

Article

Galois F_4 -Bundles, Involutions, and Fixed-Point Subvarieties

Álvaro Antón-Sancho ^{1,2,*}  and Samer R. Yaseen ³ ¹ Department of Mathematics and Experimental Science, Fray Luis de León University College of Education, C/Tirso de Molina, 44, 47010 Valladolid, Spain² Faculty of Humanities and Education, Catholic University of Ávila, C/Canteros s/n, 05005 Ávila, Spain³ Department of Mathematics, College of Education for Pure Science, Tikrit University, Tikrit 34001, Iraq; samer2017@tu.edu.iq

* Correspondence: alvaro.anton@frayluis.com or alvaro.anton@ucavila.es

Abstract

Let X be a compact Riemann surface of genus $g \geq 2$, $\sigma_X \in \text{Aut}(X)$ (the group of biholomorphic self-maps of X) an involution, and $F_4(\mathbb{C})$ (one of the five exceptional complex simple Lie groups) the fixed-point subgroup of the outer involution σ of the complex simple Lie group $E_6(\mathbb{C})$ of type E_6 . We study Galois triples (E, f, ω) : a principal $F_4(\mathbb{C})$ -bundle E with an isomorphism $f: E \xrightarrow{\sim} \sigma_X^* E$ (here $\sigma_X^* E$ denotes the pull-back bundle, whose fibre over $x \in X$ is the fibre of E over $\sigma_X(x)$) satisfying a cocycle condition, together with a nontrivial finite-order automorphism ω commuting with f . The automorphism ω determines a semisimple element $g_e \in F_4(\mathbb{C})$ and forces a reduction to the centraliser $Z_{F_4}(g_e)$, the Levi factor of a proper parabolic subgroup (a proper closed subgroup of $F_4(\mathbb{C})$ containing a Borel subgroup, together with its natural reductive quotient, the Levi factor); by Ramanathan's criterion, E is strictly polystable (built, via such a reduction, from stable bundles of equal slope, and in particular not itself stable). The main theorem proves that $E(V_{26})$, the vector bundle associated with E via the representation V_{26} (the smallest nontrivial irreducible representation of $F_4(\mathbb{C})$), with fibre V_{26} , decomposes into k eigenspace sub-bundles, each fixed by σ_X^* , where $2 \leq k \leq 25$ is determined by the eigenvalue pattern of g_e in the 26-dimensional representation V_{26} . Structural corollaries show that the decomposition depends only on $[g_e]$, stratify the set of Galois triples by semisimple conjugacy classes, and give the splitting $\text{Ad}(\iota_*(E)) = E(\mathfrak{f}_4) \oplus \bigoplus_j \mathcal{E}_j$ with each summand preserved by σ_X^* . Since $\text{Out}(F_4(\mathbb{C})) = \{1\}$ (i.e., every automorphism of $F_4(\mathbb{C})$ is inner, meaning it is conjugated by an element of $F_4(\mathbb{C})$ itself), there are exactly two conjugacy classes of involutions, with fixed-point subgroups $K_I = (\text{Sp}(6, \mathbb{C}) \times \text{SL}(2, \mathbb{C}))/\mu_2$ and $K_{II} = \text{Spin}(9, \mathbb{C})$; the general theorem specialises in each case in decompositions reflecting the product structure of K_I and the spinorial geometry of K_{II} .

Keywords: principal bundle; moduli space; Galois bundle; involution; fixed points; real form

MSC: 14H10; 14H60; 17B25; 22E46; 53C10



Academic Editor: Vladimir Balan

Received: 26 June 2026

Revised: 22 July 2026

Accepted: 24 July 2026

Published: 26 July 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

1. Introduction

Let X be a compact Riemann surface of genus $g \geq 2$ and G a semisimple complex Lie group (a connected complex Lie group with no nontrivial connected closed abelian normal subgroup). A principal G -bundle over X is a fibre bundle $E \rightarrow X$ equipped with a free right action of G that is transitive on fibres, so that $E/G \cong X$. Ramanathan [1–3] introduced natural

notions of stability and polystability for principal G -bundles over compact Riemann surfaces (a principal G -bundle over X is stable if every reduction of a structure group to a proper parabolic subgroup has a negative degree with respect to every dominant character, and it is polystable if, whenever this degree is zero, the bundle admits a further reduction to a Levi subgroup exhibiting it as being built from stable bundles of equal slope; see Section 3 for the precise statements), extending the classical theory of Narasimhan and Seshadri [4] for vector bundles, and proved that the set of isomorphism classes of polystable principal G -bundles over X forms a complex algebraic variety: the moduli space $M(G)$ (whose points are in natural bijection with these isomorphism classes). Further foundational contributions were made by Ramanan and Ramanathan [5] and by Ramanathan and Subramanian [6].

The study of automorphisms of $M(G)$ and their fixed-point subvarieties has attracted sustained interest. The automorphism group of moduli spaces of semistable bundles was determined by Kouvidakis and Pantev [7] for vector bundles and subsequently for other structure groups by Biswas, Gómez, and Muñoz [8,9]; Serman [10]; and Fringuelli [11]. Fixed-point subvarieties under involutions and higher-order automorphisms were studied systematically by García-Prada and Ramanan [12], and for exceptional groups in [13,14]. From a deformation-theoretic and equivariant perspective, the interaction between automorphisms of the base curve and principal bundles has also been investigated in several settings. In particular, Ref. [15] analyses equivariant deformations, cohomological descent, and deformations preserving structure group reductions, while Biswas and Parameswaran [16] establish general results of equivariant reductions of principal bundles to Levi subgroups. These developments provide part of the framework in which the present study of $F_4(\mathbb{C})$ -bundles endowed with Galois structures and finite-order automorphisms naturally fits. The notion of the Galois G -bundle—a principal bundle equipped with an isomorphism to its pull-back under an automorphism of the base curve, satisfying a cocycle condition—was first introduced by Oxbury and Ramanan in [17] for principal bundles with structure group $\mathrm{Spin}(8, \mathbb{C})$ and was later developed in full generality in [13].

A Galois triple (E, f, ω) extends this notion by adding a nontrivial finite-order automorphism ω of E commuting with the Galois structure f . The role of ω is two-fold. First, it provides a finer organisation of the fixed-point locus $M(F_4(\mathbb{C}))^{\sigma_X}$: within the subvariety of Galois $F_4(\mathbb{C})$ -bundles, the Galois triples are stratified according to conjugacy classes of nontrivial semisimple elements $g_e \in F_4(\mathbb{C})$ (the element determined by ω fibrewise), and different conjugacy classes give geometrically distinct strata distinguished by the eigenvalue pattern of g_e in the 26-dimensional representation V_{26} . Second, the automorphism ω forces E to admit a reduction of the structure group to the centraliser $Z_{F_4}(g_e)$, which is the Levi factor of a proper parabolic subgroup of $F_4(\mathbb{C})$ whenever $g_e \neq 1$; by Ramanathan's criterion [1,5], this forces E to be strictly polystable. The triple (E, f, ω) therefore identifies a natural sublocus of $M(F_4(\mathbb{C}))^{\sigma_X}$ lying entirely outside the stable locus, stratified by the combinatorial data of the eigenvalue pattern of g_e in V_{26} .

The present paper studies Galois $F_4(\mathbb{C})$ -bundles. The motivation is that $F_4(\mathbb{C})$ is the connected fixed-point subgroup of the outer involution σ of $E_6(\mathbb{C})$, and Ref. [18] established that principal $F_4(\mathbb{C})$ -bundles form one of the strata of $M(E_6(\mathbb{C}))^\sigma$ (this is established in the context of Higgs bundles, but the result is directly true for principal bundles since the moduli space of principal bundles is a closed subvariety of the moduli space of Higgs bundles with the same structure group by setting the Higgs field to 0). This stratum is indeed a copy of $M(F_4)$, since it has been recently proved that the inclusion $F_4 \hookrightarrow E_6$ induces a closed embedding $M(F_4) \rightarrow M(E_6)$ [14].

Within this stratum, the Galois triples considered in the present paper correspond, via the push-forward $\iota_*: M(F_4(\mathbb{C})) \rightarrow M(E_6(\mathbb{C}))$, to a distinguished sublocus of $M(E_6(\mathbb{C}))^{\sigma, \sigma_X}$: those $E_6(\mathbb{C})$ -bundles that lie in the $F_4(\mathbb{C})$ stratum are fixed by σ_X^* , and admit

a finite-order automorphism compatible with the Galois structure. The eigenvalue pattern of g_e in V_{26} provides a natural stratification of this sublocus: the generic pattern ($k = 25$ distinct eigenvalues) gives the open dense stratum, the involutory pattern ($k = 2$) gives the most degenerate, and the intermediate patterns interpolate between them, each corresponding to a specific Levi subgroup $Z_{F_4}(g_e)$ to which the structure group reduces. Since V_{26} is the (-1) eigenspace of the outer involution σ in the decomposition $\mathfrak{e}_6 = \mathfrak{f}_4 \oplus V_{26}$ [19,20], the decomposition of $E(V_{26})$ into σ_X^* -fixed sub-bundles yields a refined splitting

$$\mathrm{Ad}(\iota_*(E)) = E(\mathfrak{f}_4) \oplus \bigoplus_{j=1}^k \mathcal{E}_j$$

of the adjoint bundle of $\iota_*(E)$ (Corollary 4), in which $E(\mathfrak{f}_4)$ is the $d\sigma$ -fixed part and $\bigoplus_j \mathcal{E}_j$ is the $d\sigma$ -anti-fixed part, with each summand preserved by σ_X^* . The strict polystability of every bundle in this sublocus shows moreover that it lies entirely outside the stable locus of $M(E_6(\mathbb{C}))^\sigma$, providing an explicit description of part of the boundary of the $F_4(\mathbb{C})$ stratum. This places the present paper within the programme of understanding $M(E_6(\mathbb{C}))^\sigma$, connecting it to the automorphism theory of [14] and the fixed-point geometry of [18].

Furthermore, the study of the geometry of principal F_4 -bundles over Riemann surfaces and their automorphisms is becoming increasingly significant in its own right, both due to the interest generated by the geometry of exceptional groups—particularly F_4 [21]—and because of recent works interested in the geometry of F_4 -bundles [14,18]. Furthermore, several works have focused on bundles whose structure group is a real form of a complex reductive group (a real Lie group $G^\mathbb{R}$ whose complexification is the given complex group; equivalently, a real form of a complex Lie algebra $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{g}^\mathbb{R} \subset \mathfrak{g}$ with $\mathfrak{g}^\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = \mathfrak{g}$) [22,23]; in this research, extensions of these results to the context of exceptional groups are provided. Specifically, the present paper works throughout with a holomorphic involution $\sigma_X \in \mathrm{Aut}(X)$. When one extends the definition of Galois triples to the case in which σ_X is instead an antiholomorphic involution (a real structure on X), replacing $\sigma_X^* E$ with the conjugate bundle $\sigma_X^* \bar{E}$ in the cocycle condition, the resulting objects are precisely the real and pseudo-real principal $F_4(\mathbb{C})$ -bundles in the sense of Biswas, García-Prada, and Hurtubise [24]. The decomposition of $E(V_{26})$ established in this paper for the holomorphic setting suggests a parallel decomposition theory for real $F_4(\mathbb{C})$ -bundles over real algebraic curves, which we indicate as a direction for further investigation.

The paper operates at two levels. At the general level, g_e is an arbitrary nontrivial semisimple element of $F_4(\mathbb{C})$: the main theorem proves that any Galois triple (E, f, ω) is strictly polystable and that $E(V_{26})$ decomposes into k sub-bundles fixed by σ_X^* , where $2 \leq k \leq 25$ depends on the eigenvalue pattern of g_e in the 26-dimensional representation V_{26} . At the involution-specialised level, g_e is required to belong to the fixed-point subgroup K_θ of one of the two involution types of $F_4(\mathbb{C})$; the resulting theorems refine the general result with an additional structure determined by the involution type and by the specific eigenvalue patterns that arise for $g_e \in K_I$ or $g_e \in K_{II}$.

Since $\mathrm{Out}(F_4(\mathbb{C})) = \{1\}$, every involution of $F_4(\mathbb{C})$ is inner, and the Cartan classification gives exactly two conjugacy classes, corresponding to the non-compact real forms $F_4^{(4)}$ and $F_4^{(-20)}$:

- Type (I): Cartan involution of $F_4^{(4)}$, with fixed-point subgroup

$$K_I = (\mathrm{Sp}(6, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})) / \mu_2.$$

- Type (II): Cartan involution of $F_4^{(-20)}$, with fixed-point subgroup

$$K_{II} = \text{Spin}(9, \mathbb{C}).$$

These arise as restrictions of Cartan involutions of $E_6(\mathbb{C})$: type (I) from E_6^2 and type (II) from $E_6^{(-14)}$. The involution-specialised results therefore connect to the geometry of $M(E_6(\mathbb{C}))$ in a more refined way than the general theorem.

The key representation is V_{26} , the 26-dimensional irreducible $F_4(\mathbb{C})$ module of highest weight ω_4 , which arises naturally as the (-1) eigenspace of $d\sigma$ in the decomposition $\mathfrak{e}_6 = \mathfrak{f}_4 \oplus V_{26}$. Its weight system—the 24 short roots of $F_4(\mathbb{C})$ together with the zero weight of multiplicity 2—determines the eigenvalue structure for all semisimple elements.

Beyond the two main levels, we establish several structural corollaries of the general decomposition theorem. The decomposition $E(V_{26}) = \bigoplus_j \mathcal{E}_j$ depends only on the conjugacy class $[g_e]$ (Corollary 2), and the automorphism $\tilde{\omega}$ induced by ω on $E(V_{26})$ acts on each summand as scalar multiplication by the corresponding eigenvalue λ_j (Corollary 1). These facts underpin a formal stratification of the set $\mathcal{G}$ of Galois triples into pairwise disjoint strata indexed by nontrivial semisimple conjugacy classes of $F_4(\mathbb{C})$ (Corollary 3), which refines the description of the distinguished sublocus of $M(E_6(\mathbb{C}))^{\sigma, \sigma_X}$.

This paper is organised as follows: In Section 2, we classify the involutions of $F_4(\mathbb{C})$, determine the fixed-point subgroups, and establish their connection with $E_6(\mathbb{C})$. Section 3 defines Galois triples and connects the theory to $M(E_6(\mathbb{C}))^\sigma$. The eigenvalue patterns of semisimple elements of $F_4(\mathbb{C})$ in V_{26} , both for arbitrary elements and for elements in K_I and K_{II} , are determined in Section 4, which also contains the polystability lemma. The main theorem for arbitrary Galois triples and the structural corollaries on canonicity, stratification, and adjoint bundle splitting are proved in Section 5. In Section 6, we derive the specialised theorems for the cases $g_e \in K_I$ and $g_e \in K_{II}$. Section 7 illustrates some examples and applications of the results obtained in related areas of geometry and mathematical physics. Finally, Section 8 collects the conclusions and outlines directions for further research.

