

Article

Composition-Generated Higher-Order and Two-Branch Fuzzy Implications: Interaction Mechanisms and Asymptotic Behavior

Panagiotis G. Mangenakis * and Basil K. Papadopoulos

Section of Mathematics and Informatics, Department of Civil Engineering, Democritus University Thrace, 67100 Xanthi, Greece; papadob@civil.duth.gr

* Correspondence: pmangena@civil.duth.gr; Tel.: +30-6989953584

Abstract

In this paper, we study composition-generated fuzzy implications induced by strict fuzzy negations. Starting from the known Yager-type generated implication $I_N(x, y) = N(xN^{-1}(y))$, we first isolate a general monotone precomposition principle and use it to organize power-type families as structured preliminary constructions. These families clarify how order parameters act on the antecedent-side component and on the internal consequent-side scale $N^{-1}(y)$, and they provide a basis for studying saturation phenomena. The main constructional part of the paper introduces Yager-induced two-branch fuzzy implications with explicit admissibility conditions. Their n -order extensions are obtained through repeated product-composition of internal Yager-type components. We also introduce a recursive feedback mechanism based on a new auxiliary two-branch implication, where the order parameter counts feedback iterations rather than powers of an internal product. The asymptotic analysis shows that power-type constructions may converge pointwise toward the maximal truth value on large parts of the unit square, while the two-branch and recursive mechanisms may produce branch-dependent limiting behavior. Thus, the paper separates closure-based transformations, direct two-branch composition mechanisms, and recursive implication–composition within fuzzy implication theory.

Keywords: fuzzy implication; strict fuzzy negation; asymptotic behavior; two-branch implication; recursive feedback; n -order transformation; consequent-side implication; antecedent-side implication; uncertainty modeling

MSC: 03B52

Academic Editor: Marius Radulescu

Received: 22 June 2026

Revised: 22 July 2026

Accepted: 24 July 2026

Published: 26 July 2026

Copyright: © 2026 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

1. Introduction

Fuzzy implications are fundamental operators in fuzzy logic, since they provide a mathematical interpretation of conditional rules of the form “if x , then y ”. They play an important role in approximate reasoning, fuzzy rule-based systems, aggregation theory, decision-making, and uncertainty modeling. A fuzzy implication is usually defined as a binary operator on the unit interval satisfying suitable monotonicity and boundary conditions, thereby extending classical implication to a many-valued setting. Several important classes have been studied in the literature, including S -implications, R -

implications, QL -implications, generated implications, and implications induced by fuzzy negations and aggregation functions [1].

Generated implications form a significant class of fuzzy implications. Yager introduced important families of generated implication operators and studied their role in approximate reasoning [2], while their properties and relationships with other well-known classes were later analyzed by Baczyński and Jayaram [3]. Thus, generator-based fuzzy implications are a well-established part of the theory. In the present paper, the Yager-type implication $I_N(x, y) = N(xN^{-1}(y))$, $x, y \in [0, 1]$, is used as a known baseline construction. It combines the antecedent degree x with the internal consequent-side scale $N^{-1}(y)$, and then maps the product back through the strict fuzzy negation N .

The paper is organized around a clear hierarchy of constructions. First, we isolate a general monotone precomposition principle for fuzzy implications. This principle explains why power-type families such as $N(x^n N^{-1}(y))$, $N(x(N^{-1}(y))^n)$, and their unified (p, q) -order form $I_{p,q}(x, y) = N(x^p (N^{-1}(y))^q)$ preserve the basic fuzzy implication axioms under the corresponding assumptions. These families are presented as structured preliminary constructions, not as the principal novelty of the paper. Their role is to clarify how order parameters act on the antecedent-side component x , on the internal consequent-side scale $N^{-1}(y)$, or on both components simultaneously.

Several construction methods for fuzzy implications have already been proposed, including methods based on t -norms, t -conorms, fuzzy negations, aggregation operators, generated functions, overlap functions, copulas, quasi-copulas, and related transformations [4–12]. Two-branch and piecewise constructions are also related to the broader study of structured fuzzy operators and their interaction with aggregation-type mechanisms [13,14]. Although the present paper does not address statistical or machine-learning calibration, formal implication operators remain relevant in symbolic and rule-based models of uncertainty, where connections with computational learning and uncertainty quantification have been explored from different perspectives [15–17].

The main constructional part of the paper begins after the closure-based preliminary framework. We introduce two Yager-induced two-branch fuzzy implication constructions. These operators are generated from the same baseline implication I_N , but they are defined by different analytic expressions on the two regions $x \leq y$ and $y \leq x$.

The first construction uses the Yager-type component along transformed pairs of the form $(N(t), t)$, whereas the second inserts the Yager-type component directly into the two branches. In both cases, continuity along the diagonal is imposed, and explicit admissibility conditions are provided to ensure the fuzzy implication axioms.

We then develop n -order extensions of these two-branch constructions through repeated product-composition of the corresponding internal Yager-type components. This makes the n -order two-branch constructions conceptually different from the elementary precomposition-based power families. While the power-type families often exhibit saturation toward the maximal truth value 1, the two-branch extensions may lead to branch-dependent limiting expressions.

The paper further studies recursive feedback mechanisms based on a new auxiliary two-branch fuzzy implication. Here, the order parameter counts feedback iterations rather than powers of an internal product. One recursion inserts the previous implication value into the consequent position, while the other first transforms the previous value through N^{-1} and then inserts it into the antecedent position. This allows us to compare internal product-composition with recursive implication-composition and to examine branch switching and fixed-point type behavior.

The asymptotic analysis is an essential part of the paper. For the unified power-type family, we give precise pointwise limiting statements as $p \rightarrow \infty$, as $q \rightarrow \infty$, and jointly along arbitrary sequences (p_k, q_k) satisfying $\min(p_k, q_k) \rightarrow \infty$. The joint limiting

behavior distinguishes the point (1,0), where the value remains 0, from the rest of the unit square. The convergence is understood pointwise, and no claim of uniform convergence on the whole square is made.

The terminology used in the paper is purely mathematical. The transformation of $N^{-1}(y)$ is referred to as a consequent-side transformation or as a transformation of the internal consequent-side scale. It is not used in the sense of statistical or machine-learning calibration, and no empirical performance claim is made.

The main contributions of the paper are as follows:

1. A general monotone precomposition principle is stated and used to organize power-type fuzzy implication families as structured preliminary constructions.
2. The unified (p, q) -order family is studied in order to compare antecedent-side and consequent-side transformations and to analyze their saturation behavior.
3. Two Yager-induced two-branch fuzzy implications are introduced, together with explicit admissibility conditions ensuring the fuzzy implication axioms.
4. n -order extensions of the two-branch constructions are developed through repeated product-composition of internal Yager-type components.
5. A new auxiliary two-branch implication is used to define recursive feedback mechanisms, where the order parameter counts feedback iterations.
6. The asymptotic and structural behavior of the proposed mechanisms is compared, with emphasis on saturation, branch-dependent limits, boundary cases, and loss of secondary logical properties.

The rest of the paper is organized as follows. Section 2 recalls the necessary preliminaries on fuzzy negations, fuzzy implications, and generated implication functions. Section 3 develops the proposed constructions, beginning with the general closure principle and the associated power-type families, and then proceeding to the Yager-induced two-branch constructions, their n -order extensions, and the recursive feedback mechanisms. Section 4 discusses the relationship between the different mechanisms, with emphasis on admissibility assumptions, pointwise asymptotic behavior, structural properties, and branch-dependent phenomena. Section 5 concludes the paper and outlines directions for further research.

2. Materials and Methods

In this section, we recall the basic notions and preliminary results that will be used throughout the paper. We mainly focus on fuzzy negations, fuzzy implications, standard properties of fuzzy implications, negation-generated implication functions, and elementary power transformations on the unit interval.

Definition 1 ([2,13]). A mapping $N: [0,1] \rightarrow [0,1]$ is called a fuzzy negation if it satisfies the following conditions:

$$N(0) = 1, N(1) = 0, \quad (1)$$

and

$$x \leq y \Rightarrow N(x) \geq N(y), \quad (2)$$

for all $x, y \in [0,1]$. Thus, a fuzzy negation is a non-increasing mapping on the unit interval which reverses the boundary values 0 and 1.

A fuzzy negation N is called strict if it is continuous and strictly decreasing on $[0,1]$.

Moreover, a fuzzy negation N is called strong if it is involutive; that is,

$$N(N(x)) = x, \quad (3)$$

for every $x \in [0,1]$.

Definition 2 ([2,13]). A fuzzy implication is a mapping $I: [0,1]^2 \rightarrow [0,1]$ satisfying the following properties:

$$x_1 \leq x_2 \Rightarrow I(x_1, y) \geq I(x_2, y), \quad (4)$$

for all $x_1, x_2, y \in [0,1]$; that is, I is non-increasing with respect to its first variable, and

$$y_1 \leq y_2 \Rightarrow I(x, y_1) \leq I(x, y_2), \quad (5)$$

for all $x, y_1, y_2 \in [0,1]$; that is, I is non-decreasing with respect to its second variable.

In addition, I satisfies the boundary conditions

$$I(0,0) = 1, I(0,1) = 1, I(1,1) = 1, I(1,0) = 0. \quad (6)$$

The value $I(x, y)$ is interpreted as the truth degree of the fuzzy implication $x \Rightarrow y$.

In this interpretation, x represents the truth degree, or activation degree, of the antecedent, while y represents the truth degree, or satisfaction degree, of the consequent. Hence, $I(x, y)$ measures the degree to which the implication from x to y is satisfied.

Definition 3 ([1,2]). Several additional properties of fuzzy implications are often studied in the literature. We recall the most relevant ones for the present work.

A fuzzy implication I satisfies the left neutrality principle if

$$I(1, y) = y, \quad (7)$$

for all $y \in [0,1]$.

It satisfies the identity principle if

$$I(x, x) = 1, \quad (8)$$

and it satisfies the ordering property if

$$I(x, y) = 1 \Leftrightarrow x \leq y, \quad (9)$$

for all $x, y \in [0,1]$.

Given a fuzzy negation N , the implication I satisfies the law of contraposition with respect to N if

$$I(x, y) = I(N(y), N(x)), \quad (10)$$

for all $x, y \in [0,1]$.

Finally, I satisfies the exchange principle if

$$I(x, I(y, z)) = I(y, I(x, z)), \quad (11)$$

for all $x, y, z \in [0,1]$. These properties provide useful criteria for structural analysis and comparison of fuzzy implications. In the following sections, they will serve as reference properties for the implication functions generated by the proposed constructions.

Theorem 1 ([2]). Let $f: [0,1] \rightarrow [0,1]$ be a continuous and strictly decreasing function such that

$$f(0) = 1, f(1) = 0. \quad (12)$$

Assume also that f has a continuous inverse f^{-1} . Define the function $I: [0,1]^2 \rightarrow [0,1]$ by

$$I(x, y) = f(x f^{-1}(y)), x, y \in [0,1]. \quad (13)$$

Then, I is a fuzzy implication. Throughout the Results section, this fuzzy implication will be denoted by

$$I_N(x, y) = N(x N^{-1}(y)). \quad (14)$$

Theorem 2 ([13]). Let $f: [0,1] \rightarrow [0,1]$ be a continuous, well-defined and decreasing function such that

$$f(0) = 1, f(1) = 0.$$

Assume that f^{-1} exists and is also well-defined on $[0,1]$. Let $\varepsilon \in [0,1]$ be the equilibrium point of f ; that is,

$f(\varepsilon) = \varepsilon$. Define the function $I_A: [0,1]^2 \rightarrow [0,1]$ by

$$I_A(x, y) = \begin{cases} I_{A_1}(x, y) = f(x) \cdot f(f^{-1}(y) \cdot x), & 0 \leq x \leq \varepsilon, 0 \leq y \leq 1, \\ I_{A_2}(x, y) = y \cdot f(f^{-1}(y) \cdot x), & \varepsilon < x \leq 1, 0 \leq y \leq 1. \end{cases} \quad (15)$$

Then, I_A is a fuzzy implication. Throughout the Results section, this fuzzy implication will be denoted by:

$$I_A(x, y) = \begin{cases} I_{A_1}(x, y) = N(x) \cdot N(N^{-1}(y) \cdot x), & 0 \leq x \leq \varepsilon, 0 \leq y \leq 1, \\ I_{A_2}(x, y) = y \cdot N(N^{-1}(y) \cdot x), & \varepsilon < x \leq 1, 0 \leq y \leq 1. \end{cases} \quad (16)$$

3. Results

The purpose of this section is to introduce composition-generated fuzzy implication mechanisms induced by strict fuzzy negations. The results are arranged in increasing structural complexity. We first state a general monotone precomposition principle, which isolates an elementary closure mechanism for fuzzy implications. Starting from the known Yager-type generated implication $I_N(x, y) = N(xN^{-1}(y))$, we then derive antecedent-side, consequent-side, and unified (p, q) -order power-type families as structured consequences of this principle. These families form the preliminary part of the section: they are used mainly to organize the role of order parameters and to describe saturation effects.

The main constructional contribution begins with the Yager-induced two-branch fuzzy implications. These are not obtained merely by monotone precomposition; they are defined by two analytic expressions on the regions $x \leq y$ and $y \leq x$, and their validity is ensured by explicit admissibility conditions. Their n -order extensions are obtained through repeated product-composition of internal Yager-type components.

Throughout the paper, “higher-order” does not mean that the implication operators have arity higher than two. All operators remain binary maps from $[0,1]^2$ to $[0,1]$. The term refers only to order parameters in power-type transformations, product-composition extensions, or recursive feedback iterations. Finally, recursive two-branch feedback mechanisms are studied, where the order parameter counts feedback steps rather than powers of an internal product.

3.1. A General Transformation Principle

Before introducing the specific families, we recall a general closure principle for fuzzy implications under monotone endpoint-preserving transformations. This result will be used to present the power-type families as structured corollaries, rather than as the main novelty of the paper.

Proposition 1. Let $I: [0,1]^2 \rightarrow [0,1]$ be a fuzzy implication. Let $\alpha, \beta: [0,1] \rightarrow [0,1]$ be non-decreasing functions satisfying $\alpha(0) = 0, \alpha(1) = 1$, and $\beta(0) = 0, \beta(1) = 1$.

Define

$$J(x, y) = I(\alpha(x), \beta(y)), x, y \in [0,1]. \quad (17)$$

Then, J is a fuzzy implication.

Proof. Let $x_1, x_2 \in [0,1]$ with $x_1 \leq x_2$. Since α is non-decreasing, we have $\alpha(x_1) \leq \alpha(x_2)$. Because I is non-increasing in its first variable, $I(\alpha(x_1), \beta(y)) \geq I(\alpha(x_2), \beta(y))$. Hence, $J(x_1, y) \geq J(x_2, y)$, so J is non-increasing with respect to its first variable. Similarly, let $y_1, y_2 \in [0,1]$ with $y_1 \leq y_2$. Since β is non-decreasing, $\beta(y_1) \leq \beta(y_2)$. Because I is non-decreasing in its second variable, $I(\alpha(x), \beta(y_1)) \leq I(\alpha(x), \beta(y_2))$. Therefore, $J(x, y_1) \leq J(x, y_2)$, so J is non-decreasing with respect to its second variable. Finally, using the endpoint conditions, we obtain

$$J(0,0) = I(\alpha(0), \beta(0)) = I(0,0) = 1, J(0,1) = I(\alpha(0), \beta(1)) = I(0,1) = 1,$$

$$J(1,0) = I(\alpha(1), \beta(0)) = I(1,0) = 0, \text{ and } J(1,1) = I(\alpha(1), \beta(1)) = I(1,1) = 1.$$

Thus, J satisfies the monotonicity conditions and the four boundary conditions of a fuzzy implication. Therefore, J is a fuzzy implication. $\square$

Proposition 1 shows that preservation of the fuzzy implication axioms under monotone endpoint-preserving precomposition is a general closure phenomenon. Therefore, the power-type families introduced in Sections 3.2–3.4 are treated as preliminary structured corollaries of this principle. Their role is to provide a compact setting for comparing order-dependent transformations and saturation behavior. The main new constructional part begins with the Yager-induced two-branch constructions and their n -order and recursive feedback extensions.

3.2. Antecedent-Side Transformed n -Order Fuzzy Implications

We first consider the antecedent-side transformation of the Yager-type implication $I_N(x, y) = N(xN^{-1}(y))$. For $n \in \mathbb{N}$, the map $x \mapsto x^n$ is non-decreasing and endpoint-preserving on $[0,1]$. Since $x^n \leq x$, it attenuates the internal antecedent-side factor before evaluation by the decreasing negation N . Thus, the possible increase occurs at the final implication value, not at the internal factor.

Corollary 1. Let $N: [0,1] \rightarrow [0,1]$ be a strict fuzzy negation with inverse N^{-1} . For every $n \in \mathbb{N}$, define $I_R^{(n)}: [0,1]^2 \rightarrow [0,1]$ by

$$I_R^{(n)}(x, y) = N(x^n N^{-1}(y)), x, y \in [0,1]. \quad (18)$$

Then, $I_R^{(n)}$ is a fuzzy implication.

Proof. Consider the Yager-type generated implication $I_N(x, y) = N(xN^{-1}(y))$. For fixed $n \in \mathbb{N}$, define $\alpha_n(x) = x^n, \beta(y) = y$. Both α_n and β are non-decreasing and endpoint-preserving on $[0,1]$. Moreover,

$$I_N(\alpha_n(x), \beta(y)) = I_N(x^n, y) = N(x^n N^{-1}(y)) = I_R^{(n)}(x, y).$$

Therefore, by Proposition 1, $I_R^{(n)}$ is a fuzzy implication. $\square$

The previous corollary shows that the antecedent-side transformed family is a direct consequence of the general closure principle. The following lemma records its compositional origin.

Lemma 1. Let $I_N(x, y) = N(xN^{-1}(y))$ be the Yager-type generated implication. Define recursively $R_1(x, y) = I_N(x, y)$, and, for every $n \geq 1$, $R_{n+1}(x, y) = I_N(x, R_n(x, y))$. Then $R_n(x, y) = N(x^n N^{-1}(y)), x, y \in [0,1]$. Consequently,

$$R_n(x, y) = I_R^{(n)}(x, y). \quad (19)$$

Proof. For $n = 1$, we have $R_1(x, y) = I_N(x, y) = N(xN^{-1}(y))$. Assume that $R_n(x, y) = N(x^n N^{-1}(y))$. Then, $R_{n+1}(x, y) = I_N(x, R_n(x, y)) = N(xN^{-1}(R_n(x, y)))$.

Using the induction hypothesis, we obtain $N^{-1}(R_n(x, y)) = N^{-1}(N(x^n N^{-1}(y))) = x^n N^{-1}(y)$. Hence, $R_{n+1}(x, y) = N(x \cdot x^n N^{-1}(y)) = N(x^{n+1} N^{-1}(y))$. Therefore, by induction,

$$R_n(x, y) = N(x^n N^{-1}(y)) \text{ for every } n \in \mathbb{N}. \quad \square$$

Thus, n may also be interpreted as the order of antecedent-side composition, not merely as an external algebraic exponent.

