

Article

Semi-Closed-Form Pricing of Vulnerable Geometric Asian Options Under a Three-Factor Stochastic Volatility Jump-Diffusion Model with Stochastic Interest Rates

Libin Wang ^{1,2,*} and Ruonan Zhang ^{1,2}

¹ Hebei Key Laboratory of Financial Technology Application, Hebei Finance University, Baoding 071051, China; zhangruonan@hbfu.edu.cn

² School of Statistics and Data Science, Hebei Finance University, Baoding 071051, China

* Correspondence: wanglibin@hbfu.edu.cn

Abstract

This paper develops a semi-closed-form pricing framework for vulnerable geometric Asian options under a three-factor stochastic volatility jump-diffusion model with stochastic interest rates. To the best of our knowledge, this is the new framework to simultaneously accommodate common and idiosyncratic volatility, co-jumps, stochastic rates, and counterparty default risk in a 2D-FFT setting. The theoretical contribution is threefold: (i) a fully flexible correlation structure among asset returns, volatility factors, and the short rate; (ii) a default intensity jump component correlated with asset jumps; and (iii) a two-dimensional fast Fourier transform (2D-FFT) algorithm that computes prices and all Greeks in a single execution. Numerical results demonstrate speedup factors of 22.6 to 114.2 over Monte Carlo simulation at comparable accuracy (APE below 0.02%), with near-constant time scaling for portfolios of up to 4,096 options. Degeneracy tests confirm the framework correctly recovers known closed-form solutions and published benchmarks. The proposed method provides a computationally efficient pricing tool for real-time risk management of credit-risk-embedded derivatives.

Keywords: three-factor stochastic volatility; 2D FFT; co-jump; stochastic interest rate; efficient pricing

MSC: 91G20; 91G60



Academic Editor: J.E.
Trinidad-Segovia

Received: 17 June 2026

Revised: 20 July 2026

Accepted: 21 July 2026

Published: 23 July 2026

Copyright: © 2026 by the authors.
Licensee MDPI, Basel, Switzerland.
This article is an open access article
distributed under the terms and
conditions of the [Creative Commons
Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

1. Introduction

Over-the-counter (OTC) derivatives markets have grown substantially since the 2000s, with the notional amount outstanding exceeding \$600 trillion at its peak. Unlike exchange-traded options, OTC derivatives expose market participants to **counterparty credit risk**—the risk that the option writer defaults before the contract matures. The 2008 global financial crisis starkly illustrated this risk, when the collapse of Lehman Brothers rendered numerous outstanding derivative contracts uncollectible, causing cascading losses across the financial system.

Among OTC products, Asian options are particularly popular in commodities, currency, and interest rate markets. Their payoff depends on the average price of the underlying asset over the life of the contract. The averaging feature makes these options attractive for corporate hedging applications—for example, airlines hedging fuel costs and multinational corporations managing foreign exchange exposure—because they are

less susceptible to price manipulation at maturity and are typically cheaper than standard European options.

In the presence of counterparty default risk, these options become vulnerable options [1], where the payoff is partially or fully lost upon the option writer's default. The accurate pricing of such instruments requires a modeling framework that simultaneously captures: (i) the path-dependent nature of the payoff; (ii) the stochastic dynamics of asset prices, including volatility and jumps; (iii) the randomness of interest rates; and (iv) the correlation between the underlying asset and the counterparty's asset value. This paper addresses this challenge by developing a unified framework that integrates these four features.

The literature on vulnerable options and Asian options has developed along several distinct yet converging research trajectories. We structure our review around these trajectories to identify the specific gaps that motivate our model. The foundational work on Asian options under the Black–Scholes framework includes [2–4], who developed analytical approximations for arithmetic and geometric average options. Extensions to stochastic volatility [5] and jump-diffusion processes [6,7] have established that both volatility randomness and price discontinuities materially affect Asian option prices. However, these early models uniformly assumed away counterparty default risk. Turning to the pricing of geometric Asian options, recent methodological advances have extended the traditional affine framework in several important directions. Ref. [8] derived semi-closed-form formulas for geometric Asian options with fixed and floating strikes under the class of Volterra–Heston models, which encompasses the rough Heston model. Their approach links the pricing problem to the theory of affine Volterra processes and requires solving a Riccati–Volterra equation, representing a significant extension of the classical Heston framework to non-Markovian volatility dynamics. Ref. [9] developed a hybrid asset price model that integrates mixed fractional Brownian motion with jump diffusion to capture both long-range dependence and discontinuous shocks, deriving semi-analytical pricing formulas for geometric Asian options using the Fourier cosine series expansion (COS) method under regime-switching dynamics with liquidity risk.

Johnson and Stulz [10] pioneered the pricing of options with default risk, followed by Klein [1], who introduced the structural default framework that has become the standard approach. Subsequent extensions incorporated stochastic volatility [11], jumps [12], and multiple underlying assets [13]. However, the majority of vulnerable option models focus on European-style payoffs and do not address the path-dependent features of Asian options. Notable exceptions include Jeon et al. [14], who studied vulnerable geometric Asian options, albeit under the restrictive assumption of constant volatility.

The literature on vulnerable option pricing has continued to expand rapidly in recent years, with several important contributions that extend the foundational framework of Klein and Johnson. Ref. [13] considered vulnerable options with two underlying assets, deriving analytical solutions for options on the maximum of two underlying prices, options paying the best and cash, and options on the minimum. Ref. [15] obtained explicit pricing formulas for vulnerable path-dependent options with early counterparty credit risk, demonstrating that the timing of default has a material impact on option values. More recently, Ref. [16] provided closed-form exact formulae for two-asset vulnerable options with continuous default risk, a stochastic default boundary, and stochastic interest rates, significantly generalizing earlier structural models.

In parallel, several studies have incorporated richer dynamics into vulnerable option pricing. Ref. [17] investigated the pricing of vulnerable options under an affine stochastic volatility model with simultaneous jumps (SVSJ), where both the underlying asset price and its instantaneous variance are subject to jump processes, and the default intensity

follows a CIR process. Their Fourier-transform-based analytical solution demonstrates the importance of simultaneous jumps in both the price and variance processes for capturing volatility skewness. Ref. [18] studied vulnerable Asian options with liquidity risk under Lévy processes, showing that liquidity discount factors significantly affect option prices during market stress.

Recognizing that interest rates are not constant in practice, several studies have incorporated stochastic interest rates into option pricing [19,20]. Nielsen and Sandmann [21] and Zhang et al. [22] extended Asian option pricing to stochastic rate environments. Yet, these studies typically assume no counterparty default risk, separating the credit risk and interest rate risk channels.

Recent work has begun to combine multiple features. Xie and Deng [23] studied vulnerable European options under a Markov regime-switching SV model with stochastic rates. Li et al. [24] used double Mellin transforms to price vulnerable options under jump-diffusion. Most importantly, several authors have recently employed FFT-based methods to price vulnerable basket options under multi-asset Heston SVJD models, using Esscher-Girsanov measure changes. These works represent the closest predecessors to our study.

The decomposition of jumps into systematic (common) and idiosyncratic components has been employed in several vulnerable option studies to separate market-wide crashes from asset-specific shocks. The co-jump channel between the underlying asset and the counterparty asset has been directly examined, and we acknowledge these contributions.

The fast Fourier transform (FFT) methodology, pioneered by [25], has been extended to increasingly complex settings in recent years. Wang [26] investigated bivariate options pricing under a regime-switching stochastic volatility jump-diffusion model with stochastic intensity, stochastic interest rates, and dependent jumps, deriving analytic pricing expressions via joint discounted conditional characteristic functions and implementing the Fourier cosine expansion method. Their results demonstrate that embedding stochastic intensity, dependent co-jump structures, and Markov regime-switching into the pricing dynamics has a significant influence on option prices. Ref. [27] studied the equilibrium valuation of currency options under a discontinuous model with co-jumps, decomposing jumps into simultaneous co-jump and country-specific components, and found that co-jumps have a significant impact on option prices and implied volatility.

The closest predecessors to our work include [14], who priced vulnerable geometric Asian options under constant volatility; Ref. [13], who considered Asian options with counterparty risk under stochastic volatility models but without jumps or co-jump-default correlation; and the FFT-based vulnerable basket option studies that have emerged recently, which typically focus on European-style payoffs rather than the path-dependent geometric Asian structure. Our framework extends these benchmarks by integrating all four features within a unified, computationally tractable framework.

Despite the considerable body of existing work, no single model simultaneously addresses the following interrelated challenges. First, correlation flexibility. Existing models either assume zero correlation between interest rates and volatility, or a fixed correlation coefficient that cannot vary with market conditions. Given the documented time-varying nature of this relationship (the volatility feedback effect), this assumption is empirically questionable. Second, default intensity under jumps. In most vulnerable option models, default intensity is either constant or a deterministic function of the asset price. However, during market stress, default events are often preceded by simultaneous jumps in both the underlying and counterparty asset values—a “co-jump” that dramatically increases effective default probability. This has been discussed in the recent co-jump literature, but has not been fully integrated into a geometric Asian option framework with

stochastic volatility and stochastic rates. Third, algorithmic efficiency for path-dependent payoffs. While FFT methods have been applied to vulnerable basket options, the specific combination of geometric averaging, stochastic rates, and flexible correlation in a 2D-FFT setting remains under-explored. Existing FFT implementations typically address only subsets of these features. It is precisely these gaps that motivate the present study.

Building on the synthesis above, we identify three specific theoretical gaps:

1. **Rigid correlation structure:** Existing vulnerable option models with stochastic rates almost universally assume the correlation coefficient between interest rates and volatility to be zero or constant, neglecting the empirically documented time-varying negative correlation [28]. A key feature of our framework is that it allows a fully flexible correlation matrix among asset returns, three-dimensional variance factors, and the short rate in vulnerable geometric Asian option pricing, without sacrificing characteristic function tractability. While existing studies have considered flexible correlations in other option pricing contexts, to our knowledge, this is the first application to the vulnerable geometric Asian option setting.
2. **Default intensity misspecification under jumps:** Klein [1] and its extensions model default intensity as constant or deterministic, ignoring the fact that default intensity jumps with macroeconomic shocks and that these jumps occur **simultaneously** with asset price jumps (“co-jumps”). We introduce a **default-intensity jump component stochastically correlated with asset jumps**, causing the default probability to increase disproportionately during severe asset price declines—fundamentally altering the recovery value assessment of deep out-of-the-money vulnerable options.
3. **Computational bottleneck:** The few models incorporating all four features rely entirely on numerical methods (Monte Carlo or finite difference), making calibration time-consuming, Greeks unstable, and embedding in real-time risk systems difficult. We derive a **semi-closed-form multi-dimensional characteristic function** and design a **two-dimensional fast Fourier transform (2D-FFT) algorithm** that improves computational efficiency by several orders of magnitude over Monte Carlo simulation (see Section 5), while maintaining consistent capture of all four risk factors.

Distinction from closest competing works. We acknowledge that recent studies have employed FFT-based methods for vulnerable basket options under multi-asset Heston SVJD models, and that co-jump decomposition has been explored in the literature. Our specific contributions relative to these works are threefold:

1. **Product focus:** In contrast to most existing FFT-based vulnerable basket option studies that typically consider European-style payoffs, we focus on **geometric Asian options** with path-dependent averaging features, which are more challenging to price and more widely used in practice.
2. **Interest rate-volatility correlation:** We allow **full flexibility** in the correlation between interest rates and volatility factors—a feature absent in existing multi-asset Heston SVJD FFT implementations, which typically restrict this correlation or set it to zero.
3. **Jump-integrated default intensity:** While co-jump decomposition itself is not novel, we embed it within a **stochastic default intensity framework** that directly links the common jump component to the option writer’s default probability, thereby endogenizing the co-jump-default channel.

In summary, our contribution is not a simple aggregation of existing ingredients, but a **substantive extension** of the benchmark frameworks of Klein [1], Bates [29], and Bakshi et al. [30] along three dimensions: from fixed to flexible correlation; from deterministic default intensity to jump-correlated default intensity; and from purely numerical solution to semi-closed-form 2D-FFT implementation. The simultaneous resolution of these

three issues distinguishes this paper from all existing models, including the closest recent works on FFT-based vulnerable basket option pricing. We acknowledge that no single model can claim universal superiority; our contribution lies in the specific combination of features within a unified 2D-FFT framework, which—to our knowledge—has not been previously reported in the vulnerable geometric Asian option literature.

The remainder of this paper is organized as follows. Section 2 presents the three-factor stochastic volatility jump-diffusion model with stochastic interest rates and derives the joint characteristic function. Section 3 derives the integral pricing formulas for the four types of vulnerable geometric Asian options and their Greeks. Section 4 describes the 2D-FFT algorithm tailored to the geometric Asian option payoff structure. Section 5 presents comprehensive numerical experiments, including Monte Carlo benchmarking, degeneracy tests, computational efficiency and scalability analysis, and convergence analysis, and discusses the limitations of the proposed framework and suggests directions for future research. Section 6 concludes the paper.

2. Model Preliminary Results

In this section, we introduce a three-factors version of stochastic volatility jump-diffusion model. The prices of four kinds vulnerable geometric Asian options are dependent on the randomness of two underlying assets, the randomness associated with three volatility factors of underlying asset's return and the distributions of underlying asset's rare jump, the randomness of risk-free interest rate.

2.1. Model Design Rationale and Stylized Market Facts

Before presenting the formal model specification, it is necessary to provide a systematic economic and empirical justification for the multi-factor structure adopted in this paper. Our model is designed to accommodate salient stylized facts of financial markets, particularly during periods of stress, rather than to pursue mathematical complexity for its own sake. Below, we elaborate on the motivation for each model component from four dimensions, along with the additional economic insights that each component yields—insights that would be inaccessible in any more parsimonious specification.

(i) **Three-Factor Volatility Structure.** Empirical studies have firmly established the distinction between market-wide volatility and asset-specific volatility. In the context of vulnerable options, which involve two assets (the underlying asset and the counterparty asset), this distinction is not merely a statistical refinement but a pricing necessity: the correlation between the underlying asset and the counterparty's asset value—the key determinant of the recovery rate—depends critically on their respective exposures to market-wide versus idiosyncratic volatility shocks. Our model allows separate common and idiosyncratic volatility factors. This captures the empirically observed fact that the correlation between the underlying asset and the counterparty asset is time-varying and driven by market-wide volatility regimes. By contrast, earlier models such as Klein and its extensions assume this correlation to be constant. The additional insight enabled by the three-factor structure is that it allows us to decompose the volatility risk premium into systemic (common) and idiosyncratic (unique) components, which have distinct impacts on option prices across different moneyness levels—a feature that would be obscured in a one- or two-factor setup.

(ii) **Common and Idiosyncratic Jump Components.** The coexistence of market-wide and asset-specific jumps is a well-established empirical fact in financial econometrics. In the vulnerable option context, the co-jump structure (common jumps affecting both the underlying and the counterparty simultaneously) is essential for two reasons: (a) during market crises, both the option's payoff asset and the option writer's asset value tend to

jump downward together, amplifying the effective default risk; (b) idiosyncratic jumps, by contrast, affect only one asset and thus have a qualitatively different impact on the recovery value.

By decomposing jumps into common and idiosyncratic components, we are able to separately identify the pricing implications of systemic jump risk (which is priced in the market risk premium) and idiosyncratic jump risk (which is diversifiable and typically not priced)—a distinction that is lost in models that treat jumps as purely asset-specific. In particular, our numerical analysis reveals that, during volatile market periods (e.g., the 2008 financial crisis regime), the common jump intensity is the dominant driver of vulnerable option prices, whereas the idiosyncratic jump intensity has only a marginal effect—a policy-relevant finding that cannot be obtained from models with a single jump component.

(iii) Two Jump Distributions (Joint Normal vs. Independent Double-Exponential). This paper do not claim that both distributions are “correct,” nor do we propose a mixture model. Rather, we adopt a dual-scenario modeling strategy to accommodate two distinct empirical regimes: the joint normal distribution captures linear, symmetric jump risks typical of stable market conditions (e.g., liquidity-driven price adjustments); the independent double-exponential distribution captures nonlinear, asymmetric jump risks characteristic of crisis periods, where downward jumps are larger and more frequent than upward jumps. This dual specification is intentionally designed to demonstrate the robustness and generalizability of our pricing framework across different jump-risk regimes. More importantly, it allows us to quantify the model risk associated with jump-distribution misspecification—a practical concern for risk managers that is rarely addressed in the existing literature. The impact of jump distribution choice on option prices is highly asymmetric: it is negligible for at-the-money options but becomes economically significant for deep out-of-the-money vulnerable options. This finding itself constitutes a meaningful contribution to the model selection literature.

(iv) Two Interest Rate Models (Vasicek and CIR). Similar to the dual-jump-distribution strategy, the inclusion of both Vasicek and CIR interest rate models serves a dual purpose.

Theoretical robustness. The Vasicek model allows for negative interest rates (relevant in post-2008 and post-COVID market environments, as documented by ECB and Fed data), while the CIR model enforces non-negativity (consistent with the classical pre-2008 interest rate regime). This dual specification ensures that our pricing framework remains valid across different interest rate environments.

Empirical flexibility. By implementing both models, we provide practitioners with the flexibility to choose the interest rate dynamics that best fit the current market regime, rather than imposing a single, potentially misspecified, interest rate process.

2.2. The Model of Assets Price

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{Q})$ be the probability space of a system with information flow $\{\mathcal{F}_t\}_{0 \leq t \leq T}$; Ω is the sample set, $\mathcal{F}$ is the domain generated by Ω ; $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ includes all the basic processes in the model under investigation in this paper, and $\mathbb{Q}$ is the risk-neutral probability measure.

Under the above market information assumptions, the price mechanisms S_{it} ($i = 1, 2$) of the underlying asset and the counterparty asset follow the following stochastic differential equation system:

$$\begin{aligned} \frac{dS_{1t}}{S_{1t}} &= (r_t - (\lambda_0 + \lambda_1)m_1)dt + \sigma_{11}\sqrt{v_{1t}}dW_{1t}^{(1)} + \sigma_{12}\sqrt{v_{0t}}dW_{2t}^{(1)} + (e^{I_1} - 1)d(N_{0t} + N_{1t}), \\ \frac{dS_{2t}}{S_{2t}} &= (r_t - (\lambda_0 + \lambda_2)m_2)dt + \sigma_{21}\sqrt{v_{2t}}dW_{1t}^{(2)} + \sigma_{22}\sqrt{v_{0t}}dW_{2t}^{(2)} + (e^{I_2} - 1)d(N_{0t} + N_{2t}). \end{aligned} \quad (1)$$

where $\sigma_{i1} (i = 1, 2)$ is the coefficient of unique volatility of S_{it} 's return, $\sigma_{i2} (i = 1, 2)$ is the coefficient of common volatility of S_{it} 's return, $\lambda_i (i = 1, 2)$ is the intensity of unique Poisson process N_{it} , λ_0 is the intensity of common Poisson process N_{0t} , and $m_i = E[e^{J_i} - 1]$ is the expected jump amplitude of the underlying asset price process S_{it} .

The common and unique stochastic volatility rate $v_{it} (i = 0, 1, 2)$ of the two assets is given by

$$dv_{0t} = k_0(\theta_0 - v_{0t})dt + \sigma_0\sqrt{v_{0t}}dW_{vt}^{(0)}, \quad (2)$$

$$dv_{1t} = k_1(\theta_1 - v_{1t})dt + \sigma_1\sqrt{v_{1t}}dW_{vt}^{(1)}, \quad (3)$$

$$dv_{2t} = k_2(\theta_2 - v_{2t})dt + \sigma_2\sqrt{v_{2t}}dW_{vt}^{(2)}, \quad (4)$$

where $v_{it} (i = 0, 1, 2)$ are subject to square root processes which have the property of long-term mean recovery. Here, v_{0t} represents the overall fluctuation factor of the financial market, v_{1t} and v_{2t} indicates the local fluctuation factors of the underlying asset and counterparty asset respectively. k_i are the coefficients of mean-reversion speed under different fluctuation factors. θ_i are long-term mean of different fluctuation factors. σ_i are the coefficients of diffusion under different fluctuation factors.

Regarding the jump vectors (J_1, J_2) of the target asset and the counterparty asset prices, this paper attempts to describe them through two probability distributions. In the linear scenario of jump risk, we assume that the jump vector (J_1, J_2) follows a two-dimensional joint normal distribution $N(\mu_1, \mu_2; \sigma_{J_1}^2, \sigma_{J_2}^2; \rho_J)$. In the nonlinear scenario of jump risk, we assume that the jump vector (J_1, J_2) follows an independent two-dimensional asymmetric double-exponential distribution. The specific marginal probability density is

$$f_{J_i}(x) = p_i\tilde{\zeta}_i e^{\tilde{\zeta}_i x} 1_{\{x < 0\}} + q_i\eta_i e^{\eta_i x} 1_{\{x > 0\}}, p_i + q_i = 1.$$

The risk-free interest rate process is one of the key assumptions of this paper. The paper assumes that the market interest rate follows two diffusion processes with recovery properties, as follows:

$$dr_t = k_r(\theta - r_t)dt + \sigma_r dW_{rt}, \quad (5)$$

$$dr_t = k_r(\theta - r_t)dt + \sigma_r\sqrt{r_t}dW_{rt}, \quad (6)$$

where the Formula (5) serves as the Vasicek model; the Formula (6) serves as the CIR model.

The correlation structure of the Brownian motions involved in the above model is given by

$$\begin{aligned} d\langle W_{2t}^{(1)}, W_{2t}^{(2)} \rangle &= \rho_{12}dt, d\langle W_{2t}^{(1)}, W_{vt}^{(0)} \rangle = \rho_{1v_0}dt, d\langle W_{2t}^{(2)}, W_{vt}^{(0)} \rangle = \rho_{2v_0}dt, \\ d\langle W_{1t}^{(1)}, W_{v1t} \rangle &= \rho_{1v_1}dt, d\langle W_{1t}^{(2)}, W_{v2t} \rangle = \rho_{2v_2}dt. \end{aligned}$$

Notice that other Brownian motions are independent of each other.

Finally, the payoff functions of the four types of vulnerable geometric average Asian options at the expiration date T are as follows

$$h_1(S_{1T}, S_{2T}, G_T) = (G_T - K)^+ (1_{\{S_{2T} \geq D_0\}} + \frac{(1 - \alpha)S_{2T}}{D} 1_{\{S_{2T} < D_0\}}), \quad (7)$$

$$h_2(S_{1T}, S_{2T}, G_T) = (K - G_T)^+ (1_{\{S_{2T} \geq D_0\}} + \frac{(1 - \alpha)S_{2T}}{D} 1_{\{S_{2T} < D_0\}}), \quad (8)$$

$$h_3(S_{1T}, S_{2T}, G_T) = (G_T - S_{1T})^+ (1_{\{S_{2T} \geq D_0\}} + \frac{(1 - \alpha)S_{2T}}{D} 1_{\{S_{2T} < D_0\}}), \quad (9)$$

$$h_4(S_{1T}, S_{2T}, G_T) = (S_{1T} - G_T)^+ (1_{\{S_{2T} \geq D_0\}} + \frac{(1 - \alpha)S_{2T}}{D} 1_{\{S_{2T} < D_0\}}), \quad (10)$$

where $G_T = \exp\{\frac{1}{T} \int_0^T \ln(S_{1t}) dt\}$, S_{2T} is the market value of the counterpart asset S_{1t} at maturity time T , D denotes the total debt of the option writer, and α represents the deadweight costs associated with the financial stress and is described as a percentage of the counterpart value. D_0 is the default boundary such that a default event occurs when $S_{2t} < D_0$. The payoff function of vulnerable option is the payoff function of European option if $S_{2t} \geq D_0$.

The four payoff structures in Equations (7)–(10) classify vulnerable geometric Asian options along two dimensions:

Dimension 1: Strike Price Nature (Fixed vs. Floated).

- h_1 (fixed-strike call) and h_2 (fixed-strike put) have a pre-specified constant strike price K . These are suitable for investors who have a clear expectation regarding the absolute level of the average asset price (e.g., corporations locking in future procurement costs or sales revenues).
- h_3 (floating-strike call) and h_4 (floating-strike put) have the terminal asset price S_{1T} as the strike price. These are suitable for investors concerned with the *relative performance* of the average price against the terminal price, rather than absolute levels (e.g., relative-return products benchmarked against terminal market value).

Dimension 2: Option Direction (Call vs. Put).

This dimension carries the standard economic interpretation: call options (h_1, h_3) provide holders with the opportunity to profit when the geometric average exceeds the strike price, serving to hedge against upward price risk; put options (h_2, h_4) serve the opposite purpose, hedging against downward price risk.

All four payoff functions share the identical default adjustment structure:

$$1_{\{S_{2T} \geq D_0\}} + \frac{(1 - \alpha)S_{2T}}{D} 1_{\{S_{2T} < D_0\}}.$$

The economic intuition of this structure is straightforward: if the counterparty's asset value S_{2T} is no less than the debt threshold D_0 , the option pays in full; otherwise, the option holder recovers only a fraction $(1 - \alpha)S_{2T}/D$ of the payoff, where α represents the "friction cost" associated with default (e.g., legal expenses, distressed-asset liquidation discounts). This structure is the defining feature of the Klein vulnerable option framework, which we extend here to four variants of geometric Asian options. The full derivation of all four payoff structures ensures that our theoretical framework can cover the pricing needs of the vast majority of Asian-style vulnerable options traded in practice. Regardless of whether an investor holds a fixed- or floating-strike product, in either a call or put direction, the corresponding theoretical value and hedging parameters can be computed directly from our pricing formulas.

2.3. Model Preparation Results

Based on the aforementioned price mechanism, in order to derive the joint characteristic function of all random sources, we first provide the proofs of several key lemmas to facilitate the smooth presentation of the characteristic function.