2. Involutions of $F_4(\mathbb{C})$

The purpose of this section is to classify the conjugacy classes of involutions of $F_4(\mathbb{C})$, to determine the connected fixed-point subgroup in each case, and to establish the relationship between these involutions and the geometry of $M(E_6(\mathbb{C}))$ via the inclusion $F_4(\mathbb{C}) \subset E_6(\mathbb{C})$ as the connected fixed-point subgroup of the outer involution. The classification is entirely determined by the Cartan theory of real forms, since the triviality of the outer automorphism group of $F_4(\mathbb{C})$ forces all involutions to be inner. The subsequent identification of the two involution types as restrictions of Cartan involutions of $E_6(\mathbb{C})$ provides the algebraic connection between the specialised results of Section 6 and the broader geometry of $M(E_6(\mathbb{C}))^\sigma$.

Since the Dynkin diagram of F_4 (that is, the combinatorial diagram encoding the simple roots of F_4 and the angles/length ratios between them, which determines the Lie algebra $\mathfrak{f}_4$ up to isomorphism) admits no nontrivial symmetries,

$$\text{Out}(F_4(\mathbb{C})) = \{1\}$$

(where $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$ denotes the group of automorphisms of G modulo inner automorphisms, i.e., those of the form $x \mapsto gxg^{-1}$ for some $g \in G$) and every automorphism of $F_4(\mathbb{C})$ is inner, so the classification of conjugacy classes of involutions reduces entirely to the Cartan correspondence between involutions and non-compact real forms (the classical bijection, due to Cartan, between conjugacy classes of involutions of $F_4(\mathbb{C})$ and non-compact real forms of $\mathfrak{f}_4$) [20,25]. Recall that, for a real form $\mathfrak{g}^{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$ with

Cartan decomposition into the $+1$ - and -1 eigenspaces of the associated Cartan involution, the Cartan index is $\dim_{\mathbb{R}} \mathfrak{p} - \dim_{\mathbb{R}} \mathfrak{k}$; it is a standard label distinguishing non-isomorphic real forms of the same complex Lie algebra.

Proposition 1. *The Lie algebra $\mathfrak{f}_4$ has exactly two non-isomorphic non-compact real forms: the split form $\mathfrak{f}_4^{(4)}$ (Cartan index $+4$) and the form $\mathfrak{f}_4^{(-20)}$ (Cartan index -20). Correspondingly, there are exactly two conjugacy classes of nontrivial involutions in $F_4(\mathbb{C})$.*

Proof. By the Cartan classification of real simple Lie algebras [20,25], the only real forms of $\mathfrak{f}_4$ are the compact form and the two non-compact forms listed, which are distinguished by their Cartan indices. The Cartan correspondence gives a bijection between conjugacy classes of involutions of $F_4(\mathbb{C})$ and non-compact real forms: to each non-compact real form $\mathfrak{g}^{\mathbb{R}}$ of $\mathfrak{f}_4$, one associates the Cartan involution θ of $\mathfrak{g}^{\mathbb{R}}$, which is an involution of $\mathfrak{f}_4$ whose $+1$ eigenspace is the maximal compactly embedded subalgebra. Two real forms give conjugate Cartan involutions if and only if they are isomorphic, so the two non-compact forms give exactly two conjugacy classes. $\square$

Proposition 2. *The two conjugacy classes of nontrivial involutions of $F_4(\mathbb{C})$ have the following connected fixed-point subgroups:*

(I) *The Cartan involution θ_I of the split real form $F_4^{(4)}$, with connected fixed-point subgroup*

$$K_I = (\mathrm{Sp}(6, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})) / \mu_2, \quad \dim_{\mathbb{C}} K_I = 24, \quad (1)$$

$$\text{where } \mu_2 = \{(I_6, I_2), (-I_6, -I_2)\} \subset Z(\mathrm{Sp}(6, \mathbb{C})) \times Z(\mathrm{SL}(2, \mathbb{C})).$$

(II) *The Cartan involution θ_{II} of the real form $F_4^{(-20)}$, with connected fixed-point subgroup*

$$K_{II} = \mathrm{Spin}(9, \mathbb{C}), \quad \dim_{\mathbb{C}} K_{II} = 36. \quad (2)$$

Proof. Part (I): The maximal compact subgroup of $F_4^{(4)}$ is $(\mathrm{Sp}(3) \times \mathrm{SU}(2)) / \mu_2$ [20], of real dimension $21 + 3 = 24$. Its complexification is

$$K_I = (\mathrm{Sp}(6, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})) / \mu_2.$$

The element $(-I_6, -I_2)$ is central in $\mathrm{Sp}(6, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$:

$$-I_6 \in Z(\mathrm{Sp}(6, \mathbb{C})) = \{\pm I_6\}$$

since $(-I_6)J(-I_6)^T = J$ for any symplectic form J ;

$$-I_2 \in Z(\mathrm{SL}(2, \mathbb{C})) = \{\pm I_2\}.$$

So the quotient is well defined.

The dimension is

$$\dim_{\mathbb{C}} (\mathfrak{sp}(6, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})) = 21 + 3 = 24,$$

and the (-1) eigenspace $\mathfrak{m}_I = \ker(d\theta_I + \mathrm{id})$ has dimension $52 - 24 = 28$.

Part (II): The maximal compact subgroup of $F_4^{(-20)}$ is $\mathrm{Spin}(9)$ [20], of real dimension $\frac{9 \cdot 8}{2} = 36$. Its complexification is

$$K_{II} = \mathrm{Spin}(9, \mathbb{C}).$$

Since $F_4(\mathbb{C})$ is simply connected (it is the unique simply connected complex Lie group of type F_4), the connected subgroup with Lie algebra $\mathfrak{so}(9, \mathbb{C}) \subset \mathfrak{f}_4$ is the simply connected cover of $\mathrm{SO}(9, \mathbb{C})$, which is $\mathrm{Spin}(9, \mathbb{C})$ [25]. The dimension is

$$\dim_{\mathbb{C}} \mathfrak{so}(9, \mathbb{C}) = \frac{9 \cdot 8}{2} = 36,$$

and $\mathfrak{m}_{II}$ has dimension $52 - 36 = 16$. $\square$

Remark 1. The Cartan decompositions $\mathfrak{f}_4 = \mathfrak{k}_{\theta} \oplus \mathfrak{m}_{\theta}$ take the below explicit forms as K_{θ} modules [19,25]. For type (a): $\mathfrak{k}_I = \mathfrak{sp}(6, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ and

$$\mathfrak{m}_I \cong \Lambda_0^2(\mathbb{C}^6) \otimes \mathbb{C}^2, \quad \dim = 14 \cdot 2 = 28, \quad (3)$$

where $\Lambda_0^2(\mathbb{C}^6)$ denotes the 14-dimensional symplectic trace-free second exterior power of the standard representation of $\mathrm{Sp}(6, \mathbb{C})$. For type (II): $\mathfrak{k}_{II} = \mathfrak{so}(9, \mathbb{C})$ and

$$\mathfrak{m}_{II} \cong \Delta_9, \quad \dim = 16, \quad (4)$$

where Δ_9 is the 16-dimensional spinorial representation of $\mathrm{Spin}(9, \mathbb{C})$ [19,26]. The symmetric space $F_4^{(-20)}/\mathrm{Spin}(9)$ has real dimension 16 and a tangent space isomorphic to Δ_9 .

Remark 2. The Cartan involution θ_I of the split form acts on the Chevalley generators of $\mathfrak{f}_4$ as

$$d\theta_I(e_{\gamma}) = -e_{-\gamma}$$

and

$$d\theta_I(h) = -h$$

for all positive roots $\gamma \in \Phi_{F_4}^+$ and all $h \in \mathfrak{t}$, which is the standard formula for the Cartan involution of a split real form [20]. One verifies that $d\theta_I^2 = \mathrm{id}$:

$$d\theta_I^2(e_{\gamma}) = d\theta_I(-e_{-\gamma}) = e_{\gamma}$$

and

$$d\theta_I^2(h) = h.$$

The involution θ_{II} acts as $+\mathrm{id}$ on the $+1$ eigenspace $\mathfrak{k}_{II} = \mathfrak{so}(9, \mathbb{C})$ and as $-\mathrm{id}$ on the (-1) eigenspace $\mathfrak{m}_{II} \cong \Delta_9$.

Once we have classified the involutions of $F_4(\mathbb{C})$, we now embed them in the larger context provided by the ambient group $E_6(\mathbb{C})$. The key facts are that $F_4(\mathbb{C})$ is the connected fixed-point subgroup of the outer involution σ of $E_6(\mathbb{C})$, and that the Lie algebra $\mathfrak{e}_6$ decomposes as a direct sum of two $F_4(\mathbb{C})$ modules. This decomposition is not merely an algebraic curiosity: the second summand is precisely the representation V_{26} that governs the decomposition of the associated bundle in the main theorems. The proposition below shows moreover that both involution types of $F_4(\mathbb{C})$ arise as restrictions of Cartan involutions of $E_6(\mathbb{C})$, establishing the link between the specialised theorems of Section 6 and the geometry of $M(E_6(\mathbb{C}))^{\sigma}$.

The outer automorphism group of $E_6(\mathbb{C})$ satisfies $\mathrm{Out}(E_6(\mathbb{C})) \cong \mathbb{Z}/(2)$, generated by the outer involution σ corresponding to the unique nontrivial symmetry of the Dynkin

diagram of E_6 . Its connected fixed-point subgroup is $\text{Fix}(\sigma)^0 = F_4(\mathbb{C})$ [20,25], and the Lie algebra decomposes as an $F_4(\mathbb{C})$ module as

$$\mathfrak{e}_6 = \mathfrak{f}_4 \oplus V_{26}, \quad (5)$$

where V_{26} is the 26-dimensional irreducible $F_4(\mathbb{C})$ module of highest weight ω_4 , equal to the (-1) eigenspace of $d\sigma$ in $\mathfrak{e}_6$ [19,20,27]. Here $E_6^{(-14)}$ and E_6^2 denote the non-compact real forms of $E_6(\mathbb{C})$ with Cartan indices -14 and $+2$ respectively, following the notation of [20]; they correspond to the Cartan classes E III and E II respectively in the classification of [25].

Proposition 3. *With the notation of Proposition 2:*

- (a) *The Cartan involution of $E_6^{(-14)}$ commutes with σ and restricts to θ_{II} on $F_4(\mathbb{C})$, with fixed-point subgroup $K_{II} = \text{Spin}(9, \mathbb{C})$.*
- (b) *The Cartan involution of E_6^2 commutes with σ and restricts to θ_I on $F_4(\mathbb{C})$, with fixed-point subgroup K_I .*

Proof. Part (a): The maximal compact subgroup of $E_6^{(-14)}$ is locally isomorphic to $\text{Spin}(10) \times \text{U}(1)$ [20]. Using the decomposition (5) and the embedding $F_4(\mathbb{C}) \subset E_6(\mathbb{C})$, the intersection of $F_4(\mathbb{C})$ with the complexification of this maximal compact subgroup is computed as follows: $\text{Spin}(9, \mathbb{C})$ embeds into $\text{Spin}(10, \mathbb{C})$ as the stabiliser of a nonzero vector in the standard 10-dimensional representation, and this embedding is compatible with the inclusion

$$\text{Spin}(9, \mathbb{C}) = K_{II} \subset F_4(\mathbb{C}) \subset E_6(\mathbb{C}).$$

Hence the Cartan involution of $E_6^{(-14)}$, acting as $+\text{id}$ on the complexification of its maximal compact subalgebra and as $-\text{id}$ on its complement, restricts to $+\text{id}$ on $\mathfrak{k}_{II} = \mathfrak{so}(9, \mathbb{C})$ and to $-\text{id}$ on $\mathfrak{m}_{II}$, which is exactly θ_{II} [25].

Part (b): The maximal compact subgroup of E_6^2 is locally isomorphic to $\text{SU}(6) \times \text{SU}(2)$ [20]. The intersection of $\mathfrak{f}_4$ with the corresponding real form of $\mathfrak{su}(6) \oplus \mathfrak{su}(2)$ inside $\mathfrak{e}_6^{\mathbb{R}}$ equals $\mathfrak{sp}(3) \oplus \mathfrak{su}(2) = \mathfrak{k}_I^{\mathbb{R}}$, which is the compact real form of $\mathfrak{k}_I$ [20,25]. Hence the Cartan involution of E_6^2 restricts to $+\text{id}$ on $\mathfrak{k}_I = \mathfrak{sp}(6, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ and to $-\text{id}$ on $\mathfrak{m}_I$, which is θ_I . $\square$

Remark 3. Proposition 3 establishes the algebraic counterpart of the geometric results of [18]: both involution types of $F_4(\mathbb{C})$ arise as restrictions of Cartan involutions of non-compact real forms of $E_6(\mathbb{C})$ that commute with the outer involution σ . This provides the direct connection between the specialised results of Section 6 and the geometry of $M(E_6(\mathbb{C}))^\sigma$: a K_θ -restricted Galois triple for $F_4(\mathbb{C})$ produces, via extension of the structure group to $E_6(\mathbb{C})$, a bundle in $M(E_6(\mathbb{C}))^\sigma$ whose decomposition reflects the involution type θ .

3. The Moduli Space and Polystability

This section lays the algebraic foundations for the main theorems. We introduce the two types of Galois triple—general and K_θ -restricted—that appear in the paper, and we clarify their relationship. We then establish the connection between Galois $F_4(\mathbb{C})$ -bundles and the geometry of $M(E_6(\mathbb{C}))^\sigma$ via the inclusion

$$\iota: F_4(\mathbb{C}) \hookrightarrow E_6(\mathbb{C}).$$

The moduli space $M(F_4(\mathbb{C}))$ of polystable principal $F_4(\mathbb{C})$ -bundles over X is a complex algebraic variety constructed by Ramanathan [1–3]. For $\sigma_X \in \text{Aut}(X)$, which is an involution, pull-back by σ_X defines an algebraic automorphism of the moduli space $M(F_4(\mathbb{C}))$.

Since $\text{Out}(F_4(\mathbb{C})) = \{1\}$, every automorphism of $F_4(\mathbb{C})$ is inner, so twisting a principal $F_4(\mathbb{C})$ -bundle by any automorphism of the structure group produces an isomorphic bundle [28]. This means that for every involution θ of $F_4(\mathbb{C})$, $\theta(E) \cong E$ canonically, and the fixed-point condition $\theta(E) \cong \sigma_X^* E$ reduces simply to $E \cong \sigma_X^* E$. The subvariety

$$M(F_4(\mathbb{C}))^{\sigma_X} = \{[E] \in M(F_4(\mathbb{C})) : E \cong \sigma_X^* E\}$$

is closed algebraic by [29], and is the ambient space within which all Galois $F_4(\mathbb{C})$ -bundles live.