Example 1. Let $N(x) = \sqrt{1 - x^2}$, $x \in [0, 1]$. Then N is invertible, so $N^{-1}(x) = \sqrt{1 - x^2}$. The antecedent-side transformed family becomes $I_R^{(n)}(x, y) = N(x^n N^{-1}(y)) = \sqrt{1 - x^{2n}(1 - y^2)}$. For $x = 0.7$ and $y = 0.5$, we obtain $I_R^{(n)}(0.7, 0.5) = \sqrt{1 - 0.7^{2n}(1 - 0.5^2)}$.

For instance, $I_R^{(1)}(0.7, 0.5) \approx 0.7953$, $I_R^{(2)}(0.7, 0.5) \approx 0.9055$, and $I_R^{(3)}(0.7, 0.5) \approx 0.9549$. These values illustrate the initial increase in the implication value as the order n increases. This behavior is a consequence of the attenuation of the internal factor x^n and the fact that N is decreasing.

Remark 1. Pointwise asymptotic behavior of the antecedent-side family.

Let $I_R^{(n)}(x, y) = N(x^n N^{-1}(y))$, where N is a strict fuzzy negation. Then, the sequence $\{I_R^{(n)}\}_{n \geq 1}$ converges pointwise on $[0, 1]^2$ to the function

$$L_R(x, y) = \begin{cases} 1, & 0 \leq x < 1, \\ y, & x = 1. \end{cases} \quad (20)$$

Indeed, if $0 \leq x < 1$, then $x^n \rightarrow 0$ as $n \rightarrow +\infty$. Hence, $x^n N^{-1}(y) \rightarrow 0$. Since N is continuous and $N(0) = 1$, we obtain

$$\lim_{n \rightarrow +\infty} I_R^{(n)}(x, y) = N(0) = 1.$$

On the other hand, if $x = 1$, then $I_R^{(n)}(1, y) = N(N^{-1}(y)) = y$ for every $n \in \mathbb{N}$. Therefore,

$$\lim_{n \rightarrow +\infty} I_R^{(n)}(1, y) = y$$

Thus, the convergence is pointwise and not uniform on $[0, 1]^2$, since the limiting function is discontinuous along the boundary $x = 1$ for $y < 1$, whereas each $I_R^{(n)}$ is continuous.

Consequently, the antecedent-side transformed family exhibits pointwise saturation toward the maximal truth value 1 on the region $0 \leq x < 1$, while on the boundary line $x = 1$ it remains equal to the consequent value y .

3.3. Consequent-Side Transformed n -Order Fuzzy Implications

We now transform the internal consequent-side scale $N^{-1}(y)$ while keeping the antecedent component x unchanged. For $n \in \mathbb{N}$, the transformation $N^{-1}(y) \mapsto (N^{-1}(y))^n$ attenuates the internal consequent-side factor. Since the outer negation N is decreasing, this may increase the final implication value.

Corollary 2. Let $N: [0, 1] \rightarrow [0, 1]$ be a strict fuzzy negation with inverse N^{-1} . For every $n \in \mathbb{N}$, define $I_C^{(n)}: [0, 1]^2 \rightarrow [0, 1]$

$$\text{by } I_C^{(n)}(x, y) = N(x(N^{-1}(y))^n), x, y \in [0, 1]. \quad (21)$$

Then, $I_C^{(n)}$ is a fuzzy implication.

Proof. Consider the Yager-type generated implication $I_N(x, y) = N(xN^{-1}(y))$. For fixed $n \in \mathbb{N}$, define $\alpha(x) = x, \beta_n(y) = N((N^{-1}(y))^n)$.

The function α is non-decreasing and endpoint-preserving. Also, since N^{-1} is decreasing and $t \mapsto t^n$ is non-decreasing on $[0, 1]$, the map $y \mapsto (N^{-1}(y))^n$ is non-increasing. Since N is decreasing, it follows that β_n is non-decreasing. Moreover, $\beta_n(0) = N((N^{-1}(0))^n) = N(1) = 0$, and $\beta_n(1) = N((N^{-1}(1))^n) = N(0) = 1$. Thus, β_n is endpoint-preserving. Finally, $I_N(\alpha(x), \beta_n(y)) = I_N(x, N((xN^{-1}(y))^n))$. Since N^{-1} is the inverse of N , we obtain $N^{-1}(\beta_n(y)) = (N^{-1}(y))^n$. Therefore $I_N(\alpha(x), \beta_n(y)) = N(x(N^{-1}(y))^n) = I_C^{(n)}(x, y)$. By Proposition 1, $I_C^{(n)}$ is a fuzzy implication. $\square$

The previous corollary shows that the consequent-side transformed family is also a structured consequence of the general closure principle. The next lemma records its compositional origin.

Lemma 2. Let $I_N(x, y) = N(xN^{-1}(y))$ be the Yager-type generated implication. Define recursively $C_1(x, y) = I_N(x, y)$, and, for every $n \geq 1$, $C_{n+1}(x, y) = I_N(N^{-1}(C_n(x, y)), y)$. Then, $C_n(x, y) = N(x(N^{-1}(y))^n), x, y \in [0, 1]$. Consequently,

$$C_n(x, y) = I_C^{(n)}(x, y). \quad (22)$$

Proof. For $n = 1$, we have $C_1(x, y) = I_N(x, y) = N(xN^{-1}(y))$, which is the required formula. Assume that, for some $n \in \mathbb{N}$, $C_n(x, y) = N(x(N^{-1}(y))^n)$.

Then, $C_{n+1}(x, y) = I_N(N^{-1}(C_n(x, y)), y) = N(N^{-1}(C_n(x, y))N^{-1}(y))$. Using the induction hypothesis, we obtain $N^{-1}(C_n(x, y)) = N^{-1}(N(x(N^{-1}(y))^n)) = x(N^{-1}(y))^n$.

Hence, $C_{n+1}(x, y) = N(x(N^{-1}(y))^n N^{-1}(y)) = N(x(N^{-1}(y))^{n+1})$. Therefore, by induction, $C_n(x, y) = N(x(N^{-1}(y))^n)$ for every $n \in \mathbb{N}$. $\square$

This lemma shows that the factor $(N^{-1}(y))^n$ may also be obtained by repeated internal-scale feedback. Thus, the consequent-side family is not only a precomposition example, but also has a recursive compositional interpretation.

Example 2. Let $N(x) = \sqrt{1 - x^2}, x \in [0, 1]$. Then, N is involutive, and therefore $N^{-1}(x) = \sqrt{1 - x^2}$. The consequent-side transformed family becomes $I_C^{(n)}(x, y) = N(x(N^{-1}(y))^n) = \sqrt{1 - x^2(1 - y^2)^n}$. For $x = 0.7$ and $y = 0.5$, we obtain

$$I_C^{(n)}(0.7, 0.5) = \sqrt{1 - 0.7^2(1 - 0.5^2)^n}.$$

For instance, $I_C^{(1)}(0.7, 0.5) \approx 0.7953$, $I_C^{(2)}(0.7, 0.5) \approx 0.8511$, and $I_C^{(3)}(0.7, 0.5) \approx 0.8907$.

These values illustrate the initial increase in the implication value as the order n increases. This behavior follows from the attenuation of the internal consequent-side factor and the fact that N is decreasing.

Remark 2. Pointwise asymptotic behavior of the consequent-side family.

Let $I_C^{(n)}(x, y) = N(x(N^{-1}(y))^n)$, where N is a strict fuzzy negation. Then, the sequence $\{I_C^{(n)}\}_{n \geq 1}$ converges pointwise on $[0, 1]^2$ to the function

$$L_C(x, y) = \begin{cases} N(x), & y = 0, \\ 1, & 0 < y \leq 1. \end{cases} \quad (23)$$

Indeed, if $0 < y \leq 1$, then $0 \leq N^{-1}(y) < 1$. Hence, $(N^{-1}(y))^n \rightarrow 0$ as $n \rightarrow +\infty$, and therefore $x(N^{-1}(y))^n \rightarrow 0$. Since N is continuous and $N(0) = 1$, we obtain

$$\lim_{n \rightarrow +\infty} I_C^{(n)}(x, y) = N(0) = 1.$$

On the other hand, if $y = 0$, then $N^{-1}(0) = 1$, and therefore $I_C^{(n)}(x, 0) = N(x)$ for every $n \in \mathbb{N}$. Thus, $\lim_{n \rightarrow +\infty} I_C^{(n)}(x, 0) = N(x)$. Thus, the convergence is pointwise and not uniform on $[0, 1]^2$, since the limiting function is discontinuous along the boundary $y = 0$ whenever $N(x) \neq 1$, whereas each $I_C^{(n)}$ is continuous.

Consequently, the consequent-side transformed family exhibits pointwise saturation toward 1 on the region $0 < y \leq 1$, while on the boundary line $y = 0$ it remains equal to $N(x)$.

The antecedent-side and consequent-side constructions act on two different internal factors of the same Yager-type implication. This motivates the unified two-parameter construction introduced in the next subsection.

3.4. Unified (p, q) -Order Fuzzy Implications

After the antecedent-side and consequent-side families, we now combine the two transformations into a single two-parameter construction. The parameter p acts on the antecedent-side component x , while the parameter q acts on the internal consequent-side scale $N^{-1}(y)$. This leads to the family $I_{p,q}(x, y) = N(x^p(N^{-1}(y))^q)$, $x, y \in [0, 1]$.

As in the previous subsections, the construction is first obtained as a direct consequence of the general precomposition principle.

Corollary 3. Let $N: [0, 1] \rightarrow [0, 1]$ be a strict fuzzy negation with inverse N^{-1} . For every $p, q \in \mathbb{N}$, define $I_{p,q}: [0, 1]^2 \rightarrow [0, 1]$ by

$$I_{p,q}(x, y) = N(x^p(N^{-1}(y))^q), x, y \in [0, 1]. \quad (24)$$

Then, $I_{p,q}$ is a fuzzy implication.

Proof. Consider the Yager-type generated implication $I_N(x, y) = N(xN^{-1}(y))$. For fixed $p, q \in \mathbb{N}$, define $\alpha_p(x) = x^p$ and

$$\beta_q(y) = N((N^{-1}(y))^q).$$

The map α_p is non-decreasing and endpoint-preserving on $[0, 1]$. Moreover, since N^{-1} is decreasing and $t \mapsto t^q$ is non-decreasing on $[0, 1]$, the map $y \mapsto (N^{-1}(y))^q$ is non-increasing. Since N is decreasing, β_q is non-decreasing. Also,

$$\beta_q(0) = N((N^{-1}(0))^q) = N(1) = 0, \text{ and } \beta_q(1) = N((N^{-1}(1))^q) = N(0) = 1.$$

Thus, β_q is endpoint-preserving. Finally, $I_N(\alpha_p(x), \beta_q(y)) = I_N(x^p, N((N^{-1}(y))^q))$. Since N^{-1} is the inverse of N , we obtain $N^{-1}(\beta_q(y)) = (N^{-1}(y))^q$. Therefore,

$$I_N(\alpha_p(x), \beta_q(y)) = N(x^p(N^{-1}(y))^q) = I_{p,q}(x, y).$$

By Proposition 1, $I_{p,q}$ is a fuzzy implication. $\square$

Remark 3. Precomposition form and special cases.

Define $h_q(y) = N((N^{-1}(y))^q)$. Then $N^{-1}(h_q(y)) = (N^{-1}(y))^q$, and therefore

$$I_{p,q}(x, y) = I_N(x^p, h_q(y)). \quad (25)$$

Thus, the unified family is a structured precomposition of the baseline implication I_N . Its role is not to provide a new closure principle, but to organize the antecedent-side and consequent-side transformations in a single two-parameter form. It also contains the previous families as special cases: $I_{p,1} = I_R^{(p)}$, $I_{1,q} = I_C^{(q)}$, $I_{1,1} = I_N$.

Lemma 3. Compositional origin of the unified family.

Let $F_{a,b}(x, y) = N(x^a(N^{-1}(y))^b)$, $a, b \in \mathbb{N}$. Define the two composition operators

$R(F_{a,b})(x, y) = I_N(x, F_{a,b}(x, y))$, and $C(F_{a,b})(x, y) = I_N(N^{-1}(F_{a,b}(x, y)), y)$. Then, $R(F_{a,b})(x, y) = F_{a+1,b}(x, y)$, and $C(F_{a,b})(x, y) = F_{a,b+1}(x, y)$.

Consequently, for every $p, q \in \mathbb{N}$, the unified family $I_{p,q} = F_{p,q}$ is obtained from the baseline implication $F_{1,1} = I_N$ by applying the antecedent-side composition operator $p - 1$ times and the consequent-side composition operator $q - 1$ times.

Proof. Since $F_{a,b}(x, y) = N(x^a(N^{-1}(y))^b)$, we have $N^{-1}(F_{a,b}(x, y)) = x^a(N^{-1}(y))^b$. Therefore,

$$\begin{aligned} R(F_{a,b})(x, y) &= I_N(x, F_{a,b}(x, y)) = N(xN^{-1}(F_{a,b}(x, y))) \\ &= N(x^{a+1}(N^{-1}(y))^b) = F_{a+1,b}(x, y). \end{aligned}$$

Similarly,

$$\begin{aligned} C(F_{a,b})(x, y) &= I_N(N^{-1}(F_{a,b}(x, y)), y) = N(N^{-1}(F_{a,b}(x, y))N^{-1}(y)) \\ &= N(x^a(N^{-1}(y))^{b+1}) = F_{a,b+1}(x, y). \end{aligned}$$

Starting from $F_{1,1} = I_N$, repeated application of these two operators gives $F_{p,q} = I_{p,q}$. $\square$

Example 3. Consider the strict non-strong fuzzy negation $N(x) = 1 - x^2, x \in [0,1]$.

Then, $N^{-1}(x) = \sqrt{1-x}$. Hence, $I_{p,q}(x, y) = N(x^p(N^{-1}(y))^q) = 1 - x^{2p}(1-y)^q$.

For $x = 0.7$, $y = 0.5$, and $p, q \in \{1,2,3\}$, we obtain Table 1:

Table 1. Numerical values of the unified (p, q) -order fuzzy implication $I_{p,q}(x, y)$ for $x = 0.7$, $y = 0.5$, and $p, q \in \{1,2,3\}$.

$p \backslash q$	$q = 1$	$q = 2$	$q = 3$
$p = 1$	0.7953	0.8511	0.8907
$p = 2$	0.9055	0.9300	0.9480
$p = 3$	0.9549	0.9663	0.9749

These values show that the unified (p, q) -order framework. The first column corresponds to the antecedent-side family, since $I_{p,1} = I_R^{(p)}$, while the first row corresponds to the consequent-side family, since $I_{1,q} = I_C^{(q)}$.

The entry $p = q = 1$ gives the baseline implication value. Increasing either parameter increases the final implication value in this example, reflecting the attenuation of the corresponding internal factor before the decreasing negation N is applied, as it is presented in Figure 1.

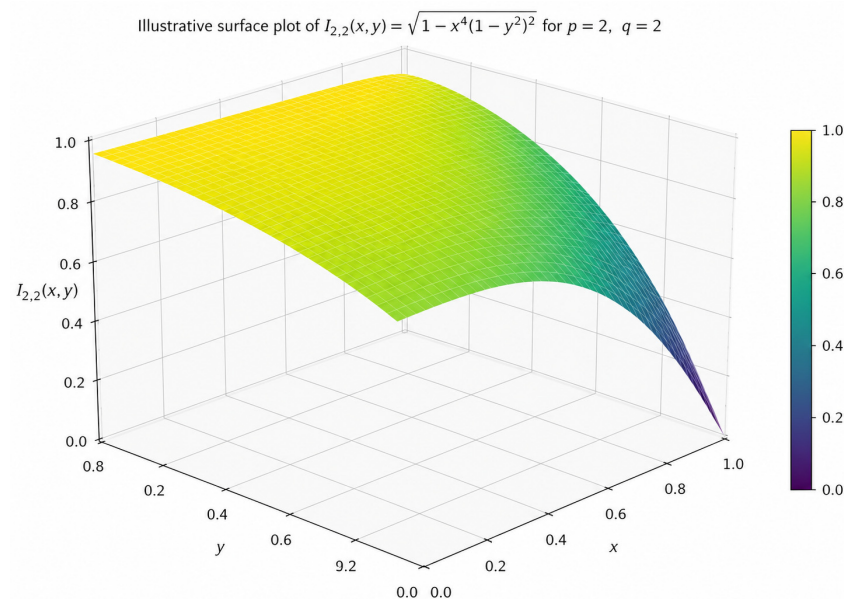


Figure 1. Surface plot of the unified (p, q) -order fuzzy implication $I_{p,q}$ generated by the strict non-strong fuzzy negation $N(t) = \sqrt{1 - t^2}$, with $N^{-1}(y) = \sqrt{1 - y^2}$. The plotted case corresponds to $p = 2$ and $q = 2$, namely $I_{2,2}(x, y) = \sqrt{1 - x^4(1 - y^2)^2}$. The figure illustrates the global behavior of the unified construction over the unit square $[0, 1]^2$ and shows the saturation tendency of the power-type family toward values close to 1 away from the boundary point $(1, 0)$, where $I_{p,q}(1, 0) = 0$ for all p, q .

Remark 4. Pointwise asymptotic behavior of the unified family.

Let $I_{p,q}(x, y) = N(x^p(N^{-1}(y))^q)$. For fixed $q \in \mathbb{N}$, we have

$$\lim_{p \rightarrow +\infty} I_{p,q}(x, y) = \begin{cases} 1, & 0 \leq x < 1, \\ N((N^{-1}(y))^q), & x = 1. \end{cases} \quad (26)$$

Similarly, for fixed $p \in \mathbb{N}$,

$$\lim_{q \rightarrow +\infty} I_{p,q}(x, y) = \begin{cases} N(x^p), & y = 0, \\ 1, & 0 < y \leq 1. \end{cases} \quad (27)$$

Moreover, for every sequence $(p_k, q_k) \subset \mathbb{N}^2$ such that $\min(p_k, q_k) \rightarrow +\infty$, we have the pointwise limit

$$\lim_{k \rightarrow +\infty} I_{p_k, q_k}(x, y) = \begin{cases} 0, & (x, y) = (1, 0), \\ 1, & \text{otherwise.} \end{cases} \quad (28)$$

Indeed, if $(x, y) \neq (1, 0)$, then either $x < 1$ or $y > 0$. If $x < 1$, then $x^{p_k} \rightarrow 0$. If $y > 0$, then $N^{-1}(y) < 1$, and therefore $(N^{-1}(y))^{q_k} \rightarrow 0$. In both cases,

$$x^{p_k}(N^{-1}(y))^{q_k} \rightarrow 0,$$

and hence

$$I_{p_k, q_k}(x, y) \rightarrow N(0) = 1.$$

At the exceptional point $(1, 0)$, we have $I_{p_k, q_k}(1, 0) = N(1^{p_k}(N^{-1}(0))^{q_k}) = N(1) = 0$ for every k . Thus, the convergence is pointwise. It is not uniform on the whole square $[0, 1]^2$, since the limiting function is discontinuous at $(1, 0)$ whereas each $I_{p,q}$ is continuous.

