

Article

Stochastic Singularity Analysis in Multi-Material Domains: Effects of Randomness in Isotropic Material Properties

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Abstract

The singularity level near a bi-material interface crack is crucial for failure prediction. Unlike a crack in a homogeneous material with a fixed eigenvalue of $1/2$, interface crack eigenvalues are complex. Their fixed real part is $1/2$, while the stochastic imaginary part depends explicitly on the material properties, governing the stress field's oscillatory nature. Consequently, deterministic approximations may underestimate the severity of the singularity. This paper presents a stochastic approximation of the eigenvalue's imaginary component for a bi-material interface crack. Both materials are isotropic, involving stochastic Young's modulus and stochastic Poisson's ratio, yielding four independent random variables. The generalized Polynomial Chaos (gPC) method is used to construct a multidimensional polynomial expansion via tensor products of orthogonal bases that are dictated by the material distributions. The deterministic coefficients are derived from evaluations at selected quadrature points. The methodology is demonstrated and directly validated on a benchmark problem of a crack between bonded half-planes, utilizing its exact closed-form analytical solution. The gPC approximation converges rapidly as the polynomial order increases, achieving relative differences below 1% in all statistical moments compared to the exact analytical solution, using only 4% of the computational effort typically required for such statistical characterization. These findings underscore that accounting for material stochasticity is essential for accurate failure criteria.

Keywords: generalized polynomial chaos; stochastic material properties; bi-material interface crack; complex eigenvalues; stochastic fracture mechanics; uncertainty quantification

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1. Introduction

The solution of linear elastic problems in the vicinity of a crack tip or a sharp re-entrant corner is well established. The eigenvalues of the asymptotic expansion in the vicinity of the singular point govern the singular level of the stress field, which in turn relates to the Stress Intensity Factors (SIFs) that are essential for failure prediction. For a homogeneous isotropic domain, the computation of these eigenvalues in the case of deterministic material properties is well known and may be performed by analytical techniques (see [1–4]) and numerical techniques (see [5–10]).

A bi-material interface is a composite of two homogeneous materials joined with continuity of tractions and displacements across the interface. The asymptotic stress and displacement fields in the vicinity of a crack lying along such an interface have been studied extensively. Williams [11] was the first to analyze the singular behavior of the stress field at

the tip of an interface crack and to derive the characteristic equation governing the eigenvalues. This behavior was further investigated in [12–16]. The dependence of the singularity order on the elastic constants of the two materials and on the geometry, including cases in which the eigenvalues form complex conjugate pairs, was studied in detail by Bogy [17]. The characteristic equation may be obtained either by employing an Airy stress function or by applying Muskhelishvili's complex potentials [18]. Because continuity of tractions and displacements across the interface must be enforced, the resulting characteristic equation, and hence the eigenvalues, depend on the material properties of both constituents and on the assumed plane-stress or plane-strain state. This dependence contrasts with a crack in a homogeneous isotropic material, where the eigenvalues are fixed at $1/2$ regardless of the material properties.

A distinctive feature of the interface crack is that its eigenvalues are complex and appear in conjugate pairs. The real part of the dominant pair remains fixed at $1/2$, while the imaginary part is governed by the elastic mismatch between the two materials. The imaginary part determines the oscillatory nature of the stress field in the vicinity of the interface crack tip. Specifically, a larger imaginary part yields higher-frequency oscillations and more severe stress concentrations, thereby increasing the likelihood of failure initiation along the interface. Consequently, the magnitude of the imaginary part is a direct measure of the severity of the singularity at the interface.

The material properties, represented by Young's modulus and Poisson's ratio in the isotropic case, depend on the manufacturing process and are determined by a measurement procedure. Even the most accurate manufacturing process and the best measurement equipment produce randomness in the measured properties, typically exhibiting uncertainties ranging from as low as 1–2% to as high as 6%, depending on the measurement method employed (for further details, see [19–24]). Although these stochastic material properties are commonly approximated as deterministic by using their mean values, such an approach can be misleading for fracture assessments. In a homogeneous isotropic material, this approximation is acceptable because the eigenvalues are independent of the material properties, leaving the singularity unaffected. However, for a bi-material interface crack, the imaginary part of the eigenvalues depends explicitly on the elastic properties of both materials. Consequently, the randomness in the material properties propagates directly into the imaginary part, noticeably altering the singularity level. Neglecting this stochastic nature by adopting a deterministic mean may therefore underestimate the actual severity of the singularity, ultimately compromising the accuracy of failure predictions.

Numerous publications have addressed the randomness of material properties in fracture mechanics, with the vast majority focusing on cracked homogeneous domains and the evaluation of stochastic Stress Intensity Factors (SIFs). This includes the utilization of Monte Carlo simulations [25–29], macroscopic approaches for brittle materials [30,31], and applications of the generalized Polynomial Chaos (gPC) method to construct polynomial representations of SIFs [32,33]. Concurrently, deterministic studies have characterized the singular stress fields and the order of the stress singularity at general multi-material corners composed of dissimilar materials, for which the eigenvalues are typically obtained through dedicated numerical procedures [34]. A shift toward quantifying uncertainties directly within the characteristic eigenvalues was recently introduced in [35] for a V-notched homogeneous isotropic domain. However, that analysis was restricted to real eigenvalues influenced solely by Poisson's ratio.

The present work extends this eigenvalue-focused approach to the more complex configuration of a cracked bi-material interface, where the singularity is governed by the imaginary part of a complex conjugate pair of eigenvalues. Here, both materials are assumed to be isotropic, with stochastic Young's modulus and Poisson's ratios, thereby

resulting in four independent random variables. The framework employs the gPC method as an efficient approach for quantifying uncertainty in stochastic problems [36–38], incurring only a fraction of the computational cost of Monte Carlo simulations. The stochastic imaginary part of the eigenvalue is approximated by a family of orthogonal polynomials, whose type is determined by the probability distribution of the material properties.

While the author has previously applied the gPC method to stochastic Stress Intensity Factors in homogeneous cracked domains [32,33] and to real eigenvalues in a homogeneous V-notched domain governed solely by Poisson's ratio [35], the present work addresses, for the first time, the stochastic complex eigenvalue governing a bi-material interface crack. This configuration introduces two key novelties: (i) the singularity is governed by the imaginary part of a complex conjugate eigenvalue pair, rather than a real eigenvalue, and (ii) four independent stochastic material properties—the Young's modulus and Poisson's ratio of each of the two bonded materials—jointly and nonlinearly determine this imaginary part, as opposed to the single-parameter dependence considered previously.

We validate the proposed method on a benchmark bi-material interface crack problem. In this setup, each material occupies a half-plane, and the crack lies along the straight interface that separates them. Since a closed-form analytical solution is available for this configuration, it serves as an ideal benchmark for directly comparing the analytical solution with the gPC approximation. The example problem emphasizes the importance of the stochastic investigation, as the singular level of the elastic solution may be more severe than expected when only deterministic material properties (mean values) are considered.

The paper is organized as follows:

- Section 2 presents the asymptotic solution near a 2D isotropic crack tip, first for a homogeneous material and then for a bi-material interface, highlighting how stochastic material properties affect the resulting eigenvalues.
- Section 3 formulates the gPC method for the stochastic imaginary part of the eigenvalue.
- Section 4 validates the method on a benchmark problem with a closed-form analytical solution.
- Section 5 concludes the paper with a summary and final remarks.

2. The Singular Level of the Elastic Asymptotic Expansion Series in the Vicinity of a Cracked Bi-Material Domain Having Stochastic Material Properties

2.1. The Eigenvalues of the Asymptotic Series in the Vicinity of a 2D Cracked Homogeneous Single-Material Isotropic Domain

The domain of interest is a 2D elastic isotropic cracked domain $\Omega \subset \mathbb{R}^2$, composed of a single material only, as presented in Figure 1. The two crack edges intersecting at the crack tip are denoted by Γ_1 and Γ_2 . The coordinates in Ω are (x_1, x_2) and will be denoted as x . Let (r, θ) be polar coordinates centered at the vertex of Ω .

The displacements in the vicinity of the crack tip, $\mathbf{u}(r, \theta) \stackrel{\text{def}}{=} (u_r \ u_\theta)^T$, are the solution of the Navier–Lamé Partial Differential Equations (PDEs) (assuming a plane strain condition and without body forces):

$$\begin{aligned} 2(\nu - 1) \left(u_r - r \frac{\partial u_r}{\partial r} - r^2 \frac{\partial^2 u_r}{\partial r^2} \right) + (1 - 2\nu) \frac{\partial^2 u_r}{\partial \theta^2} + (4\nu - 3) \frac{\partial u_\theta}{\partial \theta} + r \frac{\partial^2 u_\theta}{\partial r \partial \theta} &= 0 \\ r \frac{\partial^2 u_r}{\partial r \partial \theta} - (4\nu - 3) \frac{\partial u_r}{\partial \theta} - 2(\nu - 1) \frac{\partial^2 u_\theta}{\partial \theta^2} + (2\nu - 1) \left(u_\theta - r \frac{\partial u_\theta}{\partial r} - r^2 \frac{\partial^2 u_\theta}{\partial r^2} \right) &= 0 \end{aligned} \quad (1)$$

where ν is the material property: Poisson's ratio .