Definition 1. A Galois $F_4(\mathbb{C})$ -bundle is a pair (E, f) where E is a principal $F_4(\mathbb{C})$ -bundle over X and $f: E \xrightarrow{\sim} \sigma_X^* E$ is an isomorphism of principal bundles satisfying the cocycle condition

$$(\sigma_X^* f) \circ f = \text{id}_E. \quad (6)$$

Here $\sigma_X^* f$ denotes the isomorphism $\sigma_X^* E \rightarrow (\sigma_X^*)^2 E = E$ obtained by applying σ_X^* to f and using $\sigma_X^2 = \text{id}_X$, so the cocycle condition is an equality of automorphisms of E .

A Galois triple is a triple (E, f, ω) where (E, f) is a Galois $F_4(\mathbb{C})$ -bundle, $\omega: E \rightarrow E$ is a nontrivial automorphism of E of finite order, and ω commutes with the Galois structure in the sense that

$$\sigma_X^* \omega \circ f = f \circ \omega \quad \text{as isomorphisms } E \rightarrow \sigma_X^* E. \quad (7)$$

For $\theta \in \{\theta_I, \theta_{II}\}$ with a fixed-point subgroup $K_\theta \in \{K_I, K_{II}\}$, a Galois triple (E, f, ω) is called K_θ -restricted if the semisimple element $g_e \in F_4(\mathbb{C})$ determined by ω (see Proposition 12) has its conjugacy class intersecting K_θ .

Remark 4. Definition 1 admits a cohomological reformulation that clarifies the role of $\mathfrak{f}_4$ in the theory. Write $\Gamma = \langle \sigma_X \rangle \cong \mathbb{Z}/2\mathbb{Z}$, which acts on X and, by pull-back, on the moduli stack of principal $F_4(\mathbb{C})$ -bundles on X . Suppose $E \cong \sigma_X^* E$ and fix one isomorphism $f_0: E \xrightarrow{\sim} \sigma_X^* E$ satisfying (6), i.e., $\sigma_X^*(f_0) = f_0^{-1}$. Every other isomorphism $f: E \xrightarrow{\sim} \sigma_X^* E$ is of the form $f = f_0 \circ a$ for a unique $a \in \text{Aut}(E)$, and f satisfies (6) if and only if $a \cdot \tau(a) = \text{id}_E$, where

$$\tau(a) = f_0^{-1} \circ \sigma_X^*(a) \circ f_0$$

is the order-two group automorphism of $\text{Aut}(E)$ induced by Γ via f_0 (that $\tau^2 = \text{id}$ follows from $\sigma_X^*(f_0) = f_0^{-1}$, $\sigma_X^*(f_0^{-1}) = f_0$, and $\sigma_X^*(\sigma_X^*(a)) = a$). This is precisely the defining equation of a non-abelian 1 cocycle for Γ with values in $\text{Aut}(E)$ [30] (Chapter I). If $b \in \text{Aut}(E)$ carries $(E, f_0 \circ a)$ to $(E, f_0 \circ a')$, in the sense that $(f_0 \circ a') \circ b = \sigma_X^*(b) \circ (f_0 \circ a)$, then $a' = \tau(b) \circ a \circ b^{-1}$; substituting $c = \tau(b)$ (a bijective reparametrisation, since τ is an involution) recovers the standard twisted-conjugation equivalence $a' = c \cdot a \cdot \tau(c)^{-1}$ for non-abelian 1 cocycles. Isomorphism classes of Galois structures on E are therefore classified by the pointed set $H^1(\Gamma, \text{Aut}(E))$. Infinitesimally, the tangent space to the space of such cocycles at (E, f_0) is controlled by the hypercohomology of the complex governing Γ -equivariant deformations of E , whose degree-one term is $H^1(X, E(\mathfrak{f}_4))$, since $\mathfrak{f}_4 = \text{Lie}(F_4(\mathbb{C}))$ is the coefficient sheaf $\text{ad}(E) = E(\mathfrak{f}_4)$; this is the equivariant deformation-theoretic framework of [15]. In this sense $\mathfrak{f}_4$ enters the Galois cocycle not as data of the cocycle itself, but as the coefficient Lie algebra governing its infinitesimal deformations. The additional datum ω of a Galois triple is a further, independent piece of the structure on top of a fixed Galois structure (E, f) ; Corollary 2 shows that the eigenbundle decomposition it induces depends only on the conjugacy class $[g_e]$, not on further choices of representatives.

Remark 5. The two levels of the paper correspond to the two types of Galois triple. The general theory of Section 5 treats arbitrary Galois triples, with no restriction on g_e . The involution-

specialised theory of Section 6 treats K_θ -restricted Galois triples, where the additional constraint $g_e \in K_\theta$ refines the eigenvalue pattern of g_e in V_{26} and yields a decomposition of $E(V_{26})$ whose structure reflects the geometry of the involution type. Every K_θ -restricted Galois triple is in particular a Galois triple, so the general theorem of Section 5 applies to all of them; the specialised theorems of Section 6 provide additional information about the ranks and eigenvalue types of the summands.

The inclusion $\iota: F_4(\mathbb{C}) \hookrightarrow E_6(\mathbb{C})$ as the connected fixed-point subgroup of σ induces a morphism of moduli spaces $\iota_*: M(F_4(\mathbb{C})) \rightarrow M(E_6(\mathbb{C}))$. The following proposition shows that Galois $F_4(\mathbb{C})$ -bundles map naturally into the doubly fixed subvariety $M(E_6(\mathbb{C}))^{\sigma, \sigma_X}$, and that the Galois structure is preserved under this map. This connects the paper to the geometry of $M(E_6(\mathbb{C}))^\sigma$ established in [18].

Proposition 4. *Let (E, f, ω) be a Galois triple. Then:*

- (a) *The push-forward $\iota_*(E)$ lies in $M(E_6(\mathbb{C}))^{\sigma, \sigma_X}$, the closed subvariety of principal $E_6(\mathbb{C})$ -bundles fixed by both σ and σ_X^* .*
- (b) *The isomorphism f induces a Galois structure*

$$\iota_*(f): \iota_*(E) \xrightarrow{\sim} \sigma_X^*(\iota_*(E))$$

satisfying the cocycle condition, making $(\iota_(E), \iota_*(f))$ a Galois $E_6(\mathbb{C})$ -bundle.*

Proof. Part (a): The cocycle condition (6) implies that f is an isomorphism $E \xrightarrow{\sim} \sigma_X^* E$, so $E \cong \sigma_X^* E$, and hence $[E] \in M(F_4(\mathbb{C}))^{\sigma_X}$. Applying ι_* :

$$\iota_*(E) \cong \iota_*(\sigma_X^* E) \cong \sigma_X^*(\iota_*(E)),$$

where the first isomorphism is $\iota_*(f)$ and the second is the naturality isomorphism $\iota_*(\sigma_X^* E) \cong \sigma_X^*(\iota_*(E))$ of [28], which holds because σ_X acts on the base X and ι_* acts on the fibres. This gives

$$[\iota_*(E)] \in M(E_6(\mathbb{C}))^{\sigma_X}.$$

The inclusion

$$[\iota_*(E)] \in M(E_6(\mathbb{C}))^\sigma$$

holds because $F_4(\mathbb{C}) = \text{Fix}(\sigma)^0$, so every $F_4(\mathbb{C})$ -bundle admits a canonical reduction to $\text{Fix}(\sigma)^0$ and hence satisfies $\sigma(\iota_*(E)) \cong \iota_*(E)$; this is the content of [14,18].

Part (b): The naturality of ι_* with respect to isomorphisms of principal bundles gives a commutative diagram relating f and $\iota_*(f)$, and the naturality isomorphism of [28], from which the isomorphism

$$\iota_*(f): \iota_*(E) \xrightarrow{\sim} \sigma_X^*(\iota_*(E))$$

is obtained by composition. The cocycle condition for $\iota_*(f)$:

$$(\sigma_X^*(\iota_*(f))) \circ \iota_*(f) = \iota_*((\sigma_X^* f) \circ f) = \iota_*(\text{id}_E) = \text{id}_{\iota_*(E)},$$

using the functoriality of ι_* and σ_X^* and the cocycle condition (6) for f . $\square$

Remark 6. Proposition 4 holds for all Galois triples, regardless of the eigenvalue pattern of g_e . The additional datum ω distinguishes the various Galois triples within $M(F_4(\mathbb{C}))^{\sigma_X}$; it is the semisimple element g_e and its eigenvalue pattern in V_{26} that produce the decomposition of $E(V_{26})$ in the main theorems.

4. Semisimple Elements and Eigenvalue Decompositions

This section carries out the eigenvalue analysis that governs all the decompositions in the paper. We proceed in three stages. The first subsection treats arbitrary semisimple elements of $F_4(\mathbb{C})$, establishing the weight system of V_{26} and classifying the possible eigenvalue patterns; the results here underpin the general theorem of Section 5. The second and third subsections specialise in discussing elements in K_{II} and K_I respectively, establishing the restrictions of V_{26} to these subgroups and the resulting finer eigenvalue descriptions; the results here underpin the specialised theorems of Section 6. Throughout, we work with the root system of $F_4(\mathbb{C})$ in the coordinates of [31]: the simple roots (that is, a minimal generating subset of the roots from which every other root is obtained as a sum of simple roots with coefficients all of the same sign) are

$$\alpha_1 = e_2 - e_3, \quad \alpha_2 = e_3 - e_4, \quad \alpha_3 = e_4, \quad \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4), \quad (8)$$

where α_1, α_2 are long and α_3, α_4 are short, and a torus element

$$(t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4$$

(coordinates, relative to a fixed maximal torus

$$T \cong (\mathbb{C}^*)^4 \subset F_4(\mathbb{C}),$$

of an element of T ; every semisimple element of $F_4(\mathbb{C})$ is conjugate to such an element) acts on a weight space of weight $\sum_i n_i e_i$ (the subspace of a representation on which T acts through the character $\prod_i t_i^{n_i}$; the multiset of such weights, with multiplicities, determines the representation up to isomorphism) with eigenvalue $\prod_i t_i^{n_i}$. Section 4.1 concludes with Lemma 2, which derives the strict polystability of any Galois triple from Proposition 7 and Ramanathan's criterion.

4.1. Weights of V_{26} and Eigenvalues for Arbitrary Semisimple Elements

We begin with the complete description of the weight system of V_{26} , which is the foundation for all the eigenvalue computations that follow. The key structural fact is that the nonzero weights are precisely the short roots of F_4 , a consequence of the transitivity of the Weyl group on the set of short roots and the fact that the highest weight is itself a short root.

Proposition 5. *The irreducible $F_4(\mathbb{C})$ module V_{26} of highest weight $\omega_4 = e_1$ has the following weight system:*

- (a) *The weights $\pm e_i$ for $i = 1, 2, 3, 4$, each of multiplicity 1 (eight weights in total).*
- (b) *The weights $\frac{1}{2}(\epsilon_1 e_1 + \epsilon_2 e_2 + \epsilon_3 e_3 + \epsilon_4 e_4)$ for all $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \{+1, -1\}^4$, each of multiplicity 1 (sixteen weights in total).*
- (c) *The zero weight 0, of multiplicity 2.*

The total dimension is $8 + 16 + 2 = 26$.

Proof. The root system of F_4 consists of 24 long roots $\pm e_i \pm e_j$ ($i \neq j$) and 24 short roots: the eight $\pm e_i$ ($i = 1, 2, 3, 4$) and the sixteen

$$\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4).$$

The highest weight $\omega_4 = e_1$ is a short root, and the Weyl group $W(F_4)$ acts transitively on the set of all short roots [31], so the $W(F_4)$ orbit of ω_4 is the complete set of 24 short roots.

In an irreducible representation, the weight multiset is invariant under W , and the nonzero weights in V_{26} are exactly the 24 short roots, each of multiplicity 1, since the representation is minuscule with respect to the short roots (recall that a representation is minuscule if the Weyl group acts transitively on its set of weights, which forces every nonzero weight to have multiplicity 1); see [19,31] for the computation. The zero weight has multiplicity 2, giving the total dimension $24 + 2 = 26$. $\square$

Proposition 6. Let $g \in F_4(\mathbb{C})$ be semisimple, conjugate to the torus element $(t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4$. The eigenvalues of g in V_{26} , with their multiplicities, are:

- (a) $t_i^{\pm 1}$ for $i = 1, 2, 3, 4$, each of multiplicity 1 (eight eigenvalues).
- (b) $t_1^{\epsilon_1/2} t_2^{\epsilon_2/2} t_3^{\epsilon_3/2} t_4^{\epsilon_4/2}$ for all $(\epsilon_i) \in \{+1, -1\}^4$, each of multiplicity 1 (sixteen eigenvalues).
- (c) 1, of multiplicity 2.

The total count with multiplicity is $8 + 16 + 2 = 26$.

Proof. The eigenvalue of the torus element (t_1, t_2, t_3, t_4) on the weight space of weight $\sum_i n_i e_i$ is $\prod_i t_i^{n_i}$. Evaluating this on each weight in groups (a), (b), and (c) of Proposition 5 gives, respectively, the three families of eigenvalues listed. For the weights $\pm e_i$ in group (a): weight e_i gives t_i , and $-e_i$ gives t_i^{-1} . For the weights

$$\frac{1}{2}(\epsilon_1 e_1 + \cdots + \epsilon_4 e_4)$$

in group (b): the eigenvalue is

$$t_1^{\epsilon_1/2} t_2^{\epsilon_2/2} t_3^{\epsilon_3/2} t_4^{\epsilon_4/2}.$$

For the zero weight in group (c): the eigenvalue is $\prod_i t_i^0 = 1$, with multiplicity 2. $\square$

The following lemma is a standard fact valid in the algebraic group setting and can be found in [32] and in [25] (Chapter 3). We include the proof for completeness.

Lemma 1. Let G be a group and let $g, g' \in G$ be elements satisfying $g' = hgh^{-1}$ for some $h \in G$. Then

$$Z_G(g') = h Z_G(g) h^{-1}.$$

In particular, for semisimple elements $g_e, g'_e \in F_4(\mathbb{C})$ with $g'_e = h g_e h^{-1}$, the centralisers $Z_{F_4}(g_e)$ and $Z_{F_4}(g'_e)$ are conjugate subgroups of $F_4(\mathbb{C})$.