Consequently, very large values of p and q may lead to saturation toward 1 on most of the unit square. No parameter value is mathematically preferable without an external criterion, such as a threshold condition, an optimization objective, or an application-specific loss function.

Remark 5. Structural properties of the power-type n -order families.

We close this subsection by summarizing the logical properties satisfied by the three power-type families introduced in Corollaries 3.1–3.3. In addition to the defining axioms of fuzzy implications, we also consider several standard additional properties: the left neutrality principle, the right neutrality principle, the identity principle, the ordering principle, the exchange principle, and the contraposition principle.

For a fuzzy implication I , these properties are defined as follows:

$$\begin{aligned} \text{LNP: } I(1, y) &= y, \text{ RNP: } I(x, 1) = 1, \text{ IP: } I(x, x) = 1, \text{ OP: } I(x, y) = 1 \Leftrightarrow x \leq y, \\ \text{EP: } I(x, I(y, z)) &= I(y, I(x, z)), \text{ and, with respect to a fuzzy negation } N, \\ \text{CP(N): } I(x, y) &= I(N(y), N(x)). \end{aligned}$$

The following table summarizes the behavior of the three families in Table 2.

Table 2. Structural properties of the power-type n -order fuzzy implication families introduced in Corollaries 3.1–3.3. The entry “Generally no” means that the corresponding property is not guaranteed by the construction and may fail for admissible choices of the negation and of the parameters.

Family	FI Axioms	LNP	RNP	IP	OP	EP	CP(N)
$I_R^{(n)}(x, y) = N(x^n N^{-1}(y))$	Yes	Yes	Yes	Generally no	Generally no	Yes	Generally no
$I_C^{(n)}(x, y) = N(x(N^{-1}(y))^n)$	Yes	Only for $n = 1$	Yes	Generally no	Generally no	Only for $n = 1$	Generally no
$I_{p,q}(x, y) = N(x^p(N^{-1}(y))^q)$	Yes	Only for $q = 1$	Yes	Generally no	Generally no	Only for $q = 1$	Generally no

Remark 6. Justification of the entries in the structural table.

The positive entries follow directly from the definitions. For example,

$$I_R^{(n)}(1, y) = N(N^{-1}(y)) = y,$$

And $I_R^{(n)}(x, 1) = N(x^n N^{-1}(1)) = N(0) = 1$. Moreover, $N^{-1}(I_R^{(n)}(y, z)) = y^n N^{-1}(z)$, so $I_R^{(n)}(x, I_R^{(n)}(y, z)) = N(x^n y^n N^{-1}(z)) = I_R^{(n)}(y, I_R^{(n)}(x, z))$. For the consequent-side and unified families, left neutrality is preserved only when the exponent of $N^{-1}(y)$ is equal to 1. Thus, $I_C^{(n)}$ preserves LNP only for $n = 1$, and $I_{p,q}$ preserves LNP only for $q = 1$. Similarly, the exchange principle is preserved only when $q = 1$. For the negative entries, consider the strict non-strong negation $N(x) = 1 - x^2$, $N^{-1}(x) = \sqrt{1 - x}$. Then, $I_R^{(n)}(x, y) = 1 - x^{2n}(1 - y)$, and $I_C^{(n)}(x, y) = 1 - x^2(1 - y)^n$.

The identity principle fails, since $I_R^{(1)}\left(\frac{1}{2}, \frac{1}{2}\right) = 1 - \left(\frac{1}{2}\right)^2 \left(1 - \frac{1}{2}\right) = \frac{7}{8} \neq 1$. The ordering principle also fails. Although $0.4 \leq 0.6$, we have $I_R^{(1)}(0.4, 0.6) = 1 - (0.4)^2(1 - 0.6) = 0.936 \neq 1$. Left neutrality fails for $I_C^{(2)}$, since $I_C^{(2)}\left(1, \frac{1}{2}\right) = 1 - \left(1 - \frac{1}{2}\right)^2 = \frac{3}{4} \neq \frac{1}{2}$. The exchange principle fails for $I_C^{(2)}$. Indeed, for $x = 0.2$, $y = 0.5$, and $z = 0.5$, $I_C^{(2)}(0.2, I_C^{(2)}(0.5, 0.5)) = 0.99984375$, where $I_C^{(2)}(0.5, I_C^{(2)}(0.2, 0.5)) = 0.999975$. Thus, the two values are different. Finally, contraposition with respect to N is not generally preserved. For $I_R^{(2)}(x, y) = 1 - x^4(1 - y)$, take $x = 0.7$ and $y = 0.4$. Then, $I_R^{(2)}(0.7, 0.4) = 0.85594$. On the other hand, $N(0.4) = 0.84$, $N(0.7) = 0.51$, and

$$I_R^{(2)}(N(0.4), N(0.7)) = I_R^{(2)}(0.84, 0.51) = 0.7560430336.$$

Therefore, $I_R^{(2)}(0.7, 0.4) \neq I_R^{(2)}(N(0.4), N(0.7))$. These examples justify the negative entries in the table.

These examples show that the higher-order composition mechanisms preserve the defining axioms of fuzzy implications, but they do not automatically preserve all secondary logical principles. Thus, although the constructions are valid fuzzy implications, their additional logical properties depend on the type of composition, on the chosen strict fuzzy negation, and on the order parameters. In this sense, Corollaries 3.1–3.3 should be viewed as structured closure-based constructions whose order parameters may also modify the logical behavior of the underlying Yager-type implication. Although the transformations

$x \mapsto x^p$ and $N^{-1}(y) \mapsto (N^{-1}(y))^q$ attenuate the corresponding internal factors for $p, q \geq 1$, the final implication value may increase because these factors are evaluated through the decreasing negation N . Thus, the attenuation occurs at the level of the internal product, whereas the possible amplification occurs at the level of the final implication value.

3.5. Yager-Induced Two-Branch Fuzzy Implications and Their n -Order Extensions

In contrast with the power-type families, the two constructions introduced in this subsection are not obtained by a simple monotone precomposition of the baseline implication. They are new Yager-induced two-branch fuzzy implications, defined separately on the regions $x \leq y$ and $y \leq x$, and glued continuously along the diagonal. Their two-branch form is essential for the asymptotic analysis, because it allows the limiting behavior to depend on the active branch. Thus, these constructions provide a way to study cases where saturation toward the maximal truth value is avoided or replaced by a nontrivial branch-dependent limit. Instead of modifying only the internal product of I_N , we use the baseline implication itself to define a continuous two-branch fuzzy implication.

To this end, we introduce the following Yager-induced two-branch construction.

Remark 7. Relation with existing construction methods.

Fuzzy implications have been constructed by several standard methods, including S -implications, R -implications, QL -implications, generated implications, and transformations based on fuzzy negations, aggregation functions, copulas, quasi-copulas, and related operators. These classes and their properties are systematically studied in the literature; in particular, Yager introduced important generated classes of fuzzy implications, while Baczyński and Jayaram provided a comprehensive treatment of fuzzy implication functions and their main structural properties [2,3]. More recently, Mesiar and Kolesárová studied copula-based constructions of fuzzy implications and investigated links between fuzzy implicators, quasi-copulas and copulas [8,9].

The two-branch constructions introduced in this subsection are related to this general line of research, but they have a different structure. They are not standard S -, R -, or QL -implications, since they are not defined by a single global expression involving a t -conorm, a residual operation, or a QL -type composition. They are also not obtained only by applying a monotone endpoint-preserving precomposition of the form

$$J(x, y) = I(\alpha(x), \beta(y))$$

to an already given fuzzy implication. Moreover, they are not copula-based or quasi-copula-based constructions, because no copula or quasi-copula is used as the generating operator.

Instead, the Yager-type generated implication $I_N(x, y) = N(xN^{-1}(y))$ is inserted into two different analytic branches defined on the regions $x \leq y$ and $y \leq x$, and the two branches are glued continuously along the diagonal. Thus, the validity of the resulting operators depends on explicit admissibility conditions ensuring the monotonicity requirements of fuzzy implications.

This branch-dependent form has two main advantages. First, it separates the behavior of the implication on the two natural regions of the unit square while preserving the fuzzy implication axioms under checkable assumptions. Second, its n -order extensions may produce pointwise limiting behavior that is different from the saturation observed in the preceding power-type families. In particular, the limit may saturate to 1 on one branch and converge to a nontrivial expression on the other branch. Therefore, the two-branch Yager-induced constructions are introduced not merely as additional examples, but as a distinct composition-generated mechanism. Their distinguishing features are the use of a Yager-type internal component, the two analytic branches separated by the diagonal, the continuous gluing condition, and the explicit admissibility assumptions that guarantee the fuzzy implication axioms.

Theorem 3. Yager-induced two-branch fuzzy implication.

Let $N: [0,1] \rightarrow [0,1]$ be a strict fuzzy negation with inverse N^{-1} , and let $I_N(x,y) = N(xN^{-1}(y))$ be the associated Yager-type fuzzy implication. Define $h_N(t) = I_N(N(t), t)$, $t \in [0,1]$. Equivalently,

$$h_N(t) = N(N(t)N^{-1}(t)). \quad (29)$$

Assume that h_N is differentiable on $(0,1)$, non-decreasing on $[0,1]$, and satisfies

$$th'_N(t) \leq 1, t \in (0,1). \quad (30)$$

Define $I_{A,N}: [0,1]^2 \rightarrow [0,1]$ by

$$I_{A,N}(x,y) = \begin{cases} 1 - x + xI_N(N(y), y), & x \leq y, \\ 1 - x + yI_N(N(x), x), & y \leq x. \end{cases} \quad (31)$$

Equivalently,

$$I_{A,N}(x,y) = \begin{cases} 1 - x + xh_N(y), & x \leq y, \\ 1 - x + yh_N(x), & y \leq x. \end{cases} \quad (32)$$

Then, $I_{A,N}$ is a fuzzy implication.

Proof. First, since N is a strict fuzzy negation, we have $N(0) = 1, N(1) = 0$, and $N^{-1}(0) = 1, N^{-1}(1) = 0$. Hence, $h_N(0) = N(N(0)N^{-1}(0)) = N(1 \cdot 1) = N(1) = 0$, and $h_N(1) = N(N(1)N^{-1}(1)) = N(0 \cdot 0) = N(0) = 1$.

We first show that the two branches coincide on the diagonal. For $x = y = t$, the first branch gives $1 - t + th_N(t)$, while the second branch also gives $1 - t + th_N(t)$. Therefore, $I_{A,N}$ is well defined and continuous on the switching line $x = y$. On the switching boundary, the two analytic expressions coincide. Therefore, the construction is well defined and continuously glued along the boundary.

Now we prove that $I_{A,N}$ takes values in $[0,1]$. If $x \leq y$, then $I_{A,N}(x,y) = 1 - x + xh_N(y)$.

Since $0 \leq h_N(y) \leq 1$, we obtain $1 - x \leq I_{A,N}(x,y) \leq 1$. Hence, $0 \leq I_{A,N}(x,y) \leq 1$.

If $y \leq x$, then $I_{A,N}(x,y) = 1 - x + yh_N(x)$. Again, since $0 \leq h_N(x) \leq 1$, we have $I_{A,N}(x,y) \geq 1 - x \geq 0$. Moreover, since $y \leq x$, $yh_N(x) \leq y \leq x$. Thus,

$$I_{A,N}(x,y) = 1 - x + yh_N(x) \leq 1.$$

Therefore, $I_{A,N}(x,y) \in [0,1]$ on both branches. Next, we check the monotonicity conditions. On the branch $x \leq y$, we have $I_{A,N}(x,y) = 1 - x + xh_N(y)$. Thus, $\frac{\partial I_{A,N}}{\partial x} = -1 + h_N(y) \leq 0$, because $h_N(y) \leq 1$, and $\frac{\partial I_{A,N}}{\partial y} = xh'_N(y) \geq 0$, because h_N is non-decreasing. On the branch $y \leq x$, we have $I_{A,N}(x,y) = 1 - x + yh_N(x)$. Therefore, $\frac{\partial I_{A,N}}{\partial y} = h_N(x) \geq 0$. Also,

$$\frac{\partial I_{A,N}}{\partial x} = -1 + yh'_N(x). \text{ Since } y \leq x, \text{ we get } yh'_N(x) \leq xh'_N(x).$$

By the assumption $xh'_N(x) \leq 1$, it follows that $\frac{\partial I_{A,N}}{\partial x} = -1 + yh'_N(x) \leq 0$.

Thus, $I_{A,N}$ is non-increasing with respect to x and non-decreasing with respect to y . Finally, we verify the boundary conditions. We have $I_{A,N}(0,0) = 1, I_{A,N}(0,1) = 1, I_{A,N}(1,0) = 0$ and $I_{A,N}(1,1) = 1$. Therefore, $I_{A,N}$ satisfies all axioms of a fuzzy implication. This completes the proof. $\square$

Remark 8. The condition $th'_N(t) \leq 1, t \in (0,1)$ is not merely a technical assumption. It is directly related to the monotonicity of the second branch of the construction. Indeed, on the branch

$y \leq x$, we have $I_{A,N}(x, y) = 1 - x + yh_N(x)$. Therefore, $\frac{\partial I_{A,N}}{\partial x} = -1 + yh'_N(x)$. Since $y \leq x$, the inequality $xh'_N(x) \leq 1$ implies $yh'_N(x) \leq xh'_N(x) \leq 1$. Thus, $\frac{\partial I_{A,N}}{\partial x} \leq 0$.

Hence, the admissibility condition guarantees that the second branch is non-increasing with respect to the antecedent variable x , as required for a fuzzy implication. It is important to note that this condition is a sufficient admissibility condition. It is not claimed to be necessary, and it is not automatically satisfied by every strict fuzzy negation. Thus, Theorem 3 applies to every strict fuzzy negation satisfying this explicit condition, but it does not provide a complete classification of all strict negations that may generate valid two-branch fuzzy implications.

For instance, consider the family $N_a(t) = 1 - t^a$, $a > 0$. Then $N_a^{-1}(t) = (1 - t)^{1/a}$, and $h_{N_a}(t) = N_a(N_a(t)N_a^{-1}(t)) = 1 - (1 - t)(1 - t^a)^a$.

Consequently, $h'_{N_a}(t) = (1 - t^a)^a + a^2(1 - t)t^{a-1}(1 - t^a)^{a-1}$. Therefore, the admissibility condition becomes

$$t[(1 - t^a)^a + a^2(1 - t)t^{a-1}(1 - t^a)^{a-1}] \leq 1, t \in (0, 1).$$

Direct verification for selected parameter values shows that the admissibility condition holds, for instance, for $a = 1, 2, 3, 4, 5$, whereas it fails for some larger values, such as $a = 6$ and $a = 10$. These examples are included only to illustrate that the admissibility condition is non-vacuous but not automatic. We do not attempt to determine the exact transition threshold for the parameter a , and no complete classification of the family $N_a(t) = 1 - t^a$ is claimed.

In this paper, the phrase “not contained in the standard classes” is used in the following structural sense. The proposed two-branch operators are not given by a single global formula of the standard S -, R -, or QL -implication type, nor by a single precomposition $I(\alpha(x), \beta(y))$. They are also not generated from a copula or quasi-copula as the primary constructional object. Instead, the Yager-type internal component is inserted into two different analytic expressions, with the active formula determined by the regions $x \leq y$ and $y \leq x$. Thus, the distinguishing mathematical feature is the branch-dependent use of the Yager-type internal product together with explicit gluing and admissibility conditions.

Finally, the recursive contribution is separated into three components: the auxiliary base implication $\tilde{I}_A$, the two feedback schemes generated from it, and the convergence results describing branch switching and fixed-point type behavior.

Example 4. Let $N(t) = 1 - t^2$, $t \in [0, 1]$. Then, N is a strict fuzzy negation and $N^{-1}(t) = \sqrt{1 - t}$. The associated Yager-type fuzzy implication is $I_N(x, y) = N(xN^{-1}(y))$. Thus,

$$I_N(x, y) = N(x\sqrt{1 - y}) = 1 - x^2(1 - y). \text{ For Theorem 3, we define } h_N(t) = I_N(N(t), t).$$

Since $N(t) = 1 - t^2$, we obtain $h_N(t) = I_N(1 - t^2, t)$. Therefore, $h_N(t) = 1 - (1 - t^2)^2(1 - t)$. Equivalently, $h_N(t) = 1 - (1 - t)(1 - t^2)^2$. Moreover, $h_N(0) = 0$, $h_N(1) = 1$.

Differentiating, we get $h'_N(t) = (1 - t^2)^2 + 4t(1 - t)(1 - t^2)$. Hence, $h'_N(t) = (1 - t^2)(1 + 4t - 5t^2)$. Since $1 + 4t - 5t^2 = (1 - t)(5t + 1)$, we have $h'_N(t) = (1 - t^2)(1 - t)(5t + 1)$. Thus, $th'_N(t) = t(1 - t^2)(1 - t)(5t + 1)$. Equivalently, $th'_N(t) = t(1 - t)^2(1 + t)(5t + 1)$. A direct verification shows that $th'_N(t) \leq 1$, $t \in [0, 1]$. Therefore, the admissibility condition of Theorem 3 is satisfied. Since $h_N(0) = 0$, $h_N(1) = 1$, h_N is non-decreasing on $[0, 1]$, and $th'_N(t) \leq 1$, $t \in [0, 1]$, all the assumptions of Theorem 3 are satisfied. Therefore, by Theorem 3, the function

$$I_{A,N}(x, y) = \begin{cases} 1 - x + xh_N(y), & x \leq y, \\ 1 - x + yh_N(x), & y \leq x \end{cases}$$

is a fuzzy implication.

Substituting $h_N(t) = 1 - (1 - t)(1 - t^2)^2$, we obtain

$$I_{A,N}(x,y) = \begin{cases} 1 - x + x[1 - (1 - y)(1 - y^2)^2], & x \leq y, \\ 1 - x + y[1 - (1 - x)(1 - x^2)^2], & y \leq x. \end{cases}$$

Equivalently,

$$I_{A,N}(x,y) = \begin{cases} 1 - x(1 - y)(1 - y^2)^2, & x \leq y, \\ 1 - x + y - y(1 - x)(1 - x^2)^2, & y \leq x. \end{cases}$$

Thus, the above function is a fuzzy implication generated by the non-classical strict fuzzy negation $N(t) = 1 - t^2$ as it is presented in Figure 2.

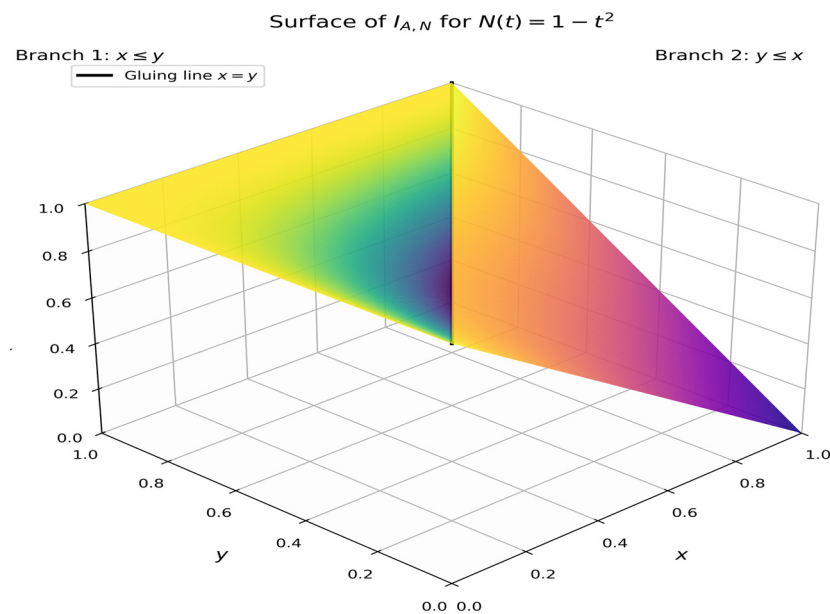


Figure 2. Surface plot of the Yager-induced two-branch fuzzy implication $I_{A,N}$ generated by the strict non-strong fuzzy negation $N(t) = 1 - t^2$. The function is defined by two analytic branches on the regions $x \leq y$ and $y \leq x$, and the black curve represents the gluing line $x = y$. The plot illustrates that the two branches are continuously joined along the diagonal while exhibiting different surface behavior on the two regions of the unit square.

Theorem 4. Direct Yager-Type Two-Branch Fuzzy Implication

Let $N: [0,1] \rightarrow [0,1]$ be a strict fuzzy negation with differentiable inverse N^{-1} . Assume that N is differentiable on $(0,1)$ and that, for every $z \in (0,1)$, $0 < y \leq N(z)$, the following admissibility conditions hold:

$$(A1) \quad 0 \leq y^2 \frac{N'(yz)}{N'(z)} \leq 1, \quad (33)$$

and

$$(A2) \quad N(yz) + yzN'(yz) \geq 0. \quad (34)$$

Define $I_{B,N}: [0,1]^2 \rightarrow [0,1]$ by

$$I_{B,N}(x,y) = \begin{cases} 1 - x + xN(xN^{-1}(y)), & x \leq y, \\ 1 - x + yN(yN^{-1}(x)), & y \leq x. \end{cases} \quad (35)$$

Equivalently, using the baseline Yager-type implication $I_N(x,y) = N(xN^{-1}(y))$, we may write

$$I_{B,N}(x,y) = \begin{cases} 1 - x + xI_N(x,y), & x \leq y, \\ 1 - x + yI_N(y,x), & y \leq x. \end{cases} \quad (36)$$

Then, $I_{B,N}$ is a fuzzy implication.

Proof. First, we show that the two branches coincide on the diagonal. For $x = y = t$, the first branch gives $1 - t + tN(tN^{-1}(t))$, whereas the second branch gives

$$1 - t + tN(tN^{-1}(t)).$$

Hence, the two expressions agree on the line $x = y$, and the function is well defined and continuous along the switching line. On the switching boundary, the two analytic expressions coincide. Therefore, the construction is well defined and continuously glued along the boundary.

We now prove that $I_{B,N}$ takes values in $[0,1]$. If $x \leq y$, then

$$I_{B,N}(x, y) = 1 - x + xN(xN^{-1}(y)).$$

Since $0 \leq N(xN^{-1}(y)) \leq 1$, we obtain $1 - x \leq I_{B,N}(x, y) \leq 1$. Thus, $0 \leq I_{B,N}(x, y) \leq 1$.

If $y \leq x$, then $I_{B,N}(x, y) = 1 - x + yN(yN^{-1}(x))$. Since $0 \leq N(yN^{-1}(x)) \leq 1$, we have $I_{B,N}(x, y) \geq 1 - x \geq 0$. Moreover, because $y \leq x$, $yN(yN^{-1}(x)) \leq y \leq x$. Therefore, $I_{B,N}(x, y) \leq 1$. Hence, $I_{B,N}(x, y) \in [0,1]$ on both branches. Next, we prove the monotonicity properties. On the first branch $x \leq y$, we have

$$I_{B,N}(x, y) = 1 - x + xN(xN^{-1}(y)).$$

The first branch satisfies the required monotonicity conditions directly from the monotonicity of N and N^{-1} . The admissibility conditions (A1) and (A2) are only needed for the second branch. For the special families considered later, the monotonicity of this branch follows directly from the explicit expression. In the present theorem, the main admissibility restrictions are imposed in order to control the second branch, where the derivative with respect to the antecedent variable is more delicate. On the second branch $y \leq x$, we have

$$I_{B,N}(x, y) = 1 - x + yN(yN^{-1}(x)).$$

We first show that this branch is non-increasing with respect to x . Differentiating with respect to x , we obtain $\frac{\partial I_{B,N}}{\partial x} = -1 + y^2 N'(yN^{-1}(x))(N^{-1})'(x)$.

Set $z = N^{-1}(x)$. Then, $x = N(z)$. Since $y \leq x$, we have $0 < y \leq N(z)$. Moreover, $(N^{-1})'(x) = \frac{1}{N'(z)}$. Hence, $\frac{\partial I_{B,N}}{\partial x} = -1 + y^2 \frac{N'(yz)}{N'(z)}$. By condition (A1), $0 \leq y^2 \frac{N'(yz)}{N'(z)} \leq 1$. Therefore, $\frac{\partial I_{B,N}}{\partial x} \leq 0$. Thus, the second branch is non-increasing with respect to x . We now show that the second branch is non-decreasing with respect to y . Differentiating with respect to y , we get $\frac{\partial I_{B,N}}{\partial y} = N(yN^{-1}(x)) + yN^{-1}(x)N'(yN^{-1}(x))$.

Using again $z = N^{-1}(x)$, we obtain $\frac{\partial I_{B,N}}{\partial y} = N(yz) + yzN'(yz)$. By condition (A2), $N(yz) + yzN'(yz) \geq 0$. Hence,

$$\frac{\partial I_{B,N}}{\partial y} \geq 0.$$

Thus, the second branch is non-decreasing with respect to y . Finally, we check the boundary conditions. We have $I_{B,N}(0,0) = 1$, $I_{B,N}(0,1) = 1$, $I_{B,N}(1,0) = 0$ and $I_{B,N}(1,1) = 1$. Therefore, $I_{B,N}$ satisfies the axioms of a fuzzy implication. This completes the proof. $\square$

Remark 9. On the role of the admissibility conditions (A1) and (A2).

The admissibility conditions (A1) and (A2) are imposed in order to control the monotonicity of the second branch of $I_{B,N}$. Indeed, the first branch

$$1 - x + xN(xN^{-1}(y)), x \leq y,$$

satisfies the required monotonicity properties directly from the monotonicity of N and N^{-1} . More precisely, differentiating with respect to x gives

$$-1 + N(xN^{-1}(y)) + xN^{-1}(y)N'(xN^{-1}(y)) \leq 0,$$

while differentiating with respect to y gives

$x^2N'(xN^{-1}(y))(N^{-1})'(y) \geq 0$, because both N' and $(N^{-1})'$ are negative. The second branch, $1 - x + yN(yN^{-1}(x)), y \leq x$, is more delicate. Differentiating with respect to x , and setting $z = N^{-1}(x)$, we obtain

$$\frac{\partial I_{B,N}}{\partial x} = -1 + y^2 \frac{N'(yz)}{N'(z)}.$$

Therefore, condition (A1), $0 \leq y^2 \frac{N'(yz)}{N'(z)} \leq 1$, guarantees that the second branch is non-increasing with respect to x . Similarly, differentiating the second branch with respect to y gives $\frac{\partial I_{B,N}}{\partial y} = N(yz) + yzN'(yz)$.

Thus, condition (A2), $N(yz) + yzN'(yz) \geq 0$, guarantees that the second branch is non-decreasing with respect to y .

Hence, (A1) and (A2) are not decorative assumptions. They are explicit and checkable sufficient admissibility conditions ensuring that the second branch preserves the monotonicity requirements of a fuzzy implication. Consequently, Theorem 4 applies to every strict fuzzy negation satisfying (A1) and (A2). No additional hidden assumptions are required once these two conditions have been verified.

Example 5. Consider the strict fuzzy negation $N(t) = 1 - t^2, t \in [0,1]$. Then, $N^{-1}(t) = \sqrt{1-t}$ and $N'(t) = -2t$. We first verify the admissibility conditions (A1) and (A2) of Theorem 4. Let $z \in (0,1), 0 < y \leq N(z) = 1 - z^2$. For condition (A1), we compute

$$y^2 \frac{N'(yz)}{N'(z)} = y^2 \frac{-2yz}{-2z} = y^3.$$

Since $0 < y \leq 1$, we have $0 \leq y^3 \leq 1$. Therefore, condition (A1) is satisfied. For condition (A2), we have $N(yz) + yzN'(yz) = 1 - y^2z^2 + yz(-2yz)$. Hence,

$$N(yz) + yzN'(yz) = 1 - 3y^2z^2.$$

Since $0 < y \leq 1 - z^2$, we get $yz \leq z(1 - z^2)$. The function $z(1 - z^2)$ attains its maximum on $[0,1]$ at $z = \frac{1}{\sqrt{3}}$, and $\max_{z \in [0,1]} z(1 - z^2) = \frac{2}{3\sqrt{3}}$. Therefore, $y^2z^2 \leq \frac{4}{27}$. Thus,

$$1 - 3y^2z^2 \geq 1 - \frac{12}{27} = \frac{5}{9} > 0.$$

So, condition (A2) is also satisfied. Consequently, all assumptions of Theorem 4 hold for the non-classical strict fuzzy negation $N(t) = 1 - t^2$. The corresponding baseline Yager-type implication is $I_N(x, y) = N(xN^{-1}(y))$. Thus, $I_N(x, y) = N(x\sqrt{1-y}) = 1 - x^2(1 - y)$.

The two-branch implication of Theorem 4 is

$$I_{B,N}(x, y) = \begin{cases} 1 - x + xN(xN^{-1}(y)), & x \leq y, \\ 1 - x + yN(yN^{-1}(x)), & y \leq x. \end{cases}$$

For $x \leq y$, we obtain $1 - x + xN(xN^{-1}(y)) = 1 - x + x[1 - x^2(1 - y)]$. Therefore, $1 - x + xN(xN^{-1}(y)) = 1 - x^3(1 - y)$. For $y \leq x$, we obtain $1 - x + yN(yN^{-1}(x)) = 1 - x + y[1 - y^2(1 - x)]$. Moreover, the boundary conditions are immediately satisfied. Indeed, $I_{B,N}(0,0) = 1, I_{B,N}(0,1) = 1, I_{B,N}(1,1) = 1$ and $I_{B,N}(1,0) = 1 - 1 + 0 - 0 = 0$. Therefore, the function satisfies the boundary conditions required for a fuzzy implication. Thus,

$$1 - x + yN(yN^{-1}(x)) = 1 - x + y - y^3(1 - x).$$

Hence, Theorem 4 gives the following fuzzy implication:

$$I_{B,N}(x, y) = \begin{cases} 1 - x^3(1 - y), & x \leq y, \\ 1 - x + y - y^3(1 - x), & y \leq x. \end{cases}$$

This example shows that Theorem 4 applies to a non-classical strict fuzzy negation and produces a continuous two-branch fuzzy implication generated directly from the baseline Yager-type implication I_N .

Before introducing the n -order version of Theorem 3, we explain the composition mechanism behind the auxiliary function. Recall that the first two-branch construction is based on $h_N(t) = N(N^{-1}(t)N(t))$. The internal product $N^{-1}(t)N(t)$ combines the inverse-negation scale $N^{-1}(t)$ with the negation value $N(t)$. To obtain an n -order version, we apply repeated product-composition to this internal component. Thus, we replace $N^{-1}(t)N(t)$ by $(N^{-1}(t)N(t))^n$. Equivalently, $(N^{-1}(t)N(t))^n = (N^{-1}(t))^n(N(t))^n$. This leads to the auxiliary function $h_{N,n}(t) = N((N^{-1}(t)N(t))^n) = N((N^{-1}(t))^n(N(t))^n)$. For $n = 1$, we recover $h_{N,1}(t) = h_N(t)$.

Remark 10. Composition mechanism behind the n -order extension

The n -order character of the following construction can be described through repeated product-composition. Let

$$s_N(t) = N^{-1}(t)N(t), t \in [0,1].$$

In Theorem 3, the first-order Yager-induced component is $h_N(t) = N(s_N(t))$.

For $n \geq 1$, define recursively $u_1(t) = s_N(t)$, and $u_{k+1}(t) = M(u_k(t), s_N(t))$, $k \geq 1$, where $M(a, b) = ab$. Then $u_2(t) = s_N(t)^2$, $u_3(t) = s_N(t)^3$, and, by induction, $u_n(t) = s_N(t)^n$. Therefore, $N(u_n(t)) = N(s_N(t)^n) = N((N^{-1}(t)N(t))^n)$. Equivalently,

$$N(u_n(t)) = N((N^{-1}(t))^n(N(t))^n).$$

Thus, the sequence of internal components has the form

$$s_N(t) \rightarrow s_N(t)^2 \rightarrow s_N(t)^3 \rightarrow \dots \rightarrow s_N(t)^n \rightarrow N(s_N(t)^n)$$

Hence, the function $h_{N,n}(t) = N((N^{-1}(t))^n(N(t))^n)$ is obtained by repeated product-composition of the first-order internal component $s_N(t)$, followed by the outer negation N . In this precise sense, the next theorem is an n -order composition extension of Theorem 3.

Theorem 5. n -Order Extension of the Yager-Induced Two-Branch Construction.

Let $N: [0,1] \rightarrow [0,1]$ be a strict fuzzy negation with inverse N^{-1} . For every $n \in \mathbb{N}$, define

$$h_{N,n}(t) = N((N^{-1}(t))^n(N(t))^n), t \in [0,1]. \quad (37)$$

Equivalently, $h_{N,n}(t) = N((N^{-1}(t)N(t))^n)$. Assume that $h_{N,n}$ is differentiable on $(0,1)$, non-decreasing on $[0,1]$, and satisfies $th'_{N,n}(t) \leq 1, t \in (0,1)$.

Define $I_{A,N}^{(n)}: [0,1]^2 \rightarrow [0,1]$ by

$$I_{A,N}^{(n)}(x, y) = \begin{cases} 1 - x + xh_{N,n}(y), & x \leq y, \\ 1 - x + yh_{N,n}(x), & y \leq x. \end{cases} \quad (38)$$

Then, $I_{A,N}^{(n)}$ is a fuzzy implication.

Proof. Since N is a strict fuzzy negation, we have $N(0) = 1, N(1) = 0$, and $N^{-1}(0) = 1, N^{-1}(1) = 0$. Therefore, $h_{N,n}(0) = N((N^{-1}(0))^n(N(0))^n)$. Using $N^{-1}(0) = 1$ and $N(0) = 1$, we obtain $h_{N,n}(0) = N(1^n \cdot 1^n) = N(1) = 0$. Similarly,

$$h_{N,n}(1) = N((N^{-1}(1))^n(N(1))^n).$$

Since $N^{-1}(1) = 0$ and $N(1) = 0$, we get $h_{N,n}(1) = N(0^n \cdot 0^n) = N(0) = 1$. Thus, $h_{N,n}(0) = 0, h_{N,n}(1) = 1$. Moreover, since $h_{N,n}$ maps $[0,1]$ into $[0,1]$, we have

$$0 \leq h_{N,n}(t) \leq 1, t \in [0,1].$$

We first show that the two branches coincide on the diagonal. For $x = y = t$, the first branch gives $1 - t + th_{N,n}(t)$, while the second branch also gives $1 - t + th_{N,n}(t)$.

Hence, $I_{A,N}^{(n)}$ is well defined and continuous on the switching line $x = y$. Next, we prove that $I_{A,N}^{(n)}$ takes values in $[0,1]$. If $x \leq y$, then $I_{A,N}^{(n)}(x, y) = 1 - x + xh_{N,n}(y)$. Since $0 \leq h_{N,n}(y) \leq 1$ we obtain $1 - x \leq I_{A,N}^{(n)}(x, y) \leq 1$. Therefore, $0 \leq I_{A,N}^{(n)}(x, y) \leq 1$.

If $y \leq x$, then $I_{A,N}^{(n)}(x, y) = 1 - x + yh_{N,n}(x)$. Again, since $0 \leq h_{N,n}(x) \leq 1$, we get

$$I_{A,N}^{(n)}(x, y) \geq 1 - x \geq 0.$$

Moreover, because $y \leq x$, $yh_{N,n}(x) \leq y \leq x$. Thus, $I_{A,N}^{(n)}(x, y) = 1 - x + yh_{N,n}(x) \leq 1$.

Therefore, $0 \leq I_{A,N}^{(n)}(x, y) \leq 1$ on both branches. On the switching boundary, the two analytic expressions coincide. Therefore, the construction is well defined and continuously glued along the boundary. We now check the monotonicity conditions. On the first branch $x \leq y$, we have $I_{A,N}^{(n)}(x, y) = 1 - x + xh_{N,n}(y)$. Differentiating with respect to x , we obtain

$$\frac{\partial I_{A,N}^{(n)}}{\partial x} = -1 + h_{N,n}(y).$$

Since $h_{N,n}(y) \leq 1$, it follows that $\frac{\partial I_{A,N}^{(n)}}{\partial x} \leq 0$. Also, $\frac{\partial I_{A,N}^{(n)}}{\partial y} = xh'_{N,n}(y)$. Since $h_{N,n}$ is non-decreasing, we have $h'_{N,n}(y) \geq 0$ where the derivative exists, and therefore $\frac{\partial I_{A,N}^{(n)}}{\partial y} \geq 0$.

Thus, the first branch is non-increasing with respect to x and non-decreasing with respect to y .

On the second branch $y \leq x$, we have $I_{A,N}^{(n)}(x, y) = 1 - x + yh_{N,n}(x)$. Differentiating with respect to y , we get $\frac{\partial I_{A,N}^{(n)}}{\partial y} = h_{N,n}(x) \geq 0$. Differentiating with respect to x , we get

$$\frac{\partial I_{A,N}^{(n)}}{\partial x} = -1 + yh'_{N,n}(x).$$

Since $y \leq x$, we have $yh'_{N,n}(x) \leq xh'_{N,n}(x)$. By the admissibility condition $xh'_{N,n}(x) \leq 1$, we obtain $yh'_{N,n}(x) \leq 1$. Consequently, $\frac{\partial I_{A,N}^{(n)}}{\partial x} = -1 + yh'_{N,n}(x) \leq 0$.

Therefore, the second branch is also non-increasing with respect to x and non-decreasing with respect to y . Finally, we verify the boundary conditions. We have

$$I_{A,N}^{(n)}(0,0) = 1, I_{A,N}^{(n)}(0,1) = 1, I_{A,N}^{(n)}(1,1) = 1, I_{A,N}^{(n)}(1,0) = 0.$$

Hence, $I_{A,N}^{(n)}$ satisfies the boundary conditions of a fuzzy implication. Combining the range condition, the two monotonicity properties, and the boundary conditions, we conclude that $I_{A,N}^{(n)}$ is a fuzzy implication. $\square$

Remark 11. On the symmetric n -order choice

The function $h_{N,n}$ is obtained by applying the same order to the two factors of the internal product $N^{-1}(t)N(t)$. More generally, one could consider

$$h_{N,p,q}(t) = N((N^{-1}(t))^p(N(t))^q), p, q \in \mathbb{N}.$$

However, both factors $N^{-1}(t)$ and $N(t)$ depend on the same variable t . Therefore, the parameters p and q do not have the same interpretation as in the unified family

$$I_{p,q}(x, y) = N(x^p(N^{-1}(y))^q),$$

where p acts on the antecedent-side component and q acts on the internal consequent-side scale. For this reason, Theorem 5 focuses on the symmetric choice $p = q = n$.

This gives $h_{N,n}(t) = N((N^{-1}(t))^n(N(t))^n) = N((N^{-1}(t)N(t))^n)$.

Thus, n measures the common product-composition order of the internal component $N^{-1}(t)N(t)$, without introducing parallel two-parameter variants with no distinct antecedent-side or consequent-side interpretation.

Example 6. Consider the strict fuzzy negation $N(t) = 1 - t, t \in [0, 1]$. Then, $N^{-1}(t) = 1 - t$. For every $n \in \mathbb{N}$, Theorem 5 defines $h_{N,n}(t) = N((N^{-1}(t)N(t))^n)$. Since $N^{-1}(t) = 1 - t$ and $N(t) = 1 - t$, we obtain $N^{-1}(t)N(t) = (1 - t)^2$. Therefore, $h_{N,n}(t) = N((1 - t)^{2n})$. Since $N(u) = 1 - u$, we get $h_{N,n}(t) = 1 - (1 - t)^{2n}$. Clearly, $h_{N,n}(0) = 0, h_{N,n}(1) = 1$. Moreover, $h'_{N,n}(t) = 2n(1 - t)^{2n-1}$. Thus, $th'_{N,n}(t) = 2nt(1 - t)^{2n-1}$.

The function $\phi_n(t) = 2nt(1 - t)^{2n-1}$ attains its maximum at $t = \frac{1}{2n}$. Hence, $\max_{t \in [0, 1]} \phi_n(t) = \left(1 - \frac{1}{2n}\right)^{2n-1} \leq 1$. Therefore, $th'_{N,n}(t) \leq 1, t \in [0, 1]$.

Thus, all assumptions of Theorem 5 are satisfied. Consequently, the n -order two-branch fuzzy implication generated by Theorem 5 is $I_{A,N}^{(n)}(x, y) = \begin{cases} 1 - x + xh_{N,n}(y), & x \leq y, \\ 1 - x + yh_{N,n}(x), & y \leq x. \end{cases}$ Substituting $h_{N,n}(t) = 1 - (1 - t)^{2n}$, we obtain $I_{A,N}^{(n)}(x, y) = \begin{cases} 1 - x + x[1 - (1 - y)^{2n}], & x \leq y, \\ 1 - x + y[1 - (1 - x)^{2n}], & y \leq x. \end{cases}$ Equivalently,

$$I_{A,N}^{(n)}(x, y) = \begin{cases} 1 - x(1 - y)^{2n}, & x \leq y, \\ 1 - x + y - y(1 - x)^{2n}, & y \leq x. \end{cases}$$

Therefore, by Theorem 5, $I_{A,N}^{(n)}$ is a fuzzy implication for every $n \in \mathbb{N}$.

Remark 12. Pointwise asymptotic behavior of Example 6

For the family in Example 6,

$$I_{A,N}^{(n)}(x, y) = \begin{cases} 1 - x(1 - y)^{2n}, & x \leq y, \\ 1 - x + y - y(1 - x)^{2n}, & y \leq x, \end{cases}$$

we have the pointwise limit

$$\lim_{n \rightarrow +\infty} I_{A,N}^{(n)}(x, y) = \begin{cases} 1, & x \leq y, \\ 1 - x + y, & y \leq x. \end{cases} \quad (39)$$

Indeed, on the branch $x \leq y$, if $y > 0$, then $(1 - y)^{2n} \rightarrow 0$ and therefore $I_{A,N}^{(n)}(x, y) \rightarrow 1$.

If $y = 0$, then $x = 0$, because $x \leq y$, and the same value is obtained. On the branch $y \leq x$, if $x > 0$, then $(1 - x)^{2n} \rightarrow 0$ and therefore $I_{A,N}^{(n)}(x, y) \rightarrow 1 - x + y$. If $x = 0$, then $y = 0$, and the same formula gives 1.

Thus, the convergence is pointwise and branch-dependent: the first branch saturates to 1, whereas the second branch converges to the non-trivial expression $1 - x + y$.

Remark 13. Composition mechanism behind the direct n -order two-branch construction

We now explain the composition mechanism behind the direct n -order extension of Theorem 4 In contrast with Theorem 5, where the order is applied to the diagonal component $N^{-1}(t)N(t)$, we now apply repeated product-composition directly to the Yager-type internal products appearing in the two branches.

For the branch $x \leq y$, set $s_{1,N}(x, y) = xN^{-1}(y)$. For the branch $y \leq x$, set

$$s_{2,N}(x, y) = yN^{-1}(x).$$

For $n \geq 1$, repeated product-composition gives $s_{1,N}(x, y)^n = (xN^{-1}(y))^n$, and $s_{2,N}(x, y)^n = (yN^{-1}(x))^n$. Therefore, the corresponding Yager-type components are $N((xN^{-1}(y))^n)$ and $N((yN^{-1}(x))^n)$. Equivalently, $N((xN^{-1}(y))^n) = N(x^n(N^{-1}(y))^n)$, and $N((yN^{-1}(x))^n) = N(y^n(N^{-1}(x))^n)$. Thus, the next theorem is a direct n -order product-composition extension of Theorem 4

Theorem 6. *n -Order Extension of the Direct Yager-Type Two-Branch Construction.*

Let $N: [0,1] \rightarrow [0,1]$ be a strict fuzzy negation with differentiable inverse N^{-1} . Assume that N is differentiable on $(0,1)$. For every $n \in \mathbb{N}$, assume that, for every

$$z \in (0,1), 0 < y \leq N(z),$$

the following admissibility conditions hold:

$$(A1_n) \quad 0 \leq ny^{n+1}z^{n-1} \frac{N'((yz)^n)}{N'(z)} \leq 1, \quad (40)$$

and

$$(A2_n) \quad N((yz)^n) + n(yz)^n N'((yz)^n) \geq 0. \quad (41)$$

Define $I_{B,N}^{(n)}: [0,1]^2 \rightarrow [0,1]$ by

$$I_{B,N}^{(n)}(x, y) = \begin{cases} 1 - x + xN(x^n(N^{-1}(y))^n), & x \leq y, \\ 1 - x + yN(y^n(N^{-1}(x))^n), & y \leq x. \end{cases} \quad (42)$$

Equivalently,

$$I_{B,N}^{(n)}(x, y) = \begin{cases} 1 - x + xN((xN^{-1}(y))^n), & x \leq y, \\ 1 - x + yN((yN^{-1}(x))^n), & y \leq x. \end{cases} \quad (43)$$

Then, $I_{B,N}^{(n)}$ is a fuzzy implication.

Proof. First, we observe that the two branches coincide on the diagonal. For $x = y = t$, the first branch gives $1 - t + tN(t^n(N^{-1}(t))^n)$, while the second branch gives the same expression, $1 - t + tN(t^n(N^{-1}(t))^n)$. Equivalently, both branches give $1 - t + tN((tN^{-1}(t))^n)$. Therefore, $I_{B,N}^{(n)}$ is well defined and continuous on the switching line $x = y$. We now check the range. Since N maps $[0,1]$ into $[0,1]$, we have $0 \leq N(x^n(N^{-1}(y))^n) \leq 1$ and $0 \leq N(y^n(N^{-1}(x))^n) \leq 1$. Thus, on the first branch,

$$I_{B,N}^{(n)}(x, y) = 1 - x + xN(x^n(N^{-1}(y))^n), \text{ and hence } 1 - x \leq I_{B,N}^{(n)}(x, y) \leq 1.$$

On the second branch, $I_{B,N}^{(n)}(x, y) = 1 - x + yN(y^n(N^{-1}(x))^n)$. Since $0 \leq N(\cdot) \leq 1$ and $y \leq x$, we get $I_{B,N}^{(n)}(x, y) \geq 1 - x \geq 0$ and $I_{B,N}^{(n)}(x, y) \leq 1 - x + y \leq 1$. Therefore,

$$0 \leq I_{B,N}^{(n)}(x, y) \leq 1$$

on both branches. We now prove the monotonicity conditions. On the first branch $x \leq y$, we have $I_{B,N}^{(n)}(x, y) = 1 - x + xN(x^n(N^{-1}(y))^n)$. Set $s = x^n(N^{-1}(y))^n$. Differentiating with respect to x , we obtain $\frac{\partial I_{B,N}^{(n)}}{\partial x} = -1 + N(s) + xN'(s) \cdot nx^{n-1}(N^{-1}(y))^n$.

Since $N(s) \leq 1$ And $N'(s) < 0$, we have $xN'(s) \cdot nx^{n-1}(N^{-1}(y))^n \leq 0$. Thus,

$$\frac{\partial I_{B,N}^{(n)}}{\partial x} \leq -1 + N(s) \leq 0.$$

Therefore, the first branch is non-increasing with respect to x . Next, differentiating with respect to y , we get $\frac{\partial I_{B,N}^{(n)}}{\partial y} = xN'(s) \cdot nx^n(N^{-1}(y))^{n-1}(N^{-1})'(y)$.

Since $N'(s) < 0$ and $(N^{-1})'(y) < 0$, it follows that $\frac{\partial I_{B,N}^{(n)}}{\partial y} \geq 0$. Thus, the first branch is non-decreasing with respect to y . Now consider the second branch $y \leq x$. We have

$$I_{B,N}^{(n)}(x, y) = 1 - x + yN(y^n(N^{-1}(x))^n).$$

Set $z = N^{-1}(x)$. Then, $x = N(z)$. Since $y \leq x$, we have $0 < y \leq N(z)$. The second branch can be written as $I_{B,N}^{(n)}(x, y) = 1 - x + yN(y^n z^n) = 1 - x + yN((yz)^n)$. Differentiating with respect to x , we get $\frac{\partial I_{B,N}^{(n)}}{\partial x} = -1 + yN'((yz)^n) \cdot n(yz)^{n-1}y(N^{-1})'(x)$.

Equivalently, $\frac{\partial I_{B,N}^{(n)}}{\partial x} = -1 + ny^{n+1}z^{n-1}N'((yz)^n)(N^{-1})'(x)$. Since $(N^{-1})'(x) = \frac{1}{N'(z)}$, we obtain

$$\frac{\partial I_{B,N}^{(n)}}{\partial x} = -1 + ny^{n+1}z^{n-1} \frac{N'((yz)^n)}{N'(z)}. \text{ By condition (A1}_n\text{), } 0 \leq ny^{n+1}z^{n-1} \frac{N'((yz)^n)}{N'(z)} \leq 1.$$

Hence, $\frac{\partial I_{B,N}^{(n)}}{\partial x} \leq 0$. Thus, the second branch is non-increasing with respect to x . Next, differentiating the second branch with respect to y , we obtain

$$\frac{\partial I_{B,N}^{(n)}}{\partial y} = N((yz)^n) + yN'((yz)^n) \cdot n(yz)^{n-1}z. \text{ Therefore}$$

$$\frac{\partial I_{B,N}^{(n)}}{\partial y} = N((yz)^n) + n(yz)^n N'((yz)^n).$$

By condition (A2_n), $N((yz)^n) + n(yz)^n N'((yz)^n) \geq 0$. Hence, $\frac{\partial I_{B,N}^{(n)}}{\partial y} \geq 0$.

Thus, the second branch is non-decreasing with respect to y .

Finally, we verify the boundary conditions. We have $I_{B,N}^{(n)}(0,0) = 1, I_{B,N}^{(n)}(0,1) = 1, I_{B,N}^{(n)}(1,1) = 1$, and $I_{B,N}^{(n)}(1,0) = 0$. Thus, $I_{B,N}^{(n)}$ satisfies the boundary conditions of a fuzzy implication. Combining the range condition, the two monotonicity properties, and the boundary conditions, we conclude that $I_{B,N}^{(n)}$ is a fuzzy implication. $\square$

Remark 14. On the symmetric choice $\mathbf{p} = \mathbf{q} = \mathbf{n}$ in Theorem 6

Theorem 6 uses the symmetric products $(xN^{-1}(y))^n$ and $(yN^{-1}(x))^n$. Equivalently,

$$(xN^{-1}(y))^n = x^n(N^{-1}(y))^n, \text{ and } (yN^{-1}(x))^n = y^n(N^{-1}(x))^n.$$

Thus, the construction may be viewed as the symmetric case $\mathbf{p} = \mathbf{q} = \mathbf{n}$ of a possible two-parameter branch-wise deformation. We focus on the symmetric case in order to keep the construction compact and directly tied to repeated product-composition of the original Yager-type internal products. This also avoids introducing several parallel two-branch families with essentially the same qualitative limiting behavior.

Example 7. For $N(t) = 1 - t^2, N^{-1}(t) = \sqrt{1-t}$, the n -order direct two-branch construction becomes

$$I_{B,N}^{(n)}(x, y) = \begin{cases} 1 - x + xN((x\sqrt{1-y})^n), & x \leq y, \\ 1 - x + yN((y\sqrt{1-x})^n), & y \leq x. \end{cases}$$

Since $N(u) = 1 - u^2$, we get $N((x\sqrt{1-y})^n) = 1 - x^{2n}(1-y)^n$, and

$N((y\sqrt{1-x})^n) = 1 - y^{2n}(1-x)^n$. Therefore,

$$I_{B,N}^{(n)}(x,y) = \begin{cases} 1 - x^{2n+1}(1-y)^n, & x \leq y, \\ 1 - x + y - y^{2n+1}(1-x)^n, & y \leq x. \end{cases}$$

For $n = 1$, this reduces to Example 5:

$$I_{B,N}(x,y) = \begin{cases} 1 - x^3(1-y), & x \leq y, \\ 1 - x + y - y^3(1-x), & y \leq x. \end{cases}$$

Remark 15. Pointwise asymptotic behavior of $I_{B,N}^{(n)}$.

For the direct n -order two-branch family

$$I_{B,N}^{(n)}(x,y) = \begin{cases} 1 - x + xN((xN^{-1}(y))^n), & x \leq y, \\ 1 - x + yN((yN^{-1}(x))^n), & y \leq x, \end{cases} \text{ we have the pointwise limit}$$

$$\lim_{n \rightarrow +\infty} I_{B,N}^{(n)}(x,y) = \begin{cases} 1, & x \leq y, \\ 1 - x + y, & y \leq x. \end{cases} \quad (44)$$

Indeed, on the branch $x \leq y$, the product $xN^{-1}(y)$ belongs to $[0,1]$. Moreover, if $xN^{-1}(y) = 1$, then necessarily $x = 1$ and $N^{-1}(y) = 1$, which would imply $y = 0$. This is incompatible with $x \leq y$, except in no admissible nontrivial point of the branch. Hence, $0 \leq xN^{-1}(y) < 1$ on the branch $x \leq y$, and therefore $(xN^{-1}(y))^n \rightarrow 0$. Thus, $N((xN^{-1}(y))^n) \rightarrow N(0) = 1$, and the first branch converges to $1 - x + x = 1$.

Similarly, on the branch $y \leq x$, $0 \leq yN^{-1}(x) < 1$ except for boundary cases that are still covered by the formula. Hence, $(yN^{-1}(x))^n \rightarrow 0$, and $N((yN^{-1}(x))^n) \rightarrow 1$. Therefore, the second branch converges to $1 - x + y$.

Thus, the convergence is pointwise and branch-dependent: the first branch saturates to 1, while the second branch converges to the non-trivial expression $1 - x + y$.

Remark 16. Structural properties of the two-branch families.

The constructions introduced in Theorems 3-6 are fuzzy implications under their corresponding admissibility assumptions. Therefore, for each admissible value of n , they satisfy the defining axioms of a fuzzy implication: they are non-increasing with respect to the first variable, non-decreasing with respect to the second variable, and satisfy the boundary conditions

$$I(0,0) = I(0,1) = I(1,1) = 1, I(1,0) = 0.$$

In addition to these defining axioms, it is useful to indicate whether the proposed families satisfy some standard additional properties usually studied for fuzzy implications, namely the left neutrality principle, the right neutrality principle, the identity principle, the ordering property, the exchange principle, and the contraposition property.

We use the following notation: $LNP: I(1,y) = y$, $RNP: I(x,1) = 1$, $IP: I(x,x) = 1$, $OP: I(x,y) = 1 \Leftrightarrow x \leq y$, $EP: I(x, I(y,z)) = I(y, I(x,z))$, and

$$CP(N): I(x,y) = I(N(y), N(x)).$$

Here, the contraposition property is considered with respect to the generating strict fuzzy negation N .

The phrase “Generally no” in Table 3 means that the property is not guaranteed by the assumptions of the corresponding theorem. In other words, the family may satisfy the property only for special choices of the negation N , or for special parameter values, but it does not satisfy it in general. This distinction is important, because the theorems are designed to guarantee the fuzzy implication axioms, not necessarily all additional logical principles.

Table 3. Structural properties of the two-branch fuzzy implication families introduced in Theorems 3–6. The column “FI axioms” refers to the defining axioms of fuzzy implications, namely monotonicity in both variables and the boundary conditions. The expression “Generally no” means that the corresponding property is not guaranteed by the assumptions of the theorem and may fail for admissible choices of the generating negation N .

Family	FI Axioms	LNP	RNP	IP	OP	EP	CP(N)
$I_{A,N}$ from Theorem 3	Yes	Yes	Yes	Generally no	Generally no	Generally no	Generally no
$I_{B,N}$ from Theorem 4	Yes	Yes	Yes	Generally no	Generally no	Generally no	Generally no
$I_{A,N}^{(n)}$ from Theorem 5	Yes, for every admissible n	Yes	Yes	Generally no	Generally no	Generally no	Generally no
$I_{B,N}^{(n)}$ from Theorem 6	Yes, for every admissible n	Yes	Yes	Generally no	Generally no	Generally no	Generally no

The expression “Generally no” means that the corresponding property is not guaranteed by the assumptions of the theorem and may fail for admissible choices of the generating negation N .

The positive entries follow directly from the definitions. For instance, in all four families the point $(1, y)$ belongs to the branch $y \leq x$, and the corresponding formula gives $I(1, y) = y$.

Similarly, since $x \leq 1$, the point $(x, 1)$ belongs to the branch $x \leq y$, and the corresponding formula gives $I(x, 1) = 1$. For the negative entries, consider the admissible strict non-strong negation $N(t) = 1 - t^2$. For the family introduced in Theorem 4, Example 5 gives

$$I_{B,N}(x, y) = \begin{cases} 1 - x^3(1 - y), & x \leq y, \\ 1 - x + y - y^3(1 - x), & y \leq x. \end{cases}$$

The identity principle fails, since $I_{B,N}(0.5, 0.5) = 0.9375 \neq 1$. The ordering property also fails. Although $0.4 \leq 0.6$, $I_{B,N}(0.4, 0.6) = 0.9744 \neq 1$. The exchange principle fails as well. For $x = 0.7$, $y = 0.4$, and $z = 0.2$, direct computation gives $I_{B,N}(0.7, I_{B,N}(0.4, 0.2)) = 0.9297536$, whereas $I_{B,N}(0.4, I_{B,N}(0.7, 0.2)) = 0.9678464$. Thus, the two values are different.

Finally, contraposition with respect to N is not generally preserved. For $x = 0.7$ and $y = 0.4$, $I_{B,N}(0.7, 0.4) = 0.6808$. On the other hand, $N(0.4) = 0.84$, $N(0.7) = 0.51$, and $I_{B,N}(N(0.4), N(0.7)) = I_{B,N}(0.84, 0.51) = 0.64877584$. Hence,

$$I_{B,N}(0.7, 0.4) \neq I_{B,N}(N(0.4), N(0.7)).$$

These examples justify the use of “Generally no” in the table. The term does not mean that the corresponding property is impossible; it means that the property is not guaranteed by the assumptions of the theorem and may fail even for admissible choices of N .

The two Yager-induced two-branch constructions introduced in this subsection extend the preceding closure-based power families in a genuinely branch-dependent direction. They are still generated from the baseline implication $I_N(x, y) = N(xN^{-1}(y))$, but they are no longer simple precomposition examples. Instead, the baseline implication is used to build continuous two-branch schemes on the regions $x \leq y$ and $y \leq x$. Their n -order extensions show that repeated product-composition may lead to branch-dependent pointwise limits: saturation on one branch and a nontrivial limiting expression on the other. This distinguishes the two-branch constructions from the earlier power-type families, where large orders mainly produce saturation toward the maximal truth value.

The admissibility assumptions used in the two-branch theorems should be understood as sufficient, proof-oriented conditions. They are designed to ensure the monotonicity requirements of fuzzy implications for the corresponding branch-wise formulas. They are not claimed to provide necessary conditions or a complete classification of all strict

fuzzy negations that may generate valid two-branch implications. The examples included above show that the assumptions are non-vacuous. Some parameter ranges are verified analytically, whereas numerical thresholds, when mentioned, are reported only as numerical evidence and are not used as proved admissibility bounds.

3.6. Two-Branch Recursive Feedback Compositions

After the Yager-induced two-branch constructions and their n -order product-composition extensions, we now consider a different composition mechanism. In the previous subsection, the order parameter n was introduced as a power inside an internal Yager-type product. In the present subsection, n counts the number of recursive feedback steps.

We first introduce an auxiliary two-branch fuzzy implication, denoted by $\tilde{I}_A(x, y)$.

This implication is generated from the same Yager-type internal term $N(xN^{-1}(y))$, but it is organized through two branches separated by the gluing curve $x = N(y)$.

The role of $\tilde{I}_A$ is to provide a base implication for the recursive feedback mechanisms studied below.

We consider two recursive schemes. In the first one, the previous implication value is inserted into the consequent position. In the second one, the previous implication value is first transformed through a decreasing endpoint-reversing map, and then inserted into the antecedent position. In the applications below, this map will be naturally chosen as N^{-1} . These recursive mechanisms are different from the direct n -order constructions of Section 3.5. They do not simply raise an internal product to a power. Instead, they repeatedly compose a fuzzy implication with its own previous output. As a result, the active branch may change during the iteration, and the limiting behavior may depend on the branch structure of the base implication. The general preservation result is stated first.

In the recursive part, the contributions should be separated from the base two-branch formula. The auxiliary implication $\tilde{I}_A$ provides the base operator. The recursive schemes $\tilde{I}_{A,R}^{(n)}$ and $\tilde{I}_{A,C}^{(n)}$ are then new feedback mechanisms generated by repeated self-composition. The convergence results and numerical trajectories are not intended to establish a universal convergence theory for all recursive schemes; rather, they show that such feedback mechanisms may produce branch switching and fixed-point type behavior. Thus, the novelty of the recursive subsection lies in the recursive use of a two-branch implication and in the resulting feedback dynamics.

Proposition 2. General Recursive Feedback Principle.

Let $I: [0,1]^2 \rightarrow [0,1]$ be a fuzzy implication. Define $\mathcal{R}_1(x, y) = I(x, y)$, and, for every $n \geq 1$,

$$\mathcal{R}_{n+1}(x, y) = I(x, \mathcal{R}_n(x, y)). \quad (45)$$

Then, $\mathcal{R}_n$ is a fuzzy implication for every $n \in \mathbb{N}$. Moreover, let $\varphi: [0,1] \rightarrow [0,1]$ be a decreasing function satisfying $\varphi(0) = 1, \varphi(1) = 0$. Define $\mathcal{C}_1(x, y) = I(x, y)$, and, for every $n \geq 1$,

$$\mathcal{C}_{n+1}(x, y) = I(\varphi(\mathcal{C}_n(x, y)), y). \quad (46)$$

Then, $\mathcal{C}_n$ is a fuzzy implication for every $n \in \mathbb{N}$.

Proof. We prove both assertions by induction. First, $\mathcal{R}_1 = I$ is a fuzzy implication. Assume that $\mathcal{R}_n$ is a fuzzy implication. We show that $\mathcal{R}_{n+1}(x, y) = I(x, \mathcal{R}_n(x, y))$ is also a fuzzy implication. The boundary conditions follow directly from those of I and $\mathcal{R}_n$: $\mathcal{R}_{n+1}(0, 0) = I(0, \mathcal{R}_n(0, 0)) = I(0, 1) = 1$, $\mathcal{R}_{n+1}(0, 1) = I(0, \mathcal{R}_n(0, 1)) = I(0, 1) = 1$, $\mathcal{R}_{n+1}(1, 0) = I(1, \mathcal{R}_n(1, 0)) = I(1, 0) = 0$, and

$$R_{n+1}(1,1) = I(1, R_n(1,1)) = I(1,1) = 1.$$

Now let $x_1 \leq x_2$. Since R_n is non-increasing in its first variable, $R_n(x_1, y) \geq R_n(x_2, y)$. Using the monotonicity of I , we obtain $I(x_1, R_n(x_1, y)) \geq I(x_2, R_n(x_1, y)) \geq I(x_2, R_n(x_2, y))$. Hence, $R_{n+1}(x_1, y) \geq R_{n+1}(x_2, y)$, so R_{n+1} is non-increasing in the first variable. Similarly, if $y_1 \leq y_2$, then $R_n(x, y_1) \leq R_n(x, y_2)$, and therefore

$$R_{n+1}(x, y_1) = I(x, R_n(x, y_1)) \leq I(x, R_n(x, y_2)) = R_{n+1}(x, y_2).$$

Thus, R_{n+1} is non-decreasing in the second variable. Hence, R_{n+1} is a fuzzy implication, and the first assertion follows by induction. We now prove the second assertion. Since $C_1 = I$, C_1 is a fuzzy implication. Assume that C_n is a fuzzy implication. We prove that $C_{n+1}(x, y) = I(\varphi(C_n(x, y)), y)$ is also a fuzzy implication. The boundary conditions are

$$\begin{aligned} C_{n+1}(0,0) &= I(\varphi(C_n(0,0)), 0) = I(\varphi(1), 0) = I(0,0) = 1, \\ C_{n+1}(0,1) &= I(\varphi(C_n(0,1)), 1) = I(\varphi(1), 1) = I(0,1) = 1, C_{n+1}(1,0) = I(\varphi(C_n(1,0)), 0) = \\ I(\varphi(0), 0) &= I(1,0) = 0, \text{ and } C_{n+1}(1,1) = I(\varphi(C_n(1,1)), 1) = I(\varphi(1), 1) = I(0,1) = 1. \end{aligned}$$

Now let $x_1 \leq x_2$. Since C_n is non-increasing in its first variable, $C_n(x_1, y) \geq C_n(x_2, y)$. Since φ is decreasing, $\varphi(C_n(x_1, y)) \leq \varphi(C_n(x_2, y))$. Using the fact that I is non-increasing in its first variable, we get

$$C_{n+1}(x_1, y) = I(\varphi(C_n(x_1, y)), y) \geq I(\varphi(C_n(x_2, y)), y) = C_{n+1}(x_2, y).$$

Thus, C_{n+1} is non-increasing in the first variable. Finally, let $y_1 \leq y_2$. Since C_n is non-decreasing in its second variable, $C_n(x, y_1) \leq C_n(x, y_2)$, and since φ is decreasing, $\varphi(C_n(x, y_1)) \geq \varphi(C_n(x, y_2))$. Therefore, using the monotonicity of I , $I(\varphi(C_n(x, y_1)), y_1) \leq I(\varphi(C_n(x, y_2)), y_1) \leq I(\varphi(C_n(x, y_2)), y_2)$. Hence,

$$C_{n+1}(x, y_1) \leq C_{n+1}(x, y_2), \text{ so } C_{n+1} \text{ is non-decreasing in the second variable.}$$

Thus, C_{n+1} is a fuzzy implication. By induction, C_n is a fuzzy implication for every $n \in \mathbb{N}$. $\square$

Remark 17. *Scope of the recursive feedback principle*

Proposition 2 shows that recursive feedback composition preserves the fuzzy implication axioms under general monotonicity assumptions. However, the proposition does not by itself determine the limiting behavior of the recursive sequences. The convergence properties depend on the specific base implication, on the chosen endpoint-reversing map φ , and on the branch structure involved in the iteration.

In the following, we apply this principle to the auxiliary two-branch implication $\tilde{I}_A$. In that setting, the recursive process may interact with the gluing curve $x = N(y)$,

and the active branch may change during the iteration. This is the main difference between the recursive feedback mechanisms studied in this subsection and the direct product-composition mechanisms of Section 3.5.

3.6.1. Consequent-Position Recursive Feedback

In order to introduce the two-branch constructions developed in the sequel, we first state and prove a useful auxiliary two-branch implication theorem. This result is a modified version of the two-branch construction recalled in the preliminaries (2.2), adapted here to the notation and framework of the present section. We include the proof in order to make clear the role of the gluing curve and to provide a self-contained basis for the recursive feedback constructions that follow.

Theorem 7. *A two-branch fuzzy implication generated by a strict fuzzy negation.*

Let $N: [0,1] \rightarrow [0,1]$ be a continuous strictly decreasing function such that $N(0) = 1, N(1) = 0$. Let N^{-1} denote its inverse. Define $\tilde{I}_A: [0,1]^2 \rightarrow [0,1]$ by

$$\tilde{I}_A(x, y) = \begin{cases} N^{-1}(x)N(xN^{-1}(y)), & 0 \leq x \leq N(y), \\ yN(xN^{-1}(y)), & N(y) < x \leq 1. \end{cases} \quad (47)$$

Then, $\tilde{I}_A$ is a fuzzy implication.

Proof. Since N is continuous and strictly decreasing with $N(0) = 1, N(1) = 0$, it is bijective from $[0,1]$ onto $[0,1]$. Therefore, its inverse $N^{-1}: [0,1] \rightarrow [0,1]$ is well defined, continuous, and strictly decreasing.

First, we check that the two branches coincide on their common boundary. The switching curve is $x = N(y)$. If $x = N(y)$, then $N^{-1}(x) = N^{-1}(N(y)) = y$. Hence, on the boundary

$x = N(y)$, the first branch gives

$$N^{-1}(x)N(xN^{-1}(y)) = yN(xN^{-1}(y)),$$

which is exactly the value of the second branch. Therefore, $\tilde{I}_A$ is well defined and continuously glued along the curve $x = N(y)$.

We now check the range. Since $N^{-1}(x) \in [0,1], y \in [0,1]$, and $N(xN^{-1}(y)) \in [0,1]$, both branches take values in $[0,1]$. Thus, $0 \leq \tilde{I}_A(x, y) \leq 1$ for all $(x, y) \in [0,1]^2$. Next, we prove that $\tilde{I}_A$ is non-increasing with respect to the first variable. On the first branch,

$$\tilde{I}_A(x, y) = N^{-1}(x)N(xN^{-1}(y)).$$

For fixed y , the factor $N^{-1}(x)$ is non-increasing in x , because N^{-1} is decreasing. Also, since $xN^{-1}(y)$ is non-decreasing in x and N is decreasing, the factor $N(xN^{-1}(y))$ is non-increasing in x . Hence, the product of two non-negative non-increasing functions is non-increasing. Therefore, the first branch is non-increasing with respect to x . On the second branch,

$$\tilde{I}_A(x, y) = yN(xN^{-1}(y)).$$

For fixed y , the factor y is constant, while $xN^{-1}(y)$ is non-decreasing in x . Since N is decreasing, $N(xN^{-1}(y))$ is non-increasing in x . Therefore, the second branch is also non-increasing with respect to x . Since the two branches coincide on the switching curve $x = N(y)$, the whole function $\tilde{I}_A$ is non-increasing with respect to x . We now prove that $\tilde{I}_A$ is non-decreasing with respect to the second variable.

On the first branch, $\tilde{I}_A(x, y) = N^{-1}(x)N(xN^{-1}(y))$. For fixed x , the factor $N^{-1}(x)$ is constant. Since N^{-1} is decreasing, the function $N^{-1}(y)$ is non-increasing in y . Hence, $xN^{-1}(y)$ is non-increasing in y . Since N is decreasing, it follows that $N(xN^{-1}(y))$ is non-decreasing in y . Therefore, the first branch is non-decreasing with respect to y . On the second branch, $\tilde{I}_A(x, y) = yN(xN^{-1}(y))$. For fixed x , the factor y is non-decreasing in y . Moreover, as shown above, $N(xN^{-1}(y))$ is also non-decreasing in y . Therefore, the product of two non-negative non-decreasing functions is non-decreasing. Hence, the second branch is non-decreasing with respect to y . Since the two branches coincide on the switching curve $x = N(y)$, the whole function $\tilde{I}_A$ is non-decreasing with respect to y . Finally, we verify the boundary conditions of a fuzzy implication.

Therefore, $\tilde{I}_A(0,0) = 1, \tilde{I}_A(0,1) = 1, \tilde{I}_A(1,1) = 1, \tilde{I}_A(1,0) = 0$. Thus, $\tilde{I}_A$ satisfies the boundary conditions, is non-increasing in its first variable, and is non-decreasing in its second variable. Hence, $\tilde{I}_A$ is a fuzzy implication. $\square$

We first apply Proposition 2. to the two-branch implication I_A . Define $I_{A,R}^{(1)}(x, y) = I_A(x, y)$, and, for every $n \geq 1$, $I_{A,R}^{(n+1)}(x, y) = I_A(x, I_{A,R}^{(n)}(x, y))$. Thus, each new order is

obtained by inserting the previous implication value into the consequent position of I_A , while keeping the antecedent variable x fixed. By Proposition 2, since I_A is a fuzzy implication, it follows that

Corollary 4. For every $n \in \mathbb{N}$, the function $\tilde{I}_{A,R}^{(n)}$ is a fuzzy implication.

Moreover, the $(n+1)$ -order member has the recursive two-branch representation

$$\tilde{I}_{A,R}^{(n+1)}(x, y) = \begin{cases} N^{-1}(x)N\left(xN^{-1}(\tilde{I}_{A,R}^{(n)}(x, y))\right), & 0 \leq x \leq N(\tilde{I}_{A,R}^{(n)}(x, y)), \\ \tilde{I}_{A,R}^{(n)}(x, y)N\left(xN^{-1}(\tilde{I}_{A,R}^{(n)}(x, y))\right), & N(\tilde{I}_{A,R}^{(n)}(x, y)) < x \leq 1. \end{cases} \quad (48)$$

The gluing curve at order $n + 1$ is $N(\tilde{I}_{A,R}^{(n)}(x, y))$.

On this curve, $N^{-1}(x) = N^{-1}\left(N(\tilde{I}_{A,R}^{(n)}(x, y))\right) = \tilde{I}_{A,R}^{(n)}(x, y)$, and therefore the two branches coincide.

Example 8. Consider the strict fuzzy negation $N(x) = 1 - x^2, x \in [0, 1]$. Then, $N^{-1}(x) = \sqrt{1 - x}$. This negation is not strong, since in general $N^{-1}(x) \neq N(x)$. For instance, $N(0.5) = 0.75, N^{-1}(0.5) = \sqrt{0.5} \approx 0.7071$. The corresponding two-branch implication is

$$\tilde{I}_A(x, y) = \begin{cases} \sqrt{1 - x}[1 - x^2(1 - y)], & 0 \leq x \leq 1 - y^2, \\ y[1 - x^2(1 - y)], & 1 - y^2 < x \leq 1. \end{cases}$$

Let $x = 0.9, y = 0.95$. For the initial pair, the gluing value is $N(y) = 1 - y^2 = 1 - (0.95)^2 = 0.0975$. Since $0.9 > 0.0975$, the second branch is used at the first step. Therefore, $\tilde{I}_{A,R}^{(1)}(0.9, 0.95) = 0.95[1 - (0.9)^2(1 - 0.95)] \approx 0.9115$. Continuing the recursion $\tilde{I}_{A,R}^{(n+1)}(0.9, 0.95) = \tilde{I}_{A,R}^{(n)}(0.9, \tilde{I}_{A,R}^{(n)}(0.9, 0.95))$, we obtain the following values in Table 4 and the plot in Figure 3.

Table 4. Numerical evolution of the consequent-position recursive feedback sequence $\tilde{I}_{A,R}^{(n)}(0.9, 0.95)$, together with the corresponding gluing value $N(\tilde{I}_{A,R}^{(n-1)}(0.9, 0.95))$ and the branch selected at each iteration. The table shows that the recursion initially remains in the second branch, crosses the gluing condition after several steps, and then enters the first branch.

n	$\tilde{I}_{A,R}^{(n)}(0.9, 0.95)$	$N(\tilde{I}_{A,R}^{(n-1)}(0.9, 0.95))$	Branch Used
1	0.9115	$N(0.95) = 0.0975$	Second
2	0.8462	$N(0.9115) \approx 0.1691$	Second
3	0.7408	$N(0.8462) \approx 0.2840$	Second
4	0.5852	$N(0.7408) \approx 0.4512$	Second

This example shows that the recursive process may remain in the second branch for several iterations and then cross the gluing condition, entering the first branch. Thus, the recursive feedback construction exhibits branch-dependent dynamics that cannot be described by a simple power transformation (Figure 3).

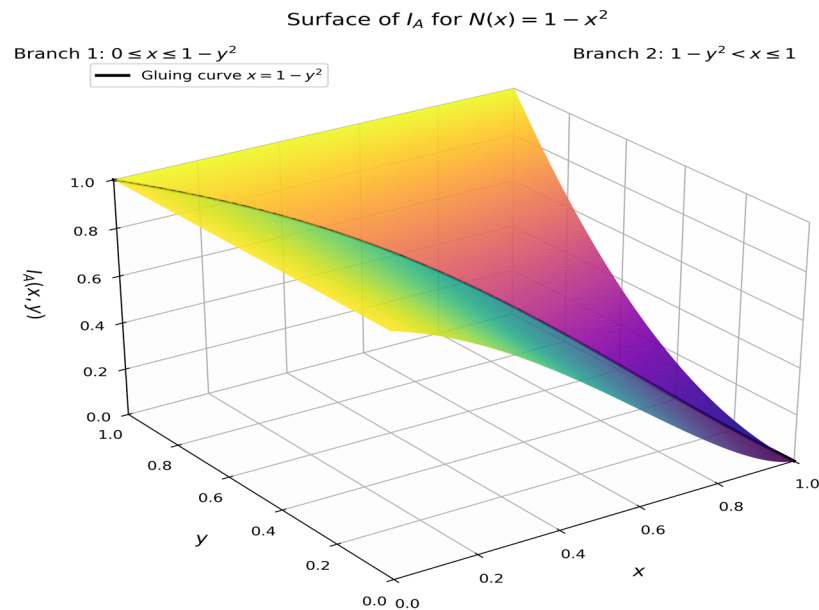


Figure 3. Surface plot of the two-branch fuzzy implication I_A generated by the strict non-strong fuzzy negation $N(x) = 1 - x^2$, with inverse $N^{-1}(x) = \sqrt{1 - x}$. The first branch is defined on the region $0 \leq x \leq 1 - y^2$, while the second branch is defined on $1 - y^2 < x \leq 1$. The black curve represents the gluing curve $x = 1 - y^2$, along which the two branches coincide continuously. The plot illustrates the global behavior of the construction over the unit square $[0,1]^2$.

Lemma 4. *Limiting Behavior of the Consequent-Position Feedback Example.*

Let $N(x) = 1 - x^2$, $N^{-1}(x) = \sqrt{1 - x}$. Fix $x \in (0,1)$ and $y \in [0,1]$. Define $z_1 = \tilde{I}_A(x, y)$ and $z_{n+1} = \tilde{I}_A(x, z_n)$, $n \geq 1$. Then, the sequence (z_n) converges and $\lim_{n \rightarrow +\infty} z_n = \frac{\sqrt{1-x}(1-x^2)}{1-x^2\sqrt{1-x}}$. Equivalently,

$$\lim_{n \rightarrow +\infty} \tilde{I}_{A,R}^{(n)}(x, y) = \frac{\sqrt{1-x}(1-x^2)}{1-x^2\sqrt{1-x}}. \quad (49)$$

For $y = 1$, one has $\tilde{I}_{A,R}^{(n)}(x, 1) = 1$ for every $n \in \mathbb{N}$.

Proof. For $N(x) = 1 - x^2$, the two-branch implication is

$$\tilde{I}_A(x, y) = \begin{cases} \sqrt{1-x}[1 - x^2(1-y)], & 0 \leq x \leq 1 - y^2, \\ y[1 - x^2(1-y)], & 1 - y^2 < x \leq 1. \end{cases}$$

Fix $x \in (0,1)$ and put $c = \sqrt{1-x}$. The first branch is used when $x \leq 1 - z_n^2$, which is equivalent to $z_n \leq \sqrt{1-x} = c$. If $z_n \leq c$, then $z_{n+1} = c[1 - x^2(1 - z_n)]$. Hence,

$z_{n+1} = c(1 - x^2) + cx^2z_n$. This is an affine recurrence $z_{n+1} = a + bz_n$, where $a = c(1 - x^2)$, $b = cx^2$. Since $0 < x < 1$ and

$0 < c < 1$, we have $0 < b < 1$. Thus, once the sequence enters the first branch, it converges to the unique fixed point L_x satisfying $L_x = c(1 - x^2) + cx^2L_x$. Solving this equation gives $L_x(1 - cx^2) = c(1 - x^2)$, and therefore $L_x = \frac{c(1-x^2)}{1-cx^2}$. Since $c = \sqrt{1-x}$, we obtain $L_x = \frac{\sqrt{1-x}(1-x^2)}{1-x^2\sqrt{1-x}}$. It remains to show that, for $y \in [0,1]$, the sequence eventually enters the first branch. If $z_n > c$, then the second branch is used and $z_{n+1} = z_n[1 - x^2(1 - z_n)]$. For $z_n \in (c, 1)$, we have $0 < 1 - x^2(1 - z_n) < 1$. Therefore, $z_{n+1} < z_n$.

Thus, while the sequence remains in the second branch, it is strictly decreasing. Suppose, by contradiction, that the sequence never enters the first branch. Then $z_n > c$ for every n , and the sequence is decreasing and bounded below by c . Hence it converges to some $\ell \geq c > 0$. Passing to the limit in the second-branch recurrence gives

$\ell = \ell[1 - x^2(1 - \ell)]$. Thus, $\ell x^2(1 - \ell) = 0$. Since $x \in (0, 1)$ and $\ell > 0$, we must have $\ell = 1$. This is impossible, because the sequence is strictly decreasing and $z_1 < 1$ whenever $y < 1$. Therefore, the sequence enters the first branch after finitely many steps.

Once the first branch is entered, it remains invariant. Indeed, if $z_n \leq c$, then

$$z_{n+1} = c(1 - x^2) + cx^2z_n \leq c(1 - x^2) + cx^2c.$$

Since $c < 1$, we get $1 - x^2 + x^2c < 1$, and hence $z_{n+1} < c$. Therefore, after entry, the sequence follows the affine recurrence and converges to $\frac{\sqrt{1-x^2(1-x^2)}}{1-x^2\sqrt{1-x^2}}$. This completes the proof. $\square$

For example, if $x = 0.9$, then $\sqrt{1-x^2} = \sqrt{0.1} \approx 0.3162$. Hence,

$$\lim_{n \rightarrow +\infty} \tilde{I}_{A,R}^{(n)}(0.9, 0.95) = \frac{\sqrt{0.1}(1 - 0.9^2)}{1 - 0.9^2\sqrt{0.1}} \approx 0.0808.$$

This agrees with the numerical behavior displayed above.

3.6.2. Inverse-Negation Recursive Feedback

We now apply the second part of Proposition 2 to the same two-branch implication I_A , with $\phi = N^{-1}$. Define $\tilde{I}_{A,C}^{(1)}(x, y) = \tilde{I}_A(x, y)$, and, for every $n \geq 1$,

$$\tilde{I}_{A,C}^{(n+1)}(x, y) = \tilde{I}_A N^{-1}(I_{A,C}^{(n)}(x, y), y).$$

In this construction, the previous implication value is first transformed through the inverse negation N^{-1} , and the result is inserted into the antecedent position of I_A . Since N^{-1} is decreasing and satisfies $N^{-1}(0) = 1, N^{-1}(1) = 0$, Proposition 2. implies that

Corollary 5. For every $n \in \mathbb{N}$, the function $\tilde{I}_{A,C}^{(n)}$ is a fuzzy implication. If

$$u_n(x, y) = N^{-1}(\tilde{I}_{A,C}^{(n)}(x, y)),$$

then the $(n+1)$ -order member has the recursive two-branch representation

$$\tilde{I}_{A,C}^{(n+1)}(x, y) = \begin{cases} N^{-1}(u_n(x, y))N(u_n(x, y)N^{-1}(y)), & 0 \leq u_n(x, y) \leq N(y), \\ yN(u_n(x, y)N^{-1}(y)), & N(y) < u_n(x, y) \leq 1. \end{cases} \quad (50)$$

The gluing condition at order $n + 1$ is $u_n(x, y) = N(y)$. On this curve,

$$N^{-1}(u_n(x, y)) = N^{-1}(N(y)) = y,$$

and hence the two branches coincide.

Example 9. Let again $N(x) = 1 - x^2, N^{-1}(x) = \sqrt{1-x}$. The inverse-negation feedback family is given by $\tilde{I}_{A,C}^{(1)}(x, y) = \tilde{I}_A(x, y)$, and

$$\tilde{I}_{A,C}^{(n+1)}(x, y) = \tilde{I}_A(\sqrt{1 - I_{A,C}^{(n)}(x, y)}, y).$$

Let $x = 0.5, y = 0.6$. The initial gluing value is $N(y) = 1 - y^2 = 1 - (0.6)^2 = 0.64$. Since $0.5 \leq 0.64$, the first branch is used for the initial value. Hence,

$$\tilde{I}_{A,C}^{(1)}(0.5, 0.6) = \sqrt{1 - 0.5}[1 - (0.5)^2(1 - 0.6)] \approx 0.6364.$$

For the next step, $N^{-1}(\tilde{I}_{A,C}^{(1)}(0.5, 0.6)) = \sqrt{1 - 0.6364} \approx 0.6030$.

Since $0.6030 \leq 0.64$, the first branch is again used. Thus, $\tilde{I}_{A,C}^{(2)}(0.5, 0.6) \approx 0.5384$.

For the third step, $N^{-1}(I_{A,C}^{(2)}(0.5,0.6)) = \sqrt{1 - 0.5384} \approx 0.6794$. Now $0.6794 > 0.64$, so the second branch is used, and $I_{A,C}^{(3)}(0.5,0.6) \approx 0.4892$.

Continuing the recursion, we obtain the following values in Table 5.

Table 5. Numerical evolution of the inverse-negation recursive feedback sequence $I_{A,C}^{(n)}(0.5,0.6)$, together with the corresponding feedback value $N^{-1}(I_{A,C}^{(n)}(0.5,0.6))$ and the branch selected at the next iteration. The table shows that the process starts in the first branch, crosses the gluing condition after the second step, and then remains in the second branch while converging toward a stable limiting value.

n	$\tilde{I}_{A,C}^{(n)}(0.5, 0.6)$	$N^{-1}(\tilde{I}_{A,C}^{(n)}(0.5, 0.6))$	Branch Used at Next Step
1	0.6364	0.6030	First
2	0.5384	0.6794	Second
3	0.4892	0.7147	Second
4	0.4774	0.7229	Second

After the second step, the second branch remains active. In the second branch, for fixed $y = 0.6$, the recurrence becomes $z_{n+1} = y[1 - (1 - z_n)(1 - y)]$. Substituting $y = 0.6$, we get $z_{n+1} = 0.6[1 - 0.4(1 - z_n)] = 0.36 + 0.24z_n$. The fixed point L_y satisfies $L_y = 0.36 + 0.24L_y$. Therefore, $0.76L_y = 0.36$, and hence

$$L_y = \frac{0.36}{0.76} \approx 0.4737. \text{ Consequently, } \lim_{n \rightarrow +\infty} I_{A,C}^{(n)}(0.5,0.6) \approx 0.4737.$$

This example shows that the inverse-negation feedback process may change branch during the iteration and then converge to a stable branch-dependent limiting value.

Lemma 5. *Limiting Behavior of the Inverse-Negation Feedback Example.*

Let $N(x) = 1 - x^2$, $N^{-1}(x) = \sqrt{1 - x}$. Fix $x \in (0,1]$, $y \in (0,1)$. Define $z_1 = \tilde{I}_A(x, y)$, and, for every $n \geq 1$, $z_{n+1} = \tilde{I}_A(\sqrt{1 - z_n}, y)$. Equivalently, $z_n = \tilde{I}_{A,C}^{(n)}(x, y)$. Then the sequence (z_n) converges and $\lim_{n \rightarrow +\infty} z_n = \frac{y^2}{1 - y + y^2}$. Equivalently,

$$\lim_{n \rightarrow +\infty} \tilde{I}_{A,C}^{(n)}(x, y) = \frac{y^2}{1 - y + y^2}. \quad (51)$$

For $y = 1$, one has $\tilde{I}_{A,C}^{(n)}(x, 1) = 1$ for every $n \in \mathbb{N}$.

Proof. For $N(x) = 1 - x^2$, $N^{-1}(x) = \sqrt{1 - x}$, the auxiliary two-branch implication is

$$\tilde{I}_A(x, y) = \begin{cases} \sqrt{1 - x} [1 - x^2(1 - y)], & 0 \leq x \leq 1 - y^2, \\ y[1 - x^2(1 - y)], & 1 - y^2 < x \leq 1. \end{cases}$$

For the inverse-negation feedback recursion, put $u_n = \sqrt{1 - z_n}$. Then, $z_{n+1} = \tilde{I}_A(u_n, y)$. The gluing condition at step $n + 1$ is $u_n \leq 1 - y^2$. Since $u_n = \sqrt{1 - z_n}$, this condition is equivalent to $z_n \geq 1 - (1 - y^2)^2$. Set $d_y = 1 - (1 - y^2)^2 = 2y^2 - y^4$.

Thus, the first branch is used when $z_n \geq d_y$, whereas the second branch is used when $z_n < d_y$. Suppose first that $z_n < d_y$. Then the second branch is active and $z_{n+1} = y[1 - u_n^2(1 - y)]$. Since $u_n^2 = 1 - z_n$, we get $z_{n+1} = y[1 - (1 - z_n)(1 - y)]$. Therefore, $z_{n+1} = y^2 + y(1 - y)z_n$. This is an affine recurrence. Its fixed point L_y satisfies $L_y = y^2 + y(1 - y)L_y$. Hence, $L_y(1 - y + y^2) = y^2$, and therefore $L_y = \frac{y^2}{1 - y + y^2}$.

Moreover, the second branch is invariant. Indeed, if $z_n \leq d_y$, then

$$z_{n+1} \leq y^2 + y(1 - y)d_y.$$

A direct computation gives $d_y - [y^2 + y(1-y)d_y] = y^2(1-y)(y^3 - y + 1) > 0$ for every $y \in (0,1)$. Hence, $z_{n+1} < d_y$. Thus, once the sequence enters the second branch, it remains there and follows the affine recurrence $z_{n+1} = y^2 + y(1-y)z_n$.

Since $0 < y(1-y) < 1$, the sequence converges to $L_y = \frac{y^2}{1-y+y^2}$.

It remains to show that the sequence enters the second branch after finitely many steps. Suppose that $z_n > d_y$. Then, the first branch is active. Put $u = \sqrt{1-z_n}$.

Since $z_n > d_y$, we have $0 < u < 1 - y^2$. The first-branch recurrence gives

$$z_{n+1} = \sqrt{1-u} [1 - (1-y)u^2],$$

whereas $z_n = 1 - u^2$. We show that $z_{n+1} < z_n$. Since all terms are non-negative, it is enough to compare squares. A direct calculation gives $(1-u)(1+u)^2 - [1 - (1-y)u^2]^2 = uP_y(u)$, where $P_y(u) = 1 + u(1-2y) - u^2 - u^3(1-y)^2$. The function P_y is concave in u , because $P_y''(u) = -2 - 6u(1-y)^2 < 0$. On the interval $0 \leq u \leq 1 - y^2$, we have $P_y(0) = 1 > 0$, and $P_y(1-y^2) = y^2(y^3 - y + 1)(y^3 - 2y^2 - y + 3) > 0$. Therefore, $P_y(u) > 0$ for every $0 < u < 1 - y^2$. It follows that $z_{n+1} < z_n$ whenever the first branch is active. Assume, by contradiction, that the sequence never enters the second branch. Then $z_n > d_y$ for every n , and the sequence is strictly decreasing and bounded below by d_y . Hence, it converges to some $\ell \geq d_y$. Passing to the limit in the first-branch recurrence gives

$$\ell = \sqrt{1 - \sqrt{1-\ell}} [1 - (1-y)(1-\ell)].$$

However, the previous inequality shows that the first-branch map satisfies

$$\tilde{I}_A(\sqrt{1-z}, y) < z \text{ for every } z \in (d_y, 1), \text{ and } \tilde{I}_A(\sqrt{1-d_y}, y) < d_y.$$

Thus no limit $\ell \geq d_y$ can satisfy the fixed-point relation above. This contradiction shows that the sequence enters the second branch after finitely many steps.

Consequently, after finitely many iterations, the sequence follows the affine recurrence $z_{n+1} = y^2 + y(1-y)z_n$, and therefore $\lim_{n \rightarrow +\infty} z_n = \frac{y^2}{1-y+y^2}$. Finally, if $y = 1$, then $\tilde{I}_A(x, 1) = 1$, and hence $\tilde{I}_{A,C}^{(n)}(x, 1) = 1$ for every $n \in \mathbb{N}$. This completes the proof. $\square$

For the numerical values in Example 9, where $y = 0.6$, Lemma 5 gives $\lim_{n \rightarrow +\infty} \tilde{I}_{A,C}^{(n)}(0.5, 0.6) = \frac{0.6^2}{1-0.6+0.6^2} = \frac{0.36}{0.76} \approx 0.4737$, which agrees with the numerical trajectory displayed in Table 6.

Table 6. Structural properties of the recursive two-branch feedback families $I_{A,R}^{(n)}$ and $I_{A,C}^{(n)}$. The entry “Generally no” means that the corresponding property is not guaranteed by the construction and may fail for admissible choices of N and of the order n .

Family	FI Axioms	LNP	RNP	IP	OP	EP	CP(N)
$I_{A,R}^{(n)}$	Yes	Generally no	Yes	Generally no	Generally no	Generally no	Generally no
$I_{A,C}^{(n)}$	Yes	Generally no	Yes	Generally no	Generally no	Generally no	Generally no

Lemmas 4 and 5 provide convergence results for the two recursive feedback mechanisms in the representative case $N(x) = 1 - x^2$. They show that, for this strict non-strong negation, both recursive schemes eventually enter a stable branch and then reduce to an explicit recurrence with a fixed point. These results do not constitute a complete convergence classification for arbitrary strict fuzzy negations. The general convergence behavior of the recursive feedback schemes for arbitrary strict negations remains open and is left for future work.

Remark 18. *Structural properties of the recursive two-branch families.*

We close this subsection by summarizing the logical properties satisfied by the two recursive two-branch feedback families introduced above. In addition to the basic axioms of fuzzy implications, we consider the left neutrality principle, the right neutrality principle, the identity principle, the ordering principle, the exchange principle, and the contraposition principle with respect to a fuzzy negation N . For a fuzzy implication I , these properties are defined as follows:

$$\begin{aligned} \text{LNP: } I(1, y) &= y, \text{ RNP: } I(x, 1) = 1, \text{ IP: } I(x, x) = 1, \text{ OP: } I(x, y) = 1 \Leftrightarrow x \leq y, \\ \text{EP: } I(x, I(y, z)) &= I(y, I(x, z)), \end{aligned}$$

and $\text{CP}(N): I(x, y) = I(N(y), N(x))$. The following Table 6 summarizes the structural behavior of the recursive two-branch families.

The positive entries follow from the recursive definitions and from the boundary behavior of the base two-branch implication $\tilde{I}_A$. Since $\tilde{I}_A(x, 1) = 1$, one obtains $\tilde{I}_{A,R}^{(n)}(x, 1) = 1$ and $\tilde{I}_{A,C}^{(n)}(x, 1) = 1$ for every $n \in \mathbb{N}$. Hence, both recursive families preserve the right neutrality principle. On the other hand, the left neutrality principle is not generally preserved. Indeed, for the base two-branch implication, $\tilde{I}_A(1, y) = yN(N^{-1}(y)) = y^2$.

Thus, for $y \in (0, 1)$, $\tilde{I}_A(1, y) \neq y$. For example, if $y = 0.5$, then $\tilde{I}_A(1, 0.5) = 0.25 \neq 0.5$.

Since $\tilde{I}_{A,R}^{(1)} = \tilde{I}_A$ and $\tilde{I}_{A,C}^{(1)} = \tilde{I}_A$, this already shows that the left neutrality principle is not guaranteed for the recursive two-branch families. Similarly, the identity principle is not generally satisfied. Consider, for instance, the strict fuzzy negation $N(x) = 1 - x^2$, $N^{-1}(x) = \sqrt{1 - x}$. Then

$$\tilde{I}_A(x, y) = \begin{cases} \sqrt{1 - x}[1 - x^2(1 - y)], & 0 \leq x \leq 1 - y^2, \\ y[1 - x^2(1 - y)], & 1 - y^2 < x \leq 1. \end{cases}$$

For $x = y = 0.5$, since $0.5 \leq 1 - (0.5)^2 = 0.75$, the first branch is used, and $\tilde{I}_A(0.5, 0.5) = \sqrt{0.5}[1 - (0.5)^2(1 - 0.5)]$. Therefore,

$$\tilde{I}_A(0.5, 0.5) = \sqrt{0.5}(0.875) \approx 0.6187 \neq 1.$$

Thus, the identity principle fails in general. The ordering principle is also not generally satisfied. Take $x = 0.4, y = 0.6$. Although $x \leq y$, we have $0.4 \leq 1 - (0.6)^2 = 0.64$, so the first branch is used. Hence, $I_A(0.4, 0.6) = \sqrt{0.6}[1 - (0.4)^2(1 - 0.6)]$.

Thus, $I_A(0.4, 0.6) = \sqrt{0.6}(0.936) \approx 0.7250 \neq 1$. Therefore, $x \leq y$ does not imply $\tilde{I}_A(x, y) = 1$, and the ordering principle is not generally satisfied. The exchange principle may also fail. For the same negation $N(x) = 1 - x^2$, take $x = 0.7, y = 0.4, z = 0.2$.

A direct computation gives $\tilde{I}_A(0.7, I_A(0.4, 0.2)) \approx 0.5680$, whereas $\tilde{I}_A(0.4, I_A(0.7, 0.2)) \approx 0.6919$. Therefore,

$$\tilde{I}_A(0.7, I_A(0.4, 0.2)) \neq I_A(0.4, I_A(0.7, 0.2)),$$

and the exchange principle is not generally valid. Finally, the contraposition principle with respect to N is not generally preserved. Again take $N(x) = 1 - x^2$, and choose $x = 0.7, y = 0.4$. Then, $I_A(0.7, 0.4) \approx 0.3867$. On the other hand, $N(0.4) = 0.84, N(0.7) = 0.51$, and $I_A(N(0.4), N(0.7)) = I_A(0.84, 0.51) \approx 0.3337$. Hence, $I_A(0.7, 0.4) \neq I_A(N(0.4), N(0.7))$. Thus, the contraposition principle with respect to N is not generally satisfied. These examples show that the recursive two-branch feedback families preserve the defining axioms of fuzzy implications, but they do not automatically preserve the usual additional logical principles. This is consistent with their purpose in the present paper: the recursive feedback constructions are introduced mainly to study repeated implication composition, branch transitions, and nontrivial asymptotic behavior, rather than to preserve all secondary logical principles.

The recursive feedback constructions studied in this subsection are different from the direct Yager-induced two-branch constructions of Section 3.5. In Section 3.5, the order parameter is introduced inside the Yager-type product of the two-branch construction. In the present subsection, the order parameter counts recursive self-compositions of an already available two-branch implication.

For both recursive feedback mechanisms, representative convergence results are provided for the strict non-strong negation $N(x) = 1 - x^2$. In both cases, the analysis shows that the recursive process eventually enters a stable branch and then reduces to an affine recurrence with an explicit fixed point. Hence, the two recursive schemes are treated at a comparable theoretical level for the representative negation considered here. A full classification of their convergence behavior for arbitrary strict negations is beyond the scope of the present paper and may be addressed in future work.

Thus, the results of Section 3 exhibit three different levels of composition-generated fuzzy implication constructions. The first level consists of the power-type families, which are structured closure-based consequences of the general monotone precomposition principle and mainly describe saturation effects. The second level consists of the Yager-induced two-branch constructions and their n -order product-composition extensions, where the branch structure may lead to pointwise limiting behavior depending on the active region of the unit square. The third level consists of the recursive feedback mechanisms, where repeated implication composition may produce branch transitions and fixed-point type convergence.

This hierarchy separates preliminary closure-based transformations from the main branch-dependent and recursive constructions developed in the paper. In particular, the novelty of the recursive subsection lies in the feedback dynamics generated by repeated implication composition, not in the base two-branch formula alone.

4. Discussion

The constructions introduced in this paper provide several composition-generated ways of producing fuzzy implications from a strict fuzzy negation. The common starting point is the Yager-type generated implication $I_N(x, y) = N(xN^{-1}(y))$, which is used here as a known baseline operator. The contribution of the paper is not the baseline implication itself, but the systematic study of several composition mechanisms derived from its internal structure.

The first part of the paper concerns closure-based power-type families obtained by transforming the two internal components of the product $xN^{-1}(y)$. The antecedent-side family $I_R^{(n)}(x, y) = N(x^n N^{-1}(y))$ acts on the antecedent-side component x , whereas the consequent-side family $I_C^{(n)}(x, y) = N(x(N^{-1}(y))^n)$ acts on the internal consequent-side scale $N^{-1}(y)$. The unified family $I_{p,q}(x, y) = N(x^p (N^{-1}(y))^q)$ combines these two transformations. These families are best understood as structured consequences of the general monotone precomposition principle: their purpose is not to provide a construction more general than that principle, but to isolate and compare the effect of order parameters on the two internal components of the Yager-type baseline implication.

The asymptotic behavior of the power-type families is mainly saturation-driven. As the order parameters increase, the internal product is often driven toward zero, and since $N(0) = 1$, the implication value tends to the maximal truth value 1 and is used on large parts of the unit square. This convergence is pointwise and must be distinguished from uniform convergence on the whole square. In particular, for the unified family $I_{p,q}$, the joint limit along sequences satisfying $\min(p, q) \rightarrow \infty$ is pointwise equal to 1 on $[0, 1]^2 \setminus \{(1, 0)\}$, while the exceptional point (1, 0) remains fixed at value 0. Thus, large order

parameters should not be interpreted as universally improving fuzzy reasoning; any preference for particular parameter values requires an external criterion.

The main constructional part of the paper is the introduction of Yager-induced two-branch fuzzy implications. These constructions are not simple precomposition examples. They are defined by two analytic expressions on the regions $x \leq y$ and $y \leq x$, with continuity imposed along the diagonal. Their validity as fuzzy implications is ensured by explicit admissibility conditions. This distinguishes them from the preliminary closure-based power families and makes them a genuinely branch-dependent construction mechanism.

The two-branch constructions also clarify why higher-order behavior does not have to be reduced to saturation. Their n -order extensions are obtained through repeated product-composition of internal Yager-type components. In these cases, the limiting behavior may depend on the active branch: one branch may saturate to 1, while the other converges to a nontrivial limiting expression. This branch-dependent pointwise convergence is one of the main reasons for introducing the two-branch framework.

The recursive feedback constructions of Section 3.6 provide a third mechanism. Here, the order parameter does not act as a power inside an internal product, but counts the number of feedback iterations. The auxiliary two-branch implication $\tilde{I}_A$ is introduced and proved as a base implication, and two recursive families are then generated from it: one inserts the previous implication value into the consequent position, while the other first transforms the previous value through N^{-1} and then inserts it into the antecedent position.

This recursive framework is structurally different from the direct n -order product-composition constructions. Because the base implication is two-branch, the active branch may change during the iteration. Thus, the recursive subsection separates the auxiliary base implication $\tilde{I}_A$, the recursive feedback schemes generated from it, and the corresponding branch-switching or fixed-point convergence behavior. The numerical examples and limiting results show that recursive implication-composition may exhibit branch switching and convergence toward fixed points determined by the active branch.

The structural property tables further show that the proposed constructions preserve the defining axioms of fuzzy implications under the stated assumptions, but they do not automatically preserve all secondary logical principles. Properties such as left neutrality, identity, ordering, exchange, and contraposition may fail depending on the construction, the chosen strict fuzzy negation, and the order parameters. For this reason, contraposition was specified with respect to the generating negation N , denoted by $CP(N)$, and explicit counterexamples were provided for the negative entries in the tables. Thus, “Generally no” means that the property is not guaranteed by the construction, although it may hold in special cases.

Overall, the paper provides a formal study of composition-generated fuzzy implications organized into three levels. The first level consists of closure-based power transformations of the Yager-type baseline implication. The second level introduces Yager-induced two-branch fuzzy implications with explicit admissibility conditions and branch-dependent n -order extensions. The third level develops recursive feedback mechanisms in which the order parameter counts self-composition steps and may lead to branch switching and fixed-point type convergence. This organization clarifies that the principal novelty of the paper lies not in the baseline implication or in the elementary closure-based families, but in the branch-dependent and recursive composition mechanisms developed from them.

5. Conclusions

In this paper, we studied composition-generated higher-order fuzzy implications associated with strict fuzzy negations. Starting from the known Yager-type generated implication $I_N(x, y) = N(xN^{-1}(y))$, which is closely related to the general theory of fuzzy implications and approximate reasoning [1–3], we developed several families of fuzzy implications based on different composition mechanisms. First, we considered power-type transformations of the internal product $xN^{-1}(y)$. The antecedent-side transformed family $I_R^{(n)}(x, y) = N(x^n N^{-1}(y))$ acts on the antecedent-side component, while the consequent-side transformed family $I_C^{(n)}(x, y) = N(x(N^{-1}(y))^n)$ acts on the internal consequent-side scale. These two constructions were unified in the (p, q) -order family $I_{p,q}(x, y) = N(x^p (N^{-1}(y))^q)$. This unified form provides a structured way to compare the effects of the two internal transformations.

Second, we introduced Yager-induced two-branch fuzzy implications and their n -order extensions. These constructions are defined on the two regions $x \leq y$ and $y \leq x$, and they are not merely simple precomposition examples. Their validity requires explicit admissibility conditions. The n -order versions arise from repeated product-composition of the relevant internal Yager-type components. In this sense, they complement existing construction methods based on generated implications, aggregation functions, copulas, and related dependence structures [3–13].

Third, we introduced a new auxiliary two-branch implication $\tilde{I}_A$ and used it to define recursive feedback families. In these families, the order parameter counts the number of feedback iterations rather than the power of an internal product. The resulting sequences may cross the gluing curve and may converge to branch-dependent fixed points. This shows that higher-order behavior may arise not only from power-type transformations, but also from recursive implication-composition.

The asymptotic behavior of the proposed families reveals an important distinction. The power-type constructions may exhibit saturation toward 1, apart from specific boundary cases. By contrast, the two-branch and recursive feedback constructions may produce nontrivial branch-dependent limiting behavior. Thus, different composition mechanisms lead to different long-term regimes.

The structural analysis confirms that the proposed operators satisfy the basic axioms of fuzzy implications under the stated hypotheses. However, additional logical properties are not automatically preserved. In particular, left neutrality, identity, ordering, exchange, and contraposition may fail for admissible examples. For this reason, the revised manuscript specifies contraposition with respect to the generating negation N and supports the negative entries in the structural tables with explicit counterexamples.

The results should be interpreted as formal contributions to fuzzy implication theory. No claim is made that increasing the order parameters p , q , or n universally improves fuzzy reasoning. Their role is to control mathematically defined transformations, saturation effects, branch-dependent behavior, and recursive feedback dynamics. Possible uses in fuzzy rule-based systems or uncertainty modeling would require application-specific criteria and validation, in line with the broader literature on fuzzy reasoning and computational models [1–3,15–17].

Future work may investigate further classes of strict fuzzy negations, sharper admissibility conditions for the two-branch constructions, and convergence properties of the recursive feedback families. It may also be useful to study how the parameters p , q , and n can be selected under explicitly defined application-dependent criteria, such as threshold conditions or optimization constraints in fuzzy rule-based systems.

Author Contributions: Conceptualization, P.G.M. and B.K.P.; methodology, P.G.M.; formal analysis, P.G.M.; investigation, P.G.M.; writing—original draft preparation, P.G.M.; writing—review and

editing, P.G.M. and B.K.P.; visualization, P.G.M.; supervision, B.K.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Baczynski, M.; Jayaram, B. *Fuzzy Implications*; Studies in Fuzziness and Soft Computing; Springer: Berlin/Heidelberg, Germany, 2008; Volume 231. <https://doi.org/10.1007/978-3-540-69082-5>.
2. Yager, R.R. On some new classes of implication operators and their role in approximate reasoning. *Inf. Sci.* **2004**, *167*, 193–216. <https://doi.org/10.1016/j.ins.2003.04.001>.
3. Baczynski, M.; Jayaram, B. Yager's classes of fuzzy implications: Some properties and intersections. *Kybernetika* **2007**, *43*, 157–182.
4. Massanet, S.; Torrens, J. An overview of construction methods of fuzzy implications. In *Advances in Fuzzy Implication Functions*; Studies in Fuzziness and Soft Computing; Springer: Berlin/Heidelberg, Germany, 2013; Volume 300, pp. 1–30. https://doi.org/10.1007/978-3-642-35677-3_1.
5. Pradera, A.; Beliakov, G.; Bustince, H.; De Baets, B. A review of the relationships between implication, negation and aggregation functions from the point of view of material implication. *Inf. Sci.* **2016**, *329*, 357–380. <https://doi.org/10.1016/j.ins.2015.09.033>.
6. Mesiar, R.; Kolesarova, A. Copulas and fuzzy implications. *Int. J. Approx. Reason.* **2020**, *117*, 52–59. <https://doi.org/10.1016/j.ijar.2019.11.006>.
7. Nelsen, R.B. *An Introduction to Copulas*, 2nd ed.; Springer: New York, NY, USA, 2006. <https://doi.org/10.1007/0-387-28678-0>.
8. Souliotis, G.; Rassias, M.T.; Papadopoulos, B. Fuzzy implications based on quasi-copula and fuzzy negations. *J. Nonlinear Sci. Appl.* **2023**, *16*, 99–110. <https://doi.org/10.22436/jnsa.016.02.03>.
9. Pinheiro, J.; Santos, H.; Dimuro, G.P.; Bedregal, B.; Santiago, R.H.N.; Fernandez, J.; Bustince, H. On fuzzy implications derived from general overlap functions and their relation to other classes. *Axioms* **2023**, *12*, 17. <https://doi.org/10.3390/axioms12010017>.
10. Makariadis, S. New fuzzy implication model consisting only of basic logical fuzzy connectives. *Axioms* **2024**, *13*, 777. <https://doi.org/10.3390/axioms13110777>.
11. González-Hidalgo, M.; Massanet, S.; Mir, A.; Riera, J.V.; De Miguel, L. On fuzzy implication functions based on admissible orders on the set of discrete fuzzy numbers. *Int. J. Comput. Intell. Syst.* **2025**, *18*, 130. <https://doi.org/10.1007/s44196-025-00874-9>.
12. Baczynski, M.; Jayaram, B.; Massanet, S.; Torrens, J. Fuzzy implications: Past, present, and future. In *Springer Handbook of Computational Intelligence*; Kacprzyk, J., Pedrycz, W., Eds.; Springer: Berlin/Heidelberg, Germany, 2015; pp. 183–202. https://doi.org/10.1007/978-3-662-43505-2_12.
13. Mangenakis, P.G.; Papadopoulos, B.K. A unified framework for constructing two-branched fuzzy implications and copulas via monotone and convex function composition. *Mathematics* **2025**, *13*, 3604. <https://doi.org/10.3390/math13223604>.
14. Giakoumakis, S.; Papadopoulos, B. Novel transformation of unimodal (a)symmetric possibility distributions into probability distributions. *Fuzzy Sets Syst.* **2024**, *476*, 108790. <https://doi.org/10.1016/j.fss.2023.108790>.
15. Guo, C.; Pleiss, G.; Sun, Y.; Weinberger, K.Q. On evaluation of modern neural networks. In *Proceedings of the 34th International Conference on Machine Learning*; PMLR: Sydney, Australia, 2017; Volume 70, pp. 1321–1330. Available online: <https://proceedings.mlr.press/v70/guo17a.html> (accessed on 12 June 2026).
16. Kumar, A.; Sarawagi, S.; Jain, U. Trainable evaluation measures for neural networks from kernel mean embeddings. In *Proceedings of the 35th International Conference on Machine Learning*; PMLR: Sydney, Australia, 2018; Volume 80, pp. 2805–2814. Available online: <https://proceedings.mlr.press/v80/kumar18a.html> (accessed on 13 June 2026).
17. Kuleshov, V.; Fenner, N.; Ermon, S. Accurate uncertainties for deep learning using calibrated regression. In *Proceedings of the 35th International Conference on Machine Learning*; PMLR: Sydney, Australia, 2018; Volume 80, pp. 2796–2804. Available online: <https://proceedings.mlr.press/v80/kuleshov18a.html> (accessed on 12 June 2026).

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.