments the ledger with an internal vector at each state, from which the ratio interface is recovered by a single constrained-minimum step. Only the resulting static minimizer is used in the main text; its dynamical refinements are deferred to

Appendix A. This section establishes existence and uniqueness of the cost minimizer under a fixed charge constraint: strict convexity and coercivity of $G = J \circ \exp$ guarantee a unique minimizer, and Jensen's inequality identifies it as the uniform configuration.

Lemma 3 (Coercivity of G and existence of a unique constrained minimizer). *Let J be an admissible cost and $G := J \circ \exp$. Then G is even, continuous, strictly convex on $\mathbb{R}$, with $G(0) = 0$, $G(t) > 0$ for $t \neq 0$, and $G(t) \rightarrow \infty$ as $|t| \rightarrow \infty$. Consequently, for every $N \geq 1$, every $Q \in \mathbb{R}$, and every $c \in \mathbb{R}_{>0}^N$ with $\sum_i \log c_i = Q$, the minimization problem*

$$\min \left\{ \sum_{i=1}^N J(c'_i) : c' \in \mathbb{R}_{>0}^N, \sum_{i=1}^N \log c'_i = Q \right\}$$

admits a unique minimizer.

Proof. Evenness of G follows from $J(x) = J(x^{-1})$; strict convexity and $G(0) = J(1) = 0$ are clauses of admissibility, and continuity follows from Remark 1. Moreover, by Lemma 1, $G(t) \geq 0$ for all t and $G(t) > 0$ for $t \neq 0$.

For coercivity, suppose for contradiction that a sequence $t_n \rightarrow \infty$ satisfies $G(t_n) \leq M < \infty$. For any fixed $s > 0$ and all n large enough that $t_n > s$, global convexity of G on $\mathbb{R}$ gives (writing $s = (s/t_n) \cdot t_n + (1 - s/t_n) \cdot 0$)

$$G(s) \leq \frac{s}{t_n} G(t_n) + \left(1 - \frac{s}{t_n}\right) G(0) \leq \frac{sM}{t_n} \xrightarrow{n \rightarrow \infty} 0,$$

contradicting $G(s) > 0$. Coercivity as $t \rightarrow -\infty$ follows by evenness.

For well-posedness, pass to log-coordinates $\xi_i := \log c'_i$. The functional $\xi \mapsto \sum_i G(\xi_i)$ is continuous and strictly convex on the closed affine hyperplane $H_Q := \{\xi \in \mathbb{R}^N : \sum_i \xi_i = Q\}$. Coercivity of G together with Lemma 1 ($G \geq 0$ pointwise) implies coercivity of the sum on H_Q : if $\sum_{j=1}^N G(\xi_j) \leq M$, then $G(\xi_i) \leq M$ for each i (by non-negativity of the remaining summands), hence each $|\xi_i|$ is bounded by coercivity of G , hence $\|\xi\|$ is bounded. By contraposition, $\sum_j G(\xi_j) \rightarrow \infty$ as $\|\xi\| \rightarrow \infty$ on H_Q . The infimum is therefore attained at a unique point ([17], Thm. 27.3); exponentiating the coordinates returns the unique minimizer $c' \in \mathbb{R}_{>0}^N$. $\square$

Definition 10 (N -channel variational update). *Fix an admissible cost J and an integer $N \geq 1$. The N -channel variational update is the map $T_N: \mathbb{R}_{>0}^N \rightarrow \mathbb{R}_{>0}^N$ that sends a configuration $c = (c_1, \dots, c_N) \in \mathbb{R}_{>0}^N$ to the unique minimizer c' (well-defined by Lemma 3) of*

$$\mathcal{D}(c') := \sum_{i=1}^N J(c'_i)$$

subject to the charge constraint

$$\Lambda(c') = \Lambda(c), \quad \Lambda(c) := \sum_{i=1}^N \log c_i.$$

The vector-level log-charge $\Lambda(c) := \sum_i \log c_i$ ($c \in \mathbb{R}_{>0}^N$) and the state-level log-charge $\text{LogCharge}(s) := \log r(s)$ ($s \in S_F$) are distinct quantities. Under hypothesis H3 they are linked by $\text{LogCharge}(s) = \Lambda(x(s))/N_s$ (Definition 11); the two values coincide exactly when $N_s = 1$ or, more generally, when $\Lambda(x(s)) = 0$ (axiom A3).

Theorem 8 (Unique constrained minimizer of the cost sum). *Let $c \in \mathbb{R}_{>0}^N$ and write $Q := \Lambda(c)$. Then*

$$T_N(c) = (e^{Q/N}, \dots, e^{Q/N}),$$

and T_N is idempotent.

Proof. Setting $\xi_i := \log c'_i$ and $G := J \circ \exp$, the objective becomes $\mathcal{D}(c') = \sum_{i=1}^N G(\xi_i)$ under the affine constraint $\sum_i \xi_i = Q$. Since G is strictly convex, the discrete Jensen inequality (see e.g., ([18], Cor. 1.3.4)) gives

$$\frac{1}{N} \sum_{i=1}^N G(\xi_i) \geq G\left(\frac{Q}{N}\right),$$

with equality if and only if all $\xi_i = Q/N$. The unique minimizer therefore has every component equal to $e^{Q/N}$; applying T_N to this output leaves it unchanged, confirming idempotency. $\square$

The static minimizer Theorem 8 is all that the main rigidity results require. Two supplementary layers, a one-step Lyapunov contraction (Lemma A1) and exponential convergence of a constrained gradient flow (Theorem A1), give the equilibrium a dynamical rather than merely algebraic interpretation; they are collected in Appendix A so as not to interrupt the main line of argument.

Hypothesis H3, Axiom A3, and the Collapse $r \equiv 1$

We now introduce the structural hypothesis and axiom that connect the variational update to the ratio interface.

Definition 11 (Positive-vector representation of r ; hypothesis H3). *A ledger F satisfies hypothesis (H3) if it is equipped with the following additional data: for every state $s \in S_F$, a designated integer $N_s \geq 1$ and a designated vector*

$$x(s) = (x_1(s), \dots, x_{N_s}(s)) \in \mathbb{R}_{>0}^{N_s},$$

(the internal vector of s), such that the ratio interface is reproduced by one application of the variational update:

$$r(s) = [T_{N_s}(x(s))]_1.$$

The internal vector is a genuine structural addition to the ledger: it is not derived from r , and it may be non-uniform. The first coordinate is used only to extract a scalar ratio from the post-update vector. Once Theorem 8 applies, all post-update coordinates are equal, so the choice of coordinate carries no hidden asymmetry. The internal vector is retained rather than replaced by a single scalar log-charge $Q(s)$ because the scalar description $r(s) = \exp(\frac{1}{N_s} \Lambda(x(s)))$ becomes equivalent to it only after the Jensen step has symmetrized the data (Theorem 8). Keeping the vector is therefore what lets a state carry genuine pre-equilibrium asymmetry; Remark 12 records how A3 then acts on that data.

Remark 11 (Equivalent geometric-mean form of H3; the variational content is motivational). *By Theorem 8, the ratio-reproduction clause of H3 is equivalent to the closed form*

$$r(s) = [T_{N_s}(x(s))]_1 = \exp\left(\frac{1}{N_s} \Lambda(x(s))\right) = \left(\prod_{i=1}^{N_s} x_i(s)\right)^{1/N_s},$$

so the operative content of H3 is exactly that the ratio interface is the geometric mean of the internal vector. Under this reading the collapse is transparent: A3 sets $\Lambda(x(s)) = 0$, and the geometric mean of a positive vector with zero log-sum is 1, whence $r \equiv 1$. We therefore regard this geometric-mean identity as the operative form of H3, and the constrained-minimization (Jensen) formulation of Definition 10 and Theorem 8 as its motivation: the variational statement explains

why the geometric mean, rather than some other scalar summary of $x(s)$, is the natural cost-based aggregate, namely because it is the unique minimizer of the admissible cost at fixed log-charge (Lemma 3 and Theorem 8). We claim no independent structural necessity for this aggregation rule; consistently with the coherence reading (Section 1), the H3–A3 step is close to definitional, and its role is to isolate the single modeling decision, geometric-mean aggregation together with charge neutrality, on which $r \equiv 1$ rests. A derivation of geometric-mean aggregation from symmetry and consistency axioms would upgrade H3 from a hypothesis to a consequence, as already anticipated in the subsection “Why these hypotheses?”; Proposition 1 carries this out up to a single modeling choice.

Proposition 1 (The geometric-mean aggregation of H3 is canonical). Let $\{M_N: \mathbb{R}_{>0}^N \rightarrow \mathbb{R}_{>0}\}_{N \geq 1}$ be a family of aggregation maps that is:

1. symmetric, continuous, strictly increasing in each argument, reflexive ($M_N(x, \dots, x) = x$), and associative in the replacement sense;
2. scale covariant: $M_N(\lambda x) = \lambda M_N(x)$ for all $\lambda > 0$;
3. reciprocally self-dual: $M_N(x^{-1}) = M_N(x)^{-1}$, where $x^{-1} := (x_1^{-1}, \dots, x_N^{-1})$.

Then $M_N(x) = (\prod_{i=1}^N x_i)^{1/N}$ for every N : the geometric mean is the unique such family.

Proof. By condition (1) and the Kolmogorov–Nagumo–de Finetti theorem [19,21–23], there is a continuous strictly monotone f with $M_N(x) = f^{-1}(\frac{1}{N} \sum_{i=1}^N f(x_i))$. By condition (2) and the classical characterization of the homogeneous quasi-arithmetic means ([19], Thm. 84) (see also [20]), M_N is a power mean $M_p(x) = (\frac{1}{N} \sum_{i=1}^N x_i^p)^{1/p}$ for some $p \neq 0$, or the geometric mean (the limit $p \rightarrow 0$). A direct computation gives $M_p(x^{-1}) = M_{-p}(x)^{-1}$, so condition (3) holds iff $M_{-p} = M_p$, i.e., iff $p = -p$, i.e., $p = 0$. Hence the unique reciprocally self-dual member of the family is the geometric mean $M_0(x) = (\prod_{i=1}^N x_i)^{1/N}$. $\square$

One might object that the H3–A3 step is “engineered by design.” The proposition answers this concern for the aggregation layer: the operative content of H3, geometric-mean aggregation (Remark 11), which is precisely the constrained cost-minimizer of Theorem 8, is pinned down by three independently standard requirements, each stated without reference to the target $r \equiv 1$, so it is not a stipulation reverse-engineered from the conclusion. Condition (3) is not ad hoc: reciprocal symmetry $J(x) = J(x^{-1})$ is the defining property of admissible costs (Definition 1), and self-duality is its aggregation-level counterpart. We claim the assembly, not the ingredients, which are classical. The residual freedom is the single modeling choice of imposing condition (2), scale covariance of the ratio interface, at the ledger level; deriving it from a ledger-level principle, and thereby eliminating H3 as an independent hypothesis, is the program recorded in Section 10.

Definition 12 (Axiom A3). Assume F satisfies (H3), and let $x(s)$ denote the internal vector of each state. Axiom (A3) asserts that for every $s \in S_F$,

$$\Lambda(x(s)) := \sum_{i=1}^{N_s} \log x_i(s) = 0.$$

Remark 12 (A3 as a condition on internal vector). Axiom A3 is imposed on the internal vector $x(s)$ supplied by H3, not on the ratio interface $r(s)$ directly. This distinction is mathematically substantive: the internal data may be genuinely non-uniform, and A3 constrains only its total log-charge to vanish. The pre-equilibrium witness in Example 3 illustrates this precisely: the internal data has non-uniform coordinates, yet the equilibrium value is uniform because the charge is neutral.

Corollary 1 ($r \equiv 1$ under H3 and A3). (See Remark 13.) Let F satisfy (A2), (H3), and (A3). Then, $r(s) = 1$ for all $s \in S_F$.

Proof. Fix $s \in S_F$ and let $x(s) \in \mathbb{R}_{>0}^{N_s}$ be the internal vector supplied by H3. Set $Q := \Lambda(x(s))$. By Theorem 8 applied to $x(s)$,

$$T_{N_s}(x(s)) = (e^{Q/N_s}, \dots, e^{Q/N_s}).$$

Axiom A3 gives $Q = 0$, so $T_{N_s}(x(s)) = (1, \dots, 1)$. By the ratio-reproduction clause of H3,

$$r(s) = [T_{N_s}(x(s))]_1 = 1. \quad \square$$

Remark 13 (Content of the H3–A3 collapse). The mathematical content of Corollary 1 is the strict-convexity/Jensen statement that at fixed vector-level log-charge Q the constrained cost minimizer is the uniform vector $(e^{Q/N}, \dots, e^{Q/N})$. Hypothesis H3 is the modeling link that reads the ledger ratio from that minimizer, and A3 is the neutrality condition $Q = 0$. Thus the conclusion $r \equiv 1$ is not an additional analytic classification theorem; it is the ledger-level consequence of combining the Jensen minimizer with the H3 ratio-reading rule and the A3 zero-charge constraint.

7. Main Rigidity Theorems

Because the three steps act on disjoint data (Remark 4), the two main theorems below are assembled by straightforward combination. Theorem 9 is the A3-free version ($\sigma = \varphi$ and $J = J_{\text{cost}}$); Theorem 10 adds H3 and A3 to give the singleton observational quotient $\text{SQ}(F) \simeq \mathbf{1}$.

Theorem 9 (Scale and cost rigidity). Let F satisfy (A2).

Assume:

- (H1) F carries a hierarchical structure (Definition 6) or an asymptotic hierarchical structure (Definition 7);
- (H2) the cost satisfies Definition 9.

Then $\sigma = \varphi$ and $J = J_{\text{cost}}$.

Proof. If H1 is given in exact form, the scale claim follows from Theorem 5; if H1 is given in asymptotic form, it follows from Theorem 6. The cost claim is Theorem 7. $\square$

Theorem 10 (Full observational rigidity). Let F satisfy (A1) and the hypotheses of Theorem 9. Assume additionally:

- (H3) positive-vector representation of r (Definition 11);
- (A3) the log-sum-zero condition on the internal vector (Definition 12).

For the three collapse conclusions below, the proof uses only the non-emptiness of S_F , the factorization $\mu_F = \bar{\mu}_F \circ r$, H3, A3, and Lemma 2 (see Remark 14). In addition to $\sigma = \varphi$ and $J = J_{\text{cost}}$ from Theorem 9:

1. $r(s) = 1$ for all $s \in S_F$;
2. the measurement map μ_F is constant on S_F ;
3. $\text{SQ}(F) \simeq \mathbf{1}$.

Proof. Note that $\sigma = \varphi$ and $J = J_{\text{cost}}$ follow from Theorem 9. Item (1) is Corollary 1. For item (2): since $\mu_F = \bar{\mu}_F \circ r$ by axiom (A2) and $r(s) = 1$ for every s by item (1), the measurement map equals $\bar{\mu}_F(1)$ identically; hence, it is constant. Since (A1) gives $S_F \neq \emptyset$, item (3) then follows from Lemma 2. $\square$

Remark 14 (The observational collapse is a corollary of $r \equiv 1$, not a separate rigidity statement). We state items (2) and (3) of Theorem 10 modestly, as immediate corollaries of the ratio collapse (1) rather than as further rigidity conclusions. Once $r \equiv 1$, constancy of $\mu_F = \bar{\mu}_F \circ r$ is a one-line consequence of the factorization, and the singleton quotient is then Lemma 2. In particular the evolution map T_F , the conservation law of Definition 4(4), and the countability of S_F play no role in this static collapse; they enter only the no-A3 level-set analysis of Section 8 (Theorem 11) and the dynamical Example 6. The genuine forced content of the paper is thus carried by the three quantities $\sigma = \varphi$, $J = J_{\text{cost}}$, and $r \equiv 1$; the constant measurement map and the singleton observational quotient are their downstream consequences.

Sharpness: Irredundancy of the Condition Package

The two main theorems are tight: dropping any single condition from the package (H1), (H2), (H3), (A3) allows a ledger that satisfies all remaining conditions yet violates the corresponding conclusion. The three propositions and the following A3 remark make this precise for the triple of forced quantities $\sigma = \varphi$, $J = J_{\text{cost}}$, $r \equiv 1$. The four conditions are not on equal logical footing: H1 and H2 act on disjoint data (the scale pair and the cost) and are mutually independent, while A3 is a constraint on the internal-vector data introduced by H3. Accordingly, “dropping H3” means deleting that internal-vector data (Definition 11), which also voids the statement of A3, whereas “dropping A3” retains H3 but releases the log-sum-zero condition. With this reading, each of the four can be removed to break a specific conclusion.

Proposition 2 (H1 is necessary for $\sigma = \varphi$). *There exists a ledger $\mathcal{G}_1$ satisfying (A1), (A2), (H2), (H3), (A3) that carries a geometric scale sequence with ratio $\sigma = 2$ but does not satisfy (H1) (so all remaining axioms and conditions needed after dropping H1 remain in force). Its scale is not forced to φ .*

Proof. Define $\mathcal{G}_1$: $S_F = \mathbb{D}_{>0}$ (positive dyadic rationals), $T_F = \text{id}$, $O_F = \{*\}$, $\mu_F \equiv \bar{\mu}_F \equiv *$, $J = J_{\text{cost}}$, $r \equiv 1$, $N_s = 1$, $x(s) = (1) \in \mathbb{R}_{>0}^1$, and designated scale data $(\sigma, \ell) = (2, k \mapsto 2^k)$.

The sequence $\ell(k) = 2^k$ satisfies uniform scaling $\ell(k+1) = 2\ell(k)$, but $\ell(2) = 4$ while $\ell(1) + \ell(0) = 3 \neq 4$: the additive-composition condition of Definition 6 fails, so the exact hierarchical structure does not hold. The asymptotic form (Definition 7) fails for this same designated data as well: although $\ell(k+1)/\ell(k) = 2 \rightarrow 2$ satisfies condition (1) with $\sigma = 2$, one has

$$\frac{\ell(k+2)}{\ell(k+1) + \ell(k)} = \frac{2^{k+2}}{2^{k+1} + 2^k} = \frac{4}{3} \not\rightarrow 1,$$

so condition (2) fails. With its designated scale data, $\mathcal{G}_1$ therefore satisfies neither disjunct of (H1) (Remark 7), so (H1) does not hold. Direct verification: (A1) $S_F \neq \emptyset$; (A2) $J = J_{\text{cost}}$ is admissible, $r \equiv 1$ is conserved under $T_F = \text{id}$, and $\mu_F = \bar{\mu}_F \circ r$; (H2) $J = J_{\text{cost}}$ satisfies Definition 9, as verified in Proposition 5; (H3) $r(s) = [T_1((1))]_1 = 1$; (A3) $\Lambda((1)) = \log 1 = 0$. Yet the designated scale is $\sigma = 2 \neq \varphi$. $\square$

Proposition 3 (H2 is necessary for $J = J_{\text{cost}}$). *There exists a ledger $\mathcal{G}_2$ satisfying (A1), (A2), (H1), (H3), (A3) whose cost is $J(x) = (\ln x)^2 \neq J_{\text{cost}}$ (so all remaining axioms and conditions needed after dropping H2 remain in force).*

Proof. Define $\mathcal{G}_2$: same data as $\mathcal{G}_1$ except $J(x) := (\ln x)^2$ and hierarchical data $(\varphi, k \mapsto \varphi^k)$.

The cost $J(x) = (\ln x)^2$ is admissible: $J(x^{-1}) = (\ln x)^2 = J(x)$; $J(1) = 0$; $G(t) = t^2$ is strictly convex. Its combiner is $P(u, v) = 2u + 2v$, the $c = 0$ degenerate case of Theorem 2, giving $P(1, 1) = 4 \neq 6$; (H2) fails. The hierarchical structure $(\varphi, k \mapsto \varphi^k)$ satisfies both conditions of Definition 6 ($\varphi^2 = \varphi + 1$), so (H1) holds. Conditions (A1), (A2), (H3), (A3) are

verified exactly as in Proposition 2, replacing $J = J_{\text{cost}}$ by $J(x) = (\ln x)^2$; admissibility of $(\ln x)^2$ is checked by the same three-axiom verification. The cost $(\ln x)^2 \neq J_{\text{cost}}$. $\square$

Proposition 4 (H3 is necessary for $r \equiv 1$). *There exists a ledger $\mathcal{G}_3$ satisfying (A1), (A2), (H1), (H2) (with neither H3 nor A3 imposed) whose ratio interface satisfies $r \equiv 2$ (so the remaining meaningful assumptions after dropping H3, namely A1, A2, H1, and H2, remain in force).*

Proof. Define $\mathcal{G}_3$: $S_F = \mathbb{D}_{>0}$, $T_F = \text{id}$, $O_F = \{*\}$, $\mu_F \equiv \bar{\mu}_F \equiv *$, $J = J_{\text{cost}}$, $r(s) := 2$ for all s , hierarchical data $(\varphi, k \mapsto \varphi^k)$.

The log-charge $\text{LogCharge}(s) = \log 2$ is constant, hence conserved under $T_F = \text{id}$. (A1) and (A2) hold. (H1) holds as in Proposition 3. (H2) holds since $J = J_{\text{cost}}$ satisfies Definition 9, as verified in Proposition 5. No hypothesis (H3) is imposed: there is no internal vector, and r is assigned directly without any variational mechanism. The ratio interface $r \equiv 2 \neq 1$ is constant but not 1; neither the Lyapunov step nor the Jensen machinery of Section 6 applies to control r . $\square$

Remark 15 (A3 is necessary; Section 8 gives the full picture). *Dropping only (A3) while retaining (H3) leaves the log-charge $\Lambda(x(s))$ unconstrained, so $r(s) = \exp(\Lambda(x(s))/N_s)$ can take any positive value. Section 8 gives a complete description: the state space partitions into the level sets $F_Q = \{s : \log r(s) = Q\}$, each preserved by the dynamics; Theorem 12 shows that (A3) is exactly the condition that forces all states into F_0 . Multiple level sets may coexist whenever (A3) fails, confirming that A3 is not redundant.*

Together, Propositions 2–4 and Remark 15 establish irredundancy in the following precise sense. We use “irredundant” in a deliberately modest sense: each condition is individually necessary for its own conclusion (H1 for $\sigma = \varphi$, H2 for $J = J_{\text{cost}}$, H3 for the collapse mechanism, A3 for $r \equiv 1$ given H3), so no condition can be deleted without losing a forced output; this is the *independence* of the package. We do *not* claim the stronger property that no logically weaker reformulation forces the same triple.

8. Structure Without A3

The full rigidity theorem requires all four structural assumptions H1, H2, H3, and A3 in addition to the foundational axioms (A1) and (A2). When A3 is dropped, H1 and H2 still force $\sigma = \varphi$ and $J = J_{\text{cost}}$, but r is no longer forced to equal 1. The results here are deliberately elementary. Since $\log r$ is conserved by T_F and every observable factors through r its level sets form an evolution-invariant partition of S_F on which μ_F is constant; this is the standard fact that the fibers of a conserved quantity form an invariant partition. The purpose is narrow: to pin down that A3 is exactly the selection of the neutral fiber $\log r = 0$ (Theorem 12), so that dropping A3 yields a clean decomposition into charge sectors rather than disorder.

Definition 13 (Level-set label and level sets of $\log r$). *For a ledger F , the level-set label of a state $s \in S_F$ is $Q_s := \log r(s)$. For $Q \in \mathbb{R}$, the Q -level set of F is*

$$F_Q := \{s \in S_F : \log r(s) = Q\}.$$

Theorem 11 (Partition by level sets). *Let F satisfy (A2). Then:*

1. $S_F = \bigsqcup_{Q \in \log r(S_F)} F_Q$;
2. each level set F_Q is preserved by the evolution T_F ;
3. the measurement map is constant on each level set: $\mu_F(s) = \bar{\mu}_F(e^Q)$ for all $s \in F_Q$.

Proof. Statement (1) is the fiber decomposition of the function $s \mapsto \log r(s)$ over its image. Statement (2) follows from log-charge conservation: $\log r(T_F(s)) = \log r(s)$, so T_F maps F_Q into itself. For statement (3), if $s \in F_Q$ then $r(s) = e^Q$, so $\mu_F(s) = \bar{\mu}_F(r(s)) = \bar{\mu}_F(e^Q)$. $\square$

Theorem 12 (A3 picks the level set $\log r = 0$). *Let F satisfy (A1), (A2), and (H3). Then: (A3) holds if and only if every state lies in $F_0 = \{s : \log r(s) = 0\}$.*

Proof. Let $\Lambda_s := \Lambda(x(s))$. Under H3, Theorem 8 gives $r(s) = \exp(\Lambda_s/N_s)$, so $\log r(s) = \Lambda_s/N_s$. Since $N_s \geq 1$, condition A3 ($\Lambda_s = 0$) is equivalent to $\log r(s) = 0$, i.e., $s \in F_0$. $\square$

Corollary 2 (Partition of S_F without A3). *Let F satisfy the hypotheses of Theorem 9 and also (H3), but do not assume (A3). Then:*

1. *the forced structural outputs remain $\sigma = \varphi$ and $J = J_{\text{cost}}$;*
2. *the state space decomposes as $S_F = \bigsqcup_{Q \in \log r(S_F)} F_Q$, and A3 is exactly the condition populating only the neutral set $F_0 = \{s : \log r(s) = 0\}$ (Theorem 12).*

Proof. The scale and cost are forced by Theorem 9 independently of A3. The level-set partition is Theorem 11, and the A3 equivalence is Theorem 12. $\square$

9. Examples

With the exception of the final one (Example 8), the ledgers below are *constructed* to satisfy the axioms; their role is to prove *non-vacuity* (H1–H3 and A3 are jointly satisfiable, so the main theorem is non-empty) and to exhibit specific phenomena: direct satisfiability ($\mathcal{F}_0$), the variational origin of $r = 1$ from genuinely non-uniform data ($\mathcal{F}_N$), an explicit cost drop ($N = 2$), a non-vacuous observational collapse and multi-sector quotient (injective observable), and free conservative dynamics.

We state this plainly: *every ledger in this section except the last is a logical witness only*. Each of $\mathcal{F}_0$, $\mathcal{F}_N$, and the sector examples is built by hand (trivial or prescribed evolution, a one-point or identity observable, a prescribed ratio or prescribed internal vectors chosen to satisfy A3), so they establish formal consistency and the joint non-vacuity of H1–H3 and A3, but they do *not* exhibit the ledger structure emerging from an independently recognizable mathematical model. The final example of this section, the Fibonacci model-set ledger $\mathcal{F}_{\text{Fib}}$ (Example 8), goes as far toward such a model as we can currently prove: it realizes the full package H1–H3 and A3 on the patch space of the Fibonacci quasicrystal, with the scale hypothesis, the non-uniform internal data, and the conserved-charge/sector architecture supplied intrinsically by the tiling, while the cost (H2) and the exact neutrality (A3) are still installed rather than emergent. Remark 16 states this boundary precisely, and the residual gap is the realization problem of Section 10.

Example 2 (Canonical static ledger). *Define the ledger $\mathcal{F}_0$ by:*

- $S_F = \mathbb{D}_{>0}$, the positive dyadic rationals; (any non-empty countable set serves; $\mathbb{D}_{>0}$ is a concrete choice.)
- $T_F = \text{id}_{S_F}$;
- $O_F = \{*\}$ and $\mu_F(s) = *$, $\bar{\mu}_F(x) = *$ for all $s \in S_F$, $x \in \mathbb{R}_{>0}$;
- ratio map $r(s) = 1$ for every s ;
- cost $J = J_{\text{cost}}$;
- hierarchical structure $(\sigma, \ell) = (\varphi, k \mapsto \varphi^k)$;
- internal channel structure $N_s = 1$, $x(s) = (1) \in \mathbb{R}_{>0}^1$ for every s .

Proposition 5 (Axioms hold on $\mathcal{F}_0$). *The canonical ledger $\mathcal{F}_0$ satisfies (A1), (A2), (A3), and (H1)–(H3).*

Proof. The state space $\mathbb{D}_{>0}$ is non-empty and countable, giving (A1) and the countability part of (A2). The distinguished ratio interface is $r \equiv 1$, and $\mu_F = \bar{\mu}_F \circ r \equiv *$. Local conservation is automatic since $T_F = \text{id}$. The cost J_{cost} is admissible by direct verification of Definition 1 (J_{cost} is reciprocally symmetric, $J_{\text{cost}}(1) = 0$, and $G(t) = \cosh t - 1$ is strictly convex), completing the verification of (A2).

For (H1): the sequence $\ell(k) = \varphi^k$ satisfies uniform scaling with ratio φ and additive composition $\varphi^2 = \varphi + 1$. For (H2): the cost J_{cost} satisfies H2 by direct verification: its combiner is $P(u, v) = 2uv + 2u + 2v$ (polynomial, satisfying $P(1, 1) = 6$), and $G(t) = \cosh t - 1$ gives $G''(0) = 1$. For (H3): the internal vector $x(s) = (1) \in \mathbb{R}_{>0}^1$ gives $T_1(x(s)) = (1)$, so $[T_1(x(s))]_1 = 1 = r(s)$. Finally, (A3) holds because $\Lambda(x(s)) = \log 1 = 0$. $\square$

Example 3 (N -channel example with non-uniform internal data). Fix $N \geq 2$ and let

$$\Theta_N := \{(\theta_1, \dots, \theta_N) \in \mathbb{D}^N : \theta_1 + \dots + \theta_N = 0\},$$

where $\mathbb{D}$ denotes the dyadic rationals (the dyadic choice is only a convenience; any countable dense additive subgroup of $\mathbb{R}$ would serve).

Θ_N contains $(0, \dots, 0)$ and, for $N \geq 2$, non-uniform elements such as $(1, -1, 0, \dots, 0) \in \mathbb{D}^N$; the latter has coordinates not all equal, so $x(s) = (e, e^{-1}, 1, \dots, 1)$ is genuinely non-uniform. Define the ledger $\mathcal{F}_N$ by:

- $S_F = \{s_\theta : \theta \in \Theta_N\}$;
- $T_F = \text{id}_{S_F}$;
- $O_F = \{*\}$ with $\mu_F \equiv *$ and $\bar{\mu}_F \equiv *$;
- $\text{cost } J = J_{\text{cost}}$;
- hierarchical structure $(\sigma, \ell) = (\varphi, k \mapsto \varphi^k)$;
- internal channel structure $N_{s_\theta} = N$ and

$$x(s_\theta) := (e^{\theta_1}, \dots, e^{\theta_N}) \in \mathbb{R}_{>0}^N;$$

- ratio map defined by the variational equilibrium:

$$r(s_\theta) := [T_N(x(s_\theta))]_1.$$

Proposition 6 (Axioms hold on $\mathcal{F}_N$; $r = 1$ from non-uniform internal data). For every $N \geq 2$, the ledger $\mathcal{F}_N$ satisfies (A1), (A2), (A3), and (H1)–(H3). Moreover, the value $r(s) = 1$ is a consequence of one Jensen step applied to a genuinely non-uniform internal vector, not a stipulation of the ratio interface.

Proof. Fix $s_\theta \in S_F$. The internal vector $x(s_\theta) = (e^{\theta_1}, \dots, e^{\theta_N})$ is non-uniform whenever the coordinates of θ are not all equal. Its log-charge is

$$\Lambda(x(s_\theta)) = \theta_1 + \dots + \theta_N = 0,$$

since $\theta \in \Theta_N$, so (A3) holds. By Theorem 8, the variational update maps $x(s_\theta)$ to the uniform equilibrium $(1, \dots, 1)$, so

$$r(s_\theta) = [T_N(x(s_\theta))]_1 = 1,$$

verifying (H3) and the induced form of the ratio interface. The remaining ledger conditions are checked as in Proposition 5: S_F is countable (indexed by $\Theta_N \subset \mathbb{D}^N$), conservation is automatic under $T_F = \text{id}$, and $\mu_F = \bar{\mu}_F \circ r$ by construction. The hierarchical structure is the

same φ -geometric one as in $\mathcal{F}_0$, giving (H1). Hypothesis (H2) holds since $J = J_{\text{cost}}$; the three sub-conditions of Definition 9 were verified explicitly in the proof of Proposition 5. $\square$

Example 4 (Explicit $N = 2$ calculation). Take $N = 2$ and $\theta = (1, -1) \in \Theta_2$. The internal vector is $x(s_\theta) = (e, e^{-1}) \in \mathbb{R}_{>0}^2$, which is genuinely non-uniform ($e \neq e^{-1}$). Its log-charge is $\log e + \log e^{-1} = 1 + (-1) = 0$, confirming (A3). By Theorem 8 with $Q = 0$ and $N = 2$,

$$T_2(e, e^{-1}) = (e^{0/2}, e^{0/2}) = (1, 1).$$

Hence $r(s_\theta) = [T_2(e, e^{-1})]_1 = 1$. The cost drop is

$$\mathcal{D}(e, e^{-1}) - \mathcal{D}(1, 1) = 2J_{\text{cost}}(e) - 2J_{\text{cost}}(1) = 2\left(\frac{e+e^{-1}}{2} - 1\right) - 0 = e + e^{-1} - 2 \approx 1.086 > 0,$$

quantifying the strict decrease from a non-uniform input to the uniform equilibrium. This illustrates Lemma A1: the cost strictly decreases in one variational step whenever the input is not already at the equilibrium.

Example 5 (Injective observable: non-vacuous collapse). The preceding examples use the one-point observable set $O_F = \{*\}$, so the singleton conclusion $\text{SQ}(F) \simeq \mathbf{1}$ holds automatically and does not test the collapse in Theorem 10. To show that the collapse is a genuine consequence of $r \equiv 1$ rather than an artifact of a trivial observable, we fix an injective reduced measurement map.

Fix the data $O_F = \mathbb{R}_{>0}$, $\bar{\mu}_F = \text{id}_{\mathbb{R}_{>0}}$ (so $\mu_F = r$), $T_F = \text{id}$, $J = J_{\text{cost}}$, and the hierarchical structure $(\varphi, k \mapsto \varphi^k)$, and take the states and internal vectors of $\mathcal{F}_N$ (Example 3). By Proposition 6 the internal data is charge-neutral, so $r \equiv 1$ and $\mu_F \equiv \bar{\mu}_F(1) = 1$, whence $\text{SQ}(F) \simeq \mathbf{1}$. Here $\bar{\mu}_F = \text{id}$ is injective and would separate any two states with distinct ratios; the singleton quotient is therefore forced by r being driven to 1, not by the observable set. The complementary no-A3 regime, in which this same injective observable yields a nontrivial multi-sector quotient, is the $T_F = \text{id}$ special case of Example 6 below.

Proposition 7 (Axioms for Example 5). The ledger of Example 5 satisfies (A1), (A2), (A3), and (H1)–(H3), with $\text{SQ}(F) \simeq \mathbf{1}$.

Proof. Every axiom except the choice of O_F and $\bar{\mu}_F$ is verified exactly as in Proposition 6; the factorization $\mu_F = \bar{\mu}_F \circ r$ holds by construction and $\text{id}_{\mathbb{R}_{>0}}$ is a legitimate reduced map into $O_F = \mathbb{R}_{>0}$. Since $r \equiv 1$, the map $\mu_F \equiv \bar{\mu}_F(1) = 1$ is constant, so Lemma 2 gives $\text{SQ}(F) \simeq \mathbf{1}$. $\square$

Example 6 (Non-trivial conservative dynamics within sectors). All earlier examples take $T_F = \text{id}$, so the conservation law (Definition 4, condition (4)) and the evolution map are never exercised. The following ledger $\mathcal{F}_{\text{dyn}}$ activates both: its evolution is fixed-point-free yet conserves the log-charge, permuting states within each level set of $\log r$.

Fix $O_F = \mathbb{R}_{>0}$, $\bar{\mu}_F = \text{id}_{\mathbb{R}_{>0}}$ (so $\mu_F = r$), $J = J_{\text{cost}}$, and the hierarchical structure $(\varphi, k \mapsto \varphi^k)$. Let $S_F = \{s_{q,n} : q \in \mathbb{Q}, n \in \mathbb{Z}\}$, and put

$$N_{s_{q,n}} = 1, \quad x(s_{q,n}) = (e^q) \in \mathbb{R}_{>0}^1, \quad r(s_{q,n}) = [T_1(x(s_{q,n}))]_1 = e^q,$$

with evolution given by the index shift $T_F(s_{q,n}) := s_{q,n+1}$. Taking instead the trivial evolution $T_F = \text{id}$ (equivalently, dropping the n index) gives the static special case: states indexed by $\mathbb{Q}$ with $r(s_q) = e^q$, on which the injective observable $\bar{\mu}_F = \text{id}$ yields the countably infinite multi-sector quotient $\text{SQ}(F) \simeq \{e^q : q \in \mathbb{Q}\}$, realizing Theorem 11 and Corollary 2.

Proposition 8 (Axioms for Example 6). *The ledger $\mathcal{F}_{\text{dyn}}$ satisfies (A1), (A2), (H1), (H2), and (H3) but not (A3). Its evolution T_F is a fixed-point-free bijection that preserves every level set $F_q = \{s_{q,n} : n \in \mathbb{Z}\}$ and conserves the log-charge; and $|\text{SQ}(F)| = \infty$.*

Proof. The state space $\mathbb{Q} \times \mathbb{Z}$ is countable, giving (A1) and the countability clause of (A2). Since $r(s_{q,n}) = e^q$ depends only on q and T_F fixes q , we have $\log r(T_F(s_{q,n})) = q = \log r(s_{q,n})$, so the conservation law holds; with $\mu_F = \bar{\mu}_F \circ r$ by construction this gives (A2). Conditions (H1) and (H2) hold as in Proposition 5 (the φ -geometric scale and $J = J_{\text{cost}}$), and (H3) holds with $N = 1$ because the only point of $\mathbb{R}_{>0}^1$ with log-charge q is (e^q) , so $T_1((e^q)) = (e^q)$ and $r(s_{q,n}) = e^q$. Axiom A3 fails for every $q \neq 0$ since $\Lambda(x(s_{q,n})) = q$. The shift $T_F(s_{q,n}) = s_{q,n+1}$ is a bijection with no fixed point and maps each F_q onto itself, realising the T_F -invariance of Theorem 11 by a free $\mathbb{Z}$ -action. Finally $\bar{\mu}_F = \text{id}$ is injective and $\mu_F(s_{q,n}) = e^q$ takes one value per q , so the observational quotient is $\text{SQ}(F) \simeq \{e^q : q \in \mathbb{Q}\}$, which is countably infinite. $\square$

Example 7 (Finite nontrivial sectors). *The previous dynamical example uses infinite level sets. The same sector mechanism already appears in a finite form. Let*

$$S_F = \{s_{q,\varepsilon} : q \in \{-1, 0, 1\}, \varepsilon \in \{0, 1\}\}, \quad O_F = \mathbb{R}_{>0}, \quad \bar{\mu}_F = \text{id}_{\mathbb{R}_{>0}},$$

take $J = J_{\text{cost}}$, hierarchical structure $(\varphi, k \mapsto \varphi^k)$, and set

$$N_{s_{q,\varepsilon}} = 1, \quad x(s_{q,\varepsilon}) = (e^q), \quad r(s_{q,\varepsilon}) = e^q, \quad \mu_F(s_{q,\varepsilon}) = e^q.$$

Define the evolution by the two-cycle

$$T_F(s_{q,0}) = s_{q,1}, \quad T_F(s_{q,1}) = s_{q,0}.$$

Then each level set

$$F_q = \{s_{q,0}, s_{q,1}\}$$

has two states, is preserved by T_F , and carries a nontrivial period-two action. The ledger satisfies (A1), (A2), (H1), (H2), and (H3), while A3 fails on the $q = \pm 1$ sectors. The observational quotient has three classes, represented by e^{-1} , 1 , and e . This example shows that the no-A3 sector decomposition is not merely a one-state-per-sector phenomenon.

Toward a Non-Constructed Realization: The Fibonacci Model-Set Ledger

The final example of this section carries out the static part of the realization program of Section 10: it equips a recognized mathematical object, the one-dimensional Fibonacci quasicrystal in its standard cut-and-project (model-set) description [67], with the full package H1–H3 and A3. Unlike the ledgers above, its scale structure and its internal data are not prescribed: they are read off from the object itself. We state the construction first, and then record precisely which of its features are intrinsic to the tiling and which remain installed (Remark 16).

Example 8 (Fibonacci model-set ledger). *Consider the ring $\mathbb{Z}[\varphi] = \{m + n\varphi : m, n \in \mathbb{Z}\}$ with the star map given by algebraic conjugation,*

$$(m + n\varphi)^* := m + n\varphi', \quad \varphi' = 1 - \varphi = -\varphi^{-1}, \quad |\varphi'| < 1,$$

and fix the half-open window $W := (-\frac{\varphi}{2}, \frac{\varphi}{2}]$, centered at the origin, so that the model set

$$\mathcal{M} := \{x \in \mathbb{Z}[\varphi] : x^* \in W\}$$

is the vertex set of a Fibonacci tiling of the line, with prototiles of lengths φ and 1 arranged according to the substitution $a \mapsto ab$, $b \mapsto a$ ([67], Chs. 4 and 7) (any half-open window of length φ produces such a tiling; the standard choice differs from the centered one only by a translation in internal space). Define the ledger $\mathcal{F}_{\text{Fib}}$ by:

- $S_F =$ the set of pointed patches $s = (x_0 < x_1 < \dots < x_{N_s})$ of $N_s \geq 1$ consecutive tiles of the tiling, $x_i \in \mathcal{M}$;
- $T_F = \text{id}_{S_F}$ (static; Remark 16 discusses the dynamical readings);
- $O_F = \mathbb{R}_{>0}$ and $\bar{\mu}_F = \text{id}_{\mathbb{R}_{>0}}$, so $\mu_F = r$ is the injective-observable configuration of Example 5;
- $\text{cost } J = J_{\text{cost}}$;
- hierarchical structure $(\sigma, \ell) = (\varphi, k \mapsto \varphi^k)$, realized by the inflation hierarchy: level- k supertiles have length φ^k times the prototile length, and the tile counts obey the exact Fibonacci recurrence of Example 1;
- internal channel structure: for a patch s with tiles based at $x_0, \dots, x_{N_s-1}$, set

$$\theta_i(s) := x_{i-1}^* - c(s), \quad c(s) := \frac{1}{N_s} \sum_{j=1}^{N_s} x_{j-1}^*, \quad x(s) := (e^{\theta_1(s)}, \dots, e^{\theta_{N_s}(s)}) \in \mathbb{R}_{>0}^{N_s},$$

i.e., the internal vector consists of the tiles' internal-space (star-map) coordinates, centered at the patch's internal centroid $c(s)$;

- ratio map $r(s) := [T_{N_s}(x(s))]_1$.

Proposition 9 (Axioms for Example 8). *The ledger $\mathcal{F}_{\text{Fib}}$ satisfies (A1), (A2), (A3), and (H1)–(H3), and $r \equiv 1$ with $\text{SQ}(\mathcal{F}_{\text{Fib}}) \simeq \mathbf{1}$ non-vacuously. Moreover the internal data are supplied by the tiling and are genuinely non-uniform: for every patch with $N_s \geq 2$ the coordinates of $x(s)$ are pairwise distinct, and the uncentered star coordinates x_{i-1}^* are confined to the bounded window W .*

Proof. The vertex set $\mathcal{M} \subset \mathbb{Z}[\varphi]$ is countable, hence so is the set of finite pointed patches, giving (A1) and the countability clause of (A2). Conservation is automatic since $T_F = \text{id}$, the factorization $\mu_F = \bar{\mu}_F \circ r$ holds by construction, and J_{cost} is admissible as in Proposition 5, completing (A2). (H1) holds for the φ -inflation hierarchy exactly as in Proposition 5; the tile counts moreover realize the asymptotic form of H1 through the Fibonacci recurrence (Example 1). (H2) is the direct verification of Proposition 5: the combiner of J_{cost} is $P(u, v) = 2uv + 2u + 2v$, with $P(1, 1) = 6$ and $(J \circ \exp)''(0) = 1$. (A3) holds exactly by the centering: $\Lambda(x(s)) = \sum_i \theta_i(s) = 0$. (H3): by Theorem 8 with $Q = \Lambda(x(s)) = 0$ we get $T_{N_s}(x(s)) = (1, \dots, 1)$, so the ratio-reproduction clause holds with $r(s) = 1$; since $\bar{\mu}_F$ is injective, Example 5 applies verbatim and the collapse $\text{SQ}(\mathcal{F}_{\text{Fib}}) \simeq \mathbf{1}$ is non-vacuous. Boundedness of the star coordinates is the defining property of the model set ($x^* \in W$). Non-uniformity: conjugation is injective on $\mathbb{Z}[\varphi]$ (if $(m + n\varphi)^* = (m' + n'\varphi)^*$ then $(m - m') + (n - n')\varphi' = 0$, forcing $m = m'$ and $n = n'$ because $\varphi' \notin \mathbb{Q}$), so distinct vertices have pairwise distinct star coordinates, and centering preserves pairwise distinctness. Hence for $N_s \geq 2$ the value $r(s) = 1$ is the output of the Jensen step (Theorem 8) on genuinely non-uniform data, exactly as in Example 3. $\square$

Remark 16 (What $\mathcal{F}_{\text{Fib}}$ realizes intrinsically, and what remains installed). *Example 8 upgrades the status of this section partway from “logical witness” to “realization,” and we record the boundary precisely.*

Intrinsic to the object: (i) The scale: φ is the Perron eigenvalue of the substitution matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, not a chosen parameter, and the tile counts obey the exact Fibonacci recurrence, so H1, including its asymptotic form, is supplied by the tiling. (ii) The internal data: the coordinates $\theta_i(s)$ are the tiling's own internal-space coordinates (in the physics of quasicrystals, the phason degree of

freedom [68]), bounded by the window and genuinely non-uniform (Proposition 9); they are read off from the object, not prescribed to make the axioms true. (iii) The charge-sector architecture: dropping the centering and reading the internal centroid as the log-charge ($\log r(s) = c(s)$ in the uncentered variant), the level sets of c partition the patch space into sectors exactly as in Theorem 11; axiom A3 selects the neutral fiber, and the centering convention of Example 8 is precisely the passage to neutral internal coordinates. (iv) Neutrality in the large-patch limit: by the uniform distribution of the star coordinates of a regular model set in its window [69], the internal centroid of a patch converges to the centroid of W , which is 0 for the centered window, as the patch grows. Axiom A3 is therefore asymptotically emergent in this object: the centering normalization removes only a finite-size deviation that vanishes as $N_s \rightarrow \infty$, rather than imposing structure that the tiling lacks.

Installed by hand: (i) The cost: nothing in the tiling forces J_{cost} . We install it, with the external motivation that $J_{\text{cost}}(x) = \cosh(\log x) - 1$ is the canonical reciprocally symmetric, strictly log-convex cost on multiplicative deviations, and the θ_i are precisely such log-deviations; but this is a choice, not an emergence. (ii) Exact neutrality for finite patches: A3 holds pointwise by the centering normalization; without it an individual finite patch has no reason to have an exactly vanishing internal sum, even though its centroid tends to 0 by item (iv) above.

Dynamics. The tiling's canonical dynamics do not conserve the internal coordinate: the shift acts on it as an irrational rotation (mixing the sectors of (iii)), and the inflation multiplies the star coordinate of every vertex by φ' , contracting the internal centroid by the factor $|\varphi'| = \varphi^{-1}$ per step. The inflation reading is therefore a non-engineered realization of the contraction to the neutral fiber studied in Appendix A, with A3's fiber as the global attractor, but it is not a conservative evolution in the sense of Definition 4(4). Indeed the shift action on the hull of the tiling is minimal, so every continuous shift-invariant function on the hull is constant, and a fixed-point-free charge-conserving evolution as in Example 6 has no natural counterpart in this object; this is why $\mathcal{F}_{\text{Fib}}$ is presented statically.

In summary, $\mathcal{F}_{\text{Fib}}$ is a semi-natural witness: the scale, the non-uniform internal data, and the conserved-charge/sector architecture come from a recognized object, while the cost and the neutrality normalization are still installed. Removing these two installations (a cost forced by the tiling rather than admitted by it, and a neutrality that is generic rather than normalized) is the remaining content of the realization problem in Section 10.

Together these examples establish that the condition package is jointly satisfiable, that $r \equiv 1$ is a genuine variational output of the Jensen step on non-uniform data (with a quantitatively strict cost drop), and that, with an injective observable, the collapse $\text{SQ}(F) \simeq 1$ is non-vacuous under A3 while a nontrivial multi-sector quotient and fixed-point-free conservative dynamics appear once A3 is dropped (Propositions 5–8; Examples 6 and 7). Beyond joint satisfiability, Example 8 adds the semi-natural witness just discussed, in the precise sense recorded in Remark 16.

10. Conclusions

10.1. Summary of Results

Reference [52] classifies all admissible costs with polynomial combiner. Once that classification is placed inside a conserved comparison ledger, the present paper shows that the ledger axioms together with H1–H3 and A3 force the three quantities (σ, J, r) to $\sigma = \varphi$, $J = J_{\text{cost}}$, and $r \equiv 1$, with the two immediate corollaries that the measurement map is constant and the observational quotient is a singleton (Theorems 9 and 10). The forcing is layered: H1 alone gives $\sigma = \varphi$; H1 and H2 together give $J = J_{\text{cost}}$; adding H3 and A3 then gives $r \equiv 1$, constant measurement, and singleton quotient. The main results have asymptotic and dynamical variants: H1 has an asymptotic form (Theorem 6), and the

constrained minimizer of B3 is the exponentially attracting fixed point of a gradient flow (Lemma A1 and Theorem A1).

10.2. Sharpness

The condition package is irredundant: each of H1, H2, H3, A3 is individually necessary for its forced output ($\sigma = \varphi$, $J = J_{\text{cost}}$, $r \equiv 1$). Dropping H1 allows $\sigma = 2$ (Proposition 2); dropping H2 allows the admissible cost $J(x) = (\ln x)^2$ (Proposition 3); dropping H3 allows $r \equiv 2$ (Proposition 4); dropping A3 produces the level-set decomposition of Section 8 in which r is not forced to 1.

10.3. Examples

Section 9 provides explicit ledgers that establish joint satisfiability of H1–H3 and A3 and the non-vacuity of each conclusion: the variational origin of $r = 1$ from non-uniform data ($\mathcal{F}_N$, with the explicit $N = 2$ cost drop $e + e^{-1} - 2 \approx 1.086$), a non-vacuous observational collapse and a countably infinite multi-sector quotient without A3 (injective observable), and free conservative dynamics within level sets ($\mathcal{F}_{\text{dyn}}$ and a finite two-point sector). The list closes with the Fibonacci model-set ledger $\mathcal{F}_{\text{Fib}}$ (Example 8), a semi-natural witness in which the tiling itself supplies the scale, the non-uniform internal data, and the charge-sector architecture (Proposition 9 and Remark 16).

10.4. Open Problems

Several directions remain unexplored.

Non-polynomial combiners. Hypothesis H2(1) posits a polynomial combiner. As Remark 10 makes precise, this is a mild restriction rather than a strong one: measurability already forces continuity of the combiner, and [59] excludes nonconstant continuous solutions for symmetric polynomial combiners of degree ≥ 3 , so the entire polynomial case collapses onto the bilinear one already treated. What remains genuinely open is therefore only the non-polynomial case: whether an analogous rigidity theorem holds for non-polynomial (e.g., transcendental) combiners is open. One concrete question: does an admissible cost with a continuous non-polynomial combiner satisfy any natural normalization that forces it to a specific form? Without polynomiality the combiner lives in an infinite-dimensional space and the d’Alembert–Wilson–Kannappan machinery [2,4] no longer yields a finite-parameter family; some regularity (measurability, monotonicity) is likely needed to exclude Hamel-type pathologies. A solution would either enlarge the admissible class (weakening rigidity) or show polynomiality follows from mild regularity (strengthening it).

Infinite-dimensional state spaces. The countability of S_F and the finite-dimensional internal vectors of H3 make the present theory largely combinatorial. An analogue for uncountable state spaces or infinite-dimensional internal data would require a measure-theoretic Jensen argument and a continuous-variable conservation law; whether the rigidity conclusions survive this extension is unknown. Countability of S_F is not used (Definition 4); the substantive issue is infinite-dimensional internal data. The static collapse plausibly survives (the integral Jensen inequality still makes the constant function the constrained minimizer), but the gradient flow of Theorem A1 becomes infinite-dimensional, and its contraction needs a spectral-gap/Bakry–Émery condition [26,27] automatic only in finite dimensions.

Characterization of ledgers realizing the forced values. The forward direction (H1 + H2 + H3 + A3 $\Rightarrow$ forced outputs) is proved here. The converse question (characterize all ledgers that achieve $\sigma = \varphi$, $J = J_{\text{cost}}$, $r \equiv 1$ without necessarily satisfying H1, H2, H3, A3) is open. In particular: is there an admissible cost $J \neq J_{\text{cost}}$ and a ledger satisfying the ledger axioms alone (A1, A2) such that $r \equiv 1$ holds without any variational mechanism? The trivial direction is clear (assigning $r \equiv 1$ by hand needs no hypothesis), so the interesting question seeks a converse under constraints that exclude hand-assignment, e.g., requiring r

to be variationally generated as in H3. This is a uniqueness question for the aggregation rule; the means/aggregation literature [19,57] characterizes quasi-arithmetic selection by symmetry and consistency axioms, and importing such a characterization would turn H3 into a derived consequence, the most promising route to fewer hypotheses. Proposition 1 already carries this program out up to a single residual assumption: what remains is to justify homogeneity (scale covariance of the ratio interface) from a ledger-level principle rather than positing it. Establishing that homogeneity, or replacing it by a weaker structural requirement, would eliminate H3 as an independent hypothesis and is, in our view, the single most valuable next step for deepening the self-contained content of the present framework.

Natural systems realizing the framework. A natural open problem is whether some physical or combinatorial system satisfies H1–H3 and A3 without being constructed for that purpose; such a non-designed realization would give the main theorem an interpretation beyond its abstract setting. Until one is known, the theorem should be read as a coherence and rigidity result for the stated axioms, not as a prediction about an independently identified object. This is the most important open problem. Its difficulty is that the conditions constrain unrelated features at once, a scale sequence (H1), a comparison cost (H2), and a charge-neutral internal aggregation (H3–A3), so a natural realization must exhibit all three coincidentally. Two search directions seem promising: self-similar substitution systems and quasicrystals, where φ appears intrinsically [10], for instance the Fibonacci substitution $a \mapsto ab$, $b \mapsto a$ of Example 8, which realizes the additive-plus-geometric scale of H1 without being engineered for it (Example 1); and charge-neutral statistical ensembles whose free-energy minimization realizes H3 with A3 imposed by a conservation law. The pool of non-designed systems in which φ itself arises intrinsically is in fact wider than the substitution/quasicrystal class: in Hamiltonian dynamics the golden-mean winding number labels the most robust invariant torus, the last to break under perturbation, by Greene’s residue criterion, because φ is the number worst approximated by rationals [70], and in condensed matter the ratio of the two lightest excitations of the transverse-field Ising chain CoNb_2O_6 near its quantum critical point has been measured to approach φ , as predicted by the emergent- E_8 spectrum [71]. Neither system carries an admissible comparison cost or a charge-neutral internal aggregation, so neither is a ledger witness; they show only that the scale output of H1 has independent, non-engineered realizations, which is what makes the completion program recorded in the next paragraph plausible rather than vacuous. Settling it either way would determine whether the framework is more than an internally consistent abstraction.

We record precisely how far the Fibonacci candidate currently reaches and what remains open. In the standard cut-and-project (model-set) description of the Fibonacci tiling [67], two of the ledger’s structural features are intrinsic rather than installed: the inflation hierarchy realizes H1 with $\sigma = \varphi$, as noted above, and the internal-space (star-map) coordinate attached to each configuration is a bounded additive quantity whose level sets partition the hull, a structure of exactly the kind used in the sector decomposition of Section 8. What the tiling does *not* supply on its own are the remaining two layers, the specific admissible cost J_{cost} and the exact pointwise neutrality of A3, nor a nontrivial *conservative* evolution in the sense of Definition 4(4), for the dynamical reasons detailed in Remark 16. The Fibonacci system is therefore a strong *partial* realization (H1 and the charge-sector architecture intrinsic; H2 and exact A3 still installed), and completing it, by exhibiting an admissible cost and a charge-neutral internal aggregation that *emerge* from the tiling’s local-configuration (patch) space rather than being installed on it, is the concrete form of this open problem. Until such an emergent completion is produced, the theorem retains the coherence reading stated in Section 1.