Proof. Let $x \in Z_G(g)$, so $xg = gx$. Multiplying on the left by h and on the right by h^{-1} :

$$(h x h^{-1})(h g h^{-1}) = (h g h^{-1})(h x h^{-1}),$$

that is,

$$(h x h^{-1}) g' = g' (h x h^{-1}).$$

Hence $h x h^{-1} \in Z_G(g')$, and

$$h Z_G(g) h^{-1} \subseteq Z_G(g').$$

Applying the same argument to h^{-1} and $g = h^{-1} g' h$ gives the reverse inclusion, so $Z_G(g') = h Z_G(g) h^{-1}$. $\square$

Proposition 7. Let $g \in F_4(\mathbb{C})$ be a nontrivial semisimple element with torus representative (t_1, t_2, t_3, t_4) . Then:

- (a) The centraliser $Z_{F_4}(g)$ is a connected reductive subgroup of $F_4(\mathbb{C})$ of rank 4, equal to the Levi factor of a parabolic subgroup of $F_4(\mathbb{C})$.

- (b) $Z_{F_4}(g)$ is a proper subgroup of $F_4(\mathbb{C})$.
 (c) The root system of $Z_{F_4}(g)$ is $\Psi(g) = \{\alpha \in \Phi(F_4) : \alpha(g) = 1\}$, and the semisimple rank of $Z_{F_4}(g)$ equals the rank of the root subsystem $\Psi(g)$.

Proof. Part (a): Since $F_4(\mathbb{C})$ is simply connected, Steinberg's theorem [33] guarantees that the centraliser of every semisimple element is connected. That $Z_{F_4}(g)$ is reductive follows from the general theory of algebraic groups [34]: the centraliser of a semisimple element in a reductive group is reductive. It has rank $4 = \text{rk}(F_4)$ because it contains the full maximal torus T (as $g \in T$ and T is abelian). The Levi factor description: The Lie algebra of $Z_{F_4}(g)$ is

$$\text{Lie}(Z_{F_4}(g)) = \mathfrak{t} \oplus \bigoplus_{\alpha(g)=1} \mathfrak{f}_4^\alpha,$$

which is precisely the Levi subalgebra of the parabolic subalgebra corresponding to the root system $\Psi(g)$ [25,34].

Part (b): Since $g \neq 1$ and $Z(F_4(\mathbb{C})) = \{1\}$, the element g is not central, so there exists a root $\alpha \in \Phi(F_4)$ such that $\alpha(g) \neq 1$. (Explicitly, if $\alpha(g) = 1$ for all $\alpha \in \Phi(F_4)$, then g acts trivially on every root space, hence trivially on all of $\mathfrak{f}_4$ by the adjoint representation; hence $g \in Z(F_4(\mathbb{C})) = \{1\}$, contradicting $g \neq 1$.) For any such α , the root space $\mathfrak{f}_4^\alpha$ is not contained in $\text{Lie}(Z_{F_4}(g))$, so $Z_{F_4}(g) \subsetneq F_4(\mathbb{C})$.

Part (c) follows directly from the Lie algebra description in part (a): $\mathfrak{f}_4^\alpha \subset \text{Lie}(Z_{F_4}(g))$ if and only if $\alpha(g) = 1$. $\square$

Definition 2. Let $g \in F_4(\mathbb{C})$ be nontrivial semisimple with a torus representative (t_1, t_2, t_3, t_4) . The eigenvalue pattern of g in V_{26} is the list of distinct eigenvalues of g in V_{26} (from Proposition 6) together with their multiplicities, with $k = k(g)$ denoting the number of distinct eigenvalues. Three qualitative cases are distinguished:

- (I) Generic pattern ($k = 25$): The 26 eigenvalues are pairwise distinct except for the two copies of 1 from group (c), which have multiplicity 2; all eigenvalues in groups (a) and (b) are distinct from each other and from 1.
- (II) Intermediate pattern ($2 \leq k < 25$): Further coincidences occur among the eigenvalues in groups (a) and (b), or between those groups and eigenvalue 1, due to special relations among t_1, t_2, t_3, t_4 .
- (III) Involutionary pattern ($k = 2$): $g^2 = 1$, which forces $\rho(g)^2 = \text{id}$ for every representation ρ , so all eigenvalues of g in V_{26} lie in $\{+1, -1\}$.

Remark 7. The eigenvalue pattern of g in V_{26} is directly related to the structure of $Z_{F_4}(g)$ via the root system $\Psi(g)$. In particular, by Lemma 1, these properties depend only on the conjugacy class of g . Specifically, two distinct eigenvalues $\lambda = \mu(g)$ and $\lambda' = \nu(g)$ of g (corresponding to weights μ and ν of V_{26}) coincide if and only if $(\mu - \nu)(g) = 1$. When the weight difference $\mu - \nu$ is a root $\alpha \in \Phi(F_4)$, this is exactly the condition $\alpha \in \Psi(g)$, i.e., α belongs to the root system of $Z_{F_4}(g)$. Thus the number k of distinct eigenvalues of g in V_{26} decreases precisely as the root system $\Psi(g)$ of $Z_{F_4}(g)$ grows. In the generic case $\Psi(g) = \emptyset$ (so $Z_{F_4}(g) = T$) and $k = 25$; in the involutory case $\Psi(g)$ is large and $k = 2$.

The following lemma applies Proposition 7 to derive the polystability consequence that underpins all the main theorems of the paper. The key point is that this argument is completely uniform; it does not depend on which subgroup of $F_4(\mathbb{C})$ the element g_e belongs to, only on its nontriviality.

Lemma 2. *Let E be a polystable principal $F_4(\mathbb{C})$ -bundle over X . If E admits a reduction of the structure group to $Z_{F_4}(g_e)$ for some nontrivial semisimple element $g_e \in F_4(\mathbb{C})$, then E is strictly polystable.*

Proof. By Proposition 7(a), the centraliser $Z_{F_4}(g_e)$ is connected reductive and is the Levi factor of a parabolic subgroup of $F_4(\mathbb{C})$. By Proposition 7(b), since g_e is nontrivial and $Z(F_4(\mathbb{C})) = \{1\}$, there exists a root $\alpha \in \Phi(F_4)$ with $\alpha(g_e) \neq 1$, so $Z_{F_4}(g_e)$ does not contain the full root space $\mathfrak{f}_4^\alpha$ and is therefore a proper subgroup of $F_4(\mathbb{C})$. Consequently, the corresponding parabolic subgroup is proper. Ramanathan's polystability criterion [1,5] states that a polystable principal G -bundle admitting a reduction to the Levi factor of a proper parabolic subgroup is strictly polystable (not stable). Applied here, this gives the strict polystability of E . $\square$

Remark 8. *Lemma 2 applies to all nontrivial semisimple elements $g_e \in F_4(\mathbb{C})$, whether $g_e \in K_I$, $g_e \in K_{II}$, or g_e belongs to no particular named subgroup. The uniformity of the argument reflects the fact that the key property—properness of $Z_{F_4}(g_e)$ —depends only on $g_e \neq 1$ and $Z(F_4(\mathbb{C})) = \{1\}$, and not on any finer structure of g_e .*

4.2. Restrictions of V_{26} and Eigenvalues for K_{II}

We now specialise in semisimple elements $h \in K_{II} = \text{Spin}(9, \mathbb{C})$. The first step is to identify the $\text{Spin}(9, \mathbb{C})$ module structure of V_{26} by decomposing the restriction, which reveals that the three weight families of Proposition 5 correspond exactly to the three summands in the decomposition of $V_{26}|_{K_{II}}$. This identification is the representation-theoretic basis for the specialised results of Section 6.2.

Proposition 8. *The restriction of V_{26} to $K_{II} = \text{Spin}(9, \mathbb{C})$ decomposes as*

$$V_{26}|_{\text{Spin}(9, \mathbb{C})} \cong \mathbf{9} \oplus \mathbf{16} \oplus \mathbf{1}, \quad (9)$$

where $\mathbf{9}$ is the standard representation of $\text{SO}(9, \mathbb{C})$ lifted to $\text{Spin}(9, \mathbb{C})$, $\mathbf{16}$ is the unique spinorial representation of $\text{Spin}(9, \mathbb{C})$ of highest weight $\omega_4^{B_4}$ and dimension $2^4 = 16$, and $\mathbf{1}$ is the trivial representation.

Proof. This follows from the identification of V_{26} with the trace-zero part of the exceptional Jordan algebra $\mathfrak{J}_3(\mathbb{O})$ of 3×3 Hermitian matrices over the octonions $\mathbb{O}$, on which $F_4(\mathbb{C})$ acts as the group of automorphisms [19,27]. Under the subgroup $\text{Spin}(9, \mathbb{C}) \subset F_4(\mathbb{C})$, the three summands correspond to the three weight families of Proposition 5: the weights $0, \pm e_i$ of groups (a) and (c) form the weight system of $\mathbf{9}$ (the standard representation of $\text{SO}(9, \mathbb{C})$ has weights $0, \pm e_i, i = 1, 2, 3, 4$); the weights

$$\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$$

of group (b) form the weight system of the spinorial representation $\mathbf{16}$ (which has all $2^4 = 16$ sign combinations); and the remaining copy of the zero weight in group (c) contributes $\mathbf{1}$. $\square$

A semisimple element $h \in \text{Spin}(9, \mathbb{C})$ is conjugate in $\text{Spin}(9, \mathbb{C})$ to an element of the maximal torus of $\text{Spin}(9, \mathbb{C})$, which we parametrise by $(t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4$ so that h projects to

$$\text{diag}(t_1, t_1^{-1}, t_2, t_2^{-1}, t_3, t_3^{-1}, t_4, t_4^{-1}, 1) \in \text{SO}(9, \mathbb{C})$$

under the covering $\text{Spin}(9) \rightarrow \text{SO}(9)$.

Proposition 9. Let $h \in K_{II} = \text{Spin}(9, \mathbb{C})$ be nontrivial semisimple with parameters (t_1, t_2, t_3, t_4) . The eigenvalues of h in V_{26} are as given in Proposition 6. The number of distinct eigenvalues satisfies $2 \leq k \leq 25$, with $k = 2$ when $h^2 = 1$ (involutory pattern) and $k = 25$ for generic h .

Proof. Since $\text{Spin}(9, \mathbb{C}) \subset F_4(\mathbb{C})$ and the torus of $\text{Spin}(9, \mathbb{C})$ maps to the torus of $F_4(\mathbb{C})$ via the same coordinates (t_1, t_2, t_3, t_4) , the eigenvalues of h in V_{26} are given directly by Proposition 6. The decomposition $V_{26}|_{K_{II}} \cong \mathbf{9} \oplus \mathbf{16} \oplus \mathbf{1}$ (Proposition 8) makes the correspondence explicit: on $\mathbf{9}$ the eigenvalues are $1, t_i^{\pm 1}$; on $\mathbf{16}$ they are $t_1^{\epsilon_1/2} t_2^{\epsilon_2/2} t_3^{\epsilon_3/2} t_4^{\epsilon_4/2}$; and on $\mathbf{1}$ the eigenvalue is 1. In the involutory case $h^2 = 1$, since $\rho_{26}(h)^2 = \rho_{26}(h^2) = \text{id}$, all eigenvalues of $\rho_{26}(h)$ are roots of unity of order dividing two, and hence are in $\{+1, -1\}$, giving $k = 2$. $\square$

Remark 9. The necessity of working with $K_{II} = \text{Spin}(9, \mathbb{C})$ rather than $\text{SO}(9, \mathbb{C})$ is reflected in the spinorial eigenvalues $t_i^{\pm 1/2}$ of group (b): these are eigenvalues of h in the spinorial representation $\mathbf{16}$, which is defined on $\text{Spin}(9, \mathbb{C})$ but not on $\text{SO}(9, \mathbb{C})$. Indeed, the central element $-1 \in \ker(\text{Spin}(9) \rightarrow \text{SO}(9)) \cong \mu_2$ acts as $-\text{id}$ on $\mathbf{16}$ (since $\mathbf{16}$ is a spinorial representation), which means that $\mathbf{16}$ is not a well-defined representation of $\text{SO}(9, \mathbb{C})$. Geometrically, this reflects the structure of the symmetric space $F_4^{(-20)}/\text{Spin}(9)$, whose tangent space is isomorphic to $\Delta_9 \cong \mathbf{16}$ (Remark 1).

4.3. Restrictions of V_{26} and Eigenvalues for K_I

We now specialise in elements

$$g_e \in K_I = (\text{Sp}(6, \mathbb{C}) \times \text{SL}(2, \mathbb{C}))/\mu_2.$$

The product structure of K_I is reflected in the decomposition of $V_{26}|_{K_I}$ into a tensor-product piece and an exterior-power piece, and the eigenvalues of g_e factor accordingly as products of the symplectic eigenvalues $\mu_i^{\pm 1}$ and the $\text{SL}(2)$ eigenvalues $\nu^{\pm 1}$. This contrasts with the type (II) case, where the spinorial eigenvalues $t_i^{\pm 1/2}$ do not factor in this way.

Proposition 10. The restriction of V_{26} to $K_I = (\text{Sp}(6, \mathbb{C}) \times \text{SL}(2, \mathbb{C}))/\mu_2$ decomposes as

$$V_{26}|_{K_I} \cong \Lambda_0^2(\mathbb{C}^6) \oplus (\mathbb{C}^6 \otimes \mathbb{C}^2), \quad (10)$$

where $\Lambda_0^2(\mathbb{C}^6)$ is the 14-dimensional symplectic trace-free second exterior power of the standard representation of $\text{Sp}(6, \mathbb{C})$, and $\mathbb{C}^6 \otimes \mathbb{C}^2$ is the 12-dimensional tensor product of the standard representations of $\text{Sp}(6, \mathbb{C})$ and $\text{SL}(2, \mathbb{C})$. The central element $(-I_6, -I_2) \in \mu_2$ acts as $+1$ on both summands, confirming that the decomposition descends to K_I .

Proof. The decomposition follows from the branching rule for the restriction $F_4 \supset K_I$ applied to V_{26} [19,25]. That $(-I_6, -I_2)$ acts trivially on both summands is verified directly: on $\Lambda_0^2(\mathbb{C}^6)$, the element $-I_6$ acts on $\Lambda^2(\mathbb{C}^6)$ by $(-1)^2 = +1$ (since it scales all vectors by -1 , and the wedge product picks up two signs); on $\mathbb{C}^6 \otimes \mathbb{C}^2$, the element $(-I_6) \otimes (-I_2)$ acts by $(-1)(-1) = +1$. Hence both representations factor through the quotient K_I and the decomposition is well defined. $\square$

A semisimple element $g_e = [(A, B)] \in K_I$ is conjugate in K_I to an element of the maximal torus of K_I , which we parametrise by $(\mu_1, \mu_2, \mu_3, \nu) \in (\mathbb{C}^*)^4$ with

$$A = \text{diag}(\mu_1, \mu_1^{-1}, \mu_2, \mu_2^{-1}, \mu_3, \mu_3^{-1}) \in \text{Sp}(6, \mathbb{C}), \quad B = \text{diag}(\nu, \nu^{-1}) \in \text{SL}(2, \mathbb{C}).$$

Proposition 11. Let $g_e = [(A, B)] \in K_I$ be nontrivial semisimple with parameters $(\mu_1, \mu_2, \mu_3, \nu)$. The eigenvalues of g_e in $V_{26} = \Lambda_0^2(\mathbb{C}^6) \oplus (\mathbb{C}^6 \otimes \mathbb{C}^2)$ are:

(a) On $\Lambda_0^2(\mathbb{C}^6)$ (dimension 14):

$$1^{(2)}, \quad \mu_i \mu_j \ (i < j), \quad \mu_i^{-1} \mu_j^{-1} \ (i < j), \quad \mu_i \mu_j^{-1} \ (i \neq j), \quad (11)$$

giving $2 + 3 + 3 + 6 = 14$ eigenvalues counted with multiplicity.