The expansion series of the displacement field in the vicinity of the crack tip is:

$$\mathbf{u}(r, \theta) = A_n r^{\alpha_n} \boldsymbol{\varphi}_i(\theta) = A_n r^{\alpha_i} \begin{pmatrix} \varphi_{r_n}(\theta) \\ \varphi_{\theta_n}(\theta) \end{pmatrix} \quad (2)$$

where $\alpha_n \geq 0$ are the eigenvalues of the displacement field, $\boldsymbol{\varphi}_n(\theta)$ are the eigenfunctions and A_n coefficients are the displacements associated SIF.

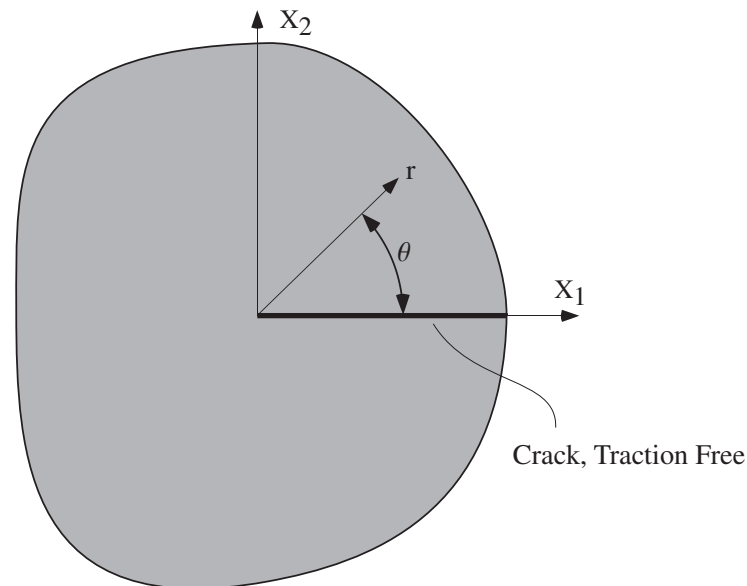


Figure 1. The 2D cracked single-material domain of interest, Ω .

By substituting (2) into Navier–Lamè system of PDEs (1), along with the traction-free boundary condition on both crack edges (Γ_1 & Γ_2), a condition is obtained on the eigenvalues:

$$[\sin(2\alpha\pi)]^2 = 0 \quad (3)$$

The first two eigenvalues that satisfy Equation (3) are zero. These zero eigenvalues are associated with rigid body motion transition in the r and θ directions and are therefore of no interest herein. The first two non-zero eigenvalues, obtained using (3), are $\alpha_1 = \alpha_2 = 1/2$, which give rise to a singular stress field at the crack tip. These two eigenvalues, α_1 and α_2 , are independent of the material properties. The stochastic variations in Young’s modulus or Poisson’s ratio will not introduce any randomness or affect these eigenvalues; they will remain constant. Therefore, the randomness of material properties will not influence the domain’s singularity level.

2.2. The Eigenvalues of the Asymptotic Series in the Vicinity of a 2D Cracked Homogeneous Bi-Material Isotropic Domain

We consider a 2D elastic cracked domain $\Omega \subset \mathbb{R}^2$, as presented in Figure 2. The domain consists of two materials: *Material 1* is an isotropic material with Young’s modulus, $E^{(1)}$, and Poisson’s ratio, $\nu^{(1)}$, occupying the range $0 \leq \theta \leq \omega^*$ of the domain. *Material 2* is an isotropic material with Young’s modulus, $E^{(2)}$, and Poisson’s ratio, $\nu^{(2)}$, which occupies the range $\omega^* \leq \theta \leq 2\pi$ of the domain. The domain contains a semi-infinite crack along the x_1 -axis, coinciding with the material interface. Both crack faces (located at $\theta = 0$ & $\theta = 2\pi$, denoted by Γ_1 & Γ_2 , respectively) are traction-free.

For each material, there exists a displacement equation, $\mathbf{u}^{(1)}(r, \theta)$ and $\mathbf{u}^{(2)}(r, \theta)$, that satisfies Navier–Lamé equations (see Equation (1)). Therefore, both displacement equations

are of the same form as $\mathbf{u}(r, \theta)$, as presented in Equation (2). The displacement fields must satisfy the traction-free boundary condition on both crack surfaces:

$$\begin{aligned}
 T|_{\Gamma_1} &= \begin{cases} \sigma_{r\theta}^{(1)}(r, \theta)|_{\theta=0} = 0 \\ \sigma_{\theta\theta}^{(1)}(r, \theta)|_{\theta=0} = 0 \end{cases} \\
 &= \begin{cases} \frac{E}{2(1+\nu)} \left(\frac{1}{r} \frac{\partial}{\partial \theta} u_r^{(1)}(r, \theta) + \frac{\partial}{\partial r} u_\theta^{(1)}(r, \theta) - \frac{1}{r} u_\theta^{(1)}(r, \theta) \right) |_{\theta=0} = 0 \\ \frac{E}{(1+\nu)(1-2\nu)} \left((1-\nu) \frac{1}{r} u_r^{(1)}(r, \theta) + \nu \frac{\partial}{\partial r} u_r^{(1)}(r, \theta) + (1-\nu) \frac{1}{r} \frac{\partial}{\partial \theta} u_\theta^{(1)}(r, \theta) \right) |_{\theta=0} = 0 \end{cases} \\
 T|_{\Gamma_2} &= \begin{cases} \sigma_{r\theta}^{(2)}(r, \theta)|_{\theta=2\pi} = 0 \\ \sigma_{\theta\theta}^{(2)}(r, \theta)|_{\theta=2\pi} = 0 \end{cases} \\
 &= \begin{cases} \frac{E}{2(1+\nu)} \left(\frac{2}{r} \frac{\partial}{\partial \theta} u_r^{(2)}(r, \theta) + \frac{\partial}{\partial r} u_\theta^{(2)}(r, \theta) - \frac{2}{r} u_\theta^{(2)}(r, \theta) \right) |_{\theta=2\pi} = 0 \\ \frac{E}{(1+\nu)(1-2\nu)} \left((1-\nu) \frac{1}{r} u_r^{(2)}(r, \theta) + \nu \frac{\partial}{\partial r} u_r^{(2)}(r, \theta) + (1-\nu) \frac{1}{r} \frac{\partial}{\partial \theta} u_\theta^{(2)}(r, \theta) \right) |_{\theta=2\pi} = 0 \end{cases}
 \end{aligned} \quad (4)$$

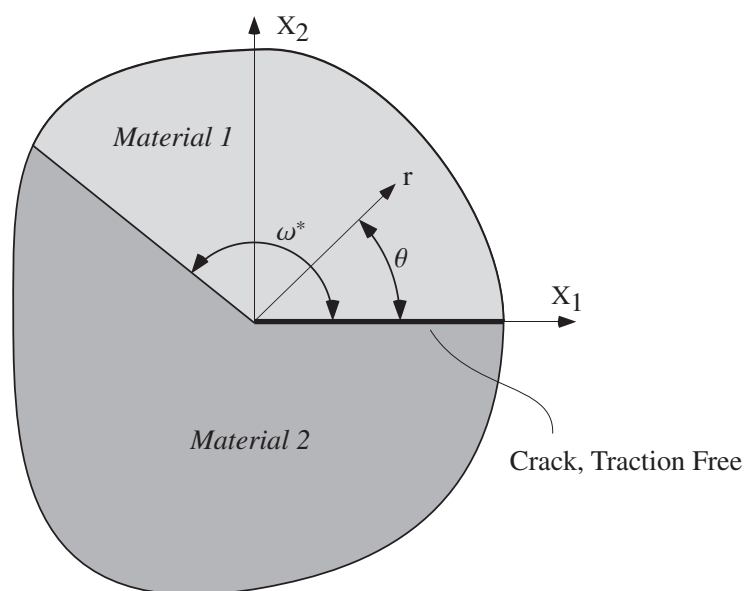


Figure 2. The 2D cracked bi-material domain of interest, Ω .

The displacement fields must also satisfy the continuity conditions at $\theta = \omega^*$. Displacement continuity at the bi-material interface:

$$\mathbf{u}^{(1)}(r, \omega^*) = \mathbf{u}^{(2)}(r, \omega^*) \quad \Rightarrow \quad \begin{cases} u_r^{(1)}(r, \omega^*) = u_r^{(2)}(r, \omega^*) \\ u_\theta^{(1)}(r, \omega^*) = u_\theta^{(2)}(r, \omega^*) \end{cases} \quad (5)$$

and traction continuity at the bi-material interface:

$$T^{(1)}|_{\theta=\omega^*} = T^{(2)}|_{\theta=\omega^*} \quad \Rightarrow \quad \begin{cases} \sigma_{r\theta}^{(1)}(r, \omega^*) = \sigma_{r\theta}^{(2)}(r, \omega^*) \\ \sigma_{\theta\theta}^{(1)}(r, \omega^*) = \sigma_{\theta\theta}^{(2)}(r, \omega^*) \end{cases} \quad (6)$$

When enforcing $\mathbf{u}^{(1)}(r, \theta)$ and $\mathbf{u}^{(2)}(r, \theta)$ to satisfy the Navier–Lamè Equation (1), together with the boundary conditions detailed in Equations (4)–(6), a condition is obtained for the eigenvalues. The condition obtained depends on the bonding angle of the two materials, ω^* , and the material properties of each of the isotropic materials that occupy the domain. In many cases, depending on the bonding angle between the materials, ω^* , analytical solutions are not available, and the eigenvalues can only be computed numerically

(the case of $\omega^* = \pi$ is exceptional and for which an analytical solution exists, as will be presented in the following.) There are two families of resulting eigenvalues for a crack at a bi-material interface:

- Integers, which are either zero or greater than 1 and therefore do not contribute to the singular behavior of the stress.
- Complex eigenvalues that come in conjugate pairs.

We focus herein on the complex eigenvalues of the expansion series:

$$\alpha_n = \alpha_n^{\Re} + i\alpha_n^{\Im}, \quad \alpha_{n+1} = \alpha_n^{\Re} - i\alpha_n^{\Im}, \quad n = 2, 4, 6, \dots \quad (7)$$

where $i \stackrel{\text{def}}{=} \sqrt{-1}$. The real part of the eigenvalues, $\alpha_n^{\Re}$, is independent of material properties. Among the complex conjugate pairs, the first pair (corresponding to $n = 2$) yields eigenvalues $\alpha_2 = \alpha_2^{\Re} + i\alpha_2^{\Im}$ and $\alpha_3 = \alpha_2^{\Re} - i\alpha_2^{\Im}$ with $\alpha_2^{\Re} = 1/2$. The subsequent pairs ($n = 4, 6, \dots$) have real parts $\alpha_n^{\Re} > 1$, which correspond to weaker singularities in the asymptotic expansion.

Since the stress field intensity in the vicinity of the crack tip is governed by the term with the strongest singularity (i.e., the smallest positive real part of the eigenvalue), we focus herein exclusively on the first complex conjugate pair with $n = 2$. This pair dominates the near-tip behavior, as the contribution from higher-order terms (with $\alpha_n^{\Re} > 1$) diminishes more rapidly with distance from the crack tip.

The imaginary part, $\alpha_2^{\Im}$, varies with the bonding angle ω^* and the material properties of both materials, and plays a crucial role in determining the oscillatory nature of the stress field near the interface crack tip. The magnitude of the imaginary part has significant consequences for the behavior of the stress field at the interface. A larger imaginary part leads to higher-frequency oscillations in the stress field, which, in turn, create more severe stress concentrations. These more rapid stress variations increase the likelihood of failure initiation at the interface, as the oscillatory nature of the stress field can promote the nucleation and growth of cracks. The severity of the stress singularity is thus directly related to the magnitude of the real and imaginary components of the eigenvalues.

2.3. The Effect of Stochastic Material Properties on the Eigenvalues of the Asymptotic Series

When characterizing isotropic materials, experimental measurements are performed to determine their fundamental material properties: Young's modulus and Poisson's ratio. Since these material property measurements inherently produce variable results, it becomes necessary to treat the properties as fundamentally stochastic. For the two isotropic materials under consideration (as presented in Figure 2), we define the stochastic properties of each material as follows: for the first material, Young's modulus $E^{(1)}(Y_1)$ and Poisson's ratio $\nu^{(1)}(Y_2)$, and for the second material, Young's modulus $E^{(2)}(Y_3)$ and Poisson's ratio $\nu^{(2)}(Y_4)$, where Y_1, Y_2, Y_3 , and Y_4 represent independent random variables. The precision and characteristics of the measurement equipment influence the statistical distribution of the four properties of the material. For the purpose of this demonstration, we assume normal distributions for all four stochastic material properties, as further discussed in Section 3. Young's modulus and Poisson's ratio of *Material 1* are expressed as:

$$\begin{aligned} E^{(1)}(Y_1) &= \mu_E^{(1)} + Y_1\sigma_E^{(1)} \quad \text{with} \quad Y_1 \sim \mathcal{N}(0, 1), \quad f_{Y_1}(t_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t_1^2}{2}} \\ \nu^{(1)}(Y_2) &= \mu_\nu^{(1)} + Y_2\sigma_\nu^{(1)} \quad \text{with} \quad Y_2 \sim \mathcal{N}(0, 1), \quad f_{Y_2}(t_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t_2^2}{2}} \end{aligned} \quad (8)$$

and for *Material 2*:

$$\begin{aligned} E^{(2)}(Y_3) &= \mu_E^{(2)} + Y_3 \sigma_E^{(2)} \quad \text{with} \quad Y_3 \sim \mathcal{N}(0, 1), \quad f_{Y_3}(t_3) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t_3^2}{2}} \\ \nu^{(2)}(Y_4) &= \mu_\nu^{(2)} + Y_4 \sigma_\nu^{(2)} \quad \text{with} \quad Y_4 \sim \mathcal{N}(0, 1), \quad f_{Y_4}(t_4) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t_4^2}{2}} \end{aligned} \quad (9)$$

where $\mu_E^{(1)}, \sigma_E^{(1)}$ and $f_{Y_1}(t_1)$ are the mean value, standard deviation and probability density function of Young's modulus of *Material 1*, and $\mu_\nu^{(1)}, \sigma_\nu^{(1)}$ and $f_{Y_2}(t_2)$ are the mean value, standard deviation and probability density function of Poisson's ratio of *Material 1*. Similarly, $\mu_E^{(2)}, \sigma_E^{(2)}, f_{Y_3}(t_3)$ and $\mu_\nu^{(2)}, \sigma_\nu^{(2)}, f_{Y_4}(t_4)$ correspond to *Material 2*.

When material properties are treated as stochastic, the real part of the eigenvalues remains unchanged and deterministic (1/2 for the first complex eigenvalues). However, the imaginary part of the eigenvalues becomes dependent on the material properties and is therefore inherently stochastic. This stochastic behavior directly influences the singularity level of the crack, potentially resulting in more severe singularities than would be anticipated when evaluating materials with deterministic properties. The first two complex eigenvalues are therefore:

$$\begin{aligned} \alpha_2(Y_1, Y_2, Y_3, Y_4) &= 1/2 + i\alpha_2^{\Im}(Y_1, Y_2, Y_3, Y_4) \\ \alpha_3(Y_1, Y_2, Y_3, Y_4) &= 1/2 - i\alpha_2^{\Im}(Y_1, Y_2, Y_3, Y_4) \end{aligned} \quad (10)$$

3. Evaluating the Stochastic Imaginary Component of the Eigenvalue, $\alpha_2^{\Im}$, via the gPC Method

In the following section, we apply the gPC method to approximate the imaginary stochastic component of the eigenvalue, $\alpha_2^{\Im}(Y_1, Y_2, Y_3, Y_4)$, which depends on the four parameters of the stochastic material. We denote this approximation by $\alpha_2^{\Im D}(Y_1, Y_2, Y_3, Y_4)$, which represents a projection of $\alpha_2^{\Im}(Y_1, Y_2, Y_3, Y_4)$ onto a finite-dimensional stochastic space spanned by the Wiener chaos expansion (see [39]). The approximation is constructed via hierarchical orthogonal polynomial projection, where the choice of polynomial family is determined by the underlying probability distribution of the stochastic variables (see [37,40]).

For the purpose of this demonstration, we assume normal distributions for all four stochastic material properties. This choice is made purely for simplicity and familiarity in illustrating the method and does not limit the generality of the methodology presented. Different distributions will yield different numerical results, just as different material parameters or bonding angles would. The gPC methodology itself remains the same, requiring only a corresponding choice of orthogonal polynomial family. Since material properties are modeled as normally distributed herein, we employ Hermite polynomials, which form the natural basis for projecting normally distributed variables in gPC theory. The stochastic approximation of the imaginary component is thus expressed as a multi-dimensional Hermite polynomial expansion, with each dimension corresponding to a single stochastic material property. Specifically, we employ a tensor product basis constructed from univariate Hermite polynomials $He_{n_k}(Y_k)$ for $k = 1, 2, 3, 4$:

$$\alpha_2^{\Im D}(Y_1, Y_2, Y_3, Y_4) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \sum_{n_3=0}^{N_3-1} \sum_{n_4=0}^{N_4-1} \alpha_{2,n_1,n_2,n_3,n_4}^{\Im D} He_{n_1}(Y_1) He_{n_2}(Y_2) He_{n_3}(Y_3) He_{n_4}(Y_4) \quad (11)$$

where $\alpha_{2,n_1,n_2,n_3,n_4}^{\Im D}$ are the deterministic projection coefficients, and N_k denotes the number of Hermite polynomials employed in the k -th dimension, corresponding to the k -th stochastic ma-

terial property. For notational convenience, we introduce the multi-index $\mathbf{n} = (n_1, n_2, n_3, n_4)$ and the shorthand notation $\Psi_{\mathbf{n}}(\mathbf{Y}) = He_{n_1}(Y_1)He_{n_2}(Y_2)He_{n_3}(Y_3)He_{n_4}(Y_4)$, where $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4)$.

We define the approximation error in the stochastic field as the difference between the exact stochastic imaginary component of the eigenvalue, $\alpha_2^{\Im}(\mathbf{Y})$, and its approximation, $\alpha_2^{\Im D}(\mathbf{Y})$:

$$\begin{aligned} e^{(\alpha_2^{\Im})}(\mathbf{Y}) &= \alpha_2^{\Im}(\mathbf{Y}) - \alpha_2^{\Im D}(\mathbf{Y}) \\ &= \alpha_2^{\Im}(\mathbf{Y}) - \sum_{\mathbf{n}} \alpha_{2,\mathbf{n}}^{\Im D} \Psi_{\mathbf{n}}(\mathbf{Y}) \end{aligned} \quad (12)$$

where the summation extends over all multi-indices satisfying $0 \leq n_k < N_k$ for $k = 1, 2, 3, 4$.

Due to the orthogonality of the Hermite polynomial basis, the error $e^{(\alpha_2^{\Im})}(\mathbf{Y})$ is orthogonal to each basis function $\Psi_j(\mathbf{Y})$. Consequently, the expected value of the product $e^{(\alpha_2^{\Im})}(\mathbf{Y})\Psi_j(\mathbf{Y})$ vanishes for all multi-indices j :

$$\mathbb{E}(e^{(\alpha_2^{\Im})}(\mathbf{Y})\Psi_j(\mathbf{Y})) \equiv 0 \quad (13)$$

where

$$\begin{aligned} \mathbb{E}(e^{(\alpha_2^{\Im})}(\mathbf{Y})\Psi_j(\mathbf{Y})) &= \int_{\mathbb{R}^4} e^{(\alpha_2^{\Im})}(\mathbf{s})\Psi_j(\mathbf{s})f_{\mathbf{Y}}(\mathbf{s})d\mathbf{s} \\ &= \int_{\mathbb{R}^4} \left(\alpha_2^{\Im}(\mathbf{s}) - \sum_{\mathbf{n}} \alpha_{2,\mathbf{n}}^{\Im D} \Psi_{\mathbf{n}}(\mathbf{s}) \right) \Psi_j(\mathbf{s})f_{\mathbf{Y}}(\mathbf{s})d\mathbf{s} \\ &= 0 \end{aligned} \quad (14)$$

Here, $\mathbf{s} = (s_1, s_2, s_3, s_4)$ and $f_{\mathbf{Y}}(\mathbf{s}) = \prod_{k=1}^4 f_{Y_k}(s_k)$, where the factorization follows from the assumed independence of the stochastic parameters. Rearranging the integral in (14) yields:

$$\int_{\mathbb{R}^4} \alpha_2^{\Im}(\mathbf{s})\Psi_j(\mathbf{s})f_{\mathbf{Y}}(\mathbf{s})d\mathbf{s} = \int_{\mathbb{R}^4} \sum_{\mathbf{n}} \alpha_{2,\mathbf{n}}^{\Im D} \Psi_{\mathbf{n}}(\mathbf{s})\Psi_j(\mathbf{s})f_{\mathbf{Y}}(\mathbf{s})d\mathbf{s} \quad (15)$$

The Left-Hand Side (LHS) of (15) is evaluated numerically using tensor product Gauss–Hermite quadrature:

$$\begin{aligned} LHS &= \int_{\mathbb{R}^4} \alpha_2^{\Im}(\mathbf{s})\Psi_j(\mathbf{s})f_{\mathbf{Y}}(\mathbf{s})d\mathbf{s} \\ &= \int_{\mathbb{R}^4} \alpha_2^{\Im}(\mathbf{s})\Psi_j(\mathbf{s}) \prod_{k=1}^4 \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{s_k^2}{2}} \right) d\mathbf{s} \\ &= \sum_{m_1=1}^{M_1} \sum_{m_2=1}^{M_2} \sum_{m_3=1}^{M_3} \sum_{m_4=1}^{M_4} \left(\prod_{k=1}^4 w_{m_k}^{(k)} e^{\frac{Y_{k,m_k}^2}{2}} \right) \alpha_2^{\Im}(Y_{1,m_1}, Y_{2,m_2}, Y_{3,m_3}, Y_{4,m_4}) \\ &\quad \times \Psi_j(Y_{1,m_1}, Y_{2,m_2}, Y_{3,m_3}, Y_{4,m_4}) \prod_{k=1}^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{Y_{k,m_k}^2}{2}} \\ &= \frac{1}{(2\pi)^2} \sum_{m_1=1}^{M_1} \sum_{m_2=1}^{M_2} \sum_{m_3=1}^{M_3} \sum_{m_4=1}^{M_4} \left(\prod_{k=1}^4 w_{m_k}^{(k)} e^{\frac{Y_{k,m_k}^2}{2}} \right) \alpha_2^{\Im}(\mathbf{Y}_{\mathbf{m}}) \Psi_j(\mathbf{Y}_{\mathbf{m}}) \end{aligned} \quad (16)$$

where Y_{k,m_k} denote the Gauss–Hermite quadrature points in the k -th dimension (i.e., the roots of the physical Hermite polynomial $H_{M_k}(Y_k)$), and $w_{m_k}^{(k)}$ are the corresponding quadrature weights. The parameter M_k represents the number of quadrature points employed in the k -th dimension. We use the compact notation $\mathbf{Y}_{\mathbf{m}} = (Y_{1,m_1}, Y_{2,m_2}, Y_{3,m_3}, Y_{4,m_4})$ with multi-index $\mathbf{m} = (m_1, m_2, m_3, m_4)$.

The Right-Hand Side (RHS) of (15) can be simplified using the orthogonality properties of Hermite polynomials:

$$\begin{aligned}
 RHS &= \int_{\mathbb{R}^4} \sum_{\mathbf{n}} \alpha_{2,\mathbf{n}}^{\mathfrak{S}D} \Psi_{\mathbf{n}}(\mathbf{s}) \Psi_{\mathbf{j}}(\mathbf{s}) f_{\mathbf{Y}}(\mathbf{s}) d\mathbf{s} \\
 &= \sum_{\mathbf{n}} \alpha_{2,\mathbf{n}}^{\mathfrak{S}D} \int_{\mathbb{R}^4} \Psi_{\mathbf{n}}(\mathbf{s}) \Psi_{\mathbf{j}}(\mathbf{s}) f_{\mathbf{Y}}(\mathbf{s}) d\mathbf{s} \\
 &= \sum_{\mathbf{n}} \alpha_{2,\mathbf{n}}^{\mathfrak{S}D} \prod_{k=1}^4 \int_{-\infty}^{\infty} He_{n_k}(s_k) He_{j_k}(s_k) \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{s_k^2}{2}} \right) ds_k \\
 &= \sum_{\mathbf{n}} \alpha_{2,\mathbf{n}}^{\mathfrak{S}D} \prod_{k=1}^4 (n_k! \delta_{n_k j_k}) \\
 &= \alpha_{2,\mathbf{j}}^{\mathfrak{S}D} \prod_{k=1}^4 j_k!
 \end{aligned} \tag{17}$$

Substituting (16) and (17) into (15) and solving for the projection coefficients yields:

$$\alpha_{2,\mathbf{j}}^{\mathfrak{S}D} = \frac{1}{(2\pi)^2 \prod_{k=1}^4 j_k!} \sum_{m_1=1}^{M_1} \sum_{m_2=1}^{M_2} \sum_{m_3=1}^{M_3} \sum_{m_4=1}^{M_4} \left(\prod_{k=1}^4 w_{m_k}^{(k)} e^{\frac{Y_{k,m_k}^2}{2}} \right) \alpha_2^{\mathfrak{S}}(\mathbf{Y}_{\mathbf{m}}) \Psi_{\mathbf{j}}(\mathbf{Y}_{\mathbf{m}}) \tag{18}$$

The values $\alpha_2^{\mathfrak{S}}(\mathbf{Y}_{\mathbf{m}})$ represent deterministic evaluations of the imaginary component, computed using the deterministic material properties corresponding to each quadrature point $\mathbf{Y}_{\mathbf{m}}$. Performing an index substitution in (18) and substituting the result into (11), we obtain the complete stochastic approximation, where the summations extend over multi-indices $\mathbf{n}$ with $0 \leq n_k < N_k$ and $\mathbf{m}$ with $1 \leq m_k \leq M_k$ for $k = 1, 2, 3, 4$. The convergence of this approximation is governed by four pairs of parameters: the numbers of Gauss-Hermite quadrature points M_k and the polynomial orders N_k . Although the approximation can be constructed with independent choices of M_k and N_k , optimal precision is achieved when $M_k = N_k$ for all dimensions (see [41,42]), rendering the approximation nonhierarchical. With $M_k = N_k$, the stochastic approximation becomes:

$$\alpha_2^{\mathfrak{S}D}(\mathbf{Y}) = \sum_{\mathbf{n}} \left(\frac{1}{(2\pi)^2 \prod_{k=1}^4 n_k!} \sum_{\mathbf{m}} \left(\prod_{k=1}^4 w_{m_k}^{(k)} e^{\frac{Y_{k,m_k}^2}{2}} \right) \alpha_2^{\mathfrak{S}}(\mathbf{Y}_{\mathbf{m}}) \Psi_{\mathbf{n}}(\mathbf{Y}_{\mathbf{m}}) \right) \Psi_{\mathbf{n}}(\mathbf{Y}) \tag{19}$$

where now $1 \leq m_k \leq N_k$ for $k = 1, 2, 3, 4$.

4. Benchmark Case

We consider a benchmark problem consisting of a crack at a 2D bi-material interface, as shown in Figure 3. This configuration corresponds to the special case where $\omega^* = \pi$ in the general formulation. Both materials are isotropic and initially assumed to have deterministic material properties. The upper material, denoted as *Material 1*, occupies the region $0 \leq \theta \leq \pi$ and is characterized by Young's modulus $E^{(1)}$ and Poisson's ratio $\nu^{(1)}$. The lower material, denoted as *Material 2*, occupies the region $\pi \leq \theta \leq 2\pi$ and is characterized by Young's modulus $E^{(2)}$ and Poisson's ratio $\nu^{(2)}$. The semi-infinite crack lies along the interface at $\theta = 0$ and $\theta = 2\pi$.

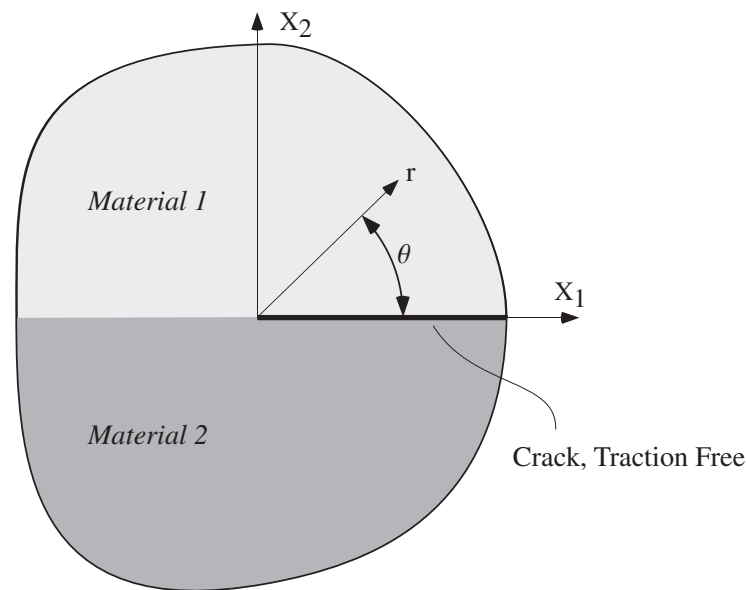


Figure 3. The 2D cracked bi-material domain of interest, Ω , for the benchmark case $\omega^* = \pi$.

This bi-material interface crack problem is a classical problem in fracture mechanics that admits a closed-form analytical solution (see [9,11–13,16]), making it an ideal benchmark for validating the stochastic analysis framework presented in this work.

This benchmark configuration is uniquely appropriate for validating the proposed stochastic gPC methodology. First, the exact closed-form analytical solution allows for a direct calculation of $\alpha_2^{\mathfrak{S}}(\mathbf{Y})$, without the need to numerically solve the characteristic equation at each realization. Consequently, it serves as an ideal reference against which the gPC approximation can be compared. Second, and crucially, the gPC framework itself is implemented independently of the analytical solution. The expansion is constructed by evaluating the characteristic equation numerically at the selected quadrature points, exactly as would be required in the general case where no closed-form solution exists. This independence ensures that the comparison provides a rigorous and unbiased validation of the proposed methodology.

4.1. Analytical Solution for the Eigenvalues

Following the procedure outlined in Section 2.2, we enforce the displacement fields $\mathbf{u}^{(1)}(r, \theta)$ and $\mathbf{u}^{(2)}(r, \theta)$ to satisfy the Navier–Lamé equations together with the boundary conditions specified in Equations (4)–(6). For the present benchmark case, the bonding angle is known and equals $\omega^* = \pi$. This specific geometric configuration yields a condition for the eigenvalues that, unlike the general case, admits a closed-form analytical solution. The eigenvalues for this bi-material interface crack configuration fall into two distinct families:

- Real eigenvalues $\alpha_n = (n - 1)/2$ for $n = 1, 3, 5, \dots$, which are either zero or greater than 1 and therefore do not contribute to the singular behavior of the stress field.
- Complex eigenvalues that appear in conjugate pairs.

As discussed previously, we focus on the first complex conjugate pair of eigenvalues ($n = 2, 3$), as these correspond to the dominant singularity near the crack tip. For a plane strain domain, the analytical solution for this conjugate pair is:

$$\begin{aligned}\alpha_2 &= 1/2 + i\alpha_2^{\mathfrak{S}} \\ \alpha_3 &= 1/2 - i\alpha_2^{\mathfrak{S}}\end{aligned}\quad (20)$$

where the imaginary component $\alpha_2^{\Im}$ is given by:

$$\alpha_2^{\Im} = \frac{1}{2\pi} \ln \left(\frac{E^{(2)}(4\nu^{(1)2} + \nu^{(1)} - 3) - E^{(1)}(1 + \nu^{(2)})}{E^{(1)}(4\nu^{(2)2} + \nu^{(2)} - 3) - E^{(2)}(1 + \nu^{(1)})} \right) \quad (21)$$

with $i \stackrel{\text{def}}{=} \sqrt{-1}$.

The real part of the eigenvalues, $\alpha^{\Re} = 1/2$, remains constant and independent of the material properties, a fundamental characteristic of the interface cracks. In contrast, the imaginary part $\alpha_2^{\Im}$ depends explicitly on all four material properties: $E^{(1)}$, $\nu^{(1)}$, $E^{(2)}$, and $\nu^{(2)}$. This dependency makes $\alpha_2^{\Im}$ sensitive to uncertainties in material properties, which motivates the stochastic analysis presented in this work.

4.2. Stochastic Formulation

The analytical expression (21) enables direct evaluation of the imaginary component when material properties are treated as random variables. Substituting the stochastic material properties defined in Equations (8) and (9) into (21), the imaginary component becomes a stochastic function:

$$\alpha_2^{\Im}(Y_1, Y_2, Y_3, Y_4) = \frac{1}{2\pi} \ln \left(\frac{E^{(2)}(Y_3)(4\nu^{(1)}(Y_2)^2 + \nu^{(1)}(Y_2) - 3) - E^{(1)}(Y_1)(1 + \nu^{(2)}(Y_4))}{E^{(1)}(Y_1)(4\nu^{(2)}(Y_4)^2 + \nu^{(2)}(Y_4) - 3) - E^{(2)}(Y_3)(1 + \nu^{(1)}(Y_2))} \right) \quad (22)$$

This closed-form expression provides the exact values of $\alpha_2^{\Im}(\mathbf{Y})$. The availability of an analytical solution makes this bi-material interface crack problem an ideal benchmark for validating the proposed stochastic methodology. Specifically, we can directly compare the gPC approximation $\alpha_2^{\Im D}(\mathbf{Y})$ obtained via Equation (19) against the exact analytical solution given by Equation (22). This comparison allows us to rigorously assess the accuracy of the stochastic approximation by computing the relative error between the gPC approximation and the analytical solution at various realizations of the random variables, examine the convergence behavior of the approximation as the polynomial order N_k increases, and evaluate the computational efficiency of the method. These quantitative measures provide a comprehensive evaluation of the method's performance and establish confidence in its applicability to more complex configurations where analytical solutions are unavailable.

4.3. Numerical Example and Validation

To validate the gPC methodology and assess its accuracy, we consider a specific numerical example with the following stochastic material properties. For *Material 1*:

$$\begin{aligned} E^{(1)}(Y_1) &= 10 + 1.6Y_1 & \text{with } Y_1 &\sim \mathcal{N}(0, 1) \\ \nu^{(1)}(Y_2) &= 0.3 + 0.04Y_2 & \text{with } Y_2 &\sim \mathcal{N}(0, 1) \end{aligned} \quad (23)$$

and for *Material 2*:

$$\begin{aligned} E^{(2)}(Y_3) &= 1 + 0.12Y_3 & \text{with } Y_3 &\sim \mathcal{N}(0, 1) \\ \nu^{(2)}(Y_4) &= 0.3 + 0.02Y_4 & \text{with } Y_4 &\sim \mathcal{N}(0, 1) \end{aligned} \quad (24)$$

This configuration represents a significant elastic mismatch between the two materials, with a mean Young's modulus ratio of approximately 10:1. Both materials have identical mean Poisson's ratios of 0.3, but *Material 1* exhibits greater stochastic variability in both properties. With all four material properties being stochastic, the problem exists in a four-dimensional stochastic space (Y_1, Y_2, Y_3, Y_4) . We compute

the imaginary stochastic component of the eigenvalue $\alpha_2^{\Im}(Y_1, Y_2, Y_3, Y_4)$ using both the exact analytical expression given by Equation (22) and the gPC approximation obtained by Equation (19). For the gPC approximation, we employ polynomial orders N_k for $k = 1, 2, 3, 4$, with $M_k = N_k$ as recommended in Section 3. At each Gauss–Hermite quadrature point $\mathbf{Y}_m = (Y_{1,m_1}, Y_{2,m_2}, Y_{3,m_3}, Y_{4,m_4})$, the corresponding material properties are obtained via Equations (23) and (24), and the characteristic equation governing the eigenvalues of the bi-material interface is numerically solved to obtain the imaginary component $\alpha_2^{\Im}(\mathbf{Y}_m)$, which is then used to compute the projection coefficients via Equation (18). Although the closed-form expression (21) is available for this benchmark configuration and is used to generate the Monte Carlo reference solution (see in the following Section 4.3.1), it is deliberately not used in constructing the gPC approximation itself, to demonstrate the applicability of the proposed framework to the general case, where there is no analytical solution and the characteristic equation must be solved numerically at every sample point. The resulting gPC approximation $\alpha_2^{\Im D}(Y_1, Y_2, Y_3, Y_4)$ is then compared with the analytical solution to assess convergence and accuracy.

4.3.1. Statistical Characterization via the Analytical Expression

Given the material property definitions in Equations (23) and (24), the imaginary stochastic component of the eigenvalue may be calculated using Equation (22). Substituting the specific material properties into the analytical expression yields the following stochastic equation for the imaginary component of the eigenvalue:

$$\alpha_2^{\Im}(\mathbf{Y}) = \frac{1}{2\pi} \ln \left(\frac{(-2.7 + 4(0.3 + 0.04Y_2)^2 + 0.04Y_2)(1 + 0.12Y_3) - (10 + 1.6Y_1)(1.3 + 0.02Y_4)}{-(1.3 + 0.04Y_2)(1 + 0.12Y_3) + (10 + 1.6Y_1)(-2.7 + 4(0.3 + 0.02Y_4)^2 + 0.02Y_4)} \right) \quad (25)$$

Due to the non-linear logarithmic form of Equation (25), analytical computation of statistical properties, such as mean and variance of $\alpha_2^{\Im}(\mathbf{Y})$, is analytically impractical. Therefore, we employ a Monte Carlo simulation to characterize these stochastic parameters numerically.

The Monte Carlo procedure consists of generating a large set of Λ samples by randomly drawing of $Y_1^{(k)}, Y_2^{(k)}, Y_3^{(k)}$ and $Y_4^{(k)}$ from their respective standard normal distributions, where $k = 1, \dots, \Lambda$. For each sample, the imaginary component is evaluated directly using Equation (26):

$$\begin{aligned} \alpha_2^{\Im(k)} &= \alpha_2^{\Im}(\mathbf{Y}^{(k)}) = \alpha_2^{\Im}(Y_1^{(k)}, Y_2^{(k)}, Y_3^{(k)}, Y_4^{(k)}) \\ &= \frac{1}{2\pi} \ln \left(\frac{(-2.7 + 4(0.3 + 0.04Y_2^{(k)})^2 + 0.04Y_2^{(k)})(1 + 0.12Y_3^{(k)}) - (10 + 1.6Y_1^{(k)})(1.3 + 0.02Y_4^{(k)})}{-(1.3 + 0.04Y_2^{(k)})(1 + 0.12Y_3^{(k)}) + (10 + 1.6Y_1^{(k)})(-2.7 + 4(0.3 + 0.02Y_4^{(k)})^2 + 0.02Y_4^{(k)})} \right) \end{aligned} \quad (26)$$

The statistical moments are then estimated from the sample ensemble as follows:

$$\begin{aligned} \mu^{(\text{MC})} &= \mathbb{E}[\alpha_2^{\Im}] \approx \frac{1}{\Lambda} \sum_{k=1}^{\Lambda} \alpha_2^{\Im(k)} \\ \text{Var}^{(\text{MC})} &= \text{Var}[\alpha_2^{\Im}] \approx \frac{1}{\Lambda} \sum_{k=1}^{\Lambda} \left(\alpha_2^{\Im(k)} - \mu^{(\text{MC})} \right)^2 \\ \text{Skew}^{(\text{MC})} &= \text{Skewness}[\alpha_2^{\Im}] \approx \frac{1}{\Lambda} \sum_{k=1}^{\Lambda} \left(\frac{\alpha_2^{\Im(k)} - \mu^{(\text{MC})}}{\sqrt{\text{Var}^{(\text{MC})}}} \right)^3 \\ \text{Kurt}^{(\text{MC})} &= \text{Kurtosis}[\alpha_2^{\Im}] \approx \frac{1}{\Lambda} \sum_{k=1}^{\Lambda} \left(\frac{\alpha_2^{\Im(k)} - \mu^{(\text{MC})}}{\sqrt{\text{Var}^{(\text{MC})}}} \right)^4 \end{aligned} \quad (27)$$

where $\mu^{(MC)}$ denotes the mean value, $\text{Var}^{(MC)}$ denotes the variance, $\text{Skew}^{(MC)}$ denotes the skewness, and $\text{Kurt}^{(MC)}$ denotes the Pearson kurtosis (without the correction -3 , where a normal distribution has kurtosis of 3).

In this study, we perform the Monte Carlo simulation with $\Lambda = 5 \times 10^5$ samples. To verify that the statistical estimates obtained with this sample size are adequately converged, we examine the evolution of the stochastic properties as a function of the sample size. The convergence is quantified through the relative difference between statistical properties computed with the full sample set ($\Lambda = 5 \times 10^5$) and those computed with a reduced number of samples $\kappa < \Lambda$:

$$\% \text{ Relative Difference } (\alpha_2^{\mathfrak{S}} |_{\gamma_\eta}^{(MC)}) = 100 \left| \frac{\gamma_\eta[\alpha_2^{\mathfrak{S}}]_{\Lambda}^{(MC)} - \gamma_\eta[\alpha_2^{\mathfrak{S}}]_{\kappa}^{(MC)}}{\gamma_\eta[\alpha_2^{\mathfrak{S}}]_{\Lambda}^{(MC)}} \right| \quad (28)$$

where γ_η are the stochastic properties:

$$\gamma_1 = \text{Mean}, \quad \gamma_2 = \text{Variance}, \quad \gamma_3 = \text{Skewness}, \quad \gamma_4 = \text{Kurtosis}$$

Figure 4 presents the relative difference as a function of κ , demonstrating that all four stochastic properties (Mean, Variance, Skewness, Kurtosis) converge well before reaching $\Lambda = 5 \times 10^5$ samples, with relative differences below 2% Skewness and below 0.4% in the Mean, Variance, and Kurtosis. This high level of precision confirms that the Monte Carlo estimates can be regarded as converged reference values for subsequent validation of the gPC approximation. The convergence behavior shown in Figure 4 is consistent with the classical Central Limit Theorem: the standard error of a Monte Carlo estimator decreases as $1/\sqrt{\kappa}$ with the number of samples κ , corresponding to an algebraic (rather than exponential) rate of convergence.

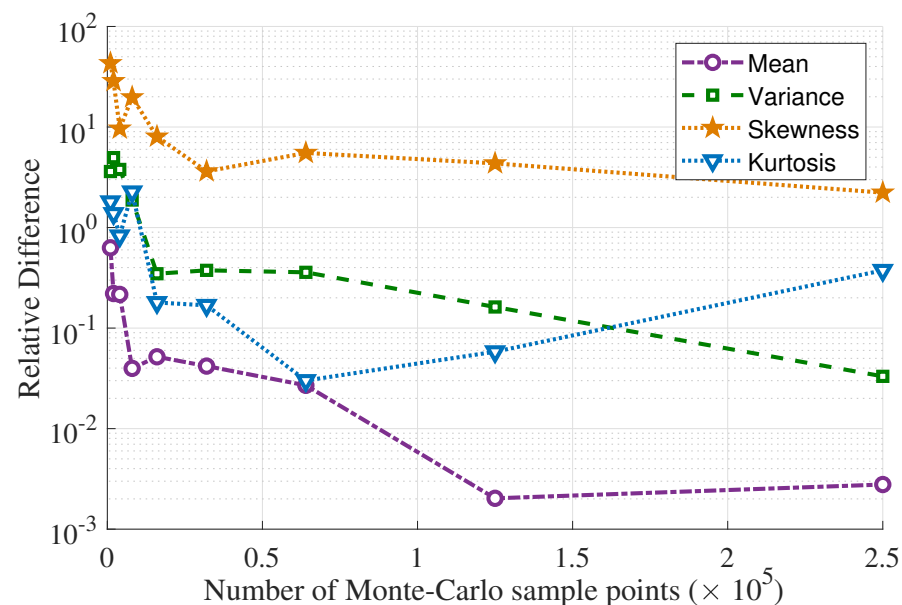


Figure 4. Relative difference of $\alpha_2^{\mathfrak{S}}(\mathbf{Y})$ stochastic properties, computed with κ samples compared to the reference solution with $\Lambda = 5 \times 10^5$ samples, as defined in (28), demonstrating Monte Carlo convergence for the bi-material interface crack with stochastic material properties defined in Equations (23) and (24).

4.3.2. Statistical Characterization via the Proposed Method, Using gPC

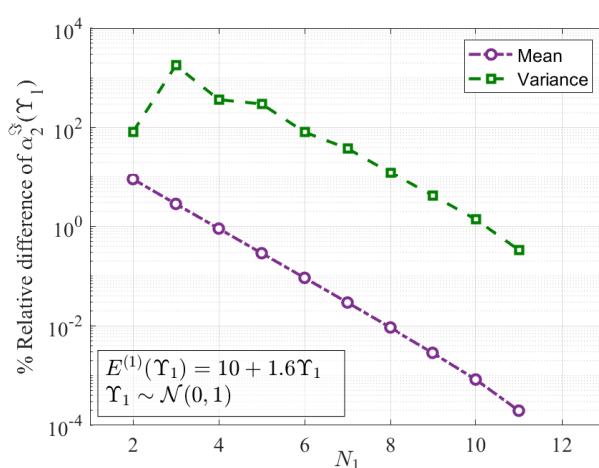
We approximate the imaginary stochastic component of the eigenvalue, $\alpha_2^{\mathfrak{S}}(\mathbf{Y})$, by applying the direct gPC method as presented in Equation (19). In this example, four in-

dependent stochastic material properties are considered: $E^{(1)}(Y_1)$, $\nu^{(1)}(Y_2)$, $E^{(2)}(Y_3)$, and $\nu^{(2)}(Y_4)$. Since these properties are statistically independent, their individual contributions to the uncertainty can be decoupled, allowing us to verify the convergence of each property separately. To this end, we perform four independent convergence tests; in each test, only one material property is treated as a stochastic variable while the remaining three are held deterministic.

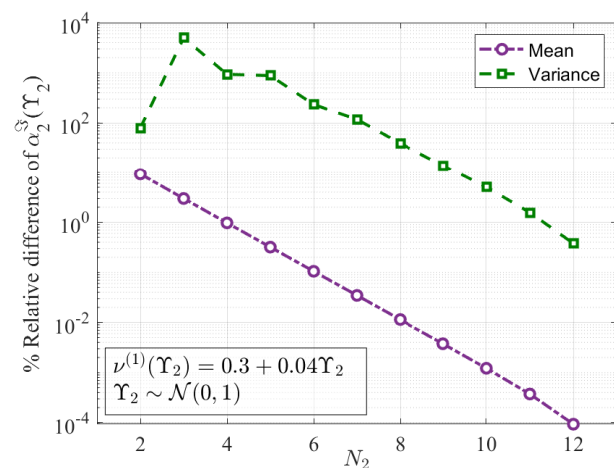
For each convergence test, only one material property is treated as a stochastic variable while the others remain deterministic. The expansion order, N_i , is increased incrementally to identify the minimum number of Hermite polynomials required to capture the stochastic behavior accurately. The convergence is monitored by evaluating the relative difference in the statistical moments (Mean and Variance) between successive approximation steps:

$$\begin{aligned} \% \text{ Relative Difference } (\alpha_2^{\mathfrak{S}}|_{\mu}^{(\text{gPC})}) &= 100 \left| \frac{\mu[\alpha_2^{\mathfrak{S}}]_{N_k=\gamma}^{(\text{gPC})} - \mu[\alpha_2^{\mathfrak{S}}]_{N_k=\gamma-1}^{(\text{gPC})}}{\mu[\alpha_2^{\mathfrak{S}}]_{N_k=\gamma}^{(\text{gPC})}} \right| \\ \% \text{ Relative Difference } (\alpha_2^{\mathfrak{S}}|_{\text{Var}}^{(\text{gPC})}) &= 100 \left| \frac{\text{Var}[\alpha_2^{\mathfrak{S}}]_{N_k=\gamma}^{(\text{gPC})} - \text{Var}[\alpha_2^{\mathfrak{S}}]_{N_k=\gamma-1}^{(\text{gPC})}}{\text{Var}[\alpha_2^{\mathfrak{S}}]_{N_k=\gamma}^{(\text{gPC})}} \right| \end{aligned} \quad (29)$$

The convergence process ends once the relative difference falls below 1%. A relative difference below 1% represents a particularly high level of accuracy in the context of stochastic statistical moments, which are inherently sensitive to the finite sample or expansion order used in their estimation. Following this procedure, the required expansion orders were determined to be $N_1 = 12$, $N_2 = 13$, $N_3 = 12$, and $N_4 = 10$. Figure 5 illustrates the convergence of these moments for each stochastic material property as a function of the number of polynomials. The markedly faster convergence observed in Figure 5, compared to the Monte Carlo convergence in Figure 4, reflects the spectral convergence property of the gPC method: for a sufficiently smooth (analytic) function of the underlying random variable, expansion in an orthogonal polynomial basis converges exponentially with the polynomial order, in contrast to the algebraic convergence rate of Monte Carlo sampling.



(a)



(b)

Figure 5. Cont.

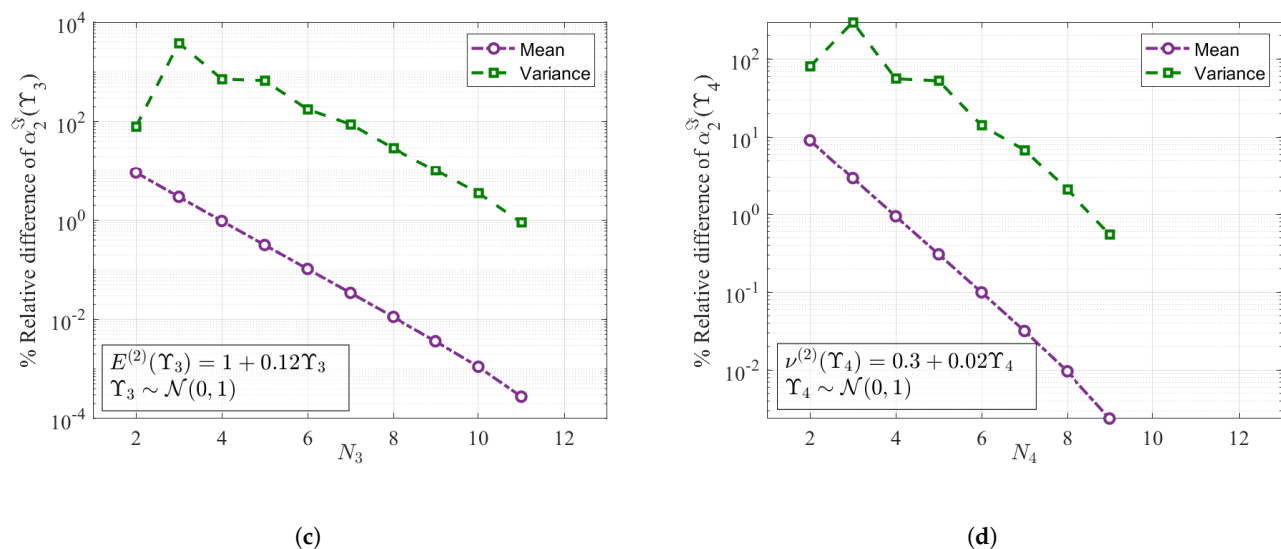


Figure 5. Relative difference of stochastic properties (mean and variance) of α_2^S computed with increasing number of Hermite polynomials and Gauss–Hermite Integration points, as defined in (29). (a) Convergence for $\alpha_2^S(Y_1)$. (b) Convergence for $\alpha_2^S(Y_2)$. (c) Convergence for $\alpha_2^S(Y_3)$. (d) Convergence for $\alpha_2^S(Y_4)$.

4.3.3. Validation of the gPC Stochastic Approximation Compared with the Analytical Expression

Following the convergence analysis, we now evaluate the full stochastic response by treating all four material properties as independent stochastic variables simultaneously. The gPC approximation for the imaginary component of the eigenvalue, $\alpha_2^S(\mathbf{Y})$, is re-computed using the optimal expansion orders previously identified: $N_1 = 12$, $N_2 = 13$, $N_3 = 12$, and $N_4 = 10$. By incorporating all stochastic dimensions into a single multidimensional expansion, we obtain the stochastic approximation of $\alpha_2^S(\mathbf{Y})$. To validate this approximation, the statistical moments (Mean, Variance, Skewness, and Kurtosis) derived from the gPC results are compared against the Monte Carlo reference values, obtained by sampling the exact analytical expression (Equation (22)). This comparison assesses whether the gPC approximation can accurately capture the statistical behavior of $\alpha_2^S(\mathbf{Y})$.

In Table 1, we present a comparison of the stochastic properties (Mean, Variance, Skewness, and Kurtosis) obtained from the two methods. The gPC approximation is compared with the Monte Carlo reference, obtained by sampling the exact analytical expression with $\Lambda = 5 \times 10^5$ samples. The relative difference between the two methods is defined as:

$$\begin{aligned}
 \% \text{ Relative Difference}(\mu) &= 100 \left| \frac{\mu[\alpha_2^S]_{\Lambda}^{(\text{MC})} - \mu[\alpha_2^S]^{(\text{gPC})}}{\mu[\alpha_2^S]_{\Lambda}^{(\text{MC})}} \right| \\
 \% \text{ Relative Difference}(\text{Var}) &= 100 \left| \frac{\text{Var}[\alpha_2^S]_{\Lambda}^{(\text{MC})} - \text{Var}[\alpha_2^S]^{(\text{gPC})}}{\text{Var}[\alpha_2^S]_{\Lambda}^{(\text{MC})}} \right| \\
 \% \text{ Relative Difference}(\text{Skew}) &= 100 \left| \frac{\text{Skew}[\alpha_2^S]_{\Lambda}^{(\text{MC})} - \text{Skew}[\alpha_2^S]^{(\text{gPC})}}{\text{Skew}[\alpha_2^S]_{\Lambda}^{(\text{MC})}} \right| \\
 \% \text{ Relative Difference}(\text{Kurt}) &= 100 \left| \frac{\text{Kurt}[\alpha_2^S]_{\Lambda}^{(\text{MC})} - \text{Kurt}[\alpha_2^S]^{(\text{gPC})}}{\text{Kurt}[\alpha_2^S]_{\Lambda}^{(\text{MC})}} \right|
 \end{aligned} \tag{30}$$

where μ = Mean, Var = Variance, Skew = Skewness, Kurt = Kurtosis.

Table 1. Comparison of statistical properties: gPC vs. Monte Carlo Reference.

Method	Mean	Variance	Skewness	Kurtosis	Evaluations
MC Reference	7.537006×10^{-2}	5.612319×10^{-5}	−0.151322	3.113254	5×10^5
Proposed gPC Method	7.536959×10^{-2}	5.631133×10^{-5}	−0.149842	3.113254	18,720
Relative Difference	$6.336026 \times 10^{-4}\%$	0.335238%	0.978125%	0.928297%	

Note: MC Reference values are obtained via Monte Carlo sampling ($\Lambda = 5 \times 10^5$) of the exact closed-form analytical solution, Equation (21).

The total number of evaluations, presented in Table 1, follows directly from the tensor-product structure of the gPC method: since $M_k = N_k$ in each dimension, the total number of quadrature points is the product $N_1 \times N_2 \times N_3 \times N_4 = 12 \times 13 \times 12 \times 10 = 18720$, where the four factors correspond, respectively, to the minimal expansion orders for $E^{(1)}$, $\nu^{(1)}$, $E^{(2)}$, and $\nu^{(2)}$, as identified in the convergence study of Figure 5. As shown in Table 1, the approximation of the proposed gPC method is in excellent agreement with the analytical solution reference, with a relative difference of less than 1% error at each of the statistical moments. This demonstrates that the gPC method can accurately capture the statistical behavior of the stochastic eigenvalue while requiring several orders of magnitude fewer function evaluations than Monte Carlo sampling.

To further illustrate the accuracy of the gPC approximation, Figure 6 presents a comparison of the probability density functions obtained from both methods. The Monte Carlo PDF (shown as bars) is constructed as a normalized histogram from $\Lambda = 5 \times 10^5$ samples, while the gPC PDF (shown as a solid black line) is derived from the stochastic approximation $\alpha_2^{\mathfrak{S}D}(\mathbf{Y})$ computed using the proposed method. This visual comparison further supports that the gPC approximation accurately captures not only the statistical moments but also the entire probability distribution of $\alpha_2^{\mathfrak{S}}(\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4)$, consistent with the quantitative agreement reported in Table 1.

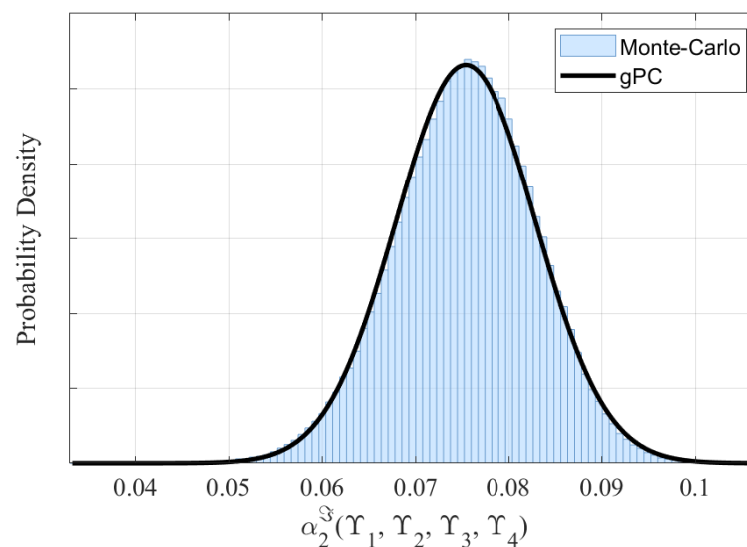


Figure 6. Comparison of probability density functions of $\alpha_2^{\mathfrak{S}}(\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4)$: Monte Carlo histogram ($\Lambda = 5 \times 10^5$ samples, bars) versus gPC approximation (solid line).

The computational efficiency of the gPC method is evident from the comparison: while the Monte Carlo simulation requires $\Lambda = 5 \times 10^5$ evaluations of Equation (25), the gPC approximation achieves comparable accuracy with only 18720 evaluations. This represents a significant reduction in computational cost, requiring less than 4% of the iterations

compared to traditional sampling methods. It should be noted that the gPC procedure itself remains unchanged regardless of the bonding angle, since it already relies on a numerical solution of the characteristic equation at each quadrature point. In contrast, the cost of computing the Monte Carlo reference depends on the bonding angle. In this benchmark case, the availability of the closed-form analytical expression (25) makes each Monte Carlo evaluation computationally inexpensive, requiring only the evaluation of a logarithmic function. However, for general interface crack configurations with arbitrary bonding angles $\omega^* \neq \pi$, there is no analytical solution for the eigenvalues. In such cases, each evaluation would require solving the characteristic equation numerically (e.g., via root-finding algorithms or eigenvalue solvers), significantly increasing the computational cost per sample. Under these circumstances, the computational advantage of the gPC method becomes even more pronounced. The 18,720 numerical eigenvalue solutions required for gPC would replace 5×10^5 such solutions in Monte Carlo, resulting in a dramatic reduction in overall computational time.

We present in Table 2 a comparison between the deterministic estimate and several representative points of the stochastic distribution, in order to further illustrate the practical implications of these findings. The deterministic estimate, obtained by substituting the mean material properties (Equations (23) and (24)) into Equation (21), represents the value that is conventionally computed when the problem is treated deterministically, without accounting for the stochastic nature of the material properties. Denoting by $\sigma_{\alpha_2^{\Im}}$ the standard deviation of the stochastic distribution of $\alpha_2^{\Im}$ itself (Table 1), we find that while the deterministic estimate closely matches the median of the stochastic distribution, values one to three standard deviations above the mean (mean + $k\sigma_{\alpha_2^{\Im}}$, $k = 1, 2, 3$) are 9.3% to 29.1% higher. Since the imaginary eigenvalue component directly governs the frequency and intensity of stress oscillations near the interface crack tip, this demonstrates that deterministic analysis based solely on mean material properties may substantially underestimate the severity of the singularity.

Table 2. Comparison of the deterministic eigenvalue estimate with the stochastic distribution.

Quantity	Value	Empirical Percentile	% Increase vs. Deterministic
Deterministic estimate (mean properties)	0.07581	51.4%	—
Stochastic mean	0.07537	50.0%	−0.6%
Mean + $1\sigma_{\alpha_2^{\Im}}$	0.08286	84.2%	+9.3%
Mean + $2\sigma_{\alpha_2^{\Im}}$	0.09036	98.1%	+19.2%
Mean + $3\sigma_{\alpha_2^{\Im}}$	0.09785	99.9%	+29.1%

These results validate the proposed gPC methodology as an accurate and efficient approach to uncertainty quantification in bi-material interface crack problems. The method successfully propagates material property uncertainties through the complex nonlinear relationship to obtain the stochastic behavior of the imaginary component of the eigenvalues.

5. Summary and Conclusions

This work presented a stochastic gPC-based framework for quantifying uncertainty in the imaginary component of the complex eigenvalue governing the singularity at a bi-material interface crack, accounting for four independent stochastic material properties (Young's modulus and Poisson's ratio of each material). Unlike a homogeneous crack, whose dominant eigenvalue is fixed and independent of material properties, the imaginary part of the interface-crack eigenvalue depends explicitly and nonlinearly on all four

properties, so that neglecting their randomness may underestimate the true severity of the singularity.

The stochastic approximation of the imaginary eigenvalue component was constructed using the gPC method, in which the material property uncertainty is represented through a multi-dimensional tensor-product expansion of orthogonal polynomials, with the polynomial family dictated by the probability distribution of each property. The deterministic expansion coefficients were obtained from evaluations of the imaginary component at Gauss–Hermite quadrature points.

The framework was validated on a benchmark bi-material interface crack problem that admits a closed-form analytical solution. The gPC approximation was constructed independently of this analytical solution, by numerically solving the characteristic equation at each quadrature point, exactly as required for a general bonding angle where no closed-form solution exists. The approximation converged rapidly with increasing polynomial order, achieving relative differences below 1% in all statistical moments (mean, variance, skewness, and kurtosis) compared with the Monte Carlo evaluation of the exact solution, using only 18,720 evaluations, a small fraction of the 5×10^5 Monte Carlo samples required for comparable statistical accuracy. This computational advantage is expected to become substantially more pronounced for general interface configurations, where each Monte Carlo sample would itself require a costly numerical solution of the characteristic equation.

While normally distributed material properties and the corresponding Hermite polynomial basis were adopted here for simplicity of presentation, the methodology extends directly to other probability distributions by choosing an appropriate orthogonal polynomial family. These results highlight the importance of accounting for material stochasticity when assessing interface crack singularities and, consequently, when formulating reliable failure criteria for bonded structures.

6. Limitations

A methodological limitation of the proposed framework concerns the non-hierarchical nature of the chosen quadrature scheme. Since $M_k = N_k$ in each stochastic dimension (Section 3), the quadrature points depend on the polynomial order itself. If, upon inspection, a higher polynomial order is found to be necessary for any single material property to further improve accuracy, for instance, the quadrature points change entirely. As a result, the imaginary eigenvalue component must be re-evaluated at an entirely new set of points, even for the same problem. Previously computed evaluations cannot be reused. This limitation could be addressed in future work by adopting hierarchical, nested quadrature schemes, such as Gauss–Kronrod rules or sparse grids with nested nodes, which allow the polynomial order to be refined incrementally without discarding earlier evaluations.

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Abbreviations

The following abbreviations are used in this manuscript:

gPC	Generalized Polynomial Chaos
SIFs	Stress Intensity Factors
PDEs	Partial Differential Equations
LHS	Left-Hand Side

RHS	Right-Hand Side
MC	Monte Carlo
Var	Variance
Skew	Skewness
Kurt	Kurtosis
PDF	Probability Density Function

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