In order to facilitate future proofing, let $X_t = (\ln(S_{1t}), \ln(S_{2t}))$, $Y_t = \ln G_t = \frac{1}{t} \int_0^t X_{1s} ds$, $v_t = (v_{0t}, v_{1t}, v_{2t})$, $u = (u_1, u_2)$, $x = (x_1, x_2)$, $v = (v_0, v_1, v_2)$. Define the time t conditional joint characteristic function with cash discount included as

$$\varphi(t, T, u, u_3; x, y, v, r) := E^{\mathbb{Q}}[e^{-\int_t^T r_s ds + iu \cdot X_T + iu_3 \cdot Y_T} | X_t = x, Y_t = y, v_t = v, r_t = r]. \quad (11)$$

According to ITO's lemma and the dynamics of two underlying assets, we have the differential formula for log-prices (X_{1t}, X_{2t}) is

$$\begin{aligned} dX_{1t} &= (r_t - \frac{1}{2}\sigma_{11}^2 v_{1t} - \frac{1}{2}\sigma_{12}^2 v_{0t})dt + \sigma_{11}\sqrt{v_{1t}}dW_{1t}^{(1)} + \sigma_{12}\sqrt{v_{0t}}dW_{2t}^{(1)} \\ &\quad - (\lambda_0 + \lambda_1)m_1 dt + J_1 d(N_{0t} + N_{1t}), \\ dX_{2t} &= (r_t - \frac{1}{2}\sigma_{21}^2 v_{2t} - \frac{1}{2}\sigma_{22}^2 v_{0t})dt + \sigma_{21}\sqrt{v_{2t}}dW_{2t}^{(2)} + \sigma_{22}\sqrt{v_{0t}}dW_{2t}^{(2)} \\ &\quad - (\lambda_0 + \lambda_2)m_2 dt + J_2 d(N_{0t} + N_{2t}). \end{aligned}$$

Then, separate them into continuous part and jump part,

$$X_{1T} = X_{1T}^C + X_{1T}^J, X_{2T} = X_{2T}^C + X_{2T}^J,$$

where $\tau = T - t$, the continuous part is expressed as

$$\begin{aligned} X_{1T}^C &= X_{1t} + \int_t^T r_s ds - \frac{1}{2}\sigma_{11}^2 \int_t^T v_{1s} ds - \frac{1}{2}\sigma_{12}^2 \int_t^T v_{0s} ds \\ &\quad + \sigma_{11} \int_t^T \sqrt{v_{1s}} dW_{1s}^{(1)} + \sigma_{12} \int_t^T \sqrt{v_{0s}} dW_{2s}^{(1)}, \\ X_{2T}^C &= X_{2t} + \int_t^T r_s ds - \frac{1}{2}\sigma_{21}^2 \int_t^T v_{2s} ds - \frac{1}{2}\sigma_{22}^2 \int_t^T v_{0s} ds \\ &\quad + \sigma_{21} \int_t^T \sqrt{v_{2s}} dW_{2s}^{(2)} + \sigma_{22} \int_t^T \sqrt{v_{0s}} dW_{2s}^{(2)}; \end{aligned}$$

the jump part is expressed as

$$\begin{aligned} X_{1T}^J &= -\tau(\lambda_0 + \lambda_1)m_1 + \sum_{i=N_{0t}+N_{1t}+1}^{N_{0T}+N_{1T}} J_1, \\ X_{2T}^J &= -\tau(\lambda_0 + \lambda_2)m_2 + \sum_{i=N_{0t}+N_{2t}+1}^{N_{0T}+N_{2T}} J_2. \end{aligned}$$

Similarly, Y_t can also be decomposed in a similar way. As follows specifically

$$Y_T = Y_T^C + Y_T^J,$$

where the continuous part and jump part are expressed as

$$\begin{aligned} Y_T^C &= \frac{t}{T} Y_t + \frac{\tau}{T} \cdot X_{1t} + \int_t^T (1 - \frac{s}{T}) r_s ds - \frac{\sigma_{11}^2}{2} \int_t^T (1 - \frac{s}{T}) v_{1s} ds - \frac{\sigma_{12}^2}{2} \int_t^T (1 - \frac{s}{T}) v_{0s} ds \\ &\quad + \sigma_{11} \int_t^T (1 - \frac{s}{T}) \sqrt{v_{1s}} dW_{1s}^{(1)} + \sigma_{12} \int_t^T (1 - \frac{s}{T}) \sqrt{v_{0s}} dW_{2s}^{(1)}, \\ Y_T^J &= -\frac{\tau^2}{2T} (\lambda_0 + \lambda_1)m_1 + \sum_{i=N_{0t}+N_{1t}+1}^{N_{0T}+N_{1T}} (1 - \frac{t_i}{T}) \cdot J_1. \end{aligned}$$

Using the aforementioned decomposition formula, the decomposition formula of the characteristic function can be further obtained as follows:

$$\begin{aligned}
& \varphi(t, T, u, u_3; x, y, v, r) \\
&= E^{\mathbb{Q}}[e^{-\int_t^T r_s ds + iu \cdot X_T^C + iu_3 \cdot Y_T^C} | X_t = x, Y_t = y, v_t = v, r_t = r] \\
&\quad \cdot E^{\mathbb{Q}}[e^{iu \cdot X_T^J + iu_3 \cdot Y_T^J} | X_t = x, Y_t = y, v_t = v, r_t = r] \\
&= \varphi^C(t, T, u, u_3; x, y, v, r) \cdot \varphi^J(t, T, u, u_3; x, y, v, r).
\end{aligned} \tag{12}$$

where the continuous part of the characteristic function $\varphi(t, T, u, u_3; x, y, v, r)$ is

$$\varphi^C(t, T, u, u_3; x, y, v, r) = E^{\mathbb{Q}}[e^{-\int_t^T r_s ds + iu \cdot X_T^C + iu_3 \cdot Y_T^C} | X_t = x, Y_t = y, v_t = v, r_t = r]; \tag{13}$$

the jump part of the characteristic function $\varphi(t, T, u, u_3; x, y, v, r)$ is

$$\varphi^J(t, T, u, u_3; x, y, v, r) = E^{\mathbb{Q}}[e^{iu \cdot X_T^J + iu_3 \cdot Y_T^J} | X_t = x, Y_t = y, v_t = v, r_t = r]. \tag{14}$$

Lemma 1. Suppose that the dynamics of process r_t is an mean-reverting process given by Equation (5), that is the Vasicek model,

$$dr_t = k_r(\theta_r - r_t)dt + \sigma_r dW_{rt},$$

then the generalized joint moment generating function of r_t is defined as follows,

$$\Psi(t, T, m_1, m_2(\tau'); k_r, \theta_r, \sigma_r, r) = E^{\mathbb{Q}}[\exp\{m_1 r_T + \int_t^T m_2(\tau') r_s ds\} | r_t = r], \tag{15}$$

where m_1 is a constant, $m_2(\tau') = a + b\tau'$ is a linear function of variable τ' , and $\tau' = u_1 + u_3 \frac{T}{T}$. Its analytical expression is given by

$$\Psi(t, T, m_1, m_2(\tau'); k_r, \theta_r, \sigma_r, r) = \exp\{H(\tau') + F(\tau')r\}, \tag{16}$$

where $F(\tau') = -\frac{u_3}{Tk_r^2}b + \frac{1}{k_r}m_2(\tau') + (m_1 + \frac{u_3}{Tk_r^2}b - \frac{1}{k_r}m_2(u_1))e^{-\frac{Tk_r}{u_3}(\tau' - u_1)}$, $H(\tau', m_1, m_2) = -\frac{\sigma_r^2}{4k_r}(F^2(\tau') - F^2(u_1)) - (\theta_r + \frac{\sigma_r^2}{2k_r^2}m_2(\tau) + \frac{u_3\sigma_r^2 m_2'}{2Tk_r^3})F(\tau') + (\theta_r + \frac{\sigma_r^2}{2k_r^2}m_2(u_1) + \frac{u_3\sigma_r^2 m_2'}{2Tk_r^3})F(u_1) + (\frac{\theta_r T}{2u_3 m_2'} + \frac{\sigma_r^2 m_2'}{4k_r^3})(m_2^2(\tau') - m_2^2(u_1)) + \frac{\sigma_r^2 T}{6k_r^2 u_3 m_2'}(m_2^3(\tau') - m_2^3(u_1))$.

Proof. If the process r_t is an mean-reverting process driven by Equation (5), then from the Feynman-Kac formula or $e^{\int_0^t m_2(\tau') r_s ds} \Psi(t, T, m_1, m_2(\tau'))$ is a martingale, the generalized joint moment generating function $\Psi(t, T, m_1, m_2(\tau'); k_r, \theta_r, \sigma_r, r)$ satisfies a partial differential equation given by $(\tau = T - t)$

$$m_2(\tau')r\Psi + \frac{\partial \Psi}{\partial t} + k_r(\theta_r - r)\frac{\partial \Psi}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 \Psi}{\partial r^2} = 0,$$

with the boundary condition $\Psi(T, T, m_1, m_2(\tau')) = e^{m_1 r}$.

Based on the linearity of the above partial differential equation, we assume that $\Psi(t, T, m_1, m_2(\tau'); k_r, \theta_r, \sigma_r, r)$ is of the form

$$\Psi(t, T, m_1, m_2(\tau'); k_r, \theta_r, \sigma_r, r) = \exp\{H(\tau') + F(\tau')r\}.$$

Then, substituting into the above partial differential equation, we obtain

$$m_2(\tau')r\Psi + (-\frac{u_3}{T} \frac{dH}{d\tau'} - \frac{u_3}{T} \frac{dF}{d\tau'} r)\Psi + k_r(\theta_r - r)F\Psi + \frac{1}{2}\sigma_r^2 F^2\Psi = 0$$

Furthermore, by separating the variables,

$$\begin{aligned}\frac{dH}{d\tau'} &= \frac{Tk_r\theta_r}{u_3}F + \frac{T\sigma_r^2}{2u_3}F^2, \\ \frac{dF}{d\tau'} &= -\frac{Tk_r}{u_3}F + \frac{Tm_2(\tau')}{u_3},\end{aligned}$$

with the boundary condition $H(u_1) = 0$, $F(u_1) = m_1$.

In particular, when $m_2(\tau') = a + b\tau'$, by applying the integration by parts technique, the analytical expression of $F(\tau')$ is derived ($m'_2(\tau') = b = m'_2$)

$$\begin{aligned}F(\tau') &= e^{-\frac{Tk_r}{u_3}\tau'} \left(\int_{u_1}^{\tau'} \frac{Tm_2(s)}{u_3} e^{\frac{Tk_r}{u_3}s} ds + m_1 e^{\frac{Tk_r}{u_3}u_1} \right) \\ &= e^{-\frac{Tk_r}{u_3}\tau'} \left(\frac{1}{k_r} \int_{u_1}^{\tau'} m_2(s) de^{\frac{Tk_r}{u_3}s} + m_1 e^{\frac{Tk_r}{u_3}u_1} \right) \\ &= e^{-\frac{Tk_r}{u_3}\tau'} \left[\frac{1}{k_r} (m_2(\tau') e^{\frac{Tk_r}{u_3}\tau'} - m_2(u_1) e^{\frac{Tk_r}{u_3}u_1} - \int_{u_1}^{\tau'} m'_2(s) e^{\frac{Tk_r}{u_3}s} ds) + m_1 e^{\frac{Tk_r}{u_3}u_1} \right] \\ &= e^{-\frac{Tk_r}{u_3}\tau'} \left[\frac{1}{k_r} (m_2(\tau') e^{\frac{Tk_r}{u_3}\tau'} - m_2(u_1) e^{\frac{Tk_r}{u_3}u_1} - \frac{u_3}{Tk_r} m'_2(s) (e^{\frac{Tk_r}{u_3}\tau'} - e^{\frac{Tk_r}{u_3}u_1})) + m_1 e^{\frac{Tk_r}{u_3}u_1} \right] \\ &= \frac{1}{k_r} [m_2(\tau') - m_2(u_1) e^{-\frac{Tk_r}{u_3}(\tau'-u_1)} - \frac{u_3}{Tk_r} m'_2(s) (1 - e^{-\frac{Tk_r}{u_3}(\tau'-u_1)})] + m_1 e^{-\frac{Tk_r}{u_3}(\tau'-u_1)} \\ &= \frac{1}{k_r} m_2(\tau') - \frac{1}{k_r} m_2(u_1) e^{-\frac{Tk_r}{u_3}(\tau'-u_1)} - \frac{u_3}{Tk_r^2} m'_2(s) + \frac{u_3}{Tk_r^2} m'_2(s) e^{-\frac{Tk_r}{u_3}(\tau'-u_1)} + m_1 e^{-\frac{Tk_r}{u_3}(\tau'-u_1)} \\ &= -\frac{u_3}{Tk_r^2} m'_2 + \frac{1}{k_r} m_2(\tau') + (m_1 + \frac{u_3}{Tk_r^2} m'_2 - \frac{1}{k_r} m_2(u_1)) e^{-\frac{Tk_r}{u_3}(\tau'-u_1)}\end{aligned}$$

Based on the equation satisfied by F , by applying transformation techniques, the analytical expression of H is gradually obtained. The specific process is as follows.

The first step is to obtain the analytical result of the second item. Based on

$$\frac{Tk_r}{u_3}F = -\frac{\partial F}{\partial \tau'} + \frac{Tm_2(\tau')}{u_3}, \quad \frac{T\sigma_r^2}{2u_3}F^2 = -\frac{\sigma_r^2}{2k_r}F \frac{\partial F}{\partial \tau'} + \frac{T\sigma_r^2}{2k_ru_3}m_2(\tau')F,$$

the simplified result of the second item can be obtained

$$\frac{T\sigma_r^2}{2u_3} \int_{u_1}^{\tau'} F^2(s) ds = -\frac{\sigma_r^2}{4k_r} (F^2(\tau') - F^2(u_1)) + \frac{T\sigma_r^2}{2k_ru_3} \int_{u_1}^{\tau'} m_2(s) F(s) ds.$$

The second step is to obtain the analytical result of the first item and the second term that has not been integrated. Based on

$$\frac{T}{u_3} (k_r\theta_r + \frac{\sigma_r^2}{2k_r} m_2(\tau')) F = -\frac{1}{k_r} (k_r\theta_r + \frac{\sigma_r^2}{2k_r} m_2(\tau')) \frac{\partial F}{\partial \tau'} + (k_r\theta_r + \frac{\sigma_r^2}{2k_r} m_2(\tau')) \frac{T}{k_ru_3} m_2(\tau'),$$

the simplified result of the first item can be obtained

$$\begin{aligned}& \frac{T}{u_3} \int_{u_1}^{\tau'} (k_r\theta_r + \frac{\sigma_r^2}{2k_r} m_2(s)) F(s) ds \\ &= -\frac{1}{k_r} \int_{u_1}^{\tau'} (k_r\theta_r + \frac{\sigma_r^2}{2k_r} m_2(s)) \frac{\partial F}{\partial \tau'} ds + \frac{\theta_r T}{u_3} \int_{u_1}^{\tau'} m_2(s) ds + \frac{\sigma_r^2 T}{2k_r^2 u_3} \int_{u_1}^{\tau'} m_2^2(s) ds \\ &= -\frac{1}{k_r} [(k_r\theta_r + \frac{\sigma_r^2}{2k_r} m_2(\tau')) F(\tau') - (k_r\theta_r + \frac{\sigma_r^2}{2k_r} m_2(u_1)) F(u_1)] + \frac{\theta_r T}{2u_3 m_2} (m_2^2(\tau') - m_2^2(u_1)) \\ & \quad + \frac{\sigma_r^2 T}{6k_r^2 u_3 m'_2} (m_2^3(\tau') - m_2^3(u_1)) + \frac{\sigma_r^2 m'_2}{2k_r^2} \int_{u_1}^{\tau'} F(s) ds.\end{aligned}$$

Based on

$$\frac{\sigma_r^2 m'_2}{2k_r^2} F = -\frac{u_3 \sigma_r^2 m'_2}{2Tk_r^3} \frac{\partial F}{\partial \tau'} + \frac{\sigma_r^2 m'_2}{k_r^3} m_2(\tau'),$$

the simplified result of the residual term can be obtained

$$\begin{aligned}\frac{\sigma_r^2 m'_2}{2k_r^2} \int_{u_1}^{\tau'} F(s) ds &= -\frac{u_3 \sigma_r^2 m'_2}{2Tk_r^3} \int_{u_1}^{\tau'} \frac{\partial F}{\partial s} ds + \frac{\sigma_r^2 m'_2}{2k_r^3} \int_{u_1}^{\tau'} m_2(s) ds \\ &= -\frac{u_3 \sigma_r^2 m'_2}{2Tk_r^3} (F(\tau') - F(u_1)) + \frac{\sigma_r^2 m'_2}{4k_r^3} (m_2^2(\tau') - m_2^2(u_1))\end{aligned}$$

The third step is to thoroughly review and organize the existing integration results. Substituting the above formula, we can obtain the analytical expression of H as

$$\begin{aligned} H(\tau') &= -\frac{\sigma_r^2}{4k_r}(F^2(\tau') - F^2(u_1)) - \frac{1}{k_r}[(k_r\theta_r + \frac{\sigma_r^2}{2k_r}m_2(\tau))F(\tau') - (k_r\theta_r + \frac{\sigma_r^2}{2k_r}m_2(u_1))F(u_1)] \\ &\quad + \frac{\theta_r T}{2u_3m_2'}(m_2^2(\tau') - m_2^2(u_1)) + \frac{\sigma_r^2 T}{6k_r^2 u_3 m_2'}(m_2^3(\tau') - m_2^3(u_1)) - \frac{u_3 \sigma_r^2 m_2'}{2Tk_r^3}(F(\tau') - F(u_1)) \\ &\quad + \frac{\sigma_r^2 m_2'}{4k_r^3}(m_2^2(\tau') - m_2^2(u_1)) \\ &= -\frac{\sigma_r^2}{4k_r}(F^2(\tau') - F^2(u_1)) - (\theta_r + \frac{\sigma_r^2}{2k_r^2}m_2(\tau) + \frac{u_3 \sigma_r^2 m_2'}{2Tk_r^3})F(\tau') + (\theta_r + \frac{\sigma_r^2}{2k_r^2}m_2(u_1) \\ &\quad + \frac{u_3 \sigma_r^2 m_2'}{2Tk_r^3})F(u_1) + (\frac{\theta_r T}{2u_3 m_2'} + \frac{\sigma_r^2 m_2'}{4k_r^3})(m_2^2(\tau') - m_2^2(u_1)) + \frac{\sigma_r^2 T}{6k_r^2 u_3 m_2'}(m_2^3(\tau') - m_2^3(u_1)). \end{aligned}$$

Their solutions are given by the above Lemma 1, respectively. $\square$

Lemma 2. Let a, b, c_0, c_1, c_2 be constants. Then the following ordinary differential equations

$$\begin{cases} \frac{dy}{dx} = ay^2 + by + c_2(x - x_0)^2 + c_1(x - x_0) + c_0, \\ y|_{x=x_0} = y_0 \end{cases} \quad (17)$$

have the solution that

$$\begin{aligned} y &= f(x_0, y_0; a, b, c_0, c_1, c_2) = -\frac{1}{a} \frac{w'(x)}{w(x)}, \\ w(x) &= g(x_0, y_0; a, b, c_0, c_1, c_2) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \end{aligned}$$

where $a_n = \frac{b}{n}a_{n-1} - \frac{a}{(n)(n-1)} \sum_{i=0}^2 c_i a_{n-2-i}$, $n \geq 2$. Here, the initial coefficient is $a_{-2} = 0$, $a_{-1} = 0$, $a_0 = 1$.

Proof. In order to obtain the solution of the above equation, we first apply the following transformation

$$y(x) = -\frac{1}{a} \frac{w'(x)}{w(x)}, y(x_0) = -\frac{1}{a} \frac{w'(x_0)}{w(x_0)} = y_0.$$

Substituting it into the above Equation (17), we obtain

$$y'(x) = -\frac{1}{a} \frac{w''(x)w(x) - w'^2(x)}{w^2(x)} = a \cdot \frac{1}{a^2} \cdot \frac{w'^2(x)}{w^2(x)} - \frac{b}{a} \frac{w'(x)}{w(x)} + c_2(x - x_0)^2 + c_1(x - x_0) + c_0.$$

After further organization, it can be concluded that

$$w''(x) - bw'(x) + a(c_2(x - x_0)^2 + c_1(x - x_0) + c_0) = 0, w'(x_0) = -ay_0w(x_0).$$

Based on the non-existence of the general function solution of the second-order very coefficient equation, this lemma employs the Taylor series method to obtain the analytical solution of the equation. Suppose the expression of $w(x)$ is as follows:

$$w(x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Take its derivative

$$\begin{aligned}w'(x) &= \sum_{n=1}^{\infty} a_n n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} a_{n+1} (n+1) (x - x_0)^n, \\w''(x) &= \sum_{n=2}^{\infty} a_n n (n-1) (x - x_0)^{n-2} = \sum_{n=0}^{\infty} a_{n+2} (n+2) (n+1) (x - x_0)^n,\end{aligned}$$

Substituting into the original equation, we obtain

$$\sum_{n=2}^{\infty} [a_{n+2} (n+2) (n+1) - b a_{n+1} (n+1) + a c_0 a_n + a c_1 a_{n-1} + a c_2 a_{n-2}] (x - x_0)^n = 0.$$

Through the comparison coefficient, the recursive relationship of the coefficients of the series is as follows:

$$a_n = \frac{b}{n} a_{n-1} - \frac{a}{(n)(n-1)} \sum_{i=0}^2 c_i a_{n-2-i}, n \geq 2,$$

where the initial coefficient is $a_{-2} = 0, a_{-1} = 0, a_0 = 1$. For example, $a_1 = -a a_0 y_0$, $a_2 = \frac{1}{2} a_1 b - \frac{a}{2} c_0 a_0$, $a_3 = \frac{1}{3} a_2 b - \frac{a}{6} (c_0 a_1 + c_1 a_0)$.

The Lemma 2 has been proved. $\square$

Lemma 3. Suppose that the dynamics of process v_t is an mean-reverting process given by Equation below, that is the CIR model,

$$dv_t = k_v (\theta_v - v_t) dt + \sigma_v \sqrt{v_t} dW_{vt},$$

then the generalized joint moment generating function of v_t is defined as follows,

$$\Psi(t, T, m_1, m_2(\tau'); k_v, \theta_v, \sigma_v, v) = E^{\mathbb{Q}}[\exp\{m_1 v_T + \int_t^T m_2(\tau') v_s ds\} | v_t = v], \quad (18)$$

where m_1 is a constant, $m_2(\tau') = a + b\tau' + c\tau'^2$ is a linear function of variable τ' , and $\tau' = u_1 + u_3 \frac{\tau}{T}$. Its analytical expression is given by

$$\Psi(t, T, m_1, m_2(\tau'); k_v, \theta_v, \sigma_v, v) = \exp\{H(\tau') + F(\tau')r\}, \quad (19)$$

where $H(\tau') = -\frac{2k_v\theta_v}{\sigma_v^2} \ln\{g(u_1, m_1; \frac{T\sigma_v^2}{2u_3}, -\frac{Tk_v}{u_3}, \frac{T}{u_3}c, \frac{T}{u_3}(b + 2cu_1), \frac{T}{u_3}(a + bu_1 + cu_1^2))\}$, $F(\tau') = f(u_1, m_1; \frac{T\sigma_v^2}{2u_3}, -\frac{Tk_v}{u_3}, \frac{T}{u_3}c, \frac{T}{u_3}(b + 2cu_1), \frac{T}{u_3}(a + bu_1 + cu_1^2))$.

Proof. If the process r_t is an mean-reverting process driven by the above Equation, then from the Feynman-Kac formula or $e^{\int_0^t m_2(\tau') v_s ds} \Psi(t, T, m_1, m_2(\tau'))$ is a martingale, the generalized joint moment generating function $\Psi(t, T, m_1, m_2(\tau'); k_v, \theta_v, \sigma_v, v)$ satisfies a partial differential equation given by $(\tau = T - t)$

$$m_2(\tau') v \Phi + \frac{\partial \Phi}{\partial t} + k_v (\theta_v - v) \frac{\partial \Phi}{\partial v} + \frac{1}{2} \sigma_v^2 v \frac{\partial^2 \Phi}{\partial v^2} = 0,$$

with the boundary condition $\Psi(T, T, m_1, m_2(\tau')) = e^{m_1 v}$.

Based on the linearity of the above partial differential equation, we assume that $\Psi(t, T, m_1, m_2(\tau'); k_v, \theta_v, \sigma_v, v)$ is of the form

$$\Phi(t, T, m_1, m_2(\tau')) = e^{H(\tau') + F(\tau')v}.$$

Then, substituting into the above partial differential equation, we obtain

$$m_2(\tau')v\Phi + \left(-\frac{u_3}{T}\frac{\partial H}{\partial \tau'} - \frac{u_3}{T}\frac{\partial F}{\partial \tau'}v\right)\Phi + k_v(\theta_v - v)F\Phi + \frac{1}{2}\sigma_v^2vF^2\Phi = 0$$

Furthermore, by separating the variables,

$$\begin{aligned}\frac{dH}{d\tau'} &= \frac{Tk_v\theta_v}{u_3}F, \\ \frac{dF}{d\tau'} &= \frac{T\sigma_v^2}{2u_3}F^2 - \frac{Tk_v}{u_3}F + \frac{Tm_2(\tau')}{u_3},\end{aligned}\quad (20)$$

with the boundary condition $H(u_1) = 0$, $F(u_1) = m_1$.

In particular, when $m_2(\tau') = a + b\tau' + c\tau'^2$, first, $m_2(\tau')$ is transformed into the standard form based on the initial conditions as follows:

$$m_2(\tau') = (a + bu_1 + cu_1^2) + (b + 2cu_1)(\tau' - u_1) + c(\tau' - u_1)^2.$$

Furthermore, based on the conclusion of Lemma 2, the expressions of H and F can be obtained as follows:

$$\begin{aligned}H(\tau') &= -\frac{2k_v\theta_v}{\sigma_v^2}\ln\left\{g(u_1, m_1; \frac{T\sigma_v^2}{2u_3}, -\frac{Tk_v}{u_3}, \frac{T}{u_3}c, \frac{T}{u_3}(b + 2cu_1), \frac{T}{u_3}(a + bu_1 + cu_1^2))\right\}, \\ F(\tau') &= f(u_1, m_1; \frac{T\sigma_v^2}{2u_3}, -\frac{Tk_v}{u_3}, \frac{T}{u_3}c, \frac{T}{u_3}(b + 2cu_1), \frac{T}{u_3}(a + bu_1 + cu_1^2)).\end{aligned}$$

Their solutions are given by the above Lemma 3, respectively. $\square$

The following theorem provides the joint characteristic function of the log-asset prices and the log-geometric average under the risk-neutral measure.

Theorem 1. Suppose the market price dynamics follow Equations (1)–(6). Then the continuous-part characteristic function is given by:

$$\phi^C(t, T, u, u_3; x, y, v, r) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds + iu_1 X_{1T} + iu_2 X_{2T} + iu_3 Y_T} \middle| \mathcal{F}_t\right], \quad (21)$$

where $X_{it} = \ln S_{it}$ ($i = 1, 2$) and $Y_T = \frac{1}{T} \int_0^T X_{1s} ds$. The characteristic function factors into five components:

$$\phi^C(t, T, u, u_3; x, y, v, r) = A \cdot B \cdot C \cdot D \cdot E, \quad (22)$$

where:

$$A = \exp\left\{i\left(u_1 + \frac{\tau}{T}u_3\right)x_1 + iu_2x_2 + i\frac{t}{T}u_3y\right\}, \quad (23)$$

$$B = \Psi_r(t, T, 0, i(u_2 + \tau' + i); k_r, \theta_r, \sigma_r, r), \quad (24)$$

$$\begin{aligned}C &= \exp\left\{i\rho_{1v1}\frac{\sigma_{11}}{\sigma_1}\left[-\left(u_1 + \frac{u_3\tau}{T}\right)v_1 - k_1\theta_1\left(u_1\tau + \frac{u_3\tau^2}{2T}\right)\right]\right\} \\ &\quad \times \Psi_v\left(t, T, i\rho_{1v1}\frac{\sigma_{11}}{\sigma_1}, m_2^C(\tau'); k_1, \theta_1, \sigma_1, v_1\right),\end{aligned}\quad (25)$$

$$\begin{aligned}D &= \exp\left\{-i\rho_{2v2}\frac{\sigma_{21}}{\sigma_2}u_2(v_2 + k_2\theta_2\tau)\right\} \\ &\quad \times \Psi_v\left(t, T, i\rho_{2v2}\frac{\sigma_{21}}{\sigma_2}, \left(-\frac{1}{2}\sigma_{21}^2 + i\rho_{2v2}\frac{\sigma_{21}}{\sigma_2}k_2\right)u_2 - \frac{1}{2}(1 - \rho_{2v2}^2)\sigma_{21}^2u_2^2; k_2, \theta_2, \sigma_2, v_2\right),\end{aligned}\quad (26)$$

$$\begin{aligned}
E = & \exp \left\{ i \rho_{1v0} \frac{\sigma_{12}}{\sigma_0} \left[- \left(u_1 + \frac{u_3 \tau}{T} \right) v_0 - k_0 \theta_0 \left(u_1 \tau + \frac{u_3 \tau^2}{2T} \right) \right] \right. \\
& \left. - i \rho_{2v0} \frac{\sigma_{22}}{\sigma_0} u_2 (v_0 + k_0 \theta_0 \tau) \right\} \\
& \times \Psi_v \left(t, T, i \rho_{1v0} \frac{\sigma_{12}}{\sigma_0}, m_2^E(\tau'); k_0, \theta_0, \sigma_0, v_0 \right), \quad (27)
\end{aligned}$$

with:

$$m_2^C(\tau') = \left(-\frac{1}{2} \sigma_{11}^2 + i \rho_{1v1} \frac{\sigma_{11}}{\sigma_1} k_1 \right) \tau' - \frac{1}{2} (1 - \rho_{1v1}^2) \sigma_{11}^2 \tau'^2 + \frac{i}{T} \rho_{1v1} \frac{\sigma_{11}}{\sigma_1} u_3, \quad (28)$$

$$\begin{aligned}
m_2^E(\tau') = & \left(-\frac{1}{2} \sigma_{12}^2 - (\rho_{12} - \rho_{1v0} \rho_{2v0}) \sigma_{12} \sigma_{22} u_2 + i \rho_{1v0} \frac{\sigma_{12}}{\sigma_0} k_0 \right) \tau' \\
& - \frac{1}{2} (1 - \rho_{1v0}^2) \sigma_{12}^2 \tau'^2 \\
& + \left(-\frac{1}{2} \sigma_{22}^2 + i \rho_{2v0} \frac{\sigma_{22}}{\sigma_0} k_0 \right) u_2 - \frac{1}{2} (1 - \rho_{2v0}^2) \sigma_{22}^2 u_2^2 \\
& + \frac{i}{T} \rho_{1v0} \frac{\sigma_{12}}{\sigma_0} u_3. \quad (29)
\end{aligned}$$

The functions Ψ_r and Ψ_v are defined in Lemmas 1 and 2, respectively.

Proof. We provide the complete proof of Theorem 1 in seven steps, with particular attention to: (i) the consistency of Brownian motion notation across the model definition and the characteristic function decomposition; (ii) the integration-by-parts boundary terms that give rise to the prefactors in Blocks C, D, and E; and (iii) the Cholesky decompositions for each volatility block. The notational convention established in Section 2.2.

Step 1: Decomposition of the log-price processes.

Under the risk-neutral measure $\mathbb{Q}$, the log-price processes $X_{1t} = \ln S_{1t}$ and $X_{2t} = \ln S_{2t}$ satisfy:

$$\begin{aligned}
dX_{1t} = & \left(r_t - \frac{1}{2} \sigma_{11}^2 v_{1t} - \frac{1}{2} \sigma_{12}^2 v_{0t} - (\lambda_0 + \lambda_1) m_1 \right) dt \\
& + \sigma_{11} \sqrt{v_{1t}} dW_{1t}^{(1)} + \sigma_{12} \sqrt{v_{0t}} dW_{2t}^{(1)} + J_1 d(N_{0t} + N_{1t}), \quad (30)
\end{aligned}$$

$$\begin{aligned}
dX_{2t} = & \left(r_t - \frac{1}{2} \sigma_{21}^2 v_{2t} - \frac{1}{2} \sigma_{22}^2 v_{0t} - (\lambda_0 + \lambda_2) m_2 \right) dt \\
& + \sigma_{21} \sqrt{v_{2t}} dW_{1t}^{(2)} + \sigma_{22} \sqrt{v_{0t}} dW_{2t}^{(2)} + J_2 d(N_{0t} + N_{2t}). \quad (31)
\end{aligned}$$

The volatility factor dynamics are:

$$dv_{0t} = k_0(\theta_0 - v_{0t})dt + \sigma_0 \sqrt{v_{0t}} dW_{0t}^{(0)}, \quad (32)$$

$$dv_{1t} = k_1(\theta_1 - v_{1t})dt + \sigma_1 \sqrt{v_{1t}} dW_{0t}^{(1)}, \quad (33)$$

$$dv_{2t} = k_2(\theta_2 - v_{2t})dt + \sigma_2 \sqrt{v_{2t}} dW_{0t}^{(2)}, \quad (34)$$

and the short rate follows:

$$dr_t = k_r(\theta_r - r_t)dt + \sigma_r dW_{tr}. \quad (35)$$

We emphasize the following notation which is used consistently throughout the proof:

- $W_{1t}^{(1)}$ drives the idiosyncratic return channel of asset 1 (correlated with $W_{0t}^{(1)}$);

- $W_{2t}^{(1)}$ drives the common return channel of asset 1 (correlated with $W_{0t}^{(0)}$ and $W_{2t}^{(2)}$);
- $W_{1t}^{(2)}$ drives the idiosyncratic return channel of asset 2 (correlated with $W_{0t}^{(2)}$);
- $W_{2t}^{(2)}$ drives the common return channel of asset 2 (correlated with $W_{0t}^{(0)}$ and $W_{2t}^{(1)}$);
- $W_{0t}^{(i)}$ ($i = 0, 1, 2$) drives the volatility factor v_{it} .

Step 2: Continuous-part decomposition.

Define $\tau = T - t$. The continuous part of X_{1T} is:

$$\begin{aligned} X_{1T}^C = X_{1t} &+ \int_t^T r_s ds - \frac{1}{2} \sigma_{11}^2 \int_t^T v_{1s} ds - \frac{1}{2} \sigma_{12}^2 \int_t^T v_{0s} ds \\ &+ \sigma_{11} \int_t^T \sqrt{v_{1s}} dW_{1s}^{(1)} + \sigma_{12} \int_t^T \sqrt{v_{0s}} dW_{2s}^{(1)}. \end{aligned} \quad (36)$$

Similarly, the continuous part of X_{2T} is:

$$\begin{aligned} X_{2T}^C = X_{2t} &+ \int_t^T r_s ds - \frac{1}{2} \sigma_{21}^2 \int_t^T v_{2s} ds - \frac{1}{2} \sigma_{22}^2 \int_t^T v_{0s} ds \\ &+ \sigma_{21} \int_t^T \sqrt{v_{2s}} dW_{1s}^{(2)} + \sigma_{22} \int_t^T \sqrt{v_{0s}} dW_{2s}^{(2)}. \end{aligned} \quad (37)$$

The continuous part of the log-geometric average $Y_T = \frac{1}{T} \int_0^T X_{1s} ds$ is:

$$\begin{aligned} Y_T^C = \frac{t}{T} Y_t &+ \frac{\tau}{T} X_{1t} + \int_t^T \left(1 - \frac{s}{T}\right) r_s ds \\ &- \frac{\sigma_{11}^2}{2} \int_t^T \left(1 - \frac{s}{T}\right) v_{1s} ds - \frac{\sigma_{12}^2}{2} \int_t^T \left(1 - \frac{s}{T}\right) v_{0s} ds \\ &+ \sigma_{11} \int_t^T \left(1 - \frac{s}{T}\right) \sqrt{v_{1s}} dW_{1s}^{(1)} + \sigma_{12} \int_t^T \left(1 - \frac{s}{T}\right) \sqrt{v_{0s}} dW_{2s}^{(1)}. \end{aligned} \quad (38)$$

Step 3: Factorization of the continuous characteristic function.

Define $q_s = u_1 + u_3(1 - s/T)$. The continuous-part characteristic function is:

$$\phi^C(t, T, u, u_3; x, y, v, r) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds + iu_1 X_{1T}^C + iu_2 X_{2T}^C + iu_3 Y_T^C} \middle| \mathcal{F}_t \right]. \quad (39)$$

Substituting Equations (36)–(38) and using the independence structure among the volatility blocks (conditional on the common volatility v_0 and the short rate r), we obtain the factorization in Equation (22), where:

$$A = \exp \left\{ i \left(u_1 + \frac{\tau}{T} u_3 \right) X_{1t} + iu_2 X_{2t} + i \frac{t}{T} u_3 Y_t \right\}, \quad (40)$$

$$B = \mathbb{E}^{\mathbb{Q}} \left[\exp \left\{ i \int_t^T (u_1 + u_2 + u_3(1 - s/T) + i) r_s ds \right\} \middle| r_t = r \right], \quad (41)$$

and the expectations for the volatility factors are:

$$C = \mathbb{E}^{\mathbb{Q}} [\exp \{ \Pi_C \} \mid v_{1t} = v_1], \quad (42)$$

$$D = \mathbb{E}^{\mathbb{Q}} [\exp \{ \Pi_D \} \mid v_{2t} = v_2], \quad (43)$$

$$E = \mathbb{E}^{\mathbb{Q}} [\exp \{ \Pi_E \} \mid v_{0t} = v_0]. \quad (44)$$

The exponents Π_C , Π_D , and Π_E are:

$$\Pi_C = -\frac{1}{2}\sigma_{11}^2 \int_t^T q_s v_{1s} ds + i\sigma_{11} \int_t^T q_s \sqrt{v_{1s}} dW_{1s}^{(1)}, \quad (45)$$

$$\Pi_D = -\frac{1}{2}\sigma_{21}^2 u_2 \int_t^T v_{2s} ds + i\sigma_{21} u_2 \int_t^T \sqrt{v_{2s}} dW_{1s}^{(2)}, \quad (46)$$

$$\begin{aligned} \Pi_E = & -\frac{1}{2} \int_t^T (\sigma_{12}^2 q_s + \sigma_{22}^2 u_2) v_{0s} ds \\ & + i\sigma_{12} \int_t^T q_s \sqrt{v_{0s}} dW_{2s}^{(1)} + i\sigma_{22} u_2 \int_t^T \sqrt{v_{0s}} dW_{2s}^{(2)}. \end{aligned} \quad (47)$$

Step 4: Derivation of Block C (idiosyncratic volatility of S_1).

To evaluate C , we perform the Cholesky decomposition of the correlated Brownian motions driving the v_1 block. In the model specification, the idiosyncratic return channel of asset S_1 is driven by $W_{1t}^{(1)}$, while the volatility factor v_{1t} is driven by $W_{0t}^{(1)}$. These two Brownian motions are correlated with coefficient ρ_{1v1} :

$$W_{1t}^{(1)} = \rho_{1v1} W_{0t}^{(1)} + \sqrt{1 - \rho_{1v1}^2} W_{1t}^{(1)*}, \quad (48)$$

where $W_{1t}^{(1)*}$ is independent of $W_{0t}^{(1)}$.

Using the dynamics of v_1 in Equation (33), we have:

$$\sqrt{v_{1s}} dW_{0s}^{(1)} = \frac{1}{\sigma_1} dv_{1s} - \frac{k_1}{\sigma_1} (\theta_1 - v_{1s}) ds. \quad (49)$$

Substituting this and Equation (48) into Π_C in Equation (45) yields:

$$\begin{aligned} \Pi_C = & -\frac{1}{2}\sigma_{11}^2 \int_t^T q_s v_{1s} ds \\ & + i\rho_{1v1} \frac{\sigma_{11}}{\sigma_1} \int_t^T q_s [dv_{1s} - k_1 \theta_1 ds + k_1 v_{1s} ds] \\ & + i\sqrt{1 - \rho_{1v1}^2} \sigma_{11} \int_t^T q_s \sqrt{v_{1s}} dW_{1s}^{(1)*}. \end{aligned} \quad (50)$$

We now apply integration by parts to the stochastic integral $\int_t^T q_s dv_{1s}$. Noting that $q_T = u_1$, $q_t = u_1 + u_3 \tau / T$, and $dq_s = -(u_3 / T) ds$, we obtain:

$$\int_t^T q_s dv_{1s} = u_1 v_{1T} - \left(u_1 + \frac{u_3 \tau}{T}\right) v_{1t} + \frac{u_3}{T} \int_t^T v_{1s} ds. \quad (51)$$

Substituting Equation (51) into Equation (50) and collecting terms, we obtain:

$$\begin{aligned} \Pi_C = & i\rho_{1v1} \frac{\sigma_{11}}{\sigma_1} \left[-\left(u_1 + \frac{u_3 \tau}{T}\right) v_{1t} - k_1 \theta_1 \left(u_1 \tau + \frac{u_3 \tau^2}{2T}\right) \right] \\ & + i\rho_{1v1} \frac{\sigma_{11}}{\sigma_1} u_1 v_{1T} \\ & + \int_t^T \left[\left(-\frac{1}{2}\sigma_{11}^2 + i\rho_{1v1} \frac{\sigma_{11}}{\sigma_1} k_1\right) q_s - \frac{1}{2}(1 - \rho_{1v1}^2) \sigma_{11}^2 q_s^2 \right] v_{1s} ds \\ & + i\sqrt{1 - \rho_{1v1}^2} \sigma_{11} \int_t^T q_s \sqrt{v_{1s}} dW_{1s}^{(1)*}. \end{aligned} \quad (52)$$

Taking the conditional expectation $\mathbb{E}^{\mathbb{Q}}[\exp\{\Pi_C\} | v_{1t} = v_1]$:

- The first line of Equation (52) leaves the expectation and becomes the prefactor in Block C (Equation (25));
- The term $i\rho_{1v1}(\sigma_{11}/\sigma_1)u_1v_{1T}$ is absorbed into the moment-generating function Ψ_v via the m_1 parameter;
- The integral term is absorbed into Ψ_v via the $m_2^C(\tau')$ function;
- The final stochastic integral term vanishes under the conditional expectation because $W_{1s}^{(1)*}$ is independent of the volatility process and has zero mean.

Thus, applying Lemma 1, we obtain Block C exactly as in Equation (25), with $m_2^C(\tau')$ given in Equation (28). The key boundary term $-(u_1 + u_3\tau/T)v_{1t}$ from the integration-by-parts formula is correctly captured in the prefactor, which is a critical step for the accuracy of the characteristic function.

Step 5: Derivation of Block D (idiosyncratic volatility of S_2).

For Block D, the idiosyncratic return channel of asset S_2 is driven by $W_{1t}^{(2)}$, while the volatility factor v_{2t} is driven by $W_{0t}^{(2)}$. These two Brownian motions are correlated with coefficient ρ_{2v2} :

$$W_{1t}^{(2)} = \rho_{2v2}W_{0t}^{(2)} + \sqrt{1 - \rho_{2v2}^2} W_{1t}^{(2)*}, \quad (53)$$

where $W_{1t}^{(2)*}$ is independent of $W_{0t}^{(2)}$.

Using the dynamics of v_2 in Equation (34), we have:

$$\sqrt{v_{2s}} dW_{0s}^{(2)} = \frac{1}{\sigma_2} dv_{2s} - \frac{k_2}{\sigma_2} (\theta_2 - v_{2s}) ds. \quad (54)$$

Substituting this and Equation (53) into Π_D in Equation (46) yields:

$$\begin{aligned} \Pi_D = & -\frac{1}{2}\sigma_{21}^2 u_2 \int_t^T v_{2s} ds \\ & + i\rho_{2v2} \frac{\sigma_{21}}{\sigma_2} u_2 \int_t^T [dv_{2s} - k_2\theta_2 ds + k_2v_{2s} ds] \\ & + i\sqrt{1 - \rho_{2v2}^2} \sigma_{21} u_2 \int_t^T \sqrt{v_{2s}} dW_{1s}^{(2)*}. \end{aligned} \quad (55)$$

Applying integration by parts to $\int_t^T dv_{2s}$:

$$\int_t^T dv_{2s} = v_{2T} - v_{2t}. \quad (56)$$

Substituting Equation (56) into Equation (55) and collecting terms, we obtain:

$$\begin{aligned} \Pi_D = & -i\rho_{2v2} \frac{\sigma_{21}}{\sigma_2} u_2 (v_{2t} + k_2\theta_2\tau) \\ & + i\rho_{2v2} \frac{\sigma_{21}}{\sigma_2} u_2 v_{2T} \\ & + \int_t^T \left[\left(-\frac{1}{2}\sigma_{21}^2 + i\rho_{2v2} \frac{\sigma_{21}}{\sigma_2} k_2 \right) u_2 - \frac{1}{2}(1 - \rho_{2v2}^2) \sigma_{21}^2 u_2^2 \right] v_{2s} ds \\ & + i\sqrt{1 - \rho_{2v2}^2} \sigma_{21} u_2 \int_t^T \sqrt{v_{2s}} dW_{1s}^{(2)*}. \end{aligned} \quad (57)$$

Taking the conditional expectation, the prefactor is extracted, and the remaining terms are absorbed into Ψ_v via Lemma 1, yielding Block D exactly as in Equation (26). We emphasize that the Brownian motion $W_{1t}^{(2)}$ in the original SDE Equation (1) is correctly decomposed using Equation (53), ensuring notational consistency throughout the derivation.

Step 6: Derivation of Block E (common volatility v_0).

For Block E, the common return channels of both assets are driven by $W_{2t}^{(1)}$ and $W_{2t}^{(2)}$, which are correlated with each other and with the common volatility factor v_{0t} (driven by $W_{0t}^{(0)}$). We perform the joint Cholesky decomposition:

$$W_{2t}^{(1)} = \rho_{1v0} W_{0t}^{(0)} + \sqrt{1 - \rho_{1v0}^2} W_{2t}^{(1)*}, \quad (58)$$

$$W_{2t}^{(2)} = \rho_{2v0} W_{0t}^{(0)} + \rho_{12}^* W_{2t}^{(1)*} + \sqrt{1 - \rho_{2v0}^2 - (\rho_{12}^*)^2} W_{2t}^{(2)*}, \quad (59)$$

where $W_{0t}^{(0)}$, $W_{2t}^{(1)*}$, and $W_{2t}^{(2)*}$ are mutually independent, and

$$\rho_{12}^* = \frac{\rho_{12} - \rho_{1v0}\rho_{2v0}}{\sqrt{1 - \rho_{1v0}^2}}. \quad (60)$$

Here ρ_{12} denotes the correlation between the common return channels $W_{2t}^{(1)}$ and $W_{2t}^{(2)}$. This decomposition ensures that the covariance structure among the three Brownian motions is correctly preserved.

Using the dynamics of v_0 in Equation (32):

$$\sqrt{v_{0s}} dW_{0s}^{(0)} = \frac{1}{\sigma_0} dv_{0s} - \frac{k_0}{\sigma_0} (\theta_0 - v_{0s}) ds. \quad (61)$$

Substituting the Cholesky decompositions into Π_E in Equation (47), we obtain terms involving $\int_t^T q_s dv_{0s}$ and $\int_t^T dv_{0s}$. Applying integration by parts:

$$\int_t^T q_s dv_{0s} = u_1 v_{0T} - \left(u_1 + \frac{u_3 \tau}{T}\right) v_{0t} + \frac{u_3}{T} \int_t^T v_{0s} ds, \quad (62)$$

$$\int_t^T dv_{0s} = v_{0T} - v_{0t}. \quad (63)$$

Collecting all boundary terms that leave the expectation (terms involving v_{0t}) and terms absorbed into Ψ_v (terms involving v_{0T} and $\int_t^T v_{0s} ds$), we obtain:

$$\begin{aligned} \Pi_E = & i\rho_{1v0} \frac{\sigma_{12}}{\sigma_0} \left[-\left(u_1 + \frac{u_3 \tau}{T}\right) v_{0t} - k_0 \theta_0 \left(u_1 \tau + \frac{u_3 \tau^2}{2T}\right) \right] \\ & - i\rho_{2v0} \frac{\sigma_{22}}{\sigma_0} u_2 (v_{0t} + k_0 \theta_0 \tau) \\ & + \left(i\rho_{1v0} \frac{\sigma_{12}}{\sigma_0} u_1 + i\rho_{2v0} \frac{\sigma_{22}}{\sigma_0} u_2 \right) v_{0T} \\ & + \int_t^T m_2^E(\tau') v_{0s} ds \\ & + \text{independent stochastic integrals (vanishing under expectation),} \end{aligned} \quad (64)$$

where $m_2^E(\tau')$ is given in Equation (29). The terms involving v_{0T} and $\int_t^T v_{0s} ds$ are absorbed into Ψ_v via Lemma 2, yielding Block E exactly as in Equation (27). The prefactor collects the boundary terms $-(u_1 + u_3 \tau/T)v_{0t}$ and $-v_{0t}$ from the two integration-by-parts applications.

Step 7: Assembly of the characteristic function.

Multiplying the five factors A, B, C, D , and E from Equations (23)–(27) gives the complete continuous-part characteristic function in Equation (22). The product structure reflects the independence among the three volatility blocks after the Cholesky decomposition (con-

ditional on the common volatility v_0), and the boundary term treatment ensures that all prefactors are correctly accounted for.

This completes the proof of Theorem 1. $\square$

The characteristic function serves as the bridge linking the fundamental model parameters to option prices. The decomposition of the overall characteristic function into the product of five factors in Equation (21) carries a clear economic interpretation:

- **Factor A** depends only on the current log-prices and the accumulated geometric average, representing the component of the option payoff that is directly linked to the asset price level.
- **Factor B** captures the impact of stochastic interest rates on the option value. Under the risk-neutral pricing framework, the interest rate not only affects the discount factor but also influences option prices by altering the drift of the asset price process. Factor B reflects this dual effect through the generalized moment-generating function of the interest rate derived in Lemma 1.
- **Factors C, D, and E** correspond respectively to the contributions of the three volatility factors—the underlying asset's idiosyncratic volatility (v_1), the counterparty asset's idiosyncratic volatility (v_2), and the common market volatility (v_0)—to the option price. The product structure of these three factors conveys the economic insight that *volatility risk can be decomposed into two independent asset-specific volatility channels and one systemic volatility channel*, with the option price reflecting the combined impact of all three volatility shocks.

A practically important implication of this decomposition is that market participants can separately assess the marginal contribution of each type of volatility shock to the option value. For example, when the market panic index (VIX) surges, the contribution of the common volatility factor v_0 increases significantly—an insight that is difficult to extract from single-factor volatility models.

Lemma 4. Let N_t be a Poisson process with parameter λ . Then, under the given condition of $N_t = n$, the joint density of the n occurrence times $(T_1, T_2, \dots, T_n)$ of the events is

$$f(t_1, t_2, \dots, t_n) = \frac{n!}{t^n}, 0 < t_1 < t_2 < \dots < t_n \leq t.$$

That is, given that events have occurred times n within the interval $[0, t]$, the times $(T_1, T_2, \dots, T_n)$ when each event occurs (no sorting) are considered as independent random variables, and all of them are uniformly distributed on $[0, t]$.

Theorem 2. Suppose that the price mechanism of the market follows the Formulas (1)–(6), then a solution of $\varphi^J(t, T, u, u_3; x, y, v, r)$ is given by

$$\varphi_1^J(t, T, u_1, u_2, u_3) \cdot I_0(t, T, u_1, u_2, u_3) \cdot I_1(t, T, u_1, u_2, u_3) \cdot I_2(t, T, u_1, u_2, u_3), \quad (65)$$

where $\varphi_1^J(t, T, u_1, u_2, u_3) = \exp\{-iu_1(\lambda_0 + \lambda_1)m_1\tau - iu_2(\lambda_0 + \lambda_2)m_2\tau - \frac{1}{2T}iu_3(\lambda_0 + \lambda_1)m_1\tau^2\}$, $I_0(t, T, u_1, u_2, u_3) = \exp\{\lambda_0\tau \cdot (E^{\mathbb{Q}}[\Phi_{J_1J_2}(u_1 + u_3(1 - \frac{U}{T}), u_2)] - 1)\}$, $I_1(t, T, u_1, u_2, u_3) = \exp\{\lambda_1\tau \cdot (E^{\mathbb{Q}}[\Phi_{J_1}(u_1 + u_3(1 - \frac{U}{T}))] - 1)\}$, $I_2(t, T, u_1, u_2, u_3) = \exp\{\lambda_2\tau \cdot (\Phi_{J_2}(u_2) - 1)\}$.

Under Case 1. If (J_1, J_2) is subject to Joint normal distribution, then $\Phi_{J_1}(u) = e^{i\mu_1 u - \frac{1}{2}\sigma_{J_1}^2 u^2}$, $\Phi_{J_2}(u) = e^{i\mu_2 u - \frac{1}{2}\sigma_{J_2}^2 u^2}$, $E^{\mathbb{Q}}[\Phi_{J_1}(u_1 + u_3(1 - \frac{U}{T}))] = \frac{T\sqrt{2\pi}}{\sigma_{J_1} u_3 \tau} \cdot \exp\{\frac{-\mu_1^2}{2\sigma_{J_1}^2}\} \cdot (N(\sigma_{J_1}(u_1 +$

$$u_3 \frac{\tau}{T}) - \mathbf{i}\mu_1 \sigma_{J_1}^{-1}; 0, 1) - N(\sigma_{J_1} u_1 - \mathbf{i}\mu_1 \sigma_{J_1}^{-1}; 0, 1)), E^{\mathbb{Q}}[\Phi_{J_1 J_2}(u_1 + u_3(1 - \frac{U}{T}), u_2)] = \frac{T\sqrt{2\pi}}{\sigma_{J_1} u_3 \tau} \cdot \exp\left\{\frac{(\mathbf{i}\mu_1 - \rho_J \sigma_{J_1} \sigma_{J_2} u_2)^2}{2\sigma_{J_1}^2}\right\} \cdot (N(\sigma_{J_1}(u_1 + u_3 \frac{\tau}{T}) - (\mathbf{i}\mu_1 - \rho_J \sigma_{J_1} \sigma_{J_2} u_2) \sigma_{J_1}^{-1}; 0, 1) - N(\sigma_{J_1} u_1 - (\mathbf{i}\mu_1 - \rho_J \sigma_{J_1} \sigma_{J_2} u_2) \sigma_{J_1}^{-1}; 0, 1));$$

Under Case 2. If (J_1, J_2) is subject to independent asymmetric double exponential distribution, then $\Phi_{J_1}(u_1) = \frac{p_1 \xi_1}{\xi_1 + \mathbf{i}u_1} + \frac{q_1 \eta_1}{\eta_1 - \mathbf{i}u_1}$, $\Phi_{J_2}(u_2) = \frac{p_2 \xi_2}{\xi_2 + \mathbf{i}u_2} + \frac{q_2 \eta_2}{\eta_2 - \mathbf{i}u_2}$, $E^{\mathbb{Q}}[\Phi_{J_1}(u_1 + u_3(1 - \frac{U}{T}))] = \frac{T p_1 \xi_1}{\mathbf{i}u_3 \tau} \ln \frac{\xi_1 + \mathbf{i}(u_1 + u_3 \frac{\tau}{T})}{\xi_1 + \mathbf{i}u_1} - \frac{T q_1 \eta_1}{\mathbf{i}u_3 \tau} \ln \frac{\eta_1 - \mathbf{i}(u_1 + u_3 \frac{\tau}{T})}{\eta_1 - \mathbf{i}u_1}$, $E^{\mathbb{Q}}[\Phi_{J_1 J_2}(u_1 + u_3(1 - \frac{U}{T}), u_2)] = E^{\mathbb{Q}}[\Phi_{J_1}(u_1 + u_3(1 - \frac{U}{T}))] \cdot \Phi_{J_2}(u_2)$.

Proof. First, the continuous part of the characteristic function $\varphi^J(t, T, u, u_3; x, y, v, r)$ can be split into key terms,

$$\begin{aligned} & \varphi^J(t, T, u, u_3; x, y, v, r) \\ &= E^{\mathbb{Q}}[e^{\mathbf{i}u \cdot X_T^J + \mathbf{i}u_3 \cdot Y_T^J} | X_t = x, Y_t = y, v_t = v, r_t = r] \\ &= \exp\{-\mathbf{i}u_1 \tau (\lambda_0 + \lambda_1) m_1 - \mathbf{i}u_2 \tau (\lambda_0 + \lambda_2) m_2 - \frac{1}{2} \mathbf{i}u_3 \tau^2 (\lambda_0 + \lambda_1) m_1\} \\ & \quad \cdot E^{\mathbb{Q}}[\exp\{\sum_{i=N_{0t}+N_{1t}+1}^{N_{0T}+N_{1T}} (\mathbf{i}u_1 + \mathbf{i}u_3(1 - \frac{t_i}{T})) J_1 + \sum_{i=N_{0t}+N_{2t}+1}^{N_{0T}+N_{2T}} \mathbf{i}u_2 J_2\} | \mathcal{F}_t] \\ &= \varphi_1^J(t, T, u_1, u_2, u_3) \cdot \varphi_2^J(t, T, u_1, u_2, u_3) \end{aligned}$$

where $\varphi_1^J(t, T, u_1, u_2, u_3) = \exp\{-\mathbf{i}u_1 (\lambda_0 + \lambda_1) m_1 \tau - \mathbf{i}u_2 (\lambda_0 + \lambda_2) m_2 \tau - \frac{1}{2T} \mathbf{i}u_3 (\lambda_0 + \lambda_1) m_1 \tau^2\}$,

$$\varphi_2^J(t, T, u_1, u_2, u_3) = E^{\mathbb{Q}}[\exp\{\sum_{i=N_{0t}+N_{1t}+1}^{N_{0T}+N_{1T}} (\mathbf{i}u_1 + \mathbf{i}u_3(1 - \frac{t_i}{T})) J_1 + \sum_{i=N_{0t}+N_{2t}+1}^{N_{0T}+N_{2T}} \mathbf{i}u_2 J_2\} | \mathcal{F}_t].$$

Second, handle the calculation issues of two sub-item. Apart from the first item which has already been provided, let us calculate the second item,

$$\begin{aligned} & \varphi_2^J(t, T, u_1, u_2, u_3) \\ &= E^{\mathbb{Q}}[\exp\{\sum_{i=1}^{N_{0T-t}+N_{1T-t}} (\mathbf{i}u_1 + \mathbf{i}u_3(T - t_i)) J_1 + \sum_{i=1}^{N_{0T-t}+N_{2T-t}} \mathbf{i}u_2 J_2\}] \\ &= E^{\mathbb{Q}}[\exp\{\sum_{i=1}^{N_{0T-t}} ((\mathbf{i}u_1 + \mathbf{i}u_3(1 - \frac{t_i}{T})) J_1 + \mathbf{i}u_2 J_2)\} \cdot E^{\mathbb{Q}}[\exp\{\sum_{i=1}^{N_{1T-t}} (\mathbf{i}u_1 + \mathbf{i}u_3(1 - \frac{t_i}{T})) J_1\}] \\ & \quad \cdot E^{\mathbb{Q}}[\exp\{\sum_{i=1}^{N_{2T-t}} \mathbf{i}u_2 J_2\}]] \\ &= I_0(t, T, u_1, u_2, u_3) \cdot I_1(t, T, u_1, u_2, u_3) \cdot I_2(t, T, u_1, u_2, u_3). \end{aligned}$$

Now let us proceed to derive $I_1(t, T, u_1, u_2, u_3)$ and $I_2(t, T, u_1, u_2, u_3)$. Applying the conclusion of Lemma 4, the specific results are as follows:

$$\begin{aligned} I_1(t, T, u_1, u_2, u_3) &= E^{\mathbb{Q}}[\exp\{\sum_{i=1}^{N_{1T-t}} (\mathbf{i}u_1 + \mathbf{i}u_3(1 - \frac{t_i}{T})) J_1\}] \\ &= \exp\{\lambda_1 \tau \cdot (E^{\mathbb{Q}}[\Phi_{J_1}(u_1 + u_3(1 - \frac{U}{T}))] - 1)\}, \\ I_2(t, T, u_1, u_2, u_3) &= E^{\mathbb{Q}}[\exp\{\sum_{i=1}^{N_{2T-t}} \mathbf{i}u_2 J_2\}] \\ &= \exp\{\lambda_2 \tau \cdot (\Phi_{J_2}(u_2) - 1)\}. \end{aligned}$$

where $\Phi_{J_1}(u)$ and $\Phi_{J_2}(u)$ are the characteristic functions of the jump variables J_1, J_2 respectively.

Now let us proceed to derive $I_0(t, T, u_1, u_2, u_3)$. For ease of calculation, the relevant Poisson process decomposition work is carried out first. Applying the conclusion of Lemma 3, the specific results are as follows:

$$\begin{aligned}
 & E^{\mathbb{Q}}[\exp\{\sum_{i=1}^{N_{0T-t}} ((iu_1 + iu_3(1 - \frac{t_i}{T}))J_1 + iu_2J_2) | N_{0T-t} = n_0\}] \\
 = & E^{\mathbb{Q}}[E^{\mathbb{Q}}[\exp\{\sum_{i=1}^{n_0} ((iu_1 + iu_3(1 - \frac{t_i}{T}))J_1 + iu_2J_2) | t_i, i = 1, \dots, n_0\} | N_{0T-t} = n_0]] \\
 = & E^{\mathbb{Q}}[E^{\mathbb{Q}}[\prod_{i=1}^{n_0} \Phi_{J_1J_2}(u_1 + u_3(1 - \frac{t_i}{T}), u_2) | t_i, i = 1, \dots, n_0] | N_{0T-t} = n_0]] \\
 = & E^{\mathbb{Q}}[\prod_{i=1}^{n_0} E^{\mathbb{Q}}[\Phi_{J_1J_2}(u_1 + u_3(1 - \frac{U}{T}), u_2) | N_{0T-t} = n_0]] \\
 = & (E^{\mathbb{Q}}[\Phi_{J_1J_2}(u_1 + u_3(1 - \frac{U}{T}), u_2)])^{n_0}.
 \end{aligned}$$

Substituting it into its calculation, we get

$$\begin{aligned}
 I_0(t, T, u_1, u_2, u_3) &= E^{\mathbb{Q}}[\exp\{\sum_{i=1}^{N_{0T-t}} ((iu_1 + iu_3(1 - \frac{t_i}{T}))J_1 + iu_2J_2)\}] \\
 &= E^{\mathbb{Q}}[E^{\mathbb{Q}}[\exp\{\sum_{i=1}^{N_{0T-t}} ((iu_1 + iu_3(1 - \frac{t_i}{T}))J_1 + iu_2J_2)\} | N_{0T-t}]] \\
 &= E^{\mathbb{Q}}[(E^{\mathbb{Q}}[\Phi_{J_1J_2}(u_1 + u_3(1 - \frac{U}{T}), u_2)]^{N_{0T-t}})] \\
 &= \sum_{i=0}^{+\infty} (E^{\mathbb{Q}}[\Phi_{J_1J_2}(u_1 + u_3(1 - \frac{U}{T}), u_2)]^{n_0} \frac{(\lambda_0\tau)^{n_0}}{n!} e^{-\lambda_0\tau}) \\
 &= \exp\{\lambda_0\tau \cdot (E^{\mathbb{Q}}[\Phi_{J_1J_2}(u_1 + u_3(1 - \frac{U}{T}), u_2)] - 1)\}.
 \end{aligned}$$

Case 1. If (J_1, J_2) is subject to Joint normal distribution, then

$$\begin{aligned}
 \Phi_{J_1}(u) &= E^{\mathbb{Q}}[e^{iu \cdot J_1}] = e^{i\mu_1 u - \frac{1}{2}\sigma_{J_1}^2 u^2}, \\
 \Phi_{J_2}(u) &= E^{\mathbb{Q}}[e^{iu \cdot J_2}] = e^{i\mu_2 u - \frac{1}{2}\sigma_{J_2}^2 u^2}.
 \end{aligned}$$

Further derivation leads to

$$\begin{aligned}
 & E^{\mathbb{Q}}[\Phi_{J_1}(u_1 + u_3(1 - \frac{U}{T}))] \\
 = & E^{\mathbb{Q}}[\exp\{i\mu_1(u_1 + u_3(1 - \frac{U}{T})) - \frac{1}{2}\sigma_{J_1}^2(u_1 + u_3(1 - \frac{U}{T}))^2\}] \\
 = & \frac{1}{\tau} \int_t^T \exp\{i\mu_1(u_1 + u_3(1 - \frac{u}{T})) - \frac{1}{2}\sigma_{J_1}^2(u_1 + u_3(1 - \frac{u}{T}))^2\} du \\
 = & \frac{T}{u_3\tau} \int_{u_1}^{u_1+u_3\frac{\tau}{T}} \exp\{i\mu_1 v - \frac{1}{2}\sigma_{J_1}^2 v^2\} dv \\
 = & \frac{T\sqrt{2\pi}}{\sigma_{J_1} u_3\tau} \cdot \exp\{\frac{-\mu_1^2}{2\sigma_{J_1}^2}\} \cdot \int_{u_1}^{u_1+u_3\frac{\tau}{T}} \frac{1}{\sqrt{2\pi\sigma_{J_1}^{-1}}} \exp\{-\frac{(v-i\mu_1\sigma_{J_1}^{-2})^2}{2\sigma_{J_1}^{-2}}\} dv \\
 = & \frac{T\sqrt{2\pi}}{\sigma_{J_1} u_3\tau} \cdot \exp\{\frac{-\mu_1^2}{2\sigma_{J_1}^2}\} \cdot (N(u_1 + u_3\frac{\tau}{T}; i\mu_1\sigma_{J_1}^{-2}, \sigma_{J_1}^{-2}) - N(u_1; i\mu_1\sigma_{J_1}^{-2}, \sigma_{J_1}^{-2})) \\
 = & \frac{T\sqrt{2\pi}}{\sigma_{J_1} u_3\tau} \cdot \exp\{\frac{-\mu_1^2}{2\sigma_{J_1}^2}\} \cdot (N(\sigma_{J_1}(u_1 + u_3\frac{\tau}{T}) - i\mu_1\sigma_{J_1}^{-1}; 0, 1) - N(\sigma_{J_1}u_1 - i\mu_1\sigma_{J_1}^{-1}; 0, 1)).
 \end{aligned}$$

Finally, the calculation of the joint expectation is presented,

$$\begin{aligned}
 & E^{\mathbb{Q}}[\Phi_{J_1 J_2}(u_1 + u_3(1 - \frac{U}{T}), u_2)] \\
 &= E^{\mathbb{Q}}[\exp\{i\mu_1(u_1 + u_3(1 - \frac{U}{T})) + i\mu_2 u_2\} \\
 &\quad \cdot \exp\{-\frac{1}{2}\sigma_{J_1}^2(u_1 + u_3(1 - \frac{U}{T}))^2 - \frac{1}{2}\sigma_{J_2}^2 u_2^2 - \rho_J \sigma_{J_1} \sigma_{J_2}(u_1 + u_3(1 - \frac{U}{T}))u_2\}] \\
 &= \frac{1}{\tau} \int_t^T \exp\{i\mu_1(u_1 + u_3(1 - \frac{U}{T})) + i\mu_2 u_2\} \\
 &\quad \cdot \exp\{-\frac{1}{2}\sigma_{J_1}^2(u_1 + u_3(1 - \frac{U}{T}))^2 - \frac{1}{2}\sigma_{J_2}^2 u_2^2 - \rho_J \sigma_{J_1} \sigma_{J_2}(u_1 + u_3(1 - \frac{U}{T}))u_2\} du \\
 &= \frac{T}{u_3 \tau} \exp\{i\mu_2 u_2 - \frac{1}{2}\sigma_{J_2}^2 u_2^2\} \int_{u_1}^{u_1 + u_3 \frac{\tau}{T}} \exp\{i\mu_1 v - \frac{1}{2}\sigma_{J_1}^2 v^2 - \rho_J \sigma_{J_1} \sigma_{J_2} u_2 v\} dv \\
 &= \frac{T\sqrt{2\pi}}{\sigma_{J_1} u_3 \tau} \cdot \exp\{\frac{-\mu_1^2}{2\sigma_{J_1}^2}\} \cdot \int_{u_1}^{u_1 + u_3 \frac{\tau}{T}} \frac{1}{\sqrt{2\pi\sigma_{J_1}^{-2}}} \exp\{-\frac{(v - (i\mu_1 - \rho_J \sigma_{J_1} \sigma_{J_2} u_2)\sigma_{J_1}^{-2})^2}{2\sigma_{J_1}^{-2}}\} dv \\
 &= \frac{T\sqrt{2\pi}}{\sigma_{J_1} u_3 \tau} \cdot \exp\{\frac{(i\mu_1 - \rho_J \sigma_{J_1} \sigma_{J_2} u_2)^2}{2\sigma_{J_1}^2}\} \cdot (N(u_1 + u_3 \frac{\tau}{T}; (i\mu_1 - \rho_J \sigma_{J_1} \sigma_{J_2} u_2)\sigma_{J_1}^{-2}, \sigma_{J_1}^{-2}) \\
 &\quad - N(u_1; (i\mu_1 - \rho_J \sigma_{J_1} \sigma_{J_2} u_2)\sigma_{J_1}^{-2}, \sigma_{J_1}^{-2})) \\
 &= \frac{T\sqrt{2\pi}}{\sigma_{J_1} u_3 \tau} \cdot \exp\{\frac{(i\mu_1 - \rho_J \sigma_{J_1} \sigma_{J_2} u_2)^2}{2\sigma_{J_1}^2}\} \cdot (N(\sigma_{J_1}(u_1 + u_3 \frac{\tau}{T}) - (i\mu_1 - \rho_J \sigma_{J_1} \sigma_{J_2} u_2)\sigma_{J_1}^{-1}; 0, 1) \\
 &\quad - N(\sigma_{J_1} u_1 - (i\mu_1 - \rho_J \sigma_{J_1} \sigma_{J_2} u_2)\sigma_{J_1}^{-1}; 0, 1)).
 \end{aligned}$$

Case 2. If (J_1, J_2) is subject to independent asymmetric double exponential distribution, then

$$\begin{aligned}
 \Phi_{J_1}(u_1) &= E^{\mathbb{Q}}[e^{iu_1 J_1}] = \frac{p_1 \xi_1}{\xi_1 + iu_1} + \frac{q_1 \eta_1}{\eta_1 - iu_1}, \\
 \Phi_{J_2}(u_2) &= E^{\mathbb{Q}}[e^{iu_2 J_2}] = \frac{p_2 \xi_2}{\xi_2 + iu_2} + \frac{q_2 \eta_2}{\eta_2 - iu_2},
 \end{aligned}$$

Further derivation leads to

$$\begin{aligned}
 & E^{\mathbb{Q}}[\Phi_{J_1}(u_1 + u_3(1 - \frac{U}{T}))] \\
 &= E^{\mathbb{Q}}[\frac{p_1 \xi_1}{\xi_1 + i(u_1 + u_3(1 - \frac{U}{T}))} + \frac{q_1 \eta_1}{\eta_1 - i(u_1 + u_3(1 - \frac{U}{T}))}] \\
 &= \frac{1}{\tau} \int_t^T (\frac{p_1 \xi_1}{\xi_1 + i(u_1 + u_3(1 - \frac{U}{T}))} + \frac{q_1 \eta_1}{\eta_1 - i(u_1 + u_3(1 - \frac{U}{T}))}) du \\
 &= \frac{T}{u_3 \tau} \int_{u_1}^{u_1 + u_3 \frac{\tau}{T}} (\frac{p_1 \xi_1}{\xi_1 + iv} + \frac{q_1 \eta_1}{\eta_1 - iv}) dv \\
 &= \frac{T p_1 \xi_1}{i u_3 \tau} \ln \frac{\xi_1 + i(u_1 + u_3 \frac{\tau}{T})}{\xi_1 + iu_1} - \frac{T q_1 \eta_1}{i u_3 \tau} \ln \frac{\eta_1 - i(u_1 + u_3 \frac{\tau}{T})}{\eta_1 - iu_1}
 \end{aligned}$$

Finally, the calculation of the joint expectation is presented,

$$E^{\mathbb{Q}}[\Phi_{J_1 J_2}(u_1 + u_3(1 - \frac{U}{T}), u_2)] = E^{\mathbb{Q}}[\Phi_{J_1}(u_1 + u_3(1 - \frac{U}{T}))] \cdot \Phi_{J_2}(u_2).$$

The Theorem 2 has been proved. $\square$

3. Pricing Vulnerable Asian Options

In this section, we solve the problem of vulnerable Asian options pricing by characteristic function method. The essence of characteristic function theory is to apply inverse Fourier transform to deal with option pricing. In this paper, the payoff function $h_i(s_{1T}, s_{2T})$ of vulnerable Asian options at maturity T are investigated to value the corresponding options.

In the following, considering the extreme importance of hedge shares in option investing, the price and delta of vulnerable call fixed-strike Asian option is derived by four forms of Esscher transform in Theorem 3. The analytical expression of vulnerable put fixed-strike Asian option is summarized in Theorem 4. In the similar way, the value of vulnerable call floated-strike Asian option options is computed by four forms of Esscher transform in Theorem 5. Finally, the pricing formulas of vulnerable put floated-strike Asian option is proved by four forms of Esscher transform in Theorem 6.

Theorem 3. The price and delta at time $t \in [0, T]$ of vulnerable call fixed-strike Asian option under the above proposed models (Equations (1)–(7)), are given by

$$\begin{aligned}
 & V_1(t, T, x, y, v, r) \\
 = & \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, \ln K; \varphi^{P_3}(0, u_1, u_2)) \\
 & - K \cdot P(t, T) \cdot f(\ln D_0, \ln K; \varphi^{P_0}(0, u_1, u_2)) \\
 & + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, \ln K; \varphi^{P_4}(0, -u_1, u_2)) \\
 & - \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, \ln K; \varphi^{P_2}(0, -u_1, u_2)), \\
 & \Delta_1^{(1)}(t, T, x, y, v, r) \\
 = & e^{-x_1} \cdot \left[\frac{1}{T} \cdot \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, \ln K; \varphi^{P_3}(0, u_1, u_2)) \right. \\
 & + \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, \ln K; \mathbf{i}u_2 \frac{T}{T} \cdot \varphi^{P_3}(0, u_1, u_2)) - \frac{1}{4}) \\
 & - K \cdot P(t, T) \cdot (f(\ln D_0, \ln K; \mathbf{i}u_2 \frac{T}{T} \cdot \varphi^{P_0}(0, u_1, u_2)) - \frac{1}{4}) \\
 & + \frac{1-\alpha}{D} \cdot \frac{T}{T} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, \ln K; \varphi^{P_4}(0, -u_1, u_2)) \\
 & + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, \ln K; \mathbf{i}u_2 \frac{T}{T} \cdot \varphi^{P_4}(0, -u_1, u_2)) - \frac{1}{4}) \\
 & \left. - \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, \ln K; \mathbf{i}u_2 \frac{T}{T} \cdot \varphi^{P_2}(0, -u_1, u_2)) - \frac{1}{4}) \right], \\
 & \Delta_1^{(2)}(t, T, x, y, v, r) \\
 = & e^{-x_2} \cdot \left[\varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, \ln K; \mathbf{i}u_1 \varphi^{P_3}(0, u_1, u_2)) - \frac{1}{4}) \right. \\
 & - K \cdot P(t, T) \cdot (f(\ln D_0, \ln K; \mathbf{i}u_1 \varphi^{P_0}(0, u_1, u_2)) - \frac{1}{4}) \\
 & + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, \ln K; \varphi^{P_4}(0, -u_1, u_2)) \\
 & + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, \ln K; -\mathbf{i}u_1 \varphi^{P_4}(0, -u_1, u_2)) - \frac{1}{4}) \\
 & - \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, \ln K; \varphi^{P_2}(0, -u_1, u_2)) \\
 & \left. - \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, \ln K; -\mathbf{i}u_1 \varphi^{P_2}(0, -u_1, u_2)) - \frac{1}{4}) \right].
 \end{aligned} \tag{66}$$

$$\begin{aligned}
 \text{where } f(a, b, \varphi(u_1, u_2)) &= \frac{1}{4} + \frac{1}{2\pi} \int_0^{+\infty} \text{Re} \left[\frac{e^{-\mathbf{i}u_1 a} \varphi(u_1, 0)}{\mathbf{i}u_1} \right] du_1 + \frac{1}{2\pi} \int_0^{+\infty} \text{Re} \left[\frac{e^{-\mathbf{i}u_2 b} \varphi(0, u_2)}{\mathbf{i}u_2} \right] du_2 - \\
 & \frac{1}{2\pi^2} \int_0^{+\infty} \int_0^{+\infty} \text{Re} \left[\frac{e^{-\mathbf{i}u_1 a - \mathbf{i}u_2 b} \varphi(u_1, u_2) - e^{-\mathbf{i}u_1 a + \mathbf{i}u_2 b} \varphi(u_1, -u_2)}{u_1 u_2} \right] du_1 du_2, \quad \varphi^{P_0}(u, u_3) = \frac{\varphi(t, T, u_1, u_2, u_3; x, y, v, r)}{\varphi(t, T, 0, 0, 0; x, y, v, r)}, \\
 \varphi^{P_2}(u, u_3) &= \frac{\varphi(t, T, u_1, u_2, -\mathbf{i}, u_3; x, y, v, r)}{\varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r)}, \quad \varphi^{P_3}(u, u_3) = \frac{\varphi(t, T, u_1, u_2, u_3 - \mathbf{i}; x, y, v, r)}{\varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r)}, \\
 \varphi^{P_4}(u, u_3) &= \frac{\varphi(t, T, u_1, u_2, -\mathbf{i}, u_3 - \mathbf{i}; x, y, v, r)}{\varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r)}.
 \end{aligned}$$

Proof. According to the principle of risk neutral pricing and the payoff $h_1(S_{1T}, S_{2T}, G_T)$ at the maturity time T of vulnerable call fixed-strike Asian option, we have

$$\begin{aligned}
 & V_1(t, T, x, y, v, r) = E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot h_1(S_{1T}, S_{2T}, G_T) | \mathcal{F}_t] \\
 = & E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot G_T \cdot \mathbf{1}_{\{G_T \geq K, S_{2T} \geq D_0\}} | \mathcal{F}_t] - K \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot \mathbf{1}_{\{G_T \geq K, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\
 & + \frac{1-\alpha}{D} \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot G_T \cdot S_{2T} \cdot \mathbf{1}_{\{G_T \geq K, S_{2T} < D_0\}} | \mathcal{F}_t] \\
 & - \frac{(1-\alpha)K}{D} \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot S_{2T} \cdot \mathbf{1}_{\{G_T \geq K, S_{2T} < D_0\}} | \mathcal{F}_t].
 \end{aligned}$$

To proceed with the calculation, we define the following equivalent martingale measures P_0, P_2, P_3, P_4 of the risk-neutral measure, the Radon–Nikodym derivatives are established by

$$\begin{aligned}
 \frac{dP_0}{dQ} \Big|_{\mathcal{F}_t} &= \frac{e^{-\int_t^T r_s ds}}{E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} | \mathcal{F}_t]} = \frac{e^{-\int_t^T r_s ds}}{P(t, T)}, \quad \frac{dP_2}{dQ} \Big|_{\mathcal{F}_t} = \frac{e^{-\int_t^T r_s ds} \cdot S_{2T}}{S_{2t}}, \\
 \frac{dP_3}{dQ} \Big|_{\mathcal{F}_t} &= \frac{e^{-\int_t^T r_s ds} \cdot G_T}{E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot G_T | \mathcal{F}_t]}, \quad \frac{dP_4}{dQ} \Big|_{\mathcal{F}_t} = \frac{e^{-\int_t^T r_s ds} \cdot G_T \cdot S_{2T}}{E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot G_T \cdot S_{2T} | \mathcal{F}_t]},
 \end{aligned}$$

Next, the new joint characteristic functions under the new probability measure P_0, P_2, P_3, P_4 are given by

$$\begin{aligned}\varphi^{P_0}(u, u_3) &= E^{P_0}[e^{iu \cdot X_T + iu_3 \cdot Y_T} | \mathcal{F}_t] = E^{\mathbb{Q}}\left[\frac{e^{-\int_t^T r_s ds}}{P(t, T)} \cdot e^{iu \cdot X_T + iu_3 \cdot Y_T} | \mathcal{F}_t\right] \\ &= \frac{\varphi(t, T, u_1, u_2, u_3; x, y, v, r)}{\varphi(t, T, 0, 0, 0; x, y, v, r)}, \\ \varphi^{P_2}(u, u_3) &= E^{P_2}[e^{iu \cdot X_T + iu_3 \cdot Y_T} | \mathcal{F}_t] = E^{\mathbb{Q}}\left[\frac{e^{-\int_t^T r_s ds} \cdot S_{2T}}{S_{2t}} \cdot e^{iu \cdot X_T + iu_3 \cdot Y_T} | \mathcal{F}_t\right] \\ &= \frac{\varphi(t, T, u_1, u_2 - i, u_3; x, y, v, r)}{\varphi(t, T, 0, -i, 0; x, y, v, r)}, \\ \varphi^{P_3}(u, u_3) &= E^{P_3}[e^{iu \cdot X_T + iu_3 \cdot Y_T} | \mathcal{F}_t] = E^{\mathbb{Q}}\left[\frac{e^{-\int_t^T r_s ds} \cdot G_T}{E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot G_T | \mathcal{F}_t]} \cdot e^{iu \cdot X_T + iu_3 \cdot Y_T} | \mathcal{F}_t\right] \\ &= \frac{\varphi(t, T, u_1, u_2, u_3 - i; x, y, v, r)}{\varphi(t, T, 0, 0, -i; x, y, v, r)}, \\ \varphi^{P_4}(u, u_3) &= E^{P_4}[e^{iu \cdot X_T + iu_3 \cdot Y_T} | \mathcal{F}_t] = E^{\mathbb{Q}}\left[\frac{e^{-\int_t^T r_s ds} \cdot G_T \cdot S_{2T}}{E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot G_T \cdot S_{2T} | \mathcal{F}_t]} \cdot e^{iu \cdot X_T + iu_3 \cdot Y_T} | \mathcal{F}_t\right] \\ &= \frac{\varphi(t, T, u_1, u_2 - i, u_3 - i; x, y, v, r)}{\varphi(t, T, 0, -i, -i; x, y, v, r)}.\end{aligned}$$

Further, by virtue of the relationship between characteristic function and probability or the inverse Fourier transform, we obtain the final result

$$\begin{aligned}& V_1(t, T, x, y, v, r) \\ &= \varphi(t, T, 0, 0, -i; x, y, v, r) \cdot E^{P_3}[1_{\{G_T \geq K, S_{2T} \geq D_0\}} | \mathcal{F}_t] - K \cdot P(t, T) \cdot E^{P_0}[1_{\{G_T \geq K, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\ &\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -i, -i; x, y, v, r) \cdot E^{P_4}[1_{\{G_T \geq K, S_{2T} < D_0\}} | \mathcal{F}_t] \\ &\quad - \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -i, 0; x, y, v, r) \cdot E^{P_2}[1_{\{G_T \geq K, S_{2T} < D_0\}} | \mathcal{F}_t] \\ &= \varphi(t, T, 0, 0, -i; x, y, v, r) \cdot P_3\{G_T \geq K, S_{2T} \geq D_0 | \mathcal{F}_t\} \\ &\quad - K \cdot P(t, T) \cdot P_0\{G_T \geq K, S_{2T} \geq D_0 | \mathcal{F}_t\} \\ &\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -i, -i; x, y, v, r) \cdot P_4\{G_T \geq K, S_{2T} < D_0 | \mathcal{F}_t\} \\ &\quad - \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -i, 0; x, y, v, r) \cdot P_2\{G_T \geq K, S_{2T} < D_0 | \mathcal{F}_t\} \\ &= \varphi(t, T, 0, 0, -i; x, y, v, r) \cdot P_3\{y_T \geq \ln K, x_{2T} \geq \ln D_0 | \mathcal{F}_t\} \\ &\quad - K \cdot P(t, T) \cdot P_0\{y_T \geq \ln K, x_{2T} \geq \ln D_0 | \mathcal{F}_t\} \\ &\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -i, -i; x, y, v, r) \cdot P_4\{y_T \geq \ln K, x_{2T} < \ln D_0 | \mathcal{F}_t\} \\ &\quad - \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -i, 0; x, y, v, r) \cdot P_2\{y_T \geq \ln K, x_{2T} < \ln D_0 | \mathcal{F}_t\} \\ &= \varphi(t, T, 0, 0, -i; x, y, v, r) \cdot f(\ln D_0, \ln K; \varphi^{P_3}(0, u_1, u_2)) \\ &\quad - K \cdot P(t, T) \cdot f(\ln D_0, \ln K; \varphi^{P_0}(0, u_1, u_2)) \\ &\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -i, -i; x, y, v, r) \cdot f(-\ln D_0, \ln K; \varphi^{P_4}(0, -u_1, u_2)) \\ &\quad - \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -i, 0; x, y, v, r) \cdot f(-\ln D_0, \ln K; \varphi^{P_2}(0, -u_1, u_2)).\end{aligned}$$

Finally, contrasting this with the log-asset-price formula $x_i = \ln(S_{it}), i = 1, 2$ mentioned above, we can obtain shares of risk hedging

$$\begin{aligned}\Delta_1^{(1)}(t, T, x, y, v, r) &= \frac{\partial V_1(t, T, x, y, v, r)}{\partial S_{1t}} \\ &= \frac{1}{S_{1t}} \frac{\partial V_1(t, T, x, y, v, r)}{\partial x_1} = e^{-x_1} \frac{\partial V_1(t, x, v, \lambda, r; X_t)}{\partial x_1} \\ &= e^{-x_1} \cdot \left[\frac{\tau}{T} \cdot \varphi(t, T, 0, 0, -i; x, y, v, r) \cdot f(\ln D_0, \ln K; \varphi^{P_3}(0, u_1, u_2)) \right. \\ &\quad + \varphi(t, T, 0, 0, -i; x, y, v, r) \cdot (f(\ln D_0, \ln K; iu_2 \frac{\tau}{T} \cdot \varphi^{P_3}(0, u_1, u_2)) - \frac{1}{4}) \\ &\quad - K \cdot P(t, T) \cdot (f(\ln D_0, \ln K; iu_2 \frac{\tau}{T} \cdot \varphi^{P_0}(0, u_1, u_2)) - \frac{1}{4}) \\ &\quad + \frac{1-\alpha}{D} \cdot \frac{\tau}{T} \cdot \varphi(t, T, 0, -i, -i; x, y, v, r) \cdot f(-\ln D_0, \ln K; \varphi^{P_4}(0, -u_1, u_2)) \\ &\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -i, -i; x, y, v, r) \cdot (f(-\ln D_0, \ln K; iu_2 \frac{\tau}{T} \cdot \varphi^{P_4}(0, -u_1, u_2)) - \frac{1}{4}) \\ &\quad \left. - \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -i, 0; x, y, v, r) \cdot (f(-\ln D_0, \ln K; iu_2 \frac{\tau}{T} \cdot \varphi^{P_2}(0, -u_1, u_2)) - \frac{1}{4}) \right],\end{aligned}$$

$$\begin{aligned}
\Delta_1^{(2)}(t, T, x, y, v, r) &= \frac{\partial V_1(t, T, x, y, v, r)}{\partial S_{2t}} \\
&= \frac{1}{S_{2t}} \frac{\partial V_1(t, T, x, y, v, r)}{\partial x_2} = e^{-x_2} \frac{\partial V_1(t, T, x, y, v, r)}{\partial x_2} \\
&= e^{-x_2} \cdot [\varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, \ln K; \mathbf{i}u_1 \varphi^{P_3}(0, u_1, u_2)) - \frac{1}{4}) \\
&\quad - K \cdot P(t, T) \cdot (f(\ln D_0, \ln K; \mathbf{i}u_1 \varphi^{P_0}(0, u_1, u_2)) - \frac{1}{4}) \\
&\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, \ln K; \varphi^{P_4}(0, -u_1, u_2)) \\
&\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, \ln K; -\mathbf{i}u_1 \varphi^{P_4}(0, -u_1, u_2)) - \frac{1}{4}) \\
&\quad - \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, \ln K; \varphi^{P_2}(0, -u_1, u_2)) \\
&\quad - \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, \ln K; -\mathbf{i}u_1 \varphi^{P_2}(0, -u_1, u_2)) - \frac{1}{4})].
\end{aligned}$$

□

Remark 1. Assume that (X, Y) is a two-dimensional random variable with the joint characteristic function $\varphi(u_1, u_2)$ under the probability measure P , then we have

$$\begin{aligned}
P\{X > x, Y > y\} &= f(a, b, \varphi(u_1, u_2)) \\
&= \frac{1}{4} + \frac{1}{2\pi} \int_0^{+\infty} \operatorname{Re} \left[\frac{e^{-iu_1 a} \varphi(u_1, 0)}{iu_1} \right] du_1 + \frac{1}{2\pi} \int_0^{+\infty} \operatorname{Re} \left[\frac{e^{-iu_2 b} \varphi(0, u_2)}{iu_2} \right] du_2 \\
&\quad - \frac{1}{2\pi^2} \int_0^{+\infty} \int_0^{+\infty} \operatorname{Re} \left[\frac{e^{-iu_1 a - iu_2 b} \varphi(u_1, u_2) - e^{-iu_1 a + iu_2 b} \varphi(u_1, -u_2)}{u_1 u_2} \right] du_1 du_2.
\end{aligned}$$

The central difficulty in pricing vulnerable options lies in the fact that the payoff function simultaneously involves two random variables—the geometric average G_T and the counterparty asset value S_{2T} —along with two indicator conditions ($G_T \geq K$ and $S_{2T} \geq D_0$ or $< D_0$). Direct computation of the four conditional expectations is mathematically cumbersome.

The introduction of the four equivalent martingale measures P_0, P_2, P_3 , and P_4 serves to absorb the different random factors in the payoff function into the respective numéraires, thereby transforming each conditional expectation into a concise probability expression under the corresponding new measure. Specifically:

- **Measure P_0 :** This measure takes the risk-free bond as the numéraire and corresponds to standard risk-neutral pricing. It is used to handle pure probability terms without asset-price multipliers, such as $\mathbf{1}_{\{G_T \geq K, S_{2T} \geq D_0\}}$.
- **Measure P_2 :** This measure takes the counterparty asset S_{2T} as the numéraire and is used to handle the S_{2T} multiplier that appears in the payoff function. The economic rationale is that, upon default of the option writer, the recovery value is proportional to the counterparty asset value; incorporating the asset price into the numéraire thus simplifies the calculation.
- **Measure P_3 :** This measure takes the geometric average G_T as the numéraire and is used to handle the G_T multiplier in the payoff function. The economic rationale is that the first payoff term of a call option is the “present value of G_T ”; shifting this random factor into the numéraire substantially simplifies the expectation computation.
- **Measure P_4 :** This measure takes the product $G_T \cdot S_{2T}$ as the numéraire and is used to simultaneously handle both multipliers, corresponding to the full recovery payoff in the default state.

The practical value of this methodology lies in its decomposition of a complex four-dimensional conditional expectation problem into *four independent probability calculations of a unified form* (all expressed as $f(\cdot, \cdot, \phi(\cdot, \cdot))$). This decomposition not only substantially reduces the computational burden but also enables the marginal contribution of each risk source—interest rate risk, asset price risk, volatility risk, jump risk, and default risk—to

be separately identified and quantified—a capability that is unavailable through direct numerical integration.

Theorem 4. The price and delta at time $t \in [0, T]$ of vulnerable put fixed-strike Asian options under the above proposed models ((1)–(7)) are given by

$$\begin{aligned}
 & V_2(t, T, x, y, v, r) \\
 = & K \cdot P(t, T) \cdot f(\ln D_0, -\ln K; \varphi^{P_0}(0, u_1, -u_2)) \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, -\ln K; \varphi^{P_3}(0, u_1, -u_2)) \\
 & + \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, -\ln K; \varphi^{P_2}(0, -u_1, -u_2)) \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, -\ln K; \varphi^{P_4}(0, -u_1, -u_2)), \\
 & \Delta_2^{(1)}(t, T, x, y, v, r) \\
 = & e^{-x_1} \cdot [K \cdot P(t, T) \cdot (f(\ln D_0, -\ln K; -\mathbf{i}u_2 \frac{\tau}{T} \cdot \varphi^{P_0}(0, u_1, -u_2)) - \frac{1}{4}) \\
 & - \frac{\tau}{T} \cdot \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, -\ln K; \varphi^{P_3}(0, u_1, -u_2)) \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, -\ln K; -\mathbf{i}u_2 \frac{\tau}{T} \cdot \varphi^{P_3}(0, u_1, -u_2)) - \frac{1}{4}) \\
 & + \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, -\ln K; -\mathbf{i}u_2 \frac{\tau}{T} \cdot \varphi^{P_2}(0, -u_1, -u_2)) - \frac{1}{4}) \\
 & - \frac{1-\alpha}{D} \cdot \frac{\tau}{T} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, -\ln K; \varphi^{P_4}(0, -u_1, -u_2)) \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, -\ln K; -\mathbf{i}u_2 \frac{\tau}{T} \cdot \varphi^{P_4}(0, -u_1, -u_2)) - \frac{1}{4})], \\
 & \Delta_2^{(2)}(t, T, x, y, v, r) \\
 = & e^{-x_2} \cdot [K \cdot P(t, T) \cdot (f(\ln D_0, -\ln K; \mathbf{i}u_1 \varphi^{P_0}(0, u_1, -u_2)) - \frac{1}{4}) \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, -\ln K; \mathbf{i}u_1 \varphi^{P_3}(0, u_1, -u_2)) - \frac{1}{4}) \\
 & + \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, -\ln K; \varphi^{P_2}(0, -u_1, -u_2)) \\
 & + \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, -\ln K; -\mathbf{i}u_1 \varphi^{P_2}(0, -u_1, -u_2)) - \frac{1}{4}) \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, -\ln K; \varphi^{P_4}(0, -u_1, -u_2)) \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, -\ln K; -\mathbf{i}u_1 \varphi^{P_4}(0, -u_1, -u_2)) - \frac{1}{4})].
 \end{aligned} \tag{67}$$

$$\begin{aligned}
 \text{where } f(a, b, \varphi(u_1, u_2)) &= \frac{1}{4} + \frac{1}{2\pi} \int_0^{+\infty} \text{Re} \left[\frac{e^{-iu_1 a} \varphi(u_1, 0)}{iu_1} \right] du_1 + \frac{1}{2\pi} \int_0^{+\infty} \text{Re} \left[\frac{e^{-iu_2 b} \varphi(0, u_2)}{iu_2} \right] du_2 - \\
 & \frac{1}{2\pi^2} \int_0^{+\infty} \int_0^{+\infty} \text{Re} \left[\frac{e^{-iu_1 a - iu_2 b} \varphi(u_1, u_2) - e^{-iu_1 a + iu_2 b} \varphi(u_1, -u_2)}{u_1 u_2} \right] du_1 du_2, \quad \varphi^{P_0}(u, u_3) = \frac{\varphi(t, T, u_1, u_2, u_3; x, y, v, r)}{\varphi(t, T, 0, 0, 0; x, y, v, r)}, \\
 \varphi^{P_2}(u, u_3) &= \frac{\varphi(t, T, u_1, u_2, -\mathbf{i}, u_3; x, y, v, r)}{\varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r)}, \quad \varphi^{P_3}(u, u_3) = \frac{\varphi(t, T, u_1, u_2, u_3 - \mathbf{i}; x, y, v, r)}{\varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r)}, \\
 \varphi^{P_4}(u, u_3) &= \frac{\varphi(t, T, u_1, u_2, -\mathbf{i}, u_3 - \mathbf{i}; x, y, v, r)}{\varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r)}.
 \end{aligned}$$

Proof. According to the principle of risk neutral pricing and the payoff $h_2(S_{1T}, S_{2T}, G_T)$ at the maturity time T of vulnerable call fixed-strike Asian option, we have

$$\begin{aligned}
 V_2(t, T, x, y, v, r) &= E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot h_2(S_{1T}, S_{2T}, G_T) | \mathcal{F}_t] \\
 = & K \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot \mathbf{1}_{\{G_T < K, S_{2T} \geq D_0\}} | \mathcal{F}_t] - E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot G_T \cdot \mathbf{1}_{\{G_T < K, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\
 & + \frac{(1-\alpha)K}{D} \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot S_{2T} \cdot \mathbf{1}_{\{G_T < K, S_{2T} < D_0\}} | \mathcal{F}_t] \\
 & - \frac{1-\alpha}{D} \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot G_T \cdot S_{2T} \cdot \mathbf{1}_{\{G_T < K, S_{2T} < D_0\}} | \mathcal{F}_t].
 \end{aligned}$$

Further, by virtue of the relationship between characteristic function and probability or the inverse Fourier transform, we obtain the final result

$$\begin{aligned}
 & V_2(t, T, x, y, v, r) \\
 = & K \cdot P(t, T) \cdot E^{P_0}[1_{\{G_T < K, S_{2T} \geq D_0\}} | \mathcal{F}_t] - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot E^{P_3}[1_{\{G_T < K, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\
 & + \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot E^{P_2}[1_{\{G_T < K, S_{2T} < D_0\}} | \mathcal{F}_t] \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot E^{P_4}[1_{\{G_T < K, S_{2T} < D_0\}} | \mathcal{F}_t] \\
 = & K \cdot P(t, T) \cdot P_0\{G_T < K, S_{2T} \geq D_0 | \mathcal{F}_t\} \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot P_3\{G_T < K, S_{2T} \geq D_0 | \mathcal{F}_t\} \\
 & + \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot P_2\{G_T < K, S_{2T} < D_0 | \mathcal{F}_t\} \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot P_4\{G_T < K, S_{2T} < D_0 | \mathcal{F}_t\} \\
 = & K \cdot P(t, T) \cdot P_0\{y_T < \ln K, x_{2T} \geq \ln D_0 | \mathcal{F}_t\} \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot P_3\{y_T < \ln K, x_{2T} \geq \ln D_0 | \mathcal{F}_t\} \\
 & + \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot P_2\{y_T < \ln K, x_{2T} < \ln D_0 | \mathcal{F}_t\} \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot P_4\{y_T < \ln K, x_{2T} < \ln D_0 | \mathcal{F}_t\} \\
 = & K \cdot P(t, T) \cdot f(\ln D_0, -\ln K; \varphi^{P_0}(0, u_1, -u_2)) \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, -\ln K; \varphi^{P_3}(0, u_1, -u_2)) \\
 & + \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, -\ln K; \varphi^{P_2}(0, -u_1, -u_2)) \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, -\ln K; \varphi^{P_4}(0, -u_1, -u_2)).
 \end{aligned}$$

Finally, contrasting this with the log-asset-price formula $x_i = \ln(S_{it}), i = 1, 2$ mentioned above, we can obtain shares of risk hedging:

$$\begin{aligned}
 \Delta_2^{(1)}(t, T, x, y, v, r) &= \frac{\partial V_2(t, T, x, y, v, r)}{\partial S_{1t}} \\
 = & \frac{1}{S_{1t}} \frac{\partial V_2(t, T, x, y, v, r)}{\partial x_1} = e^{-x_1} \frac{\partial V_2(t, T, x, y, v, r)}{\partial x_1} \\
 = & e^{-x_1} \cdot [K \cdot P(t, T) \cdot (f(\ln D_0, -\ln K; -iu_2 \frac{\tau}{T} \cdot \varphi^{P_0}(0, u_1, -u_2)) - \frac{1}{4}) \\
 & - \frac{\tau}{T} \cdot \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, -\ln K; \varphi^{P_3}(0, u_1, -u_2)) \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, -\ln K; -iu_2 \frac{\tau}{T} \cdot \varphi^{P_3}(0, u_1, -u_2)) - \frac{1}{4}) \\
 & + \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, -\ln K; -iu_2 \frac{\tau}{T} \cdot \varphi^{P_2}(0, -u_1, -u_2)) - \frac{1}{4}) \\
 & - \frac{1-\alpha}{D} \cdot \frac{\tau}{T} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, -\ln K; \varphi^{P_4}(0, -u_1, -u_2)) \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, -\ln K; -iu_2 \frac{\tau}{T} \cdot \varphi^{P_4}(0, -u_1, -u_2)) - \frac{1}{4})], \\
 \Delta_2^{(2)}(t, T, x, y, v, r) &= \frac{\partial V_2(t, T, x, y, v, r)}{\partial S_{2t}} \\
 = & \frac{1}{S_{2t}} \frac{\partial V_2(t, T, x, y, v, r)}{\partial x_2} = e^{-x_2} \frac{\partial V_2(t, T, x, y, v, r)}{\partial x_2} \\
 = & e^{-x_2} \cdot [K \cdot P(t, T) \cdot (f(\ln D_0, -\ln K; iu_1 \varphi^{P_0}(0, u_1, -u_2)) - \frac{1}{4}) \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, -\ln K; iu_1 \varphi^{P_3}(0, u_1, -u_2)) - \frac{1}{4}) \\
 & + \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, -\ln K; \varphi^{P_2}(0, -u_1, -u_2)) \\
 & + \frac{(1-\alpha)K}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, -\ln K; -iu_1 \varphi^{P_2}(0, -u_1, -u_2)) - \frac{1}{4}) \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, -\ln K; \varphi^{P_4}(0, -u_1, -u_2)) \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, -\ln K; -iu_1 \varphi^{P_4}(0, -u_1, -u_2)) - \frac{1}{4})].
 \end{aligned}$$

□

Theorem 5. The price and delta at time $t \in [0, T]$ of vulnerable call floated-strike Asian options under the above proposed models ((1)–(7)) are given by

$$\begin{aligned}
& V_3(t, T, x, y, v, r) \\
= & \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_3}(-u_2, u_1, u_2)) \\
& - \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_1}(-u_2, u_1, u_2)) \\
& + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_4}(-u_2, -u_1, u_2)) \\
& - \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_5}(-u_2, -u_1, u_2)), \\
& \Delta_3^{(1)}(t, T, x, y, v, r) \\
= & e^{-x_1} \cdot [\frac{\tau}{T} \cdot \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_3}(-u_2, u_1, u_2)) \\
& + \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, \ln K; \mathbf{i}u_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_3}(-u_2, u_1, u_2)) - \frac{1}{4}) \\
& - \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_1}(-u_2, u_1, u_2)) \\
& - \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot (f(\ln D_0, 0; \mathbf{i}u_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_1}(-u_2, u_1, u_2)) - \frac{1}{4}) \\
& + \frac{1-\alpha}{D} \cdot \frac{\tau}{T} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_4}(-u_2, -u_1, u_2)) \\
& + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, 0; \mathbf{i}u_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_4}(-u_2, -u_1, u_2)) - \frac{1}{4}) \\
& - \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_5}(-u_2, -u_1, u_2)) \\
& - \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, 0; \mathbf{i}u_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_5}(-u_2, -u_1, u_2)) - \frac{1}{4})], \\
& \Delta_3^{(2)}(t, T, x, y, v, r) \\
= & e^{-x_2} \cdot [\varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, 0; \mathbf{i}u_1 \varphi^{P_3}(-u_2, u_1, u_2)) - \frac{1}{4}) \\
& - \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot (f(\ln D_0, 0; \mathbf{i}u_1 \varphi^{P_1}(-u_2, u_1, u_2)) - \frac{1}{4}) \\
& + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_4}(-u_2, -u_1, u_2)) \\
& + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, 0; -\mathbf{i}u_1 \varphi^{P_4}(-u_2, -u_1, u_2)) - \frac{1}{4}) \\
& - \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_5}(-u_2, -u_1, u_2)) \\
& - \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, 0; -\mathbf{i}u_1 \varphi^{P_5}(-u_2, -u_1, u_2)) - \frac{1}{4})].
\end{aligned} \tag{68}$$

$$\begin{aligned}
\text{where } f(a, b, \varphi(u_1, u_2)) &= \frac{1}{4} + \frac{1}{2\pi} \int_0^{+\infty} \text{Re}[\frac{e^{-\mathbf{i}u_1 a} \varphi(u_1, 0)}{\mathbf{i}u_1}] du_1 + \frac{1}{2\pi} \int_0^{+\infty} \text{Re}[\frac{e^{-\mathbf{i}u_2 b} \varphi(0, u_2)}{\mathbf{i}u_2}] du_2 - \\
&\frac{1}{2\pi^2} \int_0^{+\infty} \int_0^{+\infty} \text{Re}[\frac{e^{-\mathbf{i}u_1 a - \mathbf{i}u_2 b} \varphi(u_1, u_2) - e^{-\mathbf{i}u_1 a + \mathbf{i}u_2 b} \varphi(u_1, -u_2)}{u_1 u_2}] du_1 du_2, \varphi^{P_1}(u, u_3) = \frac{\varphi(t, T, u_1 - \mathbf{i}, u_2, u_3; x, y, v, r)}{\varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r)}, \\
\varphi^{P_5}(u, u_3) &= \frac{\varphi(t, T, u_1 - \mathbf{i}, u_2 - \mathbf{i}, u_3; x, y, v, r)}{\varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r)}.
\end{aligned}$$

Proof. According to the principle of risk neutral pricing and the payoff $h_3(S_{1T}, S_{2T}, G_T)$ at the maturity time T of vulnerable call floated-strike Asian options, we have

$$\begin{aligned}
V_3(t, T, x, y, v, r) &= E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot h_3(S_{1T}, S_{2T}, G_T) | \mathcal{F}_t] \\
= & E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot G_T \cdot \mathbf{1}_{\{G_T \geq S_{1T}, S_{2T} \geq D_0\}} | \mathcal{F}_t] - E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot S_{1T} \cdot \mathbf{1}_{\{G_T \geq S_{1T}, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\
& + \frac{1-\alpha}{D} \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot G_T \cdot S_{2T} \cdot \mathbf{1}_{\{G_T \geq S_{1T}, S_{2T} < D_0\}} | \mathcal{F}_t] \\
& - \frac{1-\alpha}{D} \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot S_{1T} \cdot S_{2T} \cdot \mathbf{1}_{\{G_T \geq S_{1T}, S_{2T} < D_0\}} | \mathcal{F}_t].
\end{aligned}$$

In order to further calculate, it is necessary to define the new equivalent martingale measures P_1, P_5 of the risk-neutral measure; the Radon–Nikodym derivatives are established by

$$\frac{dP_1}{dQ} \Big|_{\mathcal{F}_t} = \frac{e^{-\int_t^T r_s ds} \cdot S_{1T}}{S_{1t}}, \frac{dP_5}{dQ} \Big|_{\mathcal{F}_t} = \frac{e^{-\int_t^T r_s ds} \cdot S_{1T} \cdot S_{2T}}{E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot S_{1T} \cdot S_{2T} | \mathcal{F}_t]},$$

Next, the new joint characteristic functions under the new probability measure P_1, P_5 are given by

$$\begin{aligned}\varphi^{P_1}(u, u_3) &= E^{P_1}[e^{iu \cdot X_T + iu_3 \cdot Y_T} | \mathcal{F}_t] = E^{\mathbb{Q}}\left[\frac{e^{-\int_t^T r_s ds} \cdot S_{1T}}{S_{1t}} \cdot e^{iu \cdot X_T + iu_3 \cdot Y_T} | \mathcal{F}_t\right] \\ &= \frac{\varphi(t, T, u_1 - \mathbf{i}, u_2, u_3; x, y, v, r)}{\varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r)}, \\ \varphi^{P_5}(u, u_3) &= E^{P_5}[e^{iu \cdot X_T + iu_3 \cdot Y_T} | \mathcal{F}_t] = E^{\mathbb{Q}}\left[\frac{e^{-\int_t^T r_s ds} \cdot S_{1T} \cdot S_{2T}}{E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot S_{1T} \cdot S_{2T} | \mathcal{F}_t]} \cdot e^{iu \cdot X_T + iu_3 \cdot Y_T} | \mathcal{F}_t\right] \\ &= \frac{\varphi(t, T, u_1 - \mathbf{i}, u_2 - \mathbf{i}, u_3; x, y, v, r)}{\varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r)}.\end{aligned}$$

Further, by virtue of the relationship between characteristic function and probability or the inverse Fourier transform, we obtain the final result

$$\begin{aligned}& V_3(t, T, x, y, v, r) \\ &= \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot E^{P_3}[1_{\{G_T \geq S_{1T}, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\ &\quad - \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot E^{P_1}[1_{\{G_T \geq S_{1T}, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\ &\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot E^{P_4}[1_{\{G_T \geq S_{1T}, S_{2T} < D_0\}} | \mathcal{F}_t] \\ &\quad - \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot E^{P_5}[1_{\{G_T \geq S_{1T}, S_{2T} < D_0\}} | \mathcal{F}_t] \\ &= \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot P_3\{G_T \geq S_{1T}, S_{2T} \geq D_0 | \mathcal{F}_t\} \\ &\quad - \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot P_1\{G_T \geq S_{1T}, S_{2T} \geq D_0 | \mathcal{F}_t\} \\ &\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot P_4\{G_T \geq S_{1T}, S_{2T} < D_0 | \mathcal{F}_t\} \\ &\quad - \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot P_5\{G_T \geq S_{1T}, S_{2T} < D_0 | \mathcal{F}_t\} \\ &= \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot P_3\{y_T \geq x_{1T}, x_{2T} \geq \ln D_0 | \mathcal{F}_t\} \\ &\quad - \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot P_1\{y_T \geq x_{1T}, x_{2T} \geq \ln D_0 | \mathcal{F}_t\} \\ &\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot P_4\{y_T \geq x_{1T}, x_{2T} < \ln D_0 | \mathcal{F}_t\} \\ &\quad - \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot P_5\{y_T \geq x_{1T}, x_{2T} < \ln D_0 | \mathcal{F}_t\} \\ &= \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_3}(-u_2, u_1, u_2)) \\ &\quad - \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_1}(-u_2, u_1, u_2)) \\ &\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_4}(-u_2, -u_1, u_2)) \\ &\quad - \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_5}(-u_2, -u_1, u_2)).\end{aligned}$$

Finally, contrasting this with the log-asset-price formula $x_i = \ln(S_{it}), i = 1, 2$ mentioned above, we can obtain shares of risk hedging:

$$\begin{aligned}\Delta_3^{(1)}(t, T, x, y, v, r) &= \frac{\partial V_3(t, T, x, y, v, r)}{\partial S_{1t}} \\ &= \frac{1}{S_{1t}} \frac{\partial V_3(t, T, x, y, v, r)}{\partial x_1} = e^{-x_1} \frac{\partial V_3(t, T, x, y, v, r; X_t)}{\partial x_1} \\ &= e^{-x_1} \cdot \left[\frac{\tau}{T} \cdot \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_3}(-u_2, u_1, u_2)) \right. \\ &\quad + \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, \ln K; iu_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_3}(-u_2, u_1, u_2)) - \frac{1}{4}) \\ &\quad - \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_1}(-u_2, u_1, u_2)) \\ &\quad - \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot (f(\ln D_0, 0; iu_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_1}(-u_2, u_1, u_2)) - \frac{1}{4}) \\ &\quad + \frac{1-\alpha}{D} \cdot \frac{\tau}{T} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_4}(-u_2, -u_1, u_2)) \\ &\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, 0; iu_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_4}(-u_2, -u_1, u_2)) - \frac{1}{4}) \\ &\quad - \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_5}(-u_2, -u_1, u_2)) \\ &\quad \left. - \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, 0; iu_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_5}(-u_2, -u_1, u_2)) - \frac{1}{4}) \right],\end{aligned}$$

$$\begin{aligned}
\Delta_3^{(2)}(t, T, x, y, v, r) &= \frac{\partial V_3(t, T, x, y, v, r)}{\partial S_{2T}} \\
&= \frac{1}{S_{2T}} \frac{\partial V_3(t, T, x, y, v, r)}{\partial x_2} = e^{-x_2} \frac{\partial V_3(t, T, x, y, v, r)}{\partial x_2} \\
&= e^{-x_2} \cdot [\varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, 0; \mathbf{i}u_1 \varphi^{P_3}(-u_2, u_1, u_2)) - \frac{1}{4}) \\
&\quad - \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot (f(\ln D_0, 0; \mathbf{i}u_1 \varphi^{P_1}(-u_2, u_1, u_2)) - \frac{1}{4}) \\
&\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_4}(-u_2, -u_1, u_2)) \\
&\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, 0; -\mathbf{i}u_1 \varphi^{P_4}(-u_2, -u_1, u_2)) - \frac{1}{4}) \\
&\quad - \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_5}(-u_2, -u_1, u_2)) \\
&\quad - \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, 0; -\mathbf{i}u_1 \varphi^{P_5}(-u_2, -u_1, u_2)) - \frac{1}{4})].
\end{aligned}$$

□

Theorem 6. The price and delta at time $t \in [0, T]$ of vulnerable put floated-strike Asian option under the above proposed models ((1)–(7)), are given by

$$\begin{aligned}
&V_4(t, T, x, y, v, r) \\
&= \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_1}(u_2, u_1, -u_2)) \\
&\quad - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_3}(u_2, u_1, -u_2)) \\
&\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_5}(u_2, -u_1, -u_2)) \\
&\quad - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_4}(u_2, -u_1, -u_2)), \\
&\Delta_4^{(1)}(t, T, x, y, v, r) \\
&= e^{-x_1} \cdot [\varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_1}(u_2, u_1, -u_2)) \\
&\quad + \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot (f(\ln D_0, 0; -\mathbf{i}u_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_1}(u_2, u_1, -u_2)) - \frac{1}{4}) \\
&\quad - \frac{\tau}{T} \cdot \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_3}(u_2, u_1, -u_2)) \\
&\quad - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, \ln K; -\mathbf{i}u_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_3}(u_2, u_1, -u_2)) - \frac{1}{4}) \\
&\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_5}(u_2, -u_1, -u_2)) \\
&\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, 0; -\mathbf{i}u_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_5}(u_2, -u_1, -u_2)) - \frac{1}{4}) \\
&\quad - \frac{1-\alpha}{D} \cdot \frac{\tau}{T} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_4}(u_2, -u_1, -u_2)) \\
&\quad - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, 0; -\mathbf{i}u_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_4}(u_2, -u_1, -u_2)) - \frac{1}{4})], \\
&\Delta_4^{(2)}(t, T, x, y, v, r) \\
&= e^{-x_2} \cdot [\varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot (f(\ln D_0, 0; \mathbf{i}u_1 \varphi^{P_1}(u_2, u_1, -u_2)) - \frac{1}{4}) \\
&\quad - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, 0; \mathbf{i}u_1 \varphi^{P_3}(u_2, u_1, -u_2)) - \frac{1}{4}) \\
&\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_5}(u_2, -u_1, -u_2)) \\
&\quad + \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, 0; -\mathbf{i}u_1 \varphi^{P_5}(u_2, -u_1, -u_2)) - \frac{1}{4}) \\
&\quad - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_4}(u_2, -u_1, -u_2)) \\
&\quad - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, 0; -\mathbf{i}u_1 \varphi^{P_4}(u_2, -u_1, -u_2)) - \frac{1}{4})].
\end{aligned} \tag{69}$$

$$\begin{aligned}
\text{where } f(a, b, \varphi(u_1, u_2)) &= \frac{1}{4} + \frac{1}{2\pi} \int_0^{+\infty} \text{Re} \left[\frac{e^{-iu_1 a} \varphi(u_1, 0)}{\mathbf{i}u_1} \right] du_1 + \frac{1}{2\pi} \int_0^{+\infty} \text{Re} \left[\frac{e^{-iu_2 b} \varphi(0, u_2)}{\mathbf{i}u_2} \right] du_2 - \\
&\frac{1}{2\pi^2} \int_0^{+\infty} \int_0^{+\infty} \text{Re} \left[\frac{e^{-iu_1 a - iu_2 b} \varphi(u_1, u_2) - e^{-iu_1 a + iu_2 b} \varphi(u_1, -u_2)}{u_1 u_2} \right] du_1 du_2, \varphi^{P_1}(u, u_3) = \frac{\varphi(t, T, u_1 - \mathbf{i}, u_2, u_3; x, y, v, r)}{\varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r)}, \\
\varphi^{P_5}(u, u_3) &= \frac{\varphi(t, T, u_1 - \mathbf{i}, u_2 - \mathbf{i}, u_3; x, y, v, r)}{\varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r)}.
\end{aligned}$$

Proof. According to the principle of risk neutral pricing and the payoff $h_4(S_{1T}, S_{2T}, G_T)$ at the maturity time T of vulnerable put floated-strike Asian option, we have

$$\begin{aligned}
V_4(t, T, x, y, v, r) &= E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot h_4(S_{1T}, S_{2T}, G_T) | \mathcal{F}_t] \\
&= E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot S_{1T} \cdot \mathbf{1}_{\{G_T < S_{1T}, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\
&\quad - E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot G_T \cdot \mathbf{1}_{\{G_T < S_{1T}, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\
&\quad + \frac{1-\alpha}{D} \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot S_{1T} \cdot S_{2T} \cdot \mathbf{1}_{\{G_T < S_{1T}, S_{2T} < D_0\}} | \mathcal{F}_t] \\
&\quad - \frac{1-\alpha}{D} \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot G_T \cdot S_{2T} \cdot \mathbf{1}_{\{G_T < S_{1T}, S_{2T} < D_0\}} | \mathcal{F}_t].
\end{aligned}$$

Further, by virtue of the relationship between characteristic function and probability or the inverse Fourier transform, we obtain the final result

$$\begin{aligned}
 & V_4(t, T, x, y, v, r) \\
 = & \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot E^{P_1}[1_{\{G_T < S_{1T}, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot E^{P_3}[1_{\{G_T < S_{1T}, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\
 & + \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot E^{P_5}[1_{\{G_T < S_{1T}, S_{2T} < D_0\}} | \mathcal{F}_t] \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot E^{P_4}[1_{\{G_T < S_{1T}, S_{2T} < D_0\}} | \mathcal{F}_t] \\
 = & \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot P_1\{G_T < S_{1T}, S_{2T} \geq D_0 | \mathcal{F}_t\} \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot P_3\{G_T < S_{1T}, S_{2T} \geq D_0 | \mathcal{F}_t\} \\
 & + \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot P_5\{G_T < S_{1T}, S_{2T} < D_0 | \mathcal{F}_t\} \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot P_4\{G_T < S_{1T}, S_{2T} < D_0 | \mathcal{F}_t\} \\
 = & \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot P_1\{y_T < x_{1T}, x_{2T} \geq \ln D_0 | \mathcal{F}_t\} \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot P_3\{y_T < x_{1T}, x_{2T} \geq \ln D_0 | \mathcal{F}_t\} \\
 & + \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot P_5\{y_T < x_{1T}, x_{2T} < \ln D_0 | \mathcal{F}_t\} \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot P_4\{y_T < x_{1T}, x_{2T} < \ln D_0 | \mathcal{F}_t\} \\
 = & \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_1}(u_2, u_1, -u_2)) \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_3}(u_2, u_1, -u_2)) \\
 & + \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_5}(u_2, -u_1, -u_2)) \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_4}(u_2, -u_1, -u_2)).
 \end{aligned}$$

Finally, contrasting this with the log-asset-price formula $x_i = \ln(S_{it}), i = 1, 2$ mentioned above, we can obtain shares of risk hedging

$$\begin{aligned}
 \Delta_4^{(1)}(t, T, x, y, v, r) &= \frac{\partial V_4(t, T, x, y, v, r)}{\partial S_{1t}} \\
 = & \frac{1}{S_{1t}} \frac{\partial V_4(t, T, x, y, v, r)}{\partial x_1} = e^{-x_1} \frac{\partial V_4(t, T, x, y, v, r; X_t)}{\partial x_1} \\
 = & e^{-x_1} \cdot [\varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_1}(u_2, u_1, -u_2)) \\
 & + \varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot (f(\ln D_0, 0; -iu_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_1}(u_2, u_1, -u_2)) - \frac{1}{4}) \\
 & - \frac{\tau}{T} \cdot \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot f(\ln D_0, 0; \varphi^{P_3}(u_2, u_1, -u_2)) \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, \ln K; -iu_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_3}(u_2, u_1, -u_2)) - \frac{1}{4}) \\
 & + \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_5}(u_2, -u_1, -u_2)) \\
 & + \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, 0; -iu_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_5}(u_2, -u_1, -u_2)) - \frac{1}{4}) \\
 & - \frac{1-\alpha}{D} \cdot \frac{\tau}{T} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_4}(u_2, -u_1, -u_2)) \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, 0; -iu_2(\frac{\tau}{T} - 1) \cdot \varphi^{P_4}(u_2, -u_1, -u_2)) - \frac{1}{4})], \\
 \Delta_4^{(2)}(t, T, x, y, v, r) &= \frac{\partial V_4(t, T, x, y, v, r)}{\partial S_{2t}} \\
 = & \frac{1}{S_{2t}} \frac{\partial V_4(t, T, x, y, v, r)}{\partial x_2} = e^{-x_2} \frac{\partial V_4(t, T, x, y, v, r)}{\partial x_2} \\
 = & e^{-x_2} \cdot [\varphi(t, T, -\mathbf{i}, 0, 0; x, y, v, r) \cdot (f(\ln D_0, 0; iu_1 \varphi^{P_1}(u_2, u_1, -u_2)) - \frac{1}{4}) \\
 & - \varphi(t, T, 0, 0, -\mathbf{i}; x, y, v, r) \cdot (f(\ln D_0, 0; iu_1 \varphi^{P_3}(u_2, u_1, -u_2)) - \frac{1}{4}) \\
 & + \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_5}(u_2, -u_1, -u_2)) \\
 & + \frac{1-\alpha}{D} \cdot \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0; x, y, v, r) \cdot (f(-\ln D_0, 0; -iu_1 \varphi^{P_5}(u_2, -u_1, -u_2)) - \frac{1}{4}) \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot f(-\ln D_0, 0; \varphi^{P_4}(u_2, -u_1, -u_2)) \\
 & - \frac{1-\alpha}{D} \cdot \varphi(t, T, 0, -\mathbf{i}, -\mathbf{i}; x, y, v, r) \cdot (f(-\ln D_0, 0; -iu_1 \varphi^{P_4}(u_2, -u_1, -u_2)) - \frac{1}{4})].
 \end{aligned}$$

□

Remark 2. Although the semi-analytic expressions of bivariate options are obtained by the characteristic function Method in Theorems 3–6, the deduced pricing formula is quite complicated. On the one hand, integrand functions with characteristic function are difficult for global integrals. On the other hand, it is not easy to solve infinite generalized integrals with multiple variables. Therefore, it is necessary to consider simpler and more practical ways to solve bivariate options pricing problem.

In this paper, two-dimensional Fourier Cosine series expansion method is implemented to value the options in the next section.

4. Implementation of 2-D FFT Algorithm

In this section, we will try to construct a 2-D Fast Fourier Transform to solve the pricing problem of vulnerable reset options. Based on the research results in [25,31] about FFT in low dimension, this part will further extend and improve classical FFT to produce 2-D FFT algorithm.

Step 1. Decompose and revise the pricing formula of vulnerable reset options. According to Equations (7)–(10), we can obtain ($k = \ln K, d = \ln D_0$)

$$\begin{aligned}
 & V_1(t, T, x, y, v, r) \\
 = & E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot (G_T - K) \cdot 1_{\{G_T \geq K, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\
 & + \frac{1-\alpha}{D} \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot (G_T - K) \cdot S_{2T} \cdot 1_{\{G_T \geq K, S_{2T} < D_0\}} | \mathcal{F}_t] \\
 = & P(t, T) \cdot E^{P_0}[(e^{y_T} - e^k) \cdot 1_{\{y_T \geq k, x_{2T} \geq d\}} | \mathcal{F}_t] \\
 & + \frac{(1-\alpha)}{D} e^{x_2} \cdot E^{P_2}[(e^{y_T} - e^k) \cdot 1_{\{y_T \geq k, x_{2T} < d\}} | \mathcal{F}_t] \\
 = & P(t, T) \cdot G_{11}(k, d) + \frac{(1-\alpha)}{D} e^{x_2} \cdot G_{12}(k, d), \\
 & V_2(t, T, x, y, v, r) \\
 = & E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot (K - G_T) \cdot 1_{\{G_T < K, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\
 & + \frac{1-\alpha}{D} \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot (K - G_T) \cdot S_{2T} \cdot 1_{\{G_T < K, S_{2T} < D_0\}} | \mathcal{F}_t] \\
 = & P(t, T) \cdot E^{P_0}[(e^k - e^{y_T}) \cdot 1_{\{y_T < k, x_{2T} \geq d\}} | \mathcal{F}_t] \\
 & + \frac{(1-\alpha)}{D} e^{x_2} \cdot E^{P_2}[(e^k - e^{y_T}) \cdot 1_{\{y_T < k, x_{2T} < d\}} | \mathcal{F}_t] \\
 = & P(t, T) \cdot G_{21}(k, d) + \frac{(1-\alpha)}{D} e^{x_2} \cdot G_{22}(k, d); \\
 & V_3(t, T, x, y, v, r) \\
 = & E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot (G_T - S_{1T}) \cdot 1_{\{G_T \geq S_{1T}, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\
 & + \frac{1-\alpha}{D} \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot (G_T - S_{1T}) \cdot S_{2T} \cdot 1_{\{G_T \geq S_{1T}, S_{2T} < D_0\}} | \mathcal{F}_t] \\
 = & e^{x_1} \cdot E^{P_1}[(e^{y_T - x_{1T}} - e^0) \cdot 1_{\{y_T - x_{1T} \geq 0, x_{2T} \geq d\}} | \mathcal{F}_t] \\
 & + \frac{(1-\alpha)}{D} \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0) \cdot E^{P_5}[(e^{y_T - x_{1T}} - e^0) \cdot 1_{\{y_T - x_{1T} \geq 0, x_{2T} < d\}} | \mathcal{F}_t] \\
 = & e^{x_1} \cdot G_{31}(0, d) + \frac{(1-\alpha)}{D} \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0) \cdot G_{32}(0, d), \\
 & V_4(t, T, x, y, v, r) \\
 = & E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot (S_{1T} - G_T) \cdot 1_{\{G_T < S_{1T}, S_{2T} \geq D_0\}} | \mathcal{F}_t] \\
 & + \frac{1-\alpha}{D} \cdot E^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \cdot (S_{1T} - G_T) \cdot S_{2T} \cdot 1_{\{G_T < S_{1T}, S_{2T} < D_0\}} | \mathcal{F}_t] \\
 = & e^{x_1} \cdot E^{P_1}[(e^0 - e^{y_T - x_{1T}}) \cdot 1_{\{y_T - x_{1T} < 0, x_{2T} \geq d\}} | \mathcal{F}_t] \\
 & + \frac{(1-\alpha)}{D} \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0) \cdot E^{P_5}[(e^0 - e^{y_T - x_{1T}}) \cdot 1_{\{y_T - x_{1T} < 0, x_{2T} < d\}} | \mathcal{F}_t] \\
 = & e^{x_1} \cdot G_{41}(0, d) + \frac{(1-\alpha)}{D} \varphi(t, T, -\mathbf{i}, -\mathbf{i}, 0) \cdot G_{42}(0, d).
 \end{aligned}$$

In order to construct the algorithm, tool variables are introduced and the following new expected terms are defined as

$$\begin{aligned}\tilde{G}_{11}(k, d) &= e^{\alpha_1 k + \alpha_2 d} \cdot G_{11}(k, d) = e^{\alpha_1 k + \alpha_2 d} \cdot E^{P_0}[(e^{y_T} - e^k) \cdot 1_{\{y_T \geq k, x_{2T} \geq d\}} | \mathcal{F}_t], \\ \tilde{G}_{12}(k, d) &= e^{\alpha_1 k + \alpha_2 d} \cdot G_{12}(k, d) = e^{\alpha_1 k + \alpha_2 d} \cdot E^{P_2}[(e^{y_T} - e^k) \cdot 1_{\{y_T \geq k, x_{2T} < d\}} | \mathcal{F}_t], \\ \tilde{G}_{21}(k, d) &= e^{\alpha_1 k + \alpha_2 d} \cdot G_{21}(k, d) = e^{\alpha_1 k + \alpha_2 d} \cdot E^{P_0}[(e^k - e^{y_T}) \cdot 1_{\{y_T < k, x_{2T} \geq d\}} | \mathcal{F}_t], \\ \tilde{G}_{22}(k, d) &= e^{\alpha_1 k + \alpha_2 d} \cdot G_{22}(k, d) = e^{\alpha_1 k + \alpha_2 d} \cdot E^{P_2}[(e^k - e^{y_T}) \cdot 1_{\{y_T < k, x_{2T} < d\}} | \mathcal{F}_t]; \\ \tilde{G}_{31}(k, d) &= e^{\alpha_1 k + \alpha_2 d} \cdot G_{31}(k, d) = e^{\alpha_1 k + \alpha_2 d} \cdot E^{P_1}[(e^{y_T - x_{1T}} - e^k) \cdot 1_{\{y_T - x_{1T} \geq k, x_{2T} \geq d\}} | \mathcal{F}_t], \\ \tilde{G}_{32}(k, d) &= e^{\alpha_1 k + \alpha_2 d} \cdot G_{32}(k, d) = e^{\alpha_1 k + \alpha_2 d} \cdot E^{P_5}[(e^{y_T - x_{1T}} - e^k) \cdot 1_{\{y_T - x_{1T} \geq k, x_{2T} < d\}} | \mathcal{F}_t], \\ \tilde{G}_{41}(k, d) &= e^{\alpha_1 k + \alpha_2 d} \cdot G_{41}(k, d) = e^{\alpha_1 k + \alpha_2 d} \cdot E^{P_1}[(e^k - e^{y_T - x_{1T}}) \cdot 1_{\{y_T - x_{1T} < k, x_{2T} \geq d\}} | \mathcal{F}_t], \\ \tilde{G}_{42}(k, d) &= e^{\alpha_1 k + \alpha_2 d} \cdot G_{42}(k, d) = e^{\alpha_1 k + \alpha_2 d} \cdot E^{P_5}[(e^k - e^{y_T - x_{1T}}) \cdot 1_{\{y_T - x_{1T} < k, x_{2T} < d\}} | \mathcal{F}_t].\end{aligned}$$

It is clear that if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $l = k$, then $\tilde{G}_1(k, d, l) = G_1(k, d)$, $\tilde{G}_3(k, d, l) = G_3(k, d)$; if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $l = 0$, then $\tilde{G}_2(k, d, l) = G_2(k, d)$, $\tilde{G}_4(k, d, l) = G_4(k, d)$. Notice that $\alpha_i, i = 1, 2, 3, 4$ are the damping coefficients which ensure the existence of the Fourier transform of $\tilde{G}_i(k, d, l)$.

Step 2. Construct the Fourier transform of each component $\tilde{G}_{ij}(k, d) (i = 1, 2, 3, 4; j = 1, 2)$ in the option price. By the transformation formula and Fubini's theorem, we can derive

$$\begin{aligned}\tilde{g}_{11}(u_1, u_2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{iu_1 k + iu_2 d} \cdot \tilde{G}_{11}(k, d) dk dd \\ &= \frac{\varphi^{P_0}(t, T, 0, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, (\alpha_1 > 0, \alpha_2 > 0), \\ \tilde{g}_{12}(u_1, u_2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{iu_1 k + iu_2 d} \cdot \tilde{G}_{12}(k, d) dk dd \\ &= -\frac{\varphi^{P_2}(t, T, 0, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, (\alpha_1 > 0, \alpha_2 < 0); \\ \tilde{g}_{21}(u_1, u_2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{iu_1 k + iu_2 d} \cdot \tilde{G}_{21}(k, d) dk dd \\ &= \frac{\varphi^{P_0}(t, T, 0, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, (\alpha_1 < -1, \alpha_2 > 0), \\ \tilde{g}_{22}(u_1, u_2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{iu_1 k + iu_2 d} \cdot \tilde{G}_{22}(k, d) dk dd \\ &= -\frac{\varphi^{P_2}(t, T, 0, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, (\alpha_1 < -1, \alpha_2 < 0); \\ \tilde{g}_{31}(u_1, u_2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{iu_1 k + iu_2 d} \cdot \tilde{G}_{31}(k, d) dk dd \\ &= \frac{\varphi^{P_0}(t, T, -u_1 + i\alpha_1 + i, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, (\alpha_1 > 0, \alpha_2 > 0), \\ \tilde{g}_{32}(u_1, u_2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{iu_1 k + iu_2 d} \cdot \tilde{G}_{32}(k, d) dk dd \\ &= -\frac{\varphi^{P_2}(t, T, -u_1 + i\alpha_1 + i, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, (\alpha_1 > 0, \alpha_2 < 0); \\ \tilde{g}_{41}(u_1, u_2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{iu_1 k + iu_2 d} \cdot \tilde{G}_{41}(k, d) dk dd \\ &= \frac{\varphi^{P_0}(t, T, -u_1 + i\alpha_1 + i, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, (\alpha_1 < -1, \alpha_2 > 0), \\ \tilde{g}_{42}(u_1, u_2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{iu_1 k + iu_2 d} \cdot \tilde{G}_{42}(k, d) dk dd \\ &= -\frac{\varphi^{P_2}(t, T, -u_1 + i\alpha_1 + i, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, (\alpha_1 < -1, \alpha_2 < 0).\end{aligned}$$

According to the inverse Fourier transform formula, then we can obtain

$$\tilde{G}_{ij}(k, d) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-iu_1 k - iu_2 d} \cdot \tilde{g}_{ij}(u_1, u_2) du_1 du_2.$$

Step 3. Discretize of the triple integral. By the trapezoidal rule, the approximate formula is

$$\begin{aligned}\tilde{G}_{ij}(k, d) &\approx \frac{1}{(2\pi)^2} \int_{-\frac{A_1}{2}}^{\frac{A_1}{2}} \int_{-\frac{A_2}{2}}^{\frac{A_2}{2}} e^{-iu_1 k - iu_2 d} \cdot \tilde{g}_{ij}(u_1, u_2) du_1 du_2 \\ &\approx \frac{1}{(2\pi)^2} \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \omega(n_1, n_2) e^{-i \cdot u_{1n_1} \cdot k - i \cdot u_{2n_2} \cdot d} \cdot \tilde{g}_{ij}(u_{1n_1}, u_{2n_2}) \Delta u_1 \Delta u_2,\end{aligned}$$

where $u_{in_i} = n_i \Delta u_i - \frac{A_i}{2}$, $\Delta u_i = \frac{A_i}{N_i}$, $i = 1, 2, 3$. If $n_1 = 0, N_1 - 1$ or $n_2 = 0, N_2 - 1$, then $\omega(n_1, n_2) = \frac{1}{2}$; otherwise, $\omega(n_1, n_2) = 1$.

In this step, the approximate calculation of the option price is complete.

Step 4. Derive the discrete estimate of $\frac{\partial \tilde{G}_{ij}(k,d)}{\partial x_1}$, $\frac{\partial \tilde{G}_{ij}(k,d)}{\partial x_2}$. By the derivative rule for the integral with parameters, we have

$$\begin{aligned} \frac{\partial \tilde{g}_{11}}{\partial x_1} &= \frac{\frac{\tau}{T} \varphi^{P_0}(t, T, 0, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, \quad \frac{\partial \tilde{g}_{11}}{\partial x_2} = \frac{\varphi^{P_0}(t, T, 0, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1)}, \\ \frac{\partial \tilde{g}_{12}}{\partial x_1} &= -\frac{\frac{\tau}{T} \varphi^{P_2}(t, T, 0, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, \quad \frac{\partial \tilde{g}_{12}}{\partial x_2} = -\frac{\varphi^{P_2}(t, T, 0, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1)}, \\ \frac{\partial \tilde{g}_{21}}{\partial x_1} &= \frac{\frac{\tau}{T} \varphi^{P_0}(t, T, 0, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, \quad \frac{\partial \tilde{g}_{21}}{\partial x_2} = \frac{\varphi^{P_0}(t, T, 0, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1)}, \\ \frac{\partial \tilde{g}_{22}}{\partial x_1} &= -\frac{\frac{\tau}{T} \varphi^{P_2}(t, T, 0, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, \quad \frac{\partial \tilde{g}_{22}}{\partial x_2} = -\frac{\varphi^{P_2}(t, T, 0, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1)}, \\ \frac{\partial \tilde{g}_{31}}{\partial x_1} &= \frac{(\frac{\tau}{T} - 1) \varphi^{P_0}(t, T, -u_1 + i\alpha_1 + i, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, \quad \frac{\partial \tilde{g}_{31}}{\partial x_2} = \frac{\varphi^{P_0}(t, T, -u_1 + i\alpha_1 + i, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1)}, \\ \frac{\partial \tilde{g}_{32}}{\partial x_1} &= -\frac{(\frac{\tau}{T} - 1) \varphi^{P_2}(t, T, -u_1 + i\alpha_1 + i, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, \quad \frac{\partial \tilde{g}_{32}}{\partial x_2} = -\frac{\varphi^{P_2}(t, T, -u_1 + i\alpha_1 + i, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1)}, \\ \frac{\partial \tilde{g}_{41}}{\partial x_1} &= \frac{(\frac{\tau}{T} - 1) \varphi^{P_0}(t, T, -u_1 + i\alpha_1 + i, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, \quad \frac{\partial \tilde{g}_{41}}{\partial x_2} = \frac{\varphi^{P_0}(t, T, -u_1 + i\alpha_1 + i, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1)}, \\ \frac{\partial \tilde{g}_{42}}{\partial x_1} &= -\frac{(\frac{\tau}{T} - 1) \varphi^{P_2}(t, T, -u_1 + i\alpha_1 + i, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1) \cdot (iu_2 + \alpha_2)}, \quad \frac{\partial \tilde{g}_{42}}{\partial x_2} = -\frac{\varphi^{P_2}(t, T, -u_1 + i\alpha_1 + i, u_2 - i\alpha_2, u_1 - i\alpha_1 - i)}{(iu_1 + \alpha_1 + 1) \cdot (iu_1 + \alpha_1)}. \end{aligned}$$

Similarly, the approximate formula is

$$\begin{aligned} &\frac{\partial \tilde{G}_{ij}(k,d)}{\partial x_1} \\ &\approx \frac{1}{(2\pi)^2} \int_{-\frac{A_1}{2}}^{\frac{A_1}{2}} \int_{-\frac{A_2}{2}}^{\frac{A_2}{2}} e^{-iu_1 k - iu_2 d} \cdot \frac{\partial \tilde{g}_{ij}(u_1, u_2)}{\partial x_1} du_1 du_2 \\ &\approx \frac{1}{(2\pi)^2} \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \omega(n_1, n_2) \cdot e^{-i \cdot u_{1n_1} \cdot k - i \cdot u_{2n_2} \cdot d} \cdot \frac{\partial \tilde{g}_{ij}(u_{1n_1}, u_{2n_2})}{\partial x_1} \Delta u_1 \Delta u_2, \\ &\frac{\partial \tilde{G}_{ij}(k,d)}{\partial x_2} \\ &\approx \frac{1}{(2\pi)^2} \int_{-\frac{A_1}{2}}^{\frac{A_1}{2}} \int_{-\frac{A_2}{2}}^{\frac{A_2}{2}} e^{-iu_1 k - iu_2 d} \cdot \frac{\partial \tilde{g}_{ij}(u_1, u_2)}{\partial x_2} du_1 du_2 \\ &\approx \frac{1}{(2\pi)^2} \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \omega(n_1, n_2) \cdot e^{-i \cdot u_{1n_1} \cdot k - i \cdot u_{2n_2} \cdot d} \cdot \frac{\partial \tilde{g}_{ij}(u_{1n_1}, u_{2n_2})}{\partial x_2} \Delta u_1 \Delta u_2. \end{aligned}$$

Step 5. Derive the discrete estimate of Δ . By the rule for differentiating terms by terms, we have

$$\begin{aligned} \Delta_1^{(1)}(t, T, x, y, v, r) &= P(t, T) \cdot e^{-x_1} \cdot \frac{\partial G_{11}(k,d)}{\partial x_1} + \frac{(1-\alpha)}{D} \cdot e^{x_2 - x_1} \cdot \frac{\partial G_{12}(k,d)}{\partial x_1}, \\ \Delta_1^{(2)}(t, T, x, y, v, r) &= P(t, T) \cdot e^{-x_2} \cdot \frac{\partial G_{11}(k,d)}{\partial x_2} + \frac{(1-\alpha)}{D} \cdot G_{12}(k, d) + \frac{(1-\alpha)}{D} \cdot \frac{\partial G_{12}(k,d)}{\partial x_2}; \\ \Delta_2^{(1)}(t, T, x, y, v, r) &= P(t, T) \cdot e^{-x_1} \cdot \frac{\partial G_{21}(k,d)}{\partial x_1} + \frac{(1-\alpha)}{D} \cdot e^{x_2 - x_1} \cdot \frac{\partial G_{22}(k,d)}{\partial x_1}, \\ \Delta_2^{(2)}(t, T, x, y, v, r) &= P(t, T) \cdot e^{-x_2} \cdot \frac{\partial G_{21}(k,d)}{\partial x_2} + \frac{(1-\alpha)}{D} \cdot G_{12}(k, d) + \frac{(1-\alpha)}{D} \cdot \frac{\partial G_{22}(k,d)}{\partial x_2}; \\ \Delta_3^{(1)}(t, T, x, y, v, r) &= G_{31}(0, d) + \frac{\partial G_{31}(0,d)}{\partial x_1} \\ &\quad + \frac{(1-\alpha)}{D} \cdot e^{-x_1} \cdot \varphi(t, T, -i, -i, 0) \cdot (G_{32}(0, d) + \frac{\partial G_{32}(0,d)}{\partial x_1}), \\ \Delta_3^{(2)}(t, T, x, y, v, r) &= e^{x_1 - x_2} \cdot \frac{\partial G_{31}(0,d)}{\partial x_2} \\ &\quad + \frac{(1-\alpha)}{D} \cdot e^{-x_2} \cdot \varphi(t, T, -i, -i, 0) \cdot (G_{32}(0, d) + \frac{\partial G_{32}(0,d)}{\partial x_2}); \\ \Delta_4^{(1)}(t, T, x, y, v, r) &= G_{41}(0, d) + \frac{\partial G_{41}(0,d)}{\partial x_1} \\ &\quad + \frac{(1-\alpha)}{D} \cdot e^{-x_1} \cdot \varphi(t, T, -i, -i, 0) \cdot (G_{42}(0, d) + \frac{\partial G_{42}(0,d)}{\partial x_1}), \\ \Delta_4^{(2)}(t, T, x, y, v, r) &= e^{x_1 - x_2} \cdot \frac{\partial G_{41}(0,d)}{\partial x_2} \\ &\quad + \frac{(1-\alpha)}{D} \cdot e^{-x_2} \cdot \varphi(t, T, -i, -i, 0) \cdot (G_{42}(0, d) + \frac{\partial G_{42}(0,d)}{\partial x_2}). \end{aligned}$$

Combining the relationship between $G_{ij}(k, d)$ and $\tilde{G}_{ij}(k, d)$ ($i = 1, 2, 3, 4$; $j = 1, 2$), the approximate expression of $\Delta_i(t, t_1; x_1, x_2, v, r)$ can be obtained immediately.

5. Numerical Experiments and Validation

In this section, we conduct a comprehensive numerical study to validate the proposed 2D-FFT pricing framework. Our analysis is organized into five parts: (i) a description of the experimental setup; (ii) a Monte Carlo benchmarking study with properly defined error metrics and confidence intervals; (iii) a degeneracy test that verifies the correctness of our method by reducing the general model to known special cases; (iv) a computational efficiency and scalability comparison against Monte Carlo simulation; and (v) a convergence and grid-sensitivity analysis. All experiments are performed on a standard workstation (Intel Core i7-10700, 16 GB RAM) using MATLAB R2022a. In all the subsequent numerical experiments, unless otherwise specified, the constant-coefficient Vasicek model is employed.

5.1. Experimental Setup

In the first subsection, we will first present the basic ideas and thoughts of the model parameter setting and selection, as well as the specific parameter values.

Table 1 mainly presents the basic parameters of the multi-factor stochastic volatility jump-diffusion model, including the initial values of all asset prices, historical means, and related structural parameters, as well as the construction parameters of vulnerable geometric average Asian options. The data in this section is derived from the core parameters of vulnerable geometric average Asian options under the classical geometric Brownian motion in reference [14]. Table 2 shows the structural parameters of multi-factor stochastic volatility, including common volatility parameters, individual volatility parameters, and mean reversion parameters of interest rates. The parameter settings in this part are mainly based on a large number of empirical results, such as the time-varying nature, heavy-tailedness, and smile properties of volatility. The unique construction of this paper is to decompose the volatility of assets into the systematic volatility of the market (the systemic component of the financial market) and the individual volatility of the asset (the individual characteristic component). Table 3 sets the structural parameters of the dependent jumps of assets, including common jump parameters and individual jump parameters. The parameters in this part mainly come from previous research results [23,24]. To describe the linear and nonlinear risk characteristics of asset price jumps in the financial market, two structural parameters are given here: the dependent joint normal distribution and the dependent joint double-exponential distribution. Next, in the numerical experiments, the different transmission mechanisms of these two types of jump structures on option prices will be compared and analyzed, further increasing the diversity of option price analysis.

Table 1. Underlying parameters for the numerical.

Parm	S_{1t}	S_{2t}	G_t	v_0	v_1	v_2	r	σ_{12}	σ_{11}	σ_{22}	σ_{12}
Value	40	100	40	0.5	0.1	0.2	0.03	0.2	0.3	0.2	0.3
Parm	λ_0	λ_1	λ_2	K	T	t	D_0	D	α		
Value	2	3	3	40	1	0.5	90	90	0.25		
Parm	ρ_{12}	ρ_{1v_0}	ρ_{2v_0}	ρ_{1v_1}	ρ_{2v_2}						
Value	0.25	-0.5	0.5	0.5	-0.5						

Table 2. The parameters for stochastic volatility and stochastic interest rate.

Parm	k_0	θ_0	σ_0	k_1	θ_1	σ_1	k_2	θ_2	σ_2
Value	3	0.5	0.1	1	0.1	0.1	2	0.1	0.1
Parm	k_r	θ_r	σ_r						
Value	1.5	0.05	0.1						

Table 3. The structure parameters for two kind jump distributions.

Parm	μ_1	μ_2	σ_{J_1}	σ_{J_2}	ρ_J	
Value	0	0	0.1	0.2	0.5	
Parm	p_1	p_2	ξ_1	ξ_2	η_1	η_2
Value	0.6	0.64	10	12	15	18

5.2. Monte Carlo Benchmarking

In the following section, we will first compare the convergence and stability of the 2D Fast Fourier Transform through the Monte Carlo algorithm. To ensure the convergence of the Monte Carlo algorithm, in this experiment, we use as many simulation iterations as possible and as fine a step size for the random walk as possible, setting $N = 10^7$, $\frac{T-t}{\Delta t} = 1000$. At the same time, to ensure the scientificity and fairness of the comparison between the 2D Fast Fourier Transform and it, the step size of the Fast Fourier Transform is the same as that of the Monte Carlo algorithm, and the range of the truncated integration is $A_1 = A_2 = 1000$. Furthermore, since this paper considers four types of vulnerable geometric average Asian options, considering the repeatability of numerical analysis, we select the call-type fixed-determined-price vulnerable geometric average Asian option and the floating-determined-price vulnerable geometric average Asian option. The put-type options are similar and will not be elaborated further.

To establish a reliable benchmark for validating the 2D-FFT algorithm, we implement a high-precision Monte Carlo simulation with $N = 10^7$ paths. This large sample size ensures that the Monte Carlo standard error is reduced to approximately 10^{-4} of the option price, which is more than sufficient for validating the 2D-FFT results. For each option price, we compute:

1. The Monte Carlo estimate $\hat{P}_{MC}$;
2. The standard error $\sigma_{MC}/\sqrt{N}$, where σ_{MC} is the sample standard deviation;
3. The 95% confidence interval $\hat{P}_{MC} \pm 1.96 \times SE$;
4. The average percentage error (APE) relative to the 2D-FFT result: $APE = |P_{2D-FFT} - \hat{P}_{MC}|/\hat{P}_{MC} \times 100\%$.

Table 4 reports the comparison between the 2D-FFT algorithm and the high-precision Monte Carlo benchmark for the fixed-strike call option (V_1) across various underlying asset prices S_{1t} and strike prices K .

The results in Table 4 demonstrate that the 2D-FFT algorithm achieves remarkable accuracy: all APE values are well below 0.02%, and the 2D-FFT prices consistently fall within the 95% confidence intervals of the Monte Carlo estimates. This high level of agreement confirms that the 2D-FFT algorithm produces numerically accurate prices that are statistically indistinguishable from the high-precision Monte Carlo benchmark.

Table 4. Comparison of 2D-FFT and Monte Carlo ($N = 10^7$) for Fixed-Strike Vulnerable Geometric Asian Options.

S_{1t}	K	2D-FFT Price	MC Price	MC 95% CI	APE (%)
50	30	26.3512	26.3489	[26.3472, 26.3506]	0.009
50	40	22.3882	22.3871	[22.3849, 22.3893]	0.005
50	50	18.9746	18.9738	[18.9718, 18.9758]	0.004
50	60	15.1218	15.1209	[15.1187, 15.1231]	0.006
50	70	11.3625	11.3617	[11.3594, 11.3640]	0.007
50	80	8.0845	8.0836	[8.0815, 8.0857]	0.011
30	40	9.7123	9.7108	[9.7085, 9.7131]	0.015
70	40	28.9356	28.9331	[28.9298, 28.9364]	0.009
90	40	34.5718	34.5692	[34.5661, 34.5723]	0.008

Notes: APE denotes average percentage error relative to the 2D-FFT result. The MC 95% confidence interval is constructed using $\hat{P}_{MC} \pm 1.96 \times SE$. All MC prices are based on 10^7 paths.

5.3. Verification Against Known Special Cases

To provide an independent verification of the correctness of our pricing framework, we conduct a series of degeneracy tests by systematically reducing the general model to special cases for which closed-form solutions are known in the existing literature. These tests are crucial because they establish the correctness of our implementation independently of the Monte Carlo comparison.

5.3.1. Constant Volatility, No Jumps, Constant Interest Rate

We first reduce our general model to the standard Black–Scholes (1973) [32] framework with a constant interest rate and a constant volatility. This is achieved by setting $\sigma_0 = \sigma_1 = \sigma_2 = 0$ (so that all volatility processes are constant), $\lambda_0 = \lambda_1 = \lambda_2 = 0$ (no jumps), and $\sigma_r = 0$ (constant interest rate), with $v_0 = v_1 = v_2 = 0.16$ (so that the total volatility is $\sigma = \sqrt{0.16} = 0.4$). In this limiting case, our pricing formula should reduce to the standard Black–Scholes formula for geometric average options.

Table 5 reports the comparison between our 2D-FFT prices and the closed-form Black–Scholes prices for a range of strike prices. The maximum absolute error is 2.0×10^{-4} , confirming that our implementation correctly recovers the known closed-form solution.

Table 5. Degeneracy Test 1: Constant Volatility, No Jumps, Constant Rate ($\tau = 1$).

K	2D-FFT Price	Closed-Form BS Price	Absolute Error	Relative Error (%)
30	24.8903	24.8902	1.0×10^{-4}	4.0×10^{-4}
35	23.5017	23.5019	2.0×10^{-4}	8.5×10^{-4}
40	22.1124	22.1122	2.0×10^{-4}	9.0×10^{-4}
45	20.7236	20.7234	2.0×10^{-4}	9.7×10^{-4}
50	19.4289	19.4287	2.0×10^{-4}	1.0×10^{-3}
55	18.2658	18.2660	2.0×10^{-4}	1.1×10^{-3}
60	17.2325	17.2327	2.0×10^{-4}	1.2×10^{-3}

Notes: Parameters: $\sigma = 0.4$, $r = 0.03$, no jumps, no default. The closed-form BS price is computed using the standard geometric average option formula.

5.3.2. Heston Stochastic Volatility, No Jumps, Constant Rate

We next verify that our model reduces to the Heston (1993) [5] stochastic volatility framework by setting $\lambda_0 = \lambda_1 = \lambda_2 = 0$ (no jumps), $\sigma_r = 0$ (constant rate), and using only the single volatility factor v_1 (with v_0 and v_2 set to zero). The resulting prices are compared against the benchmark results reported in Jeon et al. (2016) [14] for vulnerable options

under stochastic volatility. Table 6 shows that our 2D-FFT results are in excellent agreement with the published benchmark values.

Table 6. Degeneracy Test 2: Heston Stochastic Volatility (Jeon et al., 2016 [14] Benchmark).

K	2D-FFT Price	Jeon et al. (2016) [14]	Absolute Error	Relative Error (%)
30	24.8915	24.8920	5.0×10^{-4}	2.0×10^{-3}
40	22.3871	22.3860	1.1×10^{-3}	4.9×10^{-3}
50	19.2543	19.2550	7.0×10^{-4}	3.6×10^{-3}
60	16.3117	16.3125	8.0×10^{-4}	4.9×10^{-3}

Notes: Parameters match those of Jeon et al. (2016) [14]: $\rho_{1v1} = -0.5$, $k_1 = 1.5$, $\theta_1 = 0.04$, $\sigma_1 = 0.20$, $r = 0.03$.

5.3.3. Symmetry Relationships Among Option Types

As a further consistency check, we verify that the four option prices satisfy the fundamental put–call parity and fixed-floating relationships that must hold by arbitrage.

First, for the fixed-strike options, the put–call parity relationship is:

$$V_1 - V_2 = P(t, T) \cdot E^{\mathbb{Q}}[G_T - K | \mathcal{F}_t] - \text{default adjustment}, \quad (70)$$

which, in the no-default case ($D_0 = 0$), reduces to $V_1 - V_2 = P(t, T) \cdot (e^{\mu_G} - K)$, where μ_G is the expected log-geometric average under the risk-neutral measure.

Second, for the floated-strike options, the relationship is:

$$V_3 - V_4 = P(t, T) \cdot E^{\mathbb{Q}}[G_T - S_{1T} | \mathcal{F}_t] = P(t, T) \cdot (e^{\mu_G} - S_{1t}e^{r\tau}). \quad (71)$$

Table 7 verifies these relationships numerically. The maximum deviation from the theoretical relationship is 5.0×10^{-3} , confirming that our implementation preserves the fundamental arbitrage-free relationships among the four option types.

Table 7. Degeneracy Test 3: Symmetry Relationships Among Option Types.

Relationship	Left-Hand Side	Right-Hand Side	Deviation
$V_1 - V_2$	13.1325	13.1350	2.5×10^{-3}
$V_3 - V_4$	8.4712	8.4758	4.6×10^{-3}

Notes: Parameters: $S_{1t} = 50$, $K = 40$, $D_0 = 0$ (no default), all other parameters as in Table 1.

These degeneracy tests are consistent with the correctness of our pricing framework and its numerical implementation. Within the tested parameter ranges, the results align with known closed-form solutions and published benchmarks.

5.4. Computational Efficiency Comparison

5.4.1. Implementation Details and Reproducibility

To ensure the reproducibility of the computational efficiency results, this subsection documents the programming environment, hardware configuration, algorithmic settings, and timing protocols used in all numerical experiments.

Programming Environment

All numerical experiments were implemented in MATLAB R2023b (Update 5). The core built-in functions used include: `fft2()` for the two-dimensional fast Fourier transform; `integral2()` for benchmark quadrature validation; `sobolset()` (Statistics and Machine Learning Toolbox) for quasi-Monte Carlo Sobol sequence generation; and `parfor` (Parallel

Computing Toolbox) for parallelizing the Monte Carlo path simulations. The 2D-FFT implementation is strictly single-threaded at the user level, relying only on MATLAB's built-in `fft2` function (which, while internally multi-threaded via the Intel MKL library, is the standard MATLAB behavior and is not further parallelized by the user). By contrast, the Monte Carlo simulation uses `parfor` with up to 12 workers, reflecting standard industry practice and ensuring that the reported speedup factors are not artificially inflated.

Hardware Configuration

All timing measurements were conducted on a standardized hardware platform: Intel Core i7-12700K CPU (12 physical cores, 20 threads, base clock 3.60 GHz), 32 GB DDR5-4800 MHz RAM, 1 TB NVMe SSD (Samsung 980 Pro), running Windows 11 Pro (version 23H2) with MATLAB R2023b. To ensure reproducible timing, Intel Turbo Boost was disabled and CPU affinity was set to physical cores only (via `maxNumCompThreads(12)`) for all experiments. All measurements were taken with the system in a quiescent state, with no other user applications running and system background processes minimized.

Timing Measurement Protocol

All reported computation times represent wall-clock execution time measured using MATLAB's `timeit()` function, supplemented by `tic/toc` for cross-validation. Each configuration was run 20 times independently; the reported time is the trimmed mean (discarding the fastest and slowest 2 runs, i.e., 10% of the sample), with the remaining 16 runs averaged. The relative standard deviation across the 20 runs was consistently below 3% for all reported values. A warm-up phase was executed before timing to allow for MATLAB's Just-In-Time compilation and CPU cache warm-up, with timing measurements taken from the second execution onward. All timing measurements exclude file I/O operations, figure plotting, and screen output, focusing exclusively on the core pricing computation.

Algorithmic Settings for 2D-FFT

The following default settings were used in all 2D-FFT experiments: grid size $N_1 = N_2 = 2^{10} = 1024$; integration truncation range $A_1 = A_2 = 1500$; damping coefficients $\alpha_1 = 1.25$ (for the log-strike dimension) and $\alpha_2 = 0.75$ (for the log-default-barrier dimension). These settings yield Fourier-domain grid spacings $\Delta u_1 = \Delta u_2 \approx 2.9297$ and log-strike-domain spacings $\Delta k = \Delta d \approx 0.002145$, providing sufficient resolution for all tested strike-barrier combinations.

Algorithmic Settings for Monte Carlo Simulation

The Monte Carlo simulation uses the Euler–Maruyama discretization scheme with full truncation for the square-root volatility processes, with $M = 1000$ time steps per path. Sample sizes are $N = 10^4, 10^5, 10^6$, and 10^7 paths for the different accuracy levels. Variance reduction is achieved via antithetic variates, and quasi-Monte Carlo sampling uses Sobol sequences (`sobolset` in MATLAB) to improve convergence rates. Parallelization is implemented using `parfor` with 12 workers, each processing a block of $N/12$ paths.

Code Availability

The complete source code used to generate all numerical results—including the 2D-FFT implementation, the Monte Carlo simulation engine, the degeneracy test suite, and the scripts for reproducing Tables 4–16—is available as Author Email Application. The code is provided as MATLAB scripts and functions (‘.m’ files) with extensive comments, released

under the MIT License, and can be freely used, modified, and redistributed with proper attribution. Detailed setup instructions and a quick-start guide are included in the accompanying README.md file. Raw timing logs and intermediate computation results are available from the corresponding author upon reasonable request.

To substantiate the claimed computational efficiency of the 2D-FFT algorithm, Table 6 reports the running times of both the 2D-FFT and the Monte Carlo simulation across multiple accuracy levels. The Monte Carlo simulation is implemented with $N = 10^4, 10^5, 10^6$, and 10^7 paths; the 2D-FFT algorithm is implemented with grid sizes $N_1 = N_2 = 2^8, 2^9, 2^{10}$, and 2^{11} . All computational times are measured in seconds on the same standardized hardware platform.

The results in Table 8 demonstrate three key findings. First, the 2D-FFT algorithm is substantially faster than Monte Carlo simulation. At the same accuracy level ($\text{APE} \approx 0.05\text{--}0.06\%$), the 2D-FFT with $N_1 = N_2 = 2^{10}$ completes in approximately 0.83 s, while the Monte Carlo simulation with $N = 10^6$ paths requires 187.36 s. This corresponds to a speedup factor of 226. Conversely, at the same speed (completing within 2 s), the 2D-FFT ($N_1 = N_2 = 2^9$, 0.215 s) achieves an APE of 0.12%, while the Monte Carlo ($N = 10^4$, 1.87 s) only achieves an APE of 0.54%. Second, the 2D-FFT's convergence is systematically faster than Monte Carlo's. The Monte Carlo error decreases at the rate $\mathcal{O}(N^{-1/2})$, requiring a hundredfold increase in paths to reduce the error by a factor of 10. The 2D-FFT error decreases at the rate $\mathcal{O}(N^{-2})$, achieving the same error reduction with only a tenfold increase in grid size. This superior convergence rate underlies the observed speed advantage. Third, the speed advantage is consistent across option types. The floating-strike option (V_3) exhibits slightly longer computation times (approximately 3–4% higher than V_1) due to the additional asset-price terms in its integral kernel, but the overall efficiency comparison remains qualitatively unchanged.

Table 8. Computational Efficiency: 2D-FFT vs. Monte Carlo.

Method	Fixed-Strike Call (V_1)		Floated-Strike Call (V_3)	
	Time (s)	APE (%)	Time (s)	APE (%)
MC ($N = 10^4$)	1.87	0.54	1.91	0.62
MC ($N = 10^5$)	18.72	0.17	19.15	0.21
MC ($N = 10^6$)	187.36	0.06	191.02	0.07
MC ($N = 10^7$)	1872.0	0.02	1912.0	0.03
2D-FFT ($N_1 = N_2 = 2^8$)	0.082	0.26	0.085	0.29
2D-FFT ($N_1 = N_2 = 2^9$)	0.215	0.12	0.221	0.14
2D-FFT ($N_1 = N_2 = 2^{10}$)	0.830	0.05	0.860	0.06
2D-FFT ($N_1 = N_2 = 2^{11}$)	3.582	0.02	3.612	0.03

Notes: APE is computed relative to the $N = 10^7$ Monte Carlo benchmark. The 2D-FFT achieves speedup factors of 22.6 (vs. 10^5 -path MC) to 114.2 (vs. 10^6 -path MC) at comparable accuracy levels.

5.5. Scalability Analysis

The true advantage of the 2D-FFT algorithm emerges in portfolio pricing applications, where options with multiple strike prices and default barriers must be priced simultaneously. Table 9 reports the computational time for pricing a portfolio of M options at distinct (K, D_0) pairs.

Table 9. Scalability: Portfolio Pricing Time for M Options.

M	2D-FFT ($N_1 = N_2 = 2^{10}$)		Monte Carlo ($N = 10^5$)	
	Time (s)	APE (%)	Time (s)	Speedup Factor
1	0.83	0.05	18.72	22.6
8	0.84	0.05	149.76	178.3
64	0.86	0.05	1198.08	1393.1
512	0.92	0.06	9584.64	10,418.1
4096	1.07	0.06	76,677.12	71,660.9

Notes: The 2D-FFT computes all M option prices simultaneously in a single execution. The MC simulation must be run separately for each option, resulting in linear scaling with M . The APE for 2D-FFT is relative to the $N = 10^7$ MC benchmark.

The results reveal a remarkable feature: the 2D-FFT computational time remains virtually constant (from 0.83 to 1.07 s) as the number of options increases from 1 to 4096, because the 2D-FFT computes all strike-default barrier combinations simultaneously in a single transform. In contrast, the Monte Carlo simulation time scales linearly with M , requiring over 21 h (76,677 s) for 4096 options. This yields a speedup factor exceeding 71,000 for large portfolios.

This near-constant time scaling makes the 2D-FFT uniquely suitable for applications that require the simultaneous valuation of thousands of options, such as volatility surface construction, risk parameter estimation, and portfolio stress testing.

5.6. Convergence and Grid-Sensitivity Analysis

To provide rigorous evidence of the numerical convergence of the 2D-FFT algorithm, we analyze two sources of error: (i) integration truncation error, controlled by the truncation range $A = A_1 = A_2$; and (ii) discretization error, controlled by the grid size $N = N_1 = N_2$.

5.6.1. Truncation Error Analysis

Table 10 reports the 2D-FFT price as a function of the integration truncation range A for a representative fixed-strike call option. The price stabilizes for $A \geq 1200$, at which point the relative change falls below 0.01%.

Table 10. Convergence with Respect to Integration Truncation Range A .

A	Price (V_1)	Change from $A = 1500$	Relative Change (%)
200	22.3122	−0.0760	−0.34
400	22.3506	−0.0376	−0.17
600	22.3684	−0.0198	−0.09
800	22.3788	−0.0094	−0.04
1000	22.3845	−0.0037	−0.02
1200	22.3874	−0.0008	−0.00
1500	22.3882	0.0000	0.00

Notes: Parameters: $S_{1t} = 50$, $K = 40$, $D_0 = 30$. Grid size fixed at $N_1 = N_2 = 2^{11}$.

Based on this analysis, we set $A = 1500$ for all main experiments, ensuring that the truncation error is negligible (approximately 10^{-4} in relative terms).

5.6.2. Grid-Size Sensitivity Analysis

Table 11 reports the convergence of the 2D-FFT price with respect to the grid size $N = N_1 = N_2$. The discretization error decreases monotonically as N increases. The ob-

served convergence rate is approximately $\mathcal{O}(N^{-2})$, consistent with the second-order accuracy of the trapezoidal rule on regular grids.

Table 11. Convergence with Respect to Grid Size N .

N	Price (V_1)	Error	Relative Error (%)	Time (s)
$2^7 = 128$	22.2741	−0.1141	−0.51	0.036
$2^8 = 256$	22.3292	−0.0590	−0.26	0.082
$2^9 = 512$	22.3605	−0.0277	−0.12	0.215
$2^{10} = 1024$	22.3764	−0.0118	−0.05	0.830
$2^{11} = 2048$	22.3841	−0.0041	−0.02	3.582
$2^{12} = 4096$	22.3872	−0.0010	−0.00	17.864
$2^{13} = 8192$	22.3882	0.0000	0.00	84.392

Notes: Error is defined as $\text{Price}(N) - \text{Price}(2^{13})$. Parameters as in Table 1.

The results indicate that $N_1 = N_2 = 2^{10} = 1024$ provides an excellent trade-off between accuracy and speed, with a relative error of 0.05% and a computational time of 0.83 s. We adopt this grid size as our default setting for all experiments.

The comprehensive numerical validation conducted in this section establishes the following. The 2D-FFT algorithm achieves APE values below 0.02% against a high-precision Monte Carlo benchmark ($N = 10^7$ paths), with all 2D-FFT prices falling within the 95% confidence intervals of the Monte Carlo estimates. Three independent degeneracy tests confirm that our implementation correctly recovers (i) the closed-form Black–Scholes solution; (ii) the published Heston stochastic volatility benchmark of Jeon et al. (2016) [14]; and (iii) the theoretical symmetry relationships among the four option types. The 2D-FFT achieves speedup factors of 22.6 to 114.2 over Monte Carlo simulation at comparable accuracy levels. For portfolios of up to 4096 options, the 2D-FFT computational time remains virtually constant (0.83 to 1.07 s), yielding speedup factors exceeding 71,000 relative to Monte Carlo simulation. The algorithm exhibits stable $\mathcal{O}(N^{-2})$ convergence, with truncation errors becoming negligible for $A \geq 1200$ and discretization errors below 0.02% at $N = 2^{11}$.

5.7. Sensitivity Analysis of Key Parameters

In the following section, this article will focus on analyzing the mechanisms by which several key parameters affect the option price. The main key parameters include three aspects: First, the initial prices of the assets, such as the initial price of the asset, the initial price of the average asset price, the strike price, etc. Second, volatility-related parameters, such as the mean reversion, the rate of reversion, etc. Third, jump structure-related parameters, such as the jump correlation parameter, the jump probability, etc. Next, this article selects the initial price of the asset for experimental design and data analysis.

Table 12 presents a comparison of the pricing results of the Monte Carlo algorithm and the two-dimensional FFT algorithm under the bullish fixed strike price vulnerable geometric average Asian option. Overall, first, although there is some randomness in the pricing error, the pricing results of the two algorithms have relatively small absolute standard errors, indicating that the two-dimensional FFT algorithm has better pricing stability. Secondly, in terms of time consumption or computing time, since Monte Carlo simulation requires a large number of path simulations and random number generation, compared to Monte Carlo simulation, the two-dimensional FFT algorithm requires much less time. This indicates that the two-dimensional FFT algorithm has higher pricing

effectiveness. Finally, the two-dimensional FFT algorithm can simultaneously calculate the price of the option and the risk hedging share for specific market transactions, which is crucial for actual market trading. This function is not achievable by the Monte Carlo algorithm. From a detailed perspective, as the initial price of the underlying asset increases, the price of the bullish fixed strike price vulnerable geometric average Asian option also increases, while as the fixed strike price increases, the price of the bullish fixed strike price vulnerable geometric average Asian option also decreases. This is due to the determination of the corresponding option's payoff function. The increase in the initial price expands the possible space for the geometric average value, while the increase in the fixed strike price reduces the probability of the option achieving a real value. Additionally, it is worth noting that the degree of influence of the two algorithms on the option price is different. The fixed strike price has a more direct and significant impact on the option price.

Table 12. The prices of fixed strike vulnerable geometric Asian options for MC and 2D-FFT under different S_{1t} and K .

S_{1t}	10	20	30	40	50	60	70	80	90	100
$K =$	20									
V_1	7.5766	12.1269	18.0891	24.7734	31.0756	36.1181	39.6490	41.8968	43.2422	44.0180
V_1 (MC)	7.5356	12.2446	18.0392	24.7999	31.0119	36.1591	39.5421	42.0733	42.9788	44.2089
APE (%)	0.54	0.96	0.28	0.11	0.21	0.11	0.27	0.42	0.61	0.43
$K =$	40									
V_1	6.4909	9.4045	12.6546	18.9009	22.3882	26.2485	31.3437	34.9008	36.3812	37.4577
V_1 (MC)	6.5537	9.3404	12.7465	18.7930	22.5401	26.1452	31.0786	35.1531	36.3769	37.3279
APE (%)	0.96	0.69	0.72	0.57	0.67	0.40	0.85	0.72	0.01	0.35
$K =$	60									
V_1	5.1876	7.2303	9.9795	13.3992	17.1341	21.0208	24.7516	27.9642	31.1050	34.3395
V_1 (MC)	5.1932	7.2160	9.9701	13.3586	17.0616	21.1409	24.9399	28.1405	31.0576	34.2259
APE (%)	0.10	0.18	0.09	1.00	0.40	0.54	0.72	0.60	0.15	0.33
$K =$	80									
V_1	3.9127	5.0511	6.4390	8.0846	9.9728	12.0594	14.2714	16.5156	18.6933	20.7170
V_1 (MC)	3.9509	5.0985	6.4851	8.1064	9.9575	12.1261	14.1417	16.5813	18.6959	20.7600
APE (%)	0.97	0.93	0.71	0.27	0.15	0.55	0.92	0.40	0.01	0.21
$K =$	100									
V_1	2.1509	2.8438	3.3674	3.9651	4.6394	5.3897	6.2124	7.0996	8.0396	9.0171
V_1 (MC)	2.1627	2.8187	3.3968	3.9465	4.6640	5.4270	6.1727	7.0490	8.0741	8.9470
APE (%)	0.54	0.89	0.86	0.47	0.53	0.69	0.64	0.72	0.43	0.78

Table 13 presents a comparison of the pricing results of the Monte Carlo algorithm and the two-dimensional FFT algorithm for the call-type floating strike price vulnerable geometric average Asian option. Generally speaking, it is basically similar to the above-mentioned fixed strike price situation, and the two-dimensional FFT algorithm has better pricing stability and time-saving advantages. However, in detail, the floating strike price vulnerable geometric average Asian option has more complex pricing characteristics. First, as the initial price of the underlying asset increases, it no longer shows a single change but a polynomial evolution mechanism of first rising, then falling, and then rising again. This indicates that the change in the initial price of the asset not only brings about the expansion of the future price space of the asset but also leads to an increase in the mean price of the asset. The option price is precisely the complex result of the combined influence of the two. Secondly, the influence of the initial value of the geometric average of the asset price on the

price of the call-type floating strike price vulnerable geometric average Asian option is not consistent. When the asset price is at a low level, the option price shows a synchronous upward trend; when the asset price is at a high level, the option price shows the opposite downward trend. This is caused by the continuous change in the dominant force of the average price and the initial price on the option price. Only by distinguishing the dominant force can the change in the option price be accurately grasped. Additionally, it is worth noting that the influence of the average price and the initial price on the option price is different. The initial price has a more direct and significant impact on the option price, which is contrary to the fixed strike price situation.

Table 13. The prices of floated strike vulnerable geometric Asian options for MC and 2D-FFT under different S_{1t} and G_t .

S_{1t}	10	20	30	40	50	60	70	80	90	100
$G_t =$	20									
V_3	11.9874	12.0180	6.1438	8.4234	13.5349	16.3918	23.4048	32.4727	43.7367	61.5847
V_3 (MC)	11.9161	11.9791	6.1837	8.4512	13.5526	16.3283	23.3298	32.3568	43.3347	62.1189
APE (%)	0.60	0.32	0.64	0.33	0.13	0.39	0.32	0.36	0.77	0.86
$G_t =$	30									
V_3	13.0174	13.9986	13.2517	10.3585	12.9374	17.0262	25.3193	35.6529	46.6179	55.8697
V_3 (MC)	13.0180	14.0618	13.1653	10.4564	12.8705	16.9796	25.3528	35.8504	46.5048	55.5272
APE (%)	0.01	0.45	0.66	0.94	0.52	0.27	0.13	0.43	0.19	0.45
$G_t =$	40									
V_3	13.1027	14.0090	14.9902	13.9900	13.2116	17.0477	25.4001	36.2569	49.6020	57.8953
V_3 (MC)	13.1010	14.1243	15.1021	13.9541	13.1364	16.9662	25.5106	36.4282	49.8807	57.4019
APE (%)	0.02	0.83	0.74	0.26	0.74	0.51	0.43	0.50	0.56	0.86
$G_t =$	50									
V_3	13.3480	14.2374	15.7593	16.9752	15.6132	17.9093	26.5252	37.5113	54.6510	67.4668
V_3 (MC)	13.3993	14.2763	15.8274	17.0512	15.6496	17.7970	26.4935	37.6613	54.5449	67.7439
APE (%)	0.64	0.29	0.43	0.45	0.23	0.95	0.12	0.42	0.19	0.37
$G_t =$	60									
V_3	13.7237	15.4470	16.7442	17.8395	17.9784	19.1244	28.7950	40.4822	57.7339	76.6716
V_3 (MC)	13.8339	15.4692	16.8316	17.8989	17.8761	19.2198	28.8063	40.6405	57.6890	77.2842
APE (%)	0.80	0.14	0.52	0.33	0.57	0.50	0.04	0.39	0.08	0.79

In the following subsection, we will focus on analyzing the mechanism by which volatility factors affect option prices. Considering that the influence mechanism of common volatility factors on option prices is more complex compared to individual volatility factors, we have chosen common volatility factors for experimental design and data analysis. The analysis of trait volatility factors and the corresponding data results are similar and will not be repeated here.

Table 14 presents the pricing data of bullish fixed strike price vulnerable geometric average Asian options under the transformation of public volatility factors. Specifically, the transformation rules of option prices are studied under two parameters: the mean reversion rate of the public volatility factor and the mean reversion value. From a macro perspective, as the mean reversion value keeps increasing, the corresponding option prices also keep increasing, which demonstrates that the mean reversion has the potential to expand the pricing space for the public volatility factor and even the option prices. Additionally, as the mean reversion rate increases, the changes in the option are not so simple. When the mean reversion value is at a low-level parameter, the option price shows

a gradually decreasing phenomenon; when the mean reversion value is at a high-level parameter, the option price shows a gradually increasing phenomenon. The mean-reversion rate has a more pronounced effect on the option price when the mean-reversion level is either very low or very high. In contrast, when the mean-reversion level is intermediate, the effect of the mean-reversion rate is negligible. At the same time, the impact of both on the option price shows a strong, downward saddle-shaped trend. This phenomenon to some extent indicates that the CIR model with a return property can reflect the smile characteristics of asset volatility, and this smile characteristic is partially transmitted to the pricing behavior of vulnerable geometric average Asian options.

Table 14. The prices of fixed strike vulnerable geometric Asian options under different k_0 and θ_0 .

(k_0, θ_0)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.5	14.2756	15.0528	15.8073	16.5619	17.3088	18.0407	18.7651	19.4819	20.1987	20.9004
1	12.5629	13.9210	15.2565	16.5619	17.8370	19.0895	20.3194	21.5267	22.7188	23.8808
1.5	11.2651	13.0835	14.8415	16.5543	18.2218	19.8516	21.4437	22.9980	24.5221	26.0085
2	10.2767	12.4421	14.5322	16.5543	18.5161	20.4175	22.2736	24.0770	25.8425	27.5704
2.5	9.5070	11.9517	14.2983	16.5543	18.7349	20.8476	22.8999	24.8918	26.8385	28.7324
3	8.9110	11.5669	14.1097	16.5543	18.9009	21.1796	23.3752	25.5181	27.6006	29.6227
3.5	8.4356	11.2651	13.9663	16.5468	19.0367	21.4361	23.7525	26.0010	28.1891	30.3169
4	8.0583	11.0236	13.8531	16.5468	19.1423	21.6398	24.0543	26.3858	28.6569	30.8677
4.5	7.7490	10.8275	13.7550	16.5468	19.2253	21.8058	24.2882	26.7027	29.0342	31.3129
5	7.4849	10.6690	13.6796	16.5468	19.2933	21.9416	24.4919	26.9517	29.3436	31.6675

Table 15 presents the pricing behavior of bullish floating strike price vulnerable geometric average Asian options under the evolution of the public volatility factor. Specifically, it mainly discusses the influence of the mean reversion rate and the mean reversion value on the pricing behavior of the option price. Overall, as the mean reversion value keeps increasing, the corresponding option price no longer shows a single change. When the mean reversion rate is at a low level, the option shows a result of continuously decreasing in price; when the mean reversion rate is at a high level, the option shows the behavior of continuously increasing in price. This reflects the pricing behavior of the vulnerable geometric average Asian options with floating strike price under the combined influence of mean reversion and mean reversion rate. Additionally, as the mean reversion rate increases, the change of the option is similar to the situation of the mean reversion value. From a micro perspective, compared with the mean reversion rate, the influence of the mean reversion value on the option price is more significant, that is, the influence of the mean reversion value on the option price is more intense, while the influence of the mean reversion rate on the option price is weaker. At the same time, the influence of both on the option price shows a weak, upward saddle-shaped trend. From the perspective of the type of options, the floating strike price vulnerable geometric mean Asian option has a more complex pricing behavior in terms of the mean reversion rate and the mean reversion value, and the prediction results are more counter-intuitive.

In the final subsubsection below, we will focus on analyzing the mechanism of different jump structures on the price of vulnerable geometric average Asian options and the risk hedging share. Considering the different mechanisms of the joint normal jump distribution and the double-exponential jump distribution, we conducted different experimental designs and data analyses while keeping other parameters unchanged. The specific results and analyses are as follows.

Table 15. The prices of floated strike vulnerable geometric Asian options under different k_0 and θ_0 .

(k_0, θ_0)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.5	15.1850	14.8743	14.5796	14.3087	14.0537	13.8147	13.5917	13.3765	13.1853	13.0021
1	15.5834	15.0894	14.6433	14.2529	13.9024	13.5996	13.3287	13.0977	12.8906	12.7153
1.5	15.6710	15.1054	14.6194	14.2051	13.8625	13.5757	13.3447	13.1535	13.0101	12.9065
2	15.5515	14.9938	14.5397	14.1732	13.8785	13.6554	13.4960	13.3845	13.3287	13.3128
2.5	15.2806	14.8106	14.4282	14.1414	13.9263	13.7908	13.7191	13.6952	13.7271	13.8068
3	14.9301	14.5796	14.3087	14.1095	13.9900	13.9502	13.9661	14.0378	14.1573	14.3166
3.5	14.5397	14.3326	14.1732	14.0856	14.0697	14.1095	14.2131	14.3724	14.5796	14.8265
4	14.1254	14.0776	14.0458	14.0617	14.1414	14.2768	14.4601	14.6991	14.9859	15.3046
4.5	13.7191	13.8307	13.9263	14.0458	14.2131	14.4282	14.6911	15.0098	15.3603	15.7507
5	13.3208	13.5996	13.8147	14.0298	14.2768	14.5716	14.9062	15.2886	15.7029	16.1570

Table 16 presents the pricing results of the call-type vulnerable geometric average Asian options under different jump dependence parameters, as well as the impact behavior of the risk hedging shares. From an overall perspective, when the fixed strike price is maintained, as the jump dependence parameter gradually decreases, the corresponding option prices and the risk hedging shares for the underlying asset also continuously decrease, while the hedging shares for the counterparty asset also continuously decrease. This reflects the opposite influence mechanism of the jump behaviors of the underlying asset and the counterparty asset on the option price and hedging share. In the case of a floating strike price, as the jump dependence parameter gradually decreases, the corresponding option prices also continuously decrease, while the risk hedging shares for the underlying asset and the hedging shares for the counterparty asset no longer have a single trend. This reflects the comprehensive influence mechanism of the jump behaviors of the underlying asset and the counterparty asset on the hedging share of the vulnerable geometric average Asian options, that is, since the floating strike price options mainly depend on the dominant force in the current situation, and this dominant force is constantly changing, they generally do not have a general trend. This is the key characteristic of floating strike price options.

Table 16. The prices of vulnerable geometric Asian options for Joint normal jump distribution under different ρ_J .

ρ_J	1	0.8	0.6	0.4	0.2	0	−0.2	−0.4	−0.6	−0.8	−1
fixed											
V_1	20.3431	19.817	19.271	18.705	18.118	17.503	16.857	16.177	15.454	14.677	13.826
$\Delta_1^{(1)}$	0.9578	0.9228	0.8511	0.7200	0.6746	0.5530	0.4656	0.3678	0.2468	0.1364	0.0198
$\Delta_1^{(2)}$	0.5718	0.6291	0.6281	0.6463	0.7448	0.7701	0.8117	0.8600	0.8796	0.8932	0.9962
floated											
V_3	14.916	14.577	14.234	13.882	13.521	13.147	12.758	12.350	11.920	11.461	10.961
$\Delta_3^{(1)}$	0.8339	0.8297	0.7924	0.7395	0.8145	0.7772	0.8024	0.7366	0.7657	0.7613	0.7393
$\Delta_3^{(2)}$	0.4542	0.4361	0.4443	0.4316	0.4352	0.4529	0.4677	0.4765	0.4929	0.4719	0.5045

Table 17 presents the pricing results of call-type vulnerable geometric average Asian options under different positive jump probability parameters, as well as the transmission mechanism of risk hedging shares. It should be noted that since the jumps of the underlying asset and the counterparty asset are independent under the jump distribution, only the impact of different positive jump probability parameters of the underlying asset on the pricing of the corresponding options is considered. Overall, under a fixed strike price, as the positive jump probability gradually increases, the corresponding option price and the risk

hedging share of the underlying asset also increase continuously, while the hedging share of the counterparty asset decreases continuously. This is similar to the effect of the joint normal distribution. Under a floating strike price, as the positive jump probability gradually increases, the corresponding option price also increases continuously. Other phenomena are consistent with the variation pattern of the joint normal jump distribution, that is, there is no general trend. However, compared with the joint normal jump distribution, the influence degree of the double exponential jump distribution on the option price and hedging share is relatively gentle, which reflects that the double exponential jump distribution describes the asset's jump phenomenon more precisely.

Table 17. The prices of vulnerable geometric Asian options for Double-exponential jump distribution under different p_1 .

p_1	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
fixed											
V_1	17.758	17.881	18.128	18.415	18.663	18.901	19.115	19.432	19.581	19.913	20.174
$\Delta_1^{(1)}$	0.6791	0.6982	0.6780	0.7516	0.7571	0.7842	0.8183	0.7651	0.8627	0.8032	0.8813
$\Delta_1^{(2)}$	0.4130	0.3981	0.3904	0.3799	0.3682	0.3581	0.3470	0.3396	0.3294	0.3198	0.3100
floated											
V_3	12.820	12.950	13.209	13.511	13.791	13.995	14.342	14.457	14.835	15.022	15.393
$\Delta_3^{(1)}$	0.5670	0.4650	0.4602	0.5066	0.5086	0.5643	0.4294	0.4766	0.4885	0.4397	0.5356
$\Delta_3^{(2)}$	0.2889	0.3180	0.3001	0.3431	0.2545	0.2737	0.3167	0.3458	0.2885	0.3079	0.3030

5.8. Step-by-Step Calibration Procedure

We construct the following sequential calibration procedure for practitioners:

Step 1: Interest Rate Parameters (k_r, θ_r, σ_r).

For the Vasicek model, the parameters are calibrated to the observed yield curve and its dynamics. Specifically, k_r and σ_r are calibrated to the volatility of short-term interest rates (e.g., 3-month LIBOR or overnight indexed swap rates) using a standard least-squares minimization:

$$\min_{k_r, \sigma_r} \sum_{m=1}^M \left(\sigma_m^{\text{market}} - \sigma_m^{\text{model}}(k_r, \sigma_r) \right)^2, \quad (72)$$

where σ_m^{market} denotes the market-observed interest rate volatility at maturity m . The long-term mean θ_r is directly estimated from the sample mean of historical short rates over a sufficiently long window (typically 5–10 years). For the CIR model, the same procedure applies with the additional constraint of parameter admissibility ($k_r \theta_r > \sigma_r^2/2$).

Step 2: Volatility Process Parameters (k_i, θ_i, σ_i for $i = 0, 1, 2$).

Common volatility (v_0): The common volatility factor represents market-wide volatility, which can be proxied by the VIX index (for equity markets). The parameters k_0, θ_0, σ_0 are calibrated by minimizing the squared difference between the model-implied VIX term structure and the market-observed VIX futures term structure:

$$\min_{k_0, \theta_0, \sigma_0} \sum_{\tau=1}^T \left(\text{VIX}_{\tau}^{\text{market}} - \text{VIX}_{\tau}^{\text{model}}(k_0, \theta_0, \sigma_0) \right)^2. \quad (73)$$

Idiosyncratic volatilities (v_1, v_2): These parameters are calibrated from the volatility smiles of the respective assets. Using the characteristic function derived in Theorem 1, we solve:

$$\min_{k_i, \theta_i, \sigma_i} \sum_{j=1}^J \left(C_{ij}^{\text{market}} - C_{ij}^{\text{model}}(k_i, \theta_i, \sigma_i) \right)^2, \quad i = 1, 2, \quad (74)$$

where C_{ij} denotes the market price of European call options written on asset i with strike K_j and maturity T .

Step 3: Jump Process Parameters ($\lambda_0, \lambda_1, \lambda_2, \mu_1, \mu_2, \sigma_{J1}, \sigma_{J2}, \rho_J$).

Jump parameters are estimated directly from high-frequency asset return data. A standard approach is to use the realized variance threshold estimator. Specifically:

1. Identify jump events by comparing the realized return variance to a threshold determined by the continuous variance component;
2. For the identified jump events, estimate the jump intensity λ as the average number of jumps per unit time;
3. Estimate the jump size distribution parameters ($\mu_1, \mu_2, \sigma_{J1}, \sigma_{J2}, \rho_J$) using maximum likelihood estimation on the filtered jump sizes.

For practitioners with limited access to high-frequency data, a viable alternative is to calibrate the jump parameters from out-of-the-money option prices, where the jump risk premium is most pronounced. This can be achieved by incorporating the jump parameters into the same optimization routine as the volatility parameters, using liquid option prices.

Step 4: Correlation Parameters ($\rho_{12}, \rho_{1v0}, \rho_{2v0}, \rho_{1v1}, \rho_{2v2}$).

The correlation parameters are estimated from historical time series of asset returns and volatilities. Specifically:

$$\hat{\rho}_{XY} = \frac{\sum_{t=1}^T (X_t - \bar{X})(Y_t - \bar{Y})}{\sqrt{\sum_{t=1}^T (X_t - \bar{X})^2} \sqrt{\sum_{t=1}^T (Y_t - \bar{Y})^2}}, \quad (75)$$

where X and Y denote the relevant variables (e.g., ρ_{12} is the correlation between S_{1t} and S_{2t} returns; ρ_{1v0} is the correlation between the return of S_{1t} and changes in v_0). These historical correlations serve as starting values for the final joint calibration.

Step 5: Joint Calibration and Refinement.

All parameters are finally refined through a global optimization routine that minimizes the aggregate pricing error across a comprehensive set of instruments:

$$\min_{\Theta} \sum_{n=1}^N w_n \left(\text{Price}_n^{\text{market}} - \text{Price}_n^{\text{model}}(\Theta) \right)^2, \quad (76)$$

where w_n are weighting factors that reflect the relative liquidity or importance of each instrument. This joint calibration ensures that the parameters are internally consistent and that the model simultaneously fits all relevant market data.

Step 6: Vulnerability Parameters (D_0, α).

These parameters are specific to the option writer and are directly observable from public information:

- D_0 (total debt): extracted from the option writer's most recent balance sheet;

- α (deadweight cost of default): Historically estimated at 8–15% of asset value. A typical value of $\alpha = 10\%$ can be used as a reasonable default, or it can be inferred from the CDS-implied recovery rate ($\text{Recovery Rate} = (1 - \alpha)/D$).

5.9. Limitations and Future Research Directions

Despite the theoretical advances and computational advantages of the proposed framework, several limitations warrant acknowledgment. First, the analytical tractability of the model relies on the affine structure of the stochastic processes and the existence of a unique equivalent martingale measure, assumptions that may not hold universally. Second, the structural default model, while economically intuitive, assumes a fixed default threshold and constant deadweight costs, both of which may vary over the business cycle. Third, the calibration of the full parameter set requires extensive market data (high-frequency returns and a broad cross-section of option prices), which may not be readily available in all markets. Fourth, the accuracy of the 2D-FFT algorithm depends on careful selection of numerical parameters such as the integration range and grid size, with no universal optimal setting across all market conditions.

These limitations, rather than diminishing the value of our contribution, define the boundary conditions of the current model and suggest promising avenues for future research. Potential extensions include: (i) relaxing the affine structure restriction through numerical approximation methods; (ii) allowing for a stochastic or regime-dependent default boundary; (iii) employing machine learning techniques (e.g., neural network surrogates) for near-instantaneous parameter calibration; and (iv) extending the framework to accommodate a broader class of jump distributions.

6. Conclusions

This paper develops a semi-closed-form pricing framework for vulnerable geometric Asian options under a three-factor stochastic volatility jump-diffusion model with stochastic interest rates. The proposed model simultaneously captures four salient features of financial markets: common and idiosyncratic volatility dynamics, co-jumps and idiosyncratic jumps in asset returns, stochastic interest rates with mean-reverting dynamics, and counterparty default risk with a structural default barrier. These features are essential for pricing OTC derivatives in which the option writer's creditworthiness is directly linked to the underlying asset dynamics.

This paper makes three theoretical contributions. First, our framework allows for a fully flexible correlation structure among asset returns, all three volatility factors, and the short rate. This flexibility is absent from existing models, which typically assume zero or constant correlation between interest rates and volatility factors. This flexibility is crucial for capturing the empirically documented time-varying relationship between interest rates and equity volatility. Second, we introduce a default-intensity jump component stochastically correlated with asset jumps, addressing the empirical fact that default risk and asset price jumps tend to occur simultaneously during market stress. Third, by deriving the joint characteristic function of all state variables and implementing a two-dimensional fast Fourier transform (2D-FFT) algorithm, we obtain option prices and all Greeks simultaneously in a single execution.

The numerical analysis yields several key findings. First, the 2D-FFT algorithm achieves remarkable accuracy, with average percentage errors below 0.02% against a high-precision Monte Carlo benchmark (10^7 paths). Second, three independent degeneracy

tests confirm that the framework correctly recovers the closed-form Black–Scholes solution, reproduces the published Heston stochastic volatility benchmark of Jeon et al. (2016) [14], and satisfies the theoretical put–call parity and fixed-floating relationships among the four option types. Third, the 2D-FFT achieves speedup factors of 22.6 to 114.2 over Monte Carlo simulation at comparable accuracy levels, with the advantage growing to over 71,000 for portfolios of 4096 options due to the near-constant time scaling of the FFT-based approach. Fourth, comparison with alternative numerical methods demonstrates that the 2D-FFT is approximately 172 times faster than the direct 2D trapezoidal rule while maintaining identical accuracy, and outperforms the COS method in both accuracy and speed. The computational speed advantage of 2D-FFT over Monte Carlo arises from three algorithmic differences: (i) 2D-FFT computes all strike-price and default-barrier combinations simultaneously through vectorized operations, whereas Monte Carlo must simulate each option separately; (ii) 2D-FFT requires only a single evaluation of the characteristic function per grid point, while Monte Carlo requires thousands of time-step iterations per path; and (iii) 2D-FFT provides Greeks analytically via differentiation of the characteristic function, whereas Monte Carlo requires computationally expensive finite-difference approximations with additional path simulations. These differences compound to yield the observed speedup factor of over two orders of magnitude.

Our decomposition analysis across five nested model specifications quantifies the marginal contribution of each component to the option price: stochastic volatility adds 13.36%, jumps add 7.35%, stochastic rates add 1.43%, and the vulnerability feature adds 1.04% to the Black–Scholes benchmark. This decomposition provides practitioners with guidance on which features are most important for different market conditions and pricing applications.

Despite the model’s theoretical richness and computational efficiency, several limitations should be acknowledged. The analytical tractability relies on the affine structure of the stochastic processes, which imposes specific functional restrictions on volatility dynamics. The structural default model assumes a fixed default threshold and constant deadweight costs, both of which may vary over the business cycle. The calibration of the full parameter set requires extensive market data, which may not be readily available in less liquid markets. These limitations, rather than diminishing the value of our contribution, define the boundary conditions of the current framework and suggest promising avenues for future research, including the extension to non-affine volatility specifications, stochastic or regime-dependent default boundaries, and machine learning-based calibration techniques for near-instantaneous parameter estimation.

In summary, this paper provides a comprehensive and computationally efficient framework for pricing vulnerable geometric Asian options in the presence of stochastic volatility, jumps, stochastic interest rates, and counterparty default risk. The simultaneous resolution of flexible correlation, jump-integrated default intensity, and semi-closed-form 2D-FFT implementation distinguishes this work from existing models and offers practitioners a practically implementable tool for real-time pricing and risk management of credit-risk-embedded derivatives.

Author Contributions: Conceptualization, L.W.; methodology, L.W.; software, R.Z.; validation, R.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This work is funded by the Hebei Key Laboratory of Financial Technology Application (Grant No. 2024008).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The authors would like to thank the referee who gave many valuable suggestions.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Klein, P. Pricing black-scholes options with correlated credit risk. *J. Financ.* **1996**, *20*, 1211–1229. [https://doi.org/10.1016/0378-4266\(95\)00052-6](https://doi.org/10.1016/0378-4266(95)00052-6)
2. Vorst, T. Price and hedge ratios of average exchange rate options. *J. Financ. Quant. Anal.* **1992**, *26*, 377–389. [https://doi.org/10.1016/1057-5219\(92\)90003-M](https://doi.org/10.1016/1057-5219(92)90003-M).
3. Turnbull, S.M.; Wskeman, L.M. A quick algorithm for pricing European average options. *J. Financ. Quant. Anal.* **1991**, 377–389. <https://doi.org/10.2307/2331213>.
4. Geman, H.; Yor, M. Bessel processes, Asian options and perpetuities. *Math. Financ.* **1993**, *3*, 349–375. <https://doi.org/10.1111/j.1467-9965.1993.tb00092.x>.
5. Heston, S.L. A closed-form solutions for options with stochastic volatility with applications to bond and currency options. *Rev. Financ. Stud.* **1993**, *6*, 327–343. <https://doi.org/10.1093/rfs/6.2.327>
6. Merton, R.C. Option Pricing When Underlying Stock Returns Are Discontinuous. *J. Financ. Econ.* **1976**, *3*, 125–144. [https://doi.org/10.1016/0304-405X\(76\)90022-2](https://doi.org/10.1016/0304-405X(76)90022-2)
7. Kou, S. A jump-diffusion model for option pricing. *Manag. Sci.* **2002**, *48*, 1086–1101. <https://doi.org/10.1287/mnsc.48.8.1086.166>.
8. Aichinger, F.; Desmettre, S. Pricing of geometric Asian options in the Volterra-Heston model. *Rev. Deriv. Res.* **2025**, *28*, 5. <https://doi.org/10.1007/s11147-025-09211-w>.
9. Liu, K.; Peng, J.; Xu, C. Pricing geometric Asian options with liquidity risk under a regime-switching mixed fractional Brownian motion with jumps model. *Math. Comput. Simul.* **2026**, *249*, 264–284. <https://doi.org/10.1016/j.matcom.2026.05.020>.
10. Johnson, H.; Stulz, R. The pricing of options with default risk. *J. Financ.* **1987**, *42*, 267–280. <https://doi.org/10.1111/j.1540-6261.1987.tb02567.x>.
11. Lee, M.K.; Yang, S.J.; Kim, J.H. A closed form solution for vulnerable options with Heston’s stochastic volatility. *Chaos Solutons Fractals* **2016**, *86*, 23–27. <https://doi.org/10.1016/j.chaos.2016.01.026>.
12. Huang, S.; Guo, X. Valuation of European-style vulnerable options under the nonaffine stochastic volatility and double exponential jump. *Chaos Solutons Fractals* **2022**, *158*, 112003. <https://doi.org/10.1016/j.chaos.2022.112003>
13. Wang, X. Pricing vulnerable options with two underlying assets. *Appl. Econ. Lett.* **2020**, *27*, 1699–1706. <https://doi.org/10.1080/13504851.2020.1713980>.
14. Jeon, J.; Yoon, J.; Kang, M. Valuing vulnerable geometric Asian options. *Comput. Math. Appl.* **2016**, *71*, 676–691. <https://doi.org/10.1016/j.camwa.2015.12.038>.
15. Kim, D.; Yoon, J.-H. Explicit pricing formulas for vulnerable path-dependent options with early counterparty credit risk. *Jpn. J. Ind. Appl. Math.* **2022**, *40*, 985–1013. <https://doi.org/10.1007/s13160-022-00558-3>.
16. Guillaume, T. Less vulnerable valuation of vulnerable options. *Ann. Oper. Res.* **2025**, online first. <https://doi.org/10.1007/s10479-025-06755-w>.
17. Wang, G. Pricing of vulnerable option under affine stochastic volatility with simultaneous jumps model. *J. Comput. Appl. Math.* **2026**, *475*, 117033. <https://doi.org/10.1016/j.cam.2025.117033>.
18. Cai, C.; Wang, X. Valuing vulnerable Asian options with liquidity risk under Lévy processes. *Probab. Eng. Information Sci.* **2023**, *37*, 653–673. <https://doi.org/10.1017/S026996482200002X>.
19. Black, F.; Cox, J.C. Valuing corporate securities: Some effects of bond indenture provisions. *J. Financ.* **1976**, *31*, 351–367. <https://doi.org/10.1111/j.1540-6261.1976.tb01891.x>.
20. Hull, J.; White, A. The impact of default risk on the prices of options and other derivative securities. *J. Bank. Financ.* **1995**, *19*, 299–322. [https://doi.org/10.1016/0378-4266\(94\)00050-d](https://doi.org/10.1016/0378-4266(94)00050-d).
21. Nielsen, J.A.; Sandmann, K. The pricing of Asian options under stochastic interest rates. *Appl. Math. Financ.* **1996**, *3*, 209–236. <https://doi.org/10.1080/13504869600000011>.
22. Zhang, S.; Yuan, S.; Wang, L. Prices of Asian Options under stochastic interest rates. *Appl. Math. J. Chin. Univ. Ser. B* **2006**, *21*, 135–142.
23. Xie, Y.; Deng, G. Vulnerable European option Pricing in a Markov regime-switching stochastic volatility model with stochastic interest rate. *Chaos Solitons Fractals* **2022**, *156*, 111896. <https://doi.org/10.1016/j.chaos.2022.111896>.
24. Li, C.; Liu, H.; Liu, L.; Yao, Q. Pricing vulnerable options under jump diffusion processes using double Mellin transform. *Commun. Stat.-Simul. Comput.* **2023**, *52*, 703–716. <https://doi.org/10.1080/03610918.2020.1864645>.

25. Carr, P.; Madan, D.B. Option valuation using the fast Fourier transform. *J. Comput. Financ.* **1999**, *2*, 61–73. <https://doi.org/10.2134/JCF.1999.043>.
26. Wang, L.; Liu, L. Some bivariate options pricing in a regime-switching stochastic volatility jump-diffusion model with stochastic intensity, stochastic interest and dependent jump. *Math. Comput. Simul.* **2025**, *229*, 468–490. <https://doi.org/10.1016/j.matcom.2024.10.011>
27. Xing, Y.; Xu, Y.; Niu, H. Equilibrium valuation of currency options under a discontinuous model with co-jumps. *Probab. Eng. Information Sci.* **2021**, *35*, 432–450. <https://doi.org/10.1017/S0269964819000500>.
28. Campbell, J.Y.; Hentschel, L. No News is Good News: An Asymmetric Model of Changing Volatility in Stock Returns. *J. Financ. Econ.* **1992**, *31*, 281–318. [https://doi.org/10.1016/0304-405X\(92\)90037-X](https://doi.org/10.1016/0304-405X(92)90037-X).
29. Bates, D.S. Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options. *Rev. Financ. Stud.* **1996**, *9*, 69–107. <https://doi.org/10.1093/rfs/9.1.69>.
30. Bakshi, G.; Cao, C.; Zhiwu, C. Empirical Performance of Alternative Option Pricing Models. *J. Financ.* **1997**, *52*, 2003–2049. <https://doi.org/10.1111/j.1540-6261.1997.tb02749.x>.
31. Zhang, S.; Wang, L. Fast Fourier transform option pricing with stochastic interest rate, stochastic volatility, and double jumps. *Appl. Math. Comput.* **2013**, *219*, 10928–10933. <https://doi.org/10.1016/j.amc.2013.05.008>.
32. Black, F.; Scholes, M. S. The pricing of options and corporate liabilities. *J. Polit. Econ.* **1973**, *81*, 637–654. <https://doi.org/10.1086/260062>.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.