(b) On $\mathbb{C}^6 \otimes \mathbb{C}^2$ (dimension 12):

$$\mu_i^{\pm 1} \nu, \quad \mu_i^{\pm 1} \nu^{-1}, \quad i = 1, 2, 3, \quad (12)$$

giving 12 eigenvalues counted with multiplicity.

The number of distinct eigenvalues satisfies $2 \leq k \leq 25$, with $k = 2$ when $g_e^2 = 1$ and $k = 25$ generically.

Proof. Label the standard basis of $\mathbb{C}^6$ as $v_1, \dots, v_6$ so that A acts with eigenvalue μ_i on v_{2i-1} and μ_i^{-1} on v_{2i} for $i = 1, 2, 3$.

Part (a): The eigenvalue of A on the basis vector $v_p \wedge v_q \in \Lambda^2(\mathbb{C}^6)$ is the product of the eigenvalues on v_p and v_q . For $p = 2i - 1, q = 2i$: eigenvalue $\mu_i \cdot \mu_i^{-1} = 1$ for each $i = 1, 2, 3$, giving three basis vectors with eigenvalue 1 in $\Lambda^2(\mathbb{C}^6)$. The symplectic direction is $\omega_J = v_1 \wedge v_2 + v_3 \wedge v_4 + v_5 \wedge v_6$, which is a fixed vector with eigenvalue 1. In $\Lambda_0^2(\mathbb{C}^6) = \ker(\Lambda^2(\mathbb{C}^6) \xrightarrow{\iota_J} \mathbb{C})$ (where ι_J is the symplectic trace), eigenvalue 1 appears with multiplicity $3 - 1 = 2$. The remaining eigenvalues are products (eigenvalue on v_p) · (eigenvalue on v_q) for $\{p, q\}$ not of the form $\{2i - 1, 2i\}$; these are:

- $\mu_i \mu_j$ for $i < j$ (from $v_{2i-1} \wedge v_{2j-1}$): three values;
- $\mu_i^{-1} \mu_j^{-1}$ for $i < j$ (from $v_{2i} \wedge v_{2j}$): three values;
- $\mu_i \mu_j^{-1}$ for $i \neq j$ (from $v_{2i-1} \wedge v_{2j}$ with $i \neq j$): six values.

Part (b): The eigenvalues of A on $\mathbb{C}^6$ are $\mu_i^{\pm 1}$ ($i = 1, 2, 3$), and those of B on $\mathbb{C}^2$ are $\nu^{\pm 1}$. The eigenvalues on the tensor product $\mathbb{C}^6 \otimes \mathbb{C}^2$ are all products $\mu_i^{\pm 1} \cdot \nu^{\pm 1}$ for $i = 1, 2, 3$, giving $6 \times 2 = 12$ eigenvalues.

For the involutory case, $g_e^2 = 1$ forces $\rho_{26}(g_e)^2 = \text{id}$, so all eigenvalues lie in $\{+1, -1\}$, giving $k = 2$. $\square$

5. The General Decomposition Theorem

This section proves the main theorem of the paper in full generality. The proof follows the following four steps: extraction of the semisimple element g_e from ω , reduction of the structure group to the centraliser $Z_{F_4}(g_e)$, application of the polystability lemma, and decomposition of $E(V_{26})$ into eigenspace sub-bundles using the commutation condition (7).

5.1. Main Results

We begin with the proposition that extracts the semisimple element and the centraliser reduction, which serves both this section and Section 6.

Proposition 12. Let E be a principal $F_4(\mathbb{C})$ -bundle over X and $\omega: E \rightarrow E$ a nontrivial automorphism of finite order. Then:

- ω acts on each fibre E_x by right multiplication by a semisimple element $g_e \in F_4(\mathbb{C})$, well defined up to conjugation in $F_4(\mathbb{C})$ and independent of $x \in X$ by connectedness.
- The element g_e is nontrivial.
- The sub-bundle $E^{g_e} = \{e \in E : \omega(e) = e \cdot g_e\}$ is a reduction of the structure group of E to $Z_{F_4}(g_e)$.

Proof. Part (a): Since ω is a G -equivariant automorphism of the principal $F_4(\mathbb{C})$ -bundle E of finite order n , it acts on each fibre $E_x \cong F_4(\mathbb{C})$ by conjugation, which in the right-multiplication convention gives $\omega(e) = e \cdot g_e$ for some $g_e \in F_4(\mathbb{C})$. Since $\omega^n = \text{id}_E$, we have $g_e^n = 1$, so g_e is semisimple. Constancy over X : The function $x \mapsto g_e(x)$ (defined up to conjugation) is continuous on the connected space X and takes values in the discrete set of conjugacy classes of elements of order dividing n in $F_4(\mathbb{C})$, so it is constant [1].

Part (b): If $g_e = 1$, then $\omega(e) = e \cdot 1 = e$ for all $e \in E$, so $\omega = \text{id}_E$, contradicting the assumption that ω is nontrivial.

Part (c): We verify that E^{g_e} is invariant under the right action of $Z_{F_4}(g_e)$. Let $e \in E^{g_e}$ (so $\omega(e) = e \cdot g_e$) and $h \in Z_{F_4}(g_e)$ (so $g_e h = h g_e$). Then:

$$\omega(e \cdot h) = \omega(e) \cdot h = (e \cdot g_e) \cdot h = e \cdot g_e h = e \cdot h g_e = (e \cdot h) \cdot g_e,$$

where the first equality uses the $F_4(\mathbb{C})$ equivariance of ω (which commutes with right multiplication), and the fourth uses $g_e h = h g_e$. Hence $e \cdot h \in E^{g_e}$, so E^{g_e} is a principal $Z_{F_4}(g_e)$ -bundle and defines a reduction of structure group of E to $Z_{F_4}(g_e)$ [1,5]. $\square$

Theorem 1. Let X be a compact Riemann surface of genus $g \geq 2$, $\sigma_X \in \text{Aut}(X)$ an involution, and (E, f, ω) a Galois triple. Then:

- (a) E is strictly polystable.
- (b) The associated bundle $E(V_{26})$ admits a decomposition

$$E(V_{26}) = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k, \quad 2 \leq k \leq 25, \quad (13)$$

into holomorphic sub-bundles with $\sigma_X^* \mathcal{E}_j \cong \mathcal{E}_j$ for all $j = 1, \dots, k$.

- (c) The sub-bundle $\mathcal{E}_j$ is the eigenspace sub-bundle of $E(V_{26})$ corresponding to the eigenvalue λ_j of g_e in V_{26} , with $\text{rk} \mathcal{E}_j$ equal to the multiplicity of λ_j . The number k and the ranks $\text{rk} \mathcal{E}_j$ are determined by the eigenvalue pattern of g_e (Definition 2): $k = 25$ in the generic case, $k = 2$ in the involutory case, and $2 \leq k \leq 25$ in intermediate cases.

Proof. By Proposition 12(a)–(b), the automorphism ω determines a nontrivial semisimple element $g_e \in F_4(\mathbb{C})$, well defined up to conjugation (Lemma 1 ensures that $Z_{F_4}(g_e)$, and hence the entire subsequent construction depends only on this conjugacy class); we fix a representative in the conjugacy class.

By Proposition 12(c), the sub-bundle E^{g_e} is a reduction of the structure group of E from $F_4(\mathbb{C})$ to the centraliser $Z_{F_4}(g_e)$.

The element g_e is nontrivial and semisimple, so by Lemma 2, E is strictly polystable.

Since g_e is semisimple, $\rho_{26}(g_e)$ acts diagonalisably on V_{26} . Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues with corresponding eigenspaces $U_1, \dots, U_k$, so that

$$V_{26} = U_1 \oplus \cdots \oplus U_k.$$

Each U_j is preserved by $Z_{F_4}(g_e)$: for any $h \in Z_{F_4}(g_e)$ and $v \in U_j$, the commutativity $h g_e = g_e h$ gives

$$\rho_{26}(g_e)(\rho_{26}(h)v) = \rho_{26}(h)(\rho_{26}(g_e)v) = \rho_{26}(h)(\lambda_j v) = \lambda_j \rho_{26}(h)v,$$

so $\rho_{26}(h)v \in U_j$. The reduction of structure group E^{g_e} (Proposition 12(c)) is a principal $Z_{F_4}(g_e)$ -bundle, and each U_j is a $Z_{F_4}(g_e)$ module via the restriction of ρ_{26} . The eigenspace sub-bundle is defined as the associated bundle

$$\mathcal{E}_j = E^{g_e} \times^{Z_{F_4}(g_e)} U_j.$$

Since $E = E^{g_e} \times^{Z_{F_4}(g_e)} F_4(\mathbb{C})$, the associativity of the associated bundle construction gives

$$E(V_{26}) = E \times^{F_4(\mathbb{C})} V_{26} = E^{g_e} \times^{Z_{F_4}(g_e)} V_{26} = \bigoplus_{j=1}^k E^{g_e} \times^{Z_{F_4}(g_e)} U_j = \bigoplus_{j=1}^k \mathcal{E}_j,$$

with $\text{rk } \mathcal{E}_j = \dim U_j$.

The lower bound $k \geq 2$: The representation $\rho_{26}: F_4(\mathbb{C}) \rightarrow \text{GL}(V_{26})$ is faithful, because $F_4(\mathbb{C})$ is simple and V_{26} is a nontrivial irreducible module, so $\ker \rho_{26} \subset Z(F_4(\mathbb{C})) = \{1\}$. Hence $\rho_{26}(g_e) \neq \text{id}_{V_{26}}$, which means g_e has at least two distinct eigenvalues in V_{26} , giving $k \geq 2$. The upper bound $k \leq 25$ and the description of k in each case follow from Proposition 6 and Definition 2.

For the σ_X^* invariance, let

$$f_{\text{vb}}: E(V_{26}) \xrightarrow{\sim} \sigma_X^*(E(V_{26}))$$

be the isomorphism of associated vector bundles induced by the Galois isomorphism $f: E \xrightarrow{\sim} \sigma_X^*E$. In terms of the associated bundle construction, $f_{\text{vb}}([e, v]) = [f(e), v]$ in local notation, so f_{vb} commutes with the action of $F_4(\mathbb{C})$: for any $g \in F_4(\mathbb{C})$,

$$f_{\text{vb}}([e \cdot g, v]) = [f(e \cdot g), v] = [f(e) \cdot g, v]$$

using the $F_4(\mathbb{C})$ equivariance of f .

Let v be a local section of $\mathcal{E}_j$, so that in the associated bundle description, the $\rho_{26}(g_e)$ eigenvalue of v is λ_j , i.e., $\rho_{26}(g_e)(v) = \lambda_j v$. Using the commutation condition (7),

$$\rho_{26}(\sigma_X^* \omega)(f_{\text{vb}}(v)) = f_{\text{vb}}(\rho_{26}(\omega)(v)) = f_{\text{vb}}(\lambda_j v) = \lambda_j f_{\text{vb}}(v),$$

where the first equality holds because f_{vb} intertwines $\rho_{26}(\omega)$ and $\rho_{26}(\sigma_X^* \omega)$ (a consequence of (7) and the equivariance of f), and the second holds because

$$\rho_{26}(\omega)(v) = \rho_{26}(g_e)(v) = \lambda_j v.$$

Thus $f_{\text{vb}}(v)$ is a λ_j eigenvector for $\rho_{26}(\sigma_X^* \omega)$ in $\sigma_X^*(E(V_{26}))$. The λ_j eigenspace of $\rho_{26}(\sigma_X^* \omega)$ in

$$\sigma_X^*(E(V_{26})) = \sigma_X^*E \times_{F_4(\mathbb{C})} V_{26}$$

is precisely $\sigma_X^*E \times_{F_4(\mathbb{C})} U_j = \sigma_X^* \mathcal{E}_j$. Hence f_{vb} restricts to an isomorphism $\mathcal{E}_j \xrightarrow{\sim} \sigma_X^* \mathcal{E}_j$ for each j . $\square$

Corollary 1. Let (E, f, ω) be a Galois triple and let $E(V_{26}) = \bigoplus_{j=1}^k \mathcal{E}_j$ be the decomposition of Theorem 1. The automorphism

$$\tilde{\omega}: E(V_{26}) \longrightarrow E(V_{26})$$

induced by ω satisfies

$$\tilde{\omega}|_{\mathcal{E}_j} = \lambda_j \text{id}_{\mathcal{E}_j} \quad \text{for each } j = 1, \dots, k,$$

where $\lambda_j \in \mathbb{C}^*$ is the eigenvalue of $\rho_{26}(g_e)$ on U_j . In particular, $\tilde{\omega}$ is semisimple as an automorphism of $E(V_{26})$, and the decomposition of Theorem 1 is precisely the eigenspace decomposition of $E(V_{26})$ with respect to $\tilde{\omega}$.

Proof. The automorphism $\omega: E \rightarrow E$ acts fibrewise by right multiplication by g_e (Proposition 12(a)), so the induced endomorphism $\tilde{\omega}$ of the associated vector bundle $E(V_{26}) = E \times^{F_4(\mathbb{C})} V_{26}$ acts on a local section $[e, v]$ by $\tilde{\omega}([e, v]) = [e, \rho_{26}(g_e)v]$. Since

$\mathcal{E}_j = E^{g_e} \times^{Z_{F_4}(g_e)} U_j$ by definition, every local section of $\mathcal{E}_j$ is of the form $[e, v]$ with $v \in U_j$, so $\rho_{26}(g_e)v = \lambda_j v$. Therefore $\tilde{\omega}([e, v]) = [e, \lambda_j v] = \lambda_j [e, v]$, giving $\tilde{\omega}|_{\mathcal{E}_j} = \lambda_j \text{id}_{\mathcal{E}_j}$. The final statement follows because g_e is semisimple, so $\rho_{26}(g_e)$ is diagonalisable and $V_{26} = \bigoplus_j U_j$ is its eigenspace decomposition; passing to associated bundles gives the stated identification. $\square$

Lemma 3. Let (E, f, ω) be a Galois triple over (X, σ_X) . For every fixed point $x_0 \in X^{\sigma_X} = \{x \in X : \sigma_X(x) = x\}$, the isomorphism f induces a self-map $f_{x_0}: E_{x_0} \xrightarrow{\sim} E_{x_0}$ on the fibre over x_0 , and

$$f_{x_0}^2 = \text{id}_{E_{x_0}}.$$

Consequently, the cocycle condition (6) is the effective descent condition for the principal bundle E along the double covering $\pi: X \rightarrow Y = X/\langle \sigma_X \rangle$.

Proof. The isomorphism $f: E \xrightarrow{\sim} \sigma_X^* E$ has the fibre map

$$f_x: E_x \longrightarrow (\sigma_X^* E)_x = E_{\sigma_X(x)}$$

at each point $x \in X$. Applying the functor σ_X^* to f yields an isomorphism

$$\sigma_X^* f: \sigma_X^* E \xrightarrow{\sim} \sigma_X^* (\sigma_X^* E) = (\sigma_X^2)^* E = E,$$

whose fibre map at x is

$$(\sigma_X^* f)_x: (\sigma_X^* E)_x = E_{\sigma_X(x)} \longrightarrow ((\sigma_X^2)^* E)_x = E_x, \quad (\sigma_X^* f)_x = f_{\sigma_X(x)},$$

where the equality $(\sigma_X^* f)_x = f_{\sigma_X(x)}$ holds under the canonical identifications $(\sigma_X^* E)_x \cong E_{\sigma_X(x)}$ and $(\sigma_X^2)^* E = E$ (using $\sigma_X^2 = \text{id}_X$). The cocycle condition (6) at the fibre over x therefore reads

$$f_{\sigma_X(x)} \circ f_x = \text{id}_{E_x}. \quad (14)$$

At a fixed point $x_0 \in X^{\sigma_X}$, we have $\sigma_X(x_0) = x_0$, so $f_{x_0}: E_{x_0} \rightarrow E_{x_0}$ is a self-map and (14) specialises to $f_{x_0} \circ f_{x_0} = \text{id}_{E_{x_0}}$, i.e., $f_{x_0}^2 = \text{id}_{E_{x_0}}$.

For the descent statement, the effective descent condition for a principal $F_4(\mathbb{C})$ -bundle along the Galois covering $\pi: X \rightarrow Y$ with Galois group $\mathbb{Z}/(2) = \langle \sigma_X \rangle$ is the existence of an isomorphism $f: E \xrightarrow{\sim} \sigma_X^* E$ satisfying the cocycle condition $(\sigma_X^* f) \circ f = \text{id}_E$ [29]. This is precisely (6), which holds by the hypothesis. $\square$

Remark 10. The condition $f_{x_0}^2 = \text{id}_{E_{x_0}}$ at fixed points is not an additional hypothesis beyond the cocycle condition (6); it is an automatic consequence of (6) evaluated at fixed points, as the proof of Lemma 3 makes explicit. In particular, Definition 1 already incorporates all the data necessary for descent, with no further condition required at the ramification locus.

Proposition 13. Let (E, f, ω) be a Galois triple over (X, σ_X) and let $\pi: X \rightarrow Y = X/\langle \sigma_X \rangle$ be the quotient map, a double covering branched over the image of X^{σ_X} . Then:

- There exists a principal $F_4(\mathbb{C})$ -bundle E_Y over Y , unique up to isomorphism, such that $\pi^* E_Y \cong E$. The bundle E_Y is polystable.
- The automorphism ω descends to a nontrivial automorphism $\omega_Y: E_Y \rightarrow E_Y$ of finite order, with $\pi^* \omega_Y = \omega$.
- The semisimple element $g_e^Y \in F_4(\mathbb{C})$ determined by ω_Y lies in the same conjugacy class of $F_4(\mathbb{C})$ as g_e . In particular, g_e^Y is nontrivial, and $E_Y(V_{26})$ decomposes into k polystable

sub-bundles over Y , where k is the number of distinct eigenvalues of g_e in V_{26} as determined by Theorem 1.

Proof. Part (a): By Lemma 3, the cocycle condition (6) constitutes an effective descent datum for E along π . By Grothendieck's descent theory for principal bundles [29], this yields a unique principal $F_4(\mathbb{C})$ -bundle E_Y over Y with $\pi^*E_Y \cong E$.

For polystability of E_Y , we first verify semistability directly. Suppose $\sigma_Y: Y \rightarrow E_Y/P$ is a reduction to a proper parabolic subgroup $P \subsetneq F_4(\mathbb{C})$ with dominant character χ . Pulling back along π yields a reduction

$$\pi^*\sigma_Y: X \rightarrow \pi^*(E_Y/P) \cong E/P,$$

where the isomorphism uses the commutativity of π^* with the associated bundle construction [1]. By the standard formula for degrees under a finite covering of degree $\deg(\pi) = 2$,

$$\deg_X((\pi^*\sigma_Y)^*E(L_\chi)) = 2 \deg_Y(\sigma_Y^*E_Y(L_\chi)). \quad (15)$$

If $\deg_Y(\sigma_Y^*E_Y(L_\chi)) > 0$, then $\deg_X((\pi^*\sigma_Y)^*E(L_\chi)) > 0$, contradicting the semistability of E . Hence $\deg_Y(\sigma_Y^*E_Y(L_\chi)) \leq 0$ for every such reduction, and E_Y is semistable. Full polystability of E_Y follows from the general result that for a finite covering of smooth projective curves, a principal G -bundle is polystable if and only if its pull-back is polystable [35]. Since $\pi^*E_Y \cong E$ is polystable by Theorem 1(a), E_Y is polystable.

Part (b): The commutativity condition (7), namely $\sigma_X^*\omega \circ f = f \circ \omega$, states precisely that ω is a morphism of pairs with the descent datum, i.e., that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & \sigma_X^*E \\ \omega \downarrow & & \downarrow \sigma_X^*\omega \\ E & \xrightarrow{f} & \sigma_X^*E \end{array}$$

commutes. By the functoriality of Grothendieck descent [29], ω therefore descends to an automorphism $\omega_Y: E_Y \rightarrow E_Y$ satisfying $\pi^*\omega_Y = \omega$. Since $\omega^n = \text{id}_E$ for some positive integer n , applying π^* gives

$$\pi^*(\omega_Y^n) = (\pi^*\omega_Y)^n = \omega^n = \text{id}_E = \pi^*(\text{id}_{E_Y}).$$

The functor π^* is faithful on morphisms of principal bundles over Y : for any morphism $\alpha: E_Y \rightarrow E_Y$, the pull-back $\pi^*\alpha$ determines α uniquely because π is surjective and the fibre of π^*E_Y at $x \in \pi^{-1}(y)$ is canonically identified with the fibre of E_Y at y . Hence $\omega_Y^n = \text{id}_{E_Y}$ and ω_Y has finite order dividing n . If $\omega_Y = \text{id}_{E_Y}$, then $\omega = \pi^*\omega_Y = \text{id}_E$, contradicting the nontriviality of ω .

Part (c): For any $y \in Y$ and any $x \in \pi^{-1}(y)$, the isomorphism $\pi^*E_Y \cong E$ identifies the fibre $(E_Y)_y$ with E_x , and under this identification, $(\omega_Y)_y$ corresponds to ω_x . By Proposition 12(a), ω_x acts on E_x by right multiplication by g_e (up to conjugation in $F_4(\mathbb{C})$), so $(\omega_Y)_y$ acts on $(E_Y)_y$ by right multiplication by an element conjugate to g_e ; hence g_e^Y and g_e lie in the same conjugacy class. (For a non-ramified point y with two preimages x and $\sigma_X(x)$, the two identifications of $(E_Y)_y$ with fibres of E differ by the isomorphism $f_x: E_x \rightarrow E_{\sigma_X(x)}$, giving conjugation by f_x ; the conjugacy class of g_e^Y is independent of the choice of preimage since conjugation preserves conjugacy classes.) In particular, $g_e^Y \neq 1$, and the eigenvalue pattern of g_e^Y in V_{26} coincides with that of g_e , so $E_Y(V_{26})$ decomposes into k sub-bundles by the same argument as Theorem 1. $\square$

Remark 11. The descended pair (E_Y, ω_Y) is a polystable principal $F_4(\mathbb{C})$ -bundle over Y equipped with a nontrivial finite-order automorphism, but it does not in general carry a Galois structure in the sense of Definition 1: the isomorphism f serves as the descent datum and, once descent is performed, it is no longer available as an additional structure on E_Y . A Galois structure on E_Y would require a separate involution σ_Y of Y and a compatible isomorphism $E_Y \xrightarrow{\sim} \sigma_Y^* E_Y$ satisfying the cocycle condition. Such a situation arises naturally when X is a cyclic cover of a curve already equipped with an involution, and is related to the study of automorphisms of $F_4(\mathbb{C})$ of order $r \geq 4$ via the Kac diagram of $\mathfrak{f}_4$, which we indicate as a direction for further investigation in Section 8.

Remark 12. The quotient map $\pi: X \rightarrow Y = X/\Gamma$ of Proposition 13 is unramified away from X^{σ_X} and simply branched at each point of X^{σ_X} , so Y is again a smooth compact Riemann surface, but the branch locus $\pi(X^{\sigma_X}) \subset Y$ carries a natural orbifold structure with local isotropy group $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$: near $y_0 = \pi(x_0)$ the local model of π is the standard $\mathbb{Z}/2$ -quotient of a disc, and Y is naturally an orbifold curve $\underline{Y}$ with isotropy $\mathbb{Z}/2\mathbb{Z}$ at the branch points. Lemma 3 shows that $f_{x_0}^2 = \text{id}_{E_{x_0}}$ at every $x_0 \in X^{\sigma_X}$; equivalently, f_{x_0} defines an order-dividing-two automorphism of the fibre E_{x_0} , which is exactly the local monodromy datum of the orbifold principal bundle $\underline{E_Y}$ over $\underline{Y}$ associated with E_Y under the standard dictionary between Γ -equivariant bundles on X and orbifold bundles on $\underline{Y}$ [29]. When $f_{x_0} \neq \text{id}_{E_{x_0}}$, this local monodromy is nontrivial and E_Y , viewed as an orbifold bundle, is twisted at y_0 ; this is the precise sense in which the branch locus of π introduces local twisting into E_Y , complementing the global descent statement of Proposition 13 with a local, orbifold-theoretic description at the ramification points.

5.2. Canonicity, Stratification, and the Adjoint Bundle

The three results of this subsection draw out the structural properties of the eigenbundle decomposition of Theorem 1 that clarify the geometric role of the decomposition within $M(E_6(\mathbb{C}))^\sigma$.

Corollary 2 (Canonicity of the decomposition). *The eigenbundle decomposition*

$$E(V_{26}) = \bigoplus_{j=1}^k \mathcal{E}_j$$

of Theorem 1 depends only on the conjugacy class $[g_e] \in F_4(\mathbb{C})$ of the semisimple element determined by ω , up to relabelling of the summands.

Proof. Let $g'_e = hg_e h^{-1}$ for some $h \in F_4(\mathbb{C})$. By Lemma 1,

$$Z_{F_4}(g'_e) = hZ_{F_4}(g_e)h^{-1}.$$

Applying the representation ρ_{26} gives

$$\rho_{26}(g'_e) = \rho_{26}(h) \rho_{26}(g_e) \rho_{26}(h)^{-1},$$

so $\rho_{26}(g_e)$ and $\rho_{26}(g'_e)$ are conjugate endomorphisms of V_{26} and therefore have identical spectra and eigenspace dimensions. The λ_j eigenspace of $\rho_{26}(g'_e)$ is $U'_j = \rho_{26}(h)(U_j)$, and the map $\rho_{26}(h): U_j \xrightarrow{\sim} U'_j$ is an isomorphism of $Z_{F_4}(g'_e)$ modules (via the conjugation isomorphism of Lemma 1). The associated eigenspace sub-bundles $E^{g_e} \times^{Z_{F_4}(g_e)} U_j$ and $E^{g'_e} \times^{Z_{F_4}(g'_e)} U'_j$ are therefore naturally isomorphic for each j . Hence the decomposition depends only on the conjugacy class $[g_e]$; different choices of representative merely permute the labels of the summands with equal eigenvalues. $\square$

Corollary 3 (Stratification by semisimple conjugacy classes). *Let $\mathcal{G}$ denote the set of isomorphism classes of Galois triples over (X, σ_X) . The assignment $(E, f, \omega) \mapsto [g_e]$ gives a well-defined map*

$$\Phi: \mathcal{G} \longrightarrow \{\text{nontrivial semisimple conjugacy classes of } F_4(\mathbb{C})\}$$

whose fibres $\mathcal{G}_{[g_e]} = \Phi^{-1}([g_e])$ give a decomposition into pairwise disjoint strata:

$$\mathcal{G} = \bigsqcup_{[g_e]} \mathcal{G}_{[g_e]}.$$

The stratum $\mathcal{G}_{[g_e]}$ consists precisely of those Galois triples for which the eigenbundle decomposition of Theorem 1 has the eigenvalue pattern determined by $[g_e]$ in V_{26} (Definition 2).

Proof. By Proposition 12(a), the element g_e is well defined up to conjugation in $F_4(\mathbb{C})$, so Φ is well defined. By Corollary 2, the eigenvalue pattern depends only on the conjugacy class, so the stratum $\mathcal{G}_{[g_e]}$ is characterised by the eigenvalue pattern as stated. The fibres are pairwise disjoint since distinct conjugacy classes are disjoint. $\square$

Remark 13. *The stratification of Corollary 3 refines the description of the distinguished sublocus of $M(E_6(\mathbb{C}))^{\sigma, \sigma_X}$ given in Proposition 4. The generic stratum ($k = 25$, corresponding to elements with $Z_{F_4}(g_e) = T$, the maximal torus) is open and dense in $\mathcal{G}$ (Proposition 7(c) and Remark 7). The most degenerate stratum ($k = 2$, involutory pattern) consists of those Galois triples whose automorphism ω has order 2; this is the closed stratum. The intermediate strata are labelled by specific Levi subgroups $Z_{F_4}(g_e)$, each with their own eigenvalue pattern, and they interpolate between the open and closed extremes.*

Remark 14. *The stratification of Corollary 3 also has a natural interpretation in terms of the local structure of $M(F_4(\mathbb{C}))^{\sigma_X}$ itself. Since $F_4(\mathbb{C})$ is centreless, a generic stable bundle has a trivial automorphism group; for $[E]$ underlying a Galois triple (E, f, ω) with ω being nontrivial, $\text{Aut}(E)$ contains the nontrivial element ω , so $[E]$ is a point at which the automorphism group of the bundle is strictly larger than that of a generic stable bundle. By the general theory underlying the GIT construction of $M(F_4(\mathbb{C}))$ [2,3], points with a larger automorphism group are the points at which the moduli space is most likely to fail to be smooth. The stratification*

$$\mathcal{G} = \bigsqcup_{[g_e]} \mathcal{G}_{[g_e]}$$

therefore sits inside, and is compatible with a stratification of $M(F_4(\mathbb{C}))^{\sigma_X}$ by automorphism-group type: the generic stratum ($k = 25$) corresponds to the smallest centraliser $Z_{F_4}(g_e) = T$ compatible with a nontrivial ω , while the involutory stratum ($k = 2$) corresponds to comparatively large centralisers $Z_{F_4}(g_e)$ among nontrivial semisimple elements. A precise identification of the local ring, or tangent cone, of $M(F_4(\mathbb{C}))^{\sigma_X}$ at points of $\mathcal{G}_{[g_e]}$ would require a full deformation-theoretic analysis via the hypercohomology of the deformation complex of (E, f) , along the lines of [15,16]. Although we do not pursue this here, Corollary 3 already identifies the discrete combinatorial data—the conjugacy class $[g_e]$ —indexing the strata of this expected singular stratification, and Proposition 7 identifies the corresponding Levi subgroups explicitly.