The *installation* step of this program is carried out in Example 8: the patch space of the tiling does carry the full package H1–H3 and A3, with the internal data read off from the star map rather than prescribed (Proposition 9). What remains open is therefore not installation but *emergence*: a mechanism internal to the tiling that forces the cost J_{cost} rather than admitting it, an exact finite-patch neutrality (the asymptotic form is already emergent: by uniform distribution of the star coordinates in the window, the internal centroid vanishes in the large-patch limit [69]), and a nontrivial conservative evolution, the last being obstructed in principle, since the shift action on the hull is minimal and the inflation contracts the internal coordinate (Remark 16). The open problem of this paragraph retains exactly this emergence content.

10.5. Relation to Prior Work

Together with [52], the present paper settles, under the stated hypotheses, the analytic and framework-level rigidity of admissible costs with polynomial combiner. Reference [52] provides the classification; the present paper provides the framework in which that classification becomes a structural constraint. The present development is logically independent of [52] outside the four imported theorems of Section 2.

Author Contributions: Conceptualization, J.W.; methodology, J.W. and S.P.-G.; validation, S.P.-G., J.W. and E.A.; formal analysis, S.P.-G., J.W. and E.A.; investigation, S.P.-G., J.W. and E.A.; writing—original draft preparation, J.W.; writing—review and editing, S.P.-G., J.W. and E.A.; funding acquisition, J.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The authors thank the anonymous reviewers for their careful reading of the manuscript and for their valuable comments and suggestions, which significantly improved the quality and clarity of the paper.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Dynamical Interpretation of the Equilibrium

This appendix records two supplementary layers of Step B3 (Section 6). Neither is used by the main rigidity results, which rest only on the static minimizer Theorem 8; their role is to exhibit the Jensen equilibrium as a dynamically attracting state in the canonical setting of J_{cost} , for which $G(t) = \cosh t - 1$ satisfies $G''(t) = \cosh t \geq 1$ everywhere.

Appendix A.1. Layer 2: One-Step Quantitative Contraction

Lemma A1 (One-step Lyapunov inequality). *Let J be admissible, $G := J \circ \exp$, and $c \in \mathbb{R}_{>0}^N$ with $\xi_i := \log c_i$ and $\bar{\xi} := \frac{1}{N} \sum_i \xi_i$. Assume there is an interval $I \subseteq \mathbb{R}$ with $\bar{\xi} \in I$ and $\xi_1, \dots, \xi_N \in I$ on which G is twice differentiable, and set $\kappa := \inf_{t \in I} G''(t)$. Then,*

$$\sum_{i=1}^N J(c_i) - \sum_{i=1}^N J(T_N(c)_i) \geq \frac{\kappa}{2} \sum_{i=1}^N (\xi_i - \bar{\xi})^2.$$

For $J = J_{\text{cost}}$, one may take $I = \mathbb{R}$ and $\kappa = 1$.

Proof. Note that κ -strong convexity of G gives, for each i , $G(\xi_i) \geq G(\bar{\xi}) + G'(\bar{\xi})(\xi_i - \bar{\xi}) + \frac{\kappa}{2}(\xi_i - \bar{\xi})^2$. Summing over i and using $\sum_i(\xi_i - \bar{\xi}) = 0$ to eliminate the linear terms yields

$$\sum_{i=1}^N G(\xi_i) \geq N G(\bar{\xi}) + \frac{\kappa}{2} \sum_{i=1}^N (\xi_i - \bar{\xi})^2.$$

Since Theorem 8 gives $T_N(c) = (e^{\bar{\xi}}, \dots, e^{\bar{\xi}})$, the right-hand minimum is $N G(\bar{\xi})$. For $J = J_{\text{cost}}$ one has $G(t) = \cosh t - 1$, whence $G''(t) = \cosh t \geq 1$. $\square$

Appendix A.2. Layer 3: Continuous-Time Convergence

Theorem A1 (Continuous-time convergence to the Jensen equilibrium). *Fix an admissible cost J , let $G := J \circ \exp$, and assume G is C^1 with locally Lipschitz derivative (for instance $G \in C^2$) and κ -strongly convex on $\mathbb{R}$ for some $\kappa > 0$. Let $N \geq 2$ and $Q \in \mathbb{R}$. The gradient flow on the affine hyperplane $\{\xi \in \mathbb{R}^N : \sum_i \xi_i = Q\}$,*

$$\dot{\xi}_i = -G'(\xi_i) + \frac{1}{N} \sum_{j=1}^N G'(\xi_j), \quad (\text{A1})$$

preserves the mean $\bar{\xi} = Q/N$ and satisfies

$$\text{Var}(\xi(t)) := \frac{1}{N} \sum_{i=1}^N (\xi_i(t) - \bar{\xi})^2 \leq e^{-2\kappa t} \text{Var}(\xi(0)).$$

Consequently, $\xi(t) \rightarrow (\bar{\xi}, \dots, \bar{\xi})$ and $c_i(t) = e^{\xi_i(t)} \rightarrow e^{Q/N}$ exponentially, reaching the same equilibrium as the variational update. For $J = J_{\text{cost}}$, one may take $\kappa = 1$.

Proof. Since G' is locally Lipschitz on $\mathbb{R}$, the right-hand side of (A1) is locally Lipschitz on $\mathbb{R}^N$; the Picard–Lindelöf theorem gives a maximal classical solution on some interval $[0, T_{\max})$.

Constraint preservation.

$$\frac{d}{dt} \sum_{i=1}^N \xi_i = - \sum_{i=1}^N G'(\xi_i) + \sum_{i=1}^N \frac{1}{N} \sum_{j=1}^N G'(\xi_j) = 0,$$

so $\bar{\xi} = Q/N$ is constant.

Variance decay. Differentiating $\text{Var}(\xi) = \frac{1}{N} \sum_i (\xi_i - \bar{\xi})^2$ and using $\sum_i (\xi_i - \bar{\xi}) = 0$ to replace $G'(\xi_i)$ by $G'(\xi_i) - G'(\bar{\xi})$,

$$\begin{aligned} \frac{d}{dt} \text{Var}(\xi) &= \frac{2}{N} \sum_{i=1}^N (\xi_i - \bar{\xi}) \dot{\xi}_i \\ &= -\frac{2}{N} \sum_{i=1}^N (\xi_i - \bar{\xi}) \left(G'(\xi_i) - G'(\bar{\xi}) \right). \end{aligned}$$

By κ -strong convexity of G , $(\xi_i - \bar{\xi})(G'(\xi_i) - G'(\bar{\xi})) \geq \kappa(\xi_i - \bar{\xi})^2$, so $\frac{d}{dt} \text{Var}(\xi) \leq -2\kappa \text{Var}(\xi)$. Grönwall's inequality ([72], p. 24) then gives

$$\text{Var}(\xi(t)) \leq e^{-2\kappa t} \text{Var}(\xi(0)) \quad \text{for all } t \in [0, T_{\max}). \quad (\text{A2})$$

Global existence. From (A2), $|\xi_i(t) - \bar{\xi}| \leq \sqrt{N \text{Var}(\xi(0))}$ for all i and all $t \in [0, T_{\max})$. Since $\bar{\xi} = Q/N$ is fixed, every component $\xi_i(t)$ remains in the bounded interval $[Q/N - R, Q/N + R]$ with $R := \sqrt{N \text{Var}(\xi(0))}$. On this interval, G' is bounded (it is continuous on a compact set),

so the right-hand side of (A1) is uniformly bounded. A uniformly bounded derivative cannot produce blow-up in finite time, hence $T_{\max} = \infty$.

Conclusion. Inequality (A2) holds for all $t \geq 0$, giving $\text{Var}(\xi(t)) \rightarrow 0$. Since $\bar{\xi}(t) = Q/N$ is constant in t and $(\xi_i(t) - Q/N)^2 \leq N \text{Var}(\xi(t))$ for each i , each $\xi_i(t) \rightarrow Q/N$, hence $c_i(t) = e^{\xi_i(t)} \rightarrow e^{Q/N}$ exponentially. For $J = J_{\text{cost}}$, $G(t) = \cosh t - 1$ is 1-strongly convex ($G''(t) = \cosh t \geq 1$), so $\kappa = 1$ and the rate is e^{-2t} . $\square$

Remark A1 (Role of the continuous-time theorem). *Theorem A1 upgrades the interpretation of Step B3: the Jensen equilibrium is an exponentially attracting fixed point of the natural constrained gradient flow on each level set of $\log r$, not merely a one-step minimizer.*

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