Corollary 4 (Adjoint bundle splitting). *Let (E, f, ω) be a Galois triple. The adjoint bundle of the induced $E_6(\mathbb{C})$ -bundle $\iota_*(E)$ decomposes as*

$$\text{Ad}(\iota_*(E)) = E(\mathfrak{f}_4) \oplus E(V_{26}) = E(\mathfrak{f}_4) \oplus \bigoplus_{j=1}^k \mathcal{E}_j. \quad (16)$$

Each summand is preserved by the Galois action induced by $f: E \xrightarrow{\sim} \sigma_X^* E$:

$$E(\mathfrak{f}_4) \cong \sigma_X^* E(\mathfrak{f}_4), \quad \mathcal{E}_j \cong \sigma_X^* \mathcal{E}_j \quad \text{for all } j.$$

In particular, $E(\mathfrak{f}_4)$ is the $d\sigma$ -fixed part and $E(V_{26}) = \bigoplus_j \mathcal{E}_j$ is the $d\sigma$ -anti-fixed part of $\text{Ad}(\iota_*(E))$.

Proof. By the Lie algebra decomposition $\mathfrak{e}_6 = \mathfrak{f}_4 \oplus V_{26}$ as $F_4(\mathbb{C})$ modules (see Equation (5) and [19,20]), and since $\iota_*(E) = E \times^{F_4(\mathbb{C})} E_6(\mathbb{C})$, the $F_4(\mathbb{C})$ equivariance of the adjoint action gives

$$\text{Ad}(\iota_*(E)) = E \times^{F_4(\mathbb{C})} \mathfrak{e}_6 = \left(E \times^{F_4(\mathbb{C})} \mathfrak{f}_4 \right) \oplus \left(E \times^{F_4(\mathbb{C})} V_{26} \right) = E(\mathfrak{f}_4) \oplus E(V_{26}),$$

using the fact that the associated bundle functor commutes with direct sums. Substituting the decomposition $E(V_{26}) = \bigoplus_j \mathcal{E}_j$ of Theorem 1(b) gives the second equality in (16).

For the σ_X^* invariance, applying the associated bundle functor to the Galois isomorphism

$$f: E \xrightarrow{\sim} \sigma_X^* E$$

with respect to the $F_4(\mathbb{C})$ module $\mathfrak{f}_4$ yields an isomorphism

$$E(\mathfrak{f}_4) \xrightarrow{\sim} \sigma_X^* E(\mathfrak{f}_4).$$

The isomorphisms $\mathcal{E}_j \xrightarrow{\sim} \sigma_X^* \mathcal{E}_j$ are the content of Theorem 1(b).

The identification of $E(\mathfrak{f}_4)$ as the $d\sigma$ -fixed part and $E(V_{26})$ as the $d\sigma$ -anti-fixed part follows from the fact that V_{26} is the (-1) eigenspace of the outer involution $d\sigma$ on $\mathfrak{e}_6$, as recorded in Section 2 (see also [19,20]). $\square$

6. Specialised Results for the Two Involution Types

This section refines Theorem 1 for K_θ -restricted Galois triples. The specialised theorems are direct applications of the general theorem, with the additional input of the eigenvalue descriptions from Propositions 9 and 11. In each case, the new content is the precise description of the ranks and eigenvalue types of the summands in the decomposition, and the interpretation of this data in terms of the corresponding involution type and its relationship to the geometry of $M(E_6(\mathbb{C}))^\sigma$.

6.1. Type (I): The Subgroup K_I

When $g_e \in K_I = (\text{Sp}(6, \mathbb{C}) \times \text{SL}(2, \mathbb{C})) / \mu_2$, the eigenvalue pattern of g_e in V_{26} is governed by Proposition 11. The eigenvalues are products of the symplectic eigenvalues $\mu_i^{\pm 1}$ of the $\text{Sp}(6, \mathbb{C})$ component and the eigenvalues $\nu^{\pm 1}$ of the $\text{SL}(2, \mathbb{C})$ component, reflecting the product structure of K_I . The decomposition

$$V_{26}|_{K_I} \cong \Lambda_0^2(\mathbb{C}^6) \oplus (\mathbb{C}^6 \otimes \mathbb{C}^2),$$

given in Proposition 10, organises the summands into two geometrically natural families, corresponding to the two summands in the Cartan complement $\mathfrak{m}_I \cong \Lambda_0^2(\mathbb{C}^6) \otimes \mathbb{C}^2$ (Remark 1).

Theorem 2. Let (E, f, ω) be a K_I -restricted Galois triple. Then E is strictly polystable and $E(V_{26})$ decomposes as in (13), where:

- Eigenvalue 1 has multiplicity ≥ 2 , equal to 2 when none of the values $\mu_i \mu_j^{\pm 1}$, $\mu_i^{\pm 1} \mu_j^{\pm 1}$ (with $i \neq j$), or $\mu_i^{\pm 1} \nu^{\pm 1}$ coincide with 1.

- The remaining eigenvalues are among $\mu_i\mu_j$, $\mu_i^{-1}\mu_j^{-1}$, $\mu_i\mu_j^{-1}$ (from $\Lambda_0^2(\mathbb{C}^6)$) and $\mu_i^{\pm 1}\nu^{\pm 1}$ (from $\mathbb{C}^6 \otimes \mathbb{C}^2$), as given by Proposition 11.
- The number k satisfies $k = 2$ in the involutory case ($g_e^2 = 1$) and $k = 25$ in the generic case.

Proof. Every K_I -restricted Galois triple is a Galois triple, so Theorem 1 applies and gives strict polystability and decomposition (13) with $\sigma_X^*\mathcal{E}_j \cong \mathcal{E}_j$. The description of the eigenvalues and multiplicities is Proposition 11, which computes the eigenvalues of $g_e \in K_I$ explicitly from decomposition (10). $\square$

Remark 15. By Proposition 3(b), the involution θ_I arises as the restriction of the Cartan involution of E_6^2 to $F_4(\mathbb{C}) \subset E_6(\mathbb{C})$. A K_I -restricted Galois triple (E, f, ω) therefore produces, via the push-forward ι_* , a Galois $E_6(\mathbb{C})$ -bundle $(\iota_*(E), \iota_*(f))$ in $M(E_6(\mathbb{C}))^{\sigma, \sigma_X}$, whose decomposition reflects the action of the Cartan involution of E_6^2 on the $\mathfrak{f}_4$ component of the adjoint bundle via the embedding $\mathfrak{f}_4 \subset \mathfrak{e}_6$ of (5).

6.2. Type (II): The Subgroup K_{II}

When $g_e \in K_{II} = \text{Spin}(9, \mathbb{C})$, the eigenvalue pattern of g_e in V_{26} is governed by Proposition 9. The key feature distinguishing this case from type (I) is the presence of the spinorial eigenvalues

$$t_1^{\pm 1/2} t_2^{\pm 1/2} t_3^{\pm 1/2} t_4^{\pm 1/2},$$

which arise from summand 16 in (9) and which only make sense in $\text{Spin}(9, \mathbb{C})$ rather than $\text{SO}(9, \mathbb{C})$. These spinorial eigenvalues reflect the non-product geometry of K_{II} , in contrast to the factored structure of the type (I) eigenvalues.

Theorem 3. Let (E, f, ω) be a K_{II} -restricted Galois triple. Then E is strictly polystable and $E(V_{26})$ decomposes as in (13), where:

- Eigenvalue 1 has multiplicity ≥ 2 , arising from both 9 (weight 0) and 1 in (9).
- The vector eigenvalues $t_i^{\pm 1}$ arise from summand 9.
- The spinorial eigenvalues $t_1^{\pm 1/2} t_2^{\pm 1/2} t_3^{\pm 1/2} t_4^{\pm 1/2}$ arise from summand 16.
- The number k satisfies $k = 2$ in the involutory case ($g_e^2 = 1$) and $k = 25$ generically.

Proof. Every K_{II} -restricted Galois triple is a Galois triple, so Theorem 1 applies. The description of eigenvalues, multiplicities, and their geometric origins in the summands 9, 16, and 1 of (9) follows from Propositions 8 and 9. $\square$

Remark 16. By Proposition 3(a), θ_{II} arises as the restriction of the Cartan involution of $E_6^{(-14)}$. A K_{II} -restricted Galois triple (E, f, ω) therefore produces a Galois $E_6(\mathbb{C})$ -bundle in $M(E_6(\mathbb{C}))^{\sigma, \sigma_X}$ whose decomposition of $\iota_*(E)(V_{26})$ reflects the structure of the Cartan involution of $E_6^{(-14)}$ restricted to the V_{26} component of $\mathfrak{e}_6 = \mathfrak{f}_4 \oplus V_{26}$. The spinorial sub-bundles $\mathcal{E}_j$ corresponding to eigenvalues

$$t_1^{\pm 1/2} t_2^{\pm 1/2} t_3^{\pm 1/2} t_4^{\pm 1/2}$$

have no counterpart in the type (I) decomposition.

Remark 17. Theorems 2 and 3 are both specialisations of Theorem 1, and in both cases the same bounds $2 \leq k \leq 25$ hold. The distinction lies entirely in the structure of the eigenvalues: in type (I), all eigenvalues are products of the parameters μ_i and ν , reflecting the product structure of

$$K_I = (\text{Sp}(6) \times \text{SL}(2))/\mu_2;$$

in type (II), the spinorial eigenvalues $t_i^{\pm 1/2}$ arise from the non-product geometry of $K_{II} = \text{Spin}(9, \mathbb{C})$ and the symmetric space $F_4^{(-20)}/\text{Spin}(9)$.

6.3. Reduction to the Involution Case for Restricted Triples

The reduction argument of Proposition 13 extends to the restricted setting in the form established in the following result.

Proposition 14. *Let $\theta \in \{\theta_I, \theta_{II}\}$ and let (E, f, ω) be a K_θ -restricted Galois triple over (X, σ_X) . Let E_Y , ω_Y , and g_e^Y be as in Proposition 13. Then the conjugacy class of g_e^Y in $F_4(\mathbb{C})$ intersects K_θ , and the decomposition of $E_Y(V_{26})$ is of the type described in Theorem 2 (if $\theta = \theta_I$) or Theorem 3 (if $\theta = \theta_{II}$).*

Proof. By Proposition 13(c), g_e^Y is conjugate to g_e in $F_4(\mathbb{C})$, so they determine the same conjugacy class. Since (E, f, ω) is K_θ -restricted, this conjugacy class intersects K_θ (Definition 1), and hence so does the conjugacy class of g_e^Y . The decomposition of $E_Y(V_{26})$ into k sub-bundles then follows from the eigenvalue analysis of Propositions 11 and 9 applied to g_e^Y by the same argument as in the proofs of Theorems 2 and 3. $\square$

7. Examples and Applications

In order to illustrate the theory developed above, this section collects examples and connections between the representation-theoretic data of the paper—the pair $(F_4(\mathbb{C}), V_{26})$ and its eigenvalue decompositions—and several areas in which this same data appears: the classical geometry of the octonionic projective and hyperbolic planes, the physics of magic supergravity theories, and the geometric Langlands programme via Hitchin systems.

7.1. The Exceptional Jordan Algebra and the Cayley Planes

Recall from the proof of Proposition 8 that V_{26} is identified with the trace-zero part of the exceptional Jordan algebra $\mathfrak{J}_3(\mathbb{O})$ of 3×3 Hermitian octonionic matrices, on which $F_4(\mathbb{C}) = \text{Aut}(\mathfrak{J}_3(\mathbb{O})_{\mathbb{C}})$ acts [19,27]. This identification connects the eigenvalue decompositions of the present paper directly to the geometry of the two Cayley planes: the compact plane $\mathbb{O}\mathbb{P}^2 = F_4/\text{Spin}(9)$ and its non-compact dual $F_4^{(-20)}/\text{Spin}(9)$, the octonionic hyperbolic plane, which are both rank-one symmetric spaces [20,26].

Proposition 15. *Under the identification*

$$V_{26}|_{\text{Spin}(9, \mathbb{C})} \cong \mathbf{9} \oplus \mathbf{16} \oplus \mathbf{1}$$

of Proposition 8, summand $\mathbf{16}$ is isomorphic, as a $\text{Spin}(9, \mathbb{C})$ module, to Δ_9 , i.e., to the complexified isotropy representation of the symmetric pair $(F_4^{(-20)}, \text{Spin}(9))$ recorded in Remark 1. Consequently, for every K_{II} -restricted Galois triple (E, f, ω) , the sub-bundle of $E(V_{26})$ carrying the spinorial eigenvalues $t_i^{\pm 1/2}$ (Theorem 3) is, fibrewise, a copy of the complexified tangent representation of the octonionic hyperbolic plane $F_4^{(-20)}/\text{Spin}(9)$ at the corresponding fixed point.

Proof. Since $\text{Spin}(9, \mathbb{C})$ is of type B_4 , it has, up to isomorphism, a unique nontrivial irreducible representation of highest weight $\omega_4^{B_4}$, the 16-dimensional spin representation; any two spinorial (as opposed to tensorial) 16-dimensional $\text{Spin}(9, \mathbb{C})$ modules are therefore isomorphic [31,34]. Summand $\mathbf{16}$ of Proposition 8 is spinorial by Remark 9, and Δ_9 is by definition the 16-dimensional spin representation of $\text{Spin}(9, \mathbb{C})$; both are therefore isomorphic to the unique irreducible module of highest weight $\omega_4^{B_4}$. The final statement follows because the spinorial eigenvalues $t_i^{\pm 1/2}$ are, by the proof of Proposition 9, exactly

the eigenvalues of g_e occurring on summand **16** of $V_{26}|_{K_I}$; by Corollary 1 the corresponding sub-bundle of $E(V_{26})$ is therefore, fibrewise, a copy of **16** $\cong \Delta_9$. $\square$

Remark 18. No analogous statement holds for type (I): the isotropy representation $\mathfrak{m}_I \cong \Lambda_0^2(\mathbb{C}^6) \otimes \mathbb{C}^2$ of $(F_4^{(4)}, K_I)$ has dimension $14 \times 2 = 28 = \dim_{\mathbb{C}} F_4 - \dim_{\mathbb{C}} K_I$ (Remark 1), whereas $V_{26}|_{K_I} \cong \Lambda_0^2(\mathbb{C}^6) \oplus (\mathbb{C}^6 \otimes \mathbb{C}^2)$ (Proposition 10) has dimension $14 + 12 = 26$. The two share the constituent $\Lambda_0^2(\mathbb{C}^6)$ but are not isomorphic, one being a direct sum and the other a tensor product of, in part, different constituents. The coincidence of Proposition 15 is thus special to the spinorial nature of the 16-dimensional $\text{Spin}(9, \mathbb{C})$ representations, and does not extend to type (I).

7.2. The Magic Square and V_{26} in Supergravity

In the physics literature on $N = 2$ Maxwell–Einstein supergravity in five spacetime dimensions, the exceptional (or octonionic) magic supergravity of Günaydin, Sierra, and Townsend [36,37] is the theory whose 26 vector-multiplet scalar fields parametrise the real form of V_{26} underlying $\mathfrak{J}_3(\mathbb{O})_0$, with global symmetry group $E_6^{(-26)}$ and the scalar manifold rank-two symmetric space $E_6^{(-26)}/F_4$, and the isotropy group F_4 (compact form) acting on the tangent space at the origin through the fundamental 26-dimensional representation. This is the same representation V_{26} indexing the eigenspace decomposition of Theorem 1, and the eigenvalue pattern of $g_e \in F_4(\mathbb{C})$ used throughout the paper coincides with the branching of this isotropy representation under $Z_{F_4}(g_e)$. This observation does not yield a new physical statement, but it identifies the pair $(F_4(\mathbb{C}), V_{26})$ studied here as the same pair governing the geometry of the exceptional magic supergravity, offering a further context for Theorems 1–3.

7.3. Hitchin Systems and the Langlands Programme

The root system of F_4 is, together with that of G_2 , self-dual among the non-simply laced types: exchanging long and short roots carries the F_4 root system to itself (up to rescaling), so the Langlands dual group of $F_4(\mathbb{C})$ is again $F_4(\mathbb{C})$ [31,32]. Donagi and Pantev proved that, for a simple complex group G , the Hitchin system attached to G and the Hitchin system attached to its Langlands dual group ${}^L G$ are dual to one another, in the sense of a fibrewise duality between the associated families of abelian varieties (the Hitchin fibres), extending to a duality of gerbes over the singular fibres [38]. For $G = F_4(\mathbb{C})$, self-duality of the root system means the $F_4(\mathbb{C})$ -Hitchin system is, in this sense, self-dual. The Higgs bundle analogue of the present theory, already indicated as a direction for further research in Section 8, would consist of Galois triples (E, f, ω, Φ) with $\Phi \in H^0(X, E(\mathfrak{f}_4) \otimes K_X)$ compatible with (6) and (7); the associated spectral data, and its behaviour under the eigenspace decomposition of $E(V_{26})$ established here, would be a natural starting point for relating Theorem 1 to the self-duality of the $F_4(\mathbb{C})$ -Hitchin system, connecting the present results to the geometric Langlands programme [38] and to the orthogonal Higgs bundle correspondences of [23].

8. Conclusions

This paper has established a two-level decomposition theory for Galois $F_4(\mathbb{C})$ -bundles over a compact Riemann surface, unifying in a single framework the general and involution-specialised aspects of the theory.

At the general level, Theorem 1 proves that any Galois triple (E, f, ω) is strictly polystable and that the associated bundle $E(V_{26})$ decomposes into k eigenspace sub-bundles fixed by σ_X^* , where $2 \leq k \leq 25$ is determined by the eigenvalue pattern of the semisimple element $g_e \in F_4(\mathbb{C})$ (Definition 2). The weight system of V_{26} —the 24 short roots of $F_4(\mathbb{C})$ together with the zero weight of multiplicity 2 (Proposition 5)—is the foundation of the

entire eigenvalue analysis. For a torus representative (t_1, t_2, t_3, t_4) , the eigenvalues fall into three families: the vector eigenvalues $t_i^{\pm 1}$, the spinorial eigenvalues $t_1^{\pm 1/2} t_2^{\pm 1/2} t_3^{\pm 1/2} t_4^{\pm 1/2}$, and eigenvalue 1 of multiplicity 2 (Proposition 6). The strict polystability is a consequence of the reduction of the structure group to the centraliser $Z_{F_4}(g_e)$, which is always the Levi factor of a proper parabolic subgroup of $F_4(\mathbb{C})$ when g_e is nontrivial (Proposition 7), and of Ramanathan's polystability criterion [1,5] (Lemma 2).

The geometric analysis is completed by the structural corollaries of Section 5. Specifically, Corollary 1 makes explicit that $\tilde{\omega}$ acts on each summand $\mathcal{E}_j$ as scalar multiplication by λ_j , so the decomposition of Theorem 1 is precisely the eigenspace decomposition of $E(V_{26})$ with respect to the induced automorphism. Corollary 2 establishes that the decomposition depends only on the conjugacy class $[g_e]$, not on the choice of representative, confirming that the eigenbundle structure is an intrinsic invariant of the Galois triple. These two facts combine in Corollary 3 to yield a formal decomposition $\mathcal{G} = \bigsqcup_{[g_e]} \mathcal{G}_{[g_e]}$ of the set of Galois triples into pairwise disjoint strata indexed by nontrivial semisimple conjugacy classes of $F_4(\mathbb{C})$.

At the involution-specialised level, the two involution types of $F_4(\mathbb{C})$ —arising from the split real form $F_4^{(4)}$ with fixed-point subgroup $K_I = (\mathrm{Sp}(6, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})) / \mu_2$ and the form $F_4^{(-20)}$ with fixed-point subgroup $K_{II} = \mathrm{Spin}(9, \mathbb{C})$ (Proposition 2)—give rise to specialised decompositions that reflect the algebraic structure of the respective fixed-point subgroups (Theorems 2 and 3). Type (I) eigenvalues are products of symplectic and $\mathrm{SL}(2)$ eigenvalues (Proposition 11), reflecting the product structure of K_I and the decomposition $V_{26}|_{K_I} \cong \Lambda_0^2(\mathbb{C}^6) \oplus (\mathbb{C}^6 \otimes \mathbb{C}^2)$ (Proposition 10); type (II) eigenvalues include the spinorial values $t_i^{\pm 1/2}$ (Proposition 9), reflecting the non-product geometry of K_{II} and the decomposition $V_{26}|_{K_{II}} \cong \mathbf{9} \oplus \mathbf{16} \oplus \mathbf{1}$ (Proposition 8). Both involution types arise as restrictions of Cartan involutions of $E_6(\mathbb{C})$ (Proposition 3), type (I) from E_6^2 and type (II) from $E_6^{(-14)}$, connecting the specialised results to the geometry of the fixed-point subvariety of $M(E_6(\mathbb{C}))$ under the outer involution.

The descent result of Proposition 13 clarifies the geometric role of the cocycle condition (6): it is precisely the effectivity condition for Grothendieck descent of E along $\pi: X \rightarrow Y = X / \langle \sigma_X \rangle$, and Lemma 3 establishes that it implies $f_{x_0}^2 = \mathrm{id}_{E_{x_0}}$ at every ramification point $x_0 \in X^{\sigma_X}$ as an automatic consequence, with no additional hypothesis on the Galois structure at the branch locus. The descended bundle E_Y is polystable, inherits a nontrivial finite-order automorphism ω_Y with semisimple element conjugate to g_e , and—when (E, f, ω) is K_θ restricted—satisfies the conclusions of the specialised theorems over Y (Proposition 14). As clarified in Remark 11, the descended pair (E_Y, ω_Y) does not carry a Galois structure over Y in general, since the isomorphism f is consumed by the descent process.

The results of this paper shed light on the internal geometry of the $F_4(\mathbb{C})$ stratum of $M(E_6(\mathbb{C}))^\sigma$. Since the morphism $\iota_*: M(F_4(\mathbb{C})) \rightarrow M(E_6(\mathbb{C}))$ is a closed embedding whose image is precisely this stratum [14], the Galois triples studied here correspond, via ι_* , to a geometrically distinguished locus within $M(E_6(\mathbb{C}))^{\sigma, \sigma_X}$ (Proposition 4): the subvariety of $E_6(\mathbb{C})$ -bundles that lie in the $F_4(\mathbb{C})$ stratum are fixed by σ_X^* and admit a finite-order automorphism compatible with the Galois structure. Within this locus, the formal stratification $\mathcal{G} = \bigsqcup_{[g_e]} \mathcal{G}_{[g_e]}$ of Corollary 3 specialises to a stratification of the distinguished sublocus of $M(E_6(\mathbb{C}))^{\sigma, \sigma_X}$ by the eigenvalue pattern of g_e (Definition 2): the generic stratum ($k = 25$, with $Z_{F_4}(g_e)$ equal to the maximal torus) is open and dense, the involutory stratum ($k = 2$) is the most degenerate, and the intermediate strata are indexed by specific Levi subgroups $Z_{F_4}(g_e)$ (Proposition 7).

The strict polystability of every bundle in this locus (Theorem 1(a)) shows that these bundles lie entirely outside the stable locus of $M(E_6(\mathbb{C}))^\sigma$, providing a concrete description

of part of the boundary of the $F_4(\mathbb{C})$ stratum. At the involution-specialised level, the two subloci corresponding to $g_e \in K_I$ and $g_e \in K_{II}$ are further distinguished by the algebraic structure of their decompositions (Theorems 2 and 3): the K_I sublocus carries decompositions of $E(V_{26})$ governed by the restriction $V_{26}|_{K_I} \cong \Lambda_0^2(\mathbb{C}^6) \oplus (\mathbb{C}^6 \otimes \mathbb{C}^2)$ (Proposition 10), while the K_{II} sublocus carries decompositions involving spinorial eigenvalues that reflect the geometry of the symmetric space $F_4^{(-20)}/\text{Spin}(9)$ (Remark 1), embedded in the $F_4(\mathbb{C})$ stratum via the restriction of the Cartan involution of $E_6^{(-14)}$ (Proposition 3(a)).

Finally, Corollary 4 gives a geometric realisation of the decomposition within $E_6(\mathbb{C})$. Specifically, the adjoint bundle of the induced bundle $\iota_*(E)$ splits as

$$\text{Ad}(\iota_*(E)) = E(\mathfrak{f}_4) \oplus \bigoplus_j \mathcal{E}_j,$$

where $E(\mathfrak{f}_4)$ is the $d\sigma$ -fixed part and $\bigoplus_j \mathcal{E}_j$ is the $d\sigma$ -anti-fixed part of $\text{Ad}(\iota_*(E))$, with each summand preserved by σ_X^* . This makes explicit how the $E_6(\mathbb{C})$ -adjoint bundle of a Galois triple in the $F_4(\mathbb{C})$ stratum splits under the combined action of σ and σ_X^* .

Several natural directions for further investigation arise from this work.

The first is the extension of the programme to the other stratum of the fixed-point subvariety of $M(E_6(\mathbb{C}))$ under the outer involution σ , which consists of principal $\text{PSp}(8, \mathbb{C})$ -bundles [18]. A parallel treatment of Galois triples for $\text{PSp}(8, \mathbb{C})$ would complete the two-strata description and provide the full analogue, in the $E_6(\mathbb{C})$ -setting, of the combined $G_2(\mathbb{C})$ and $\text{PSL}(3, \mathbb{C})$ theory.

A second direction is the study of finite-order automorphisms of $F_4(\mathbb{C})$ of order $r \geq 3$, classified by the Kac diagram of $\mathfrak{f}_4$ [25]. The Kac classification provides the fixed-point subgroups and their representations in V_{26} for each order, and the resulting eigenvalue patterns would give rise to a richer family of decompositions than those treated in Theorems 1–3.

A third direction is the Higgs bundle analogue. For principal $F_4(\mathbb{C})$ -Higgs bundles [12], a Galois triple would include a Higgs field $\Phi \in H^0(X, E(\mathfrak{f}_4) \otimes K_X)$ compatible with the Galois structure (6) and the commutativity condition (7). The resulting constraints on Φ and on the associated spectral data in V_{26} would connect the present work to the geometry of the Hitchin fibration for $F_4(\mathbb{C})$.

Finally, when σ_X is an antiholomorphic involution of X , extending the definition of a Galois triple by replacing σ_X^*E with the conjugate bundle $\sigma_X^*\bar{E}$ in the cocycle condition (6) yields real and pseudo-real principal $F_4(\mathbb{C})$ -bundles in the sense of Biswas, García-Prada, and Hurtubise [24]. Making precise the relationship between the decompositions of $E(V_{26})$ established in Theorems 1–3 and the real structures in this sense would clarify the geometric role of the spinorial eigenvalues in the context of real algebraic curves.

Author Contributions: Conceptualisation, Á.A.-S. and S.R.Y.; Methodology, Á.A.-S. and S.R.Y.; Investigation, Á.A.-S. and S.R.Y.; Formal analysis, Á.A.-S. and S.R.Y.; Writing—original draft, Á.A.-S. and S.R.Y.; Writing—review and editing, Á.A.-S. and S.R.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Ramanathan, A. Stable principal bundles on a compact Riemann surface. *Math. Ann.* **1975**, *213*, 129–152. <https://doi.org/10.1007/bf01343949>.
2. Ramanathan, A. Moduli for principal bundles over algebraic curves I. *Proc. Indian Acad. Sci. (Math. Sci.)* **1996**, *106*, 301–328. <https://doi.org/10.1007/bf02867438>.
3. Ramanathan, A. Moduli for principal bundles over algebraic curves II. *Proc. Indian Acad. Sci. (Math. Sci.)* **1996**, *106*, 421–449. <https://doi.org/10.1007/bf02837697>.
4. Narasimhan, M.S.; Seshadri, C.S. Stable and unitary bundles on a compact Riemann surface. *Ann. Math.* **1965**, *82*, 540–567. <https://doi.org/10.2307/1970710>.
5. Ramanan, S.; Ramanathan, A. Some remarks on the instability flag. *Tohoku Math. J.* **1984**, *36*, 269–291. <https://doi.org/10.2748/tmj/1178228852>.
6. Ramanathan, A.; Subramanian, S. Einstein-Hermitian connections on principal bundles and stability. *J. Reine Angew. Math.* **1988**, *390*, 21–31. <https://doi.org/10.1515/crll.1988.390.21>.
7. Kouvidakis, A.; Pantev, T. The automorphism group of the moduli space of semistable vector bundles. *Math. Ann.* **1995**, *302*, 225–268. <https://doi.org/10.1007/bf01444495>.
8. Biswas, I.; Gómez, T.L.; Muñoz, V. Automorphisms of moduli spaces of symplectic bundles. *Internat. J. Math.* **2012**, *23*, 1250052. <https://doi.org/10.1142/S0129167X12500528>.
9. Biswas, I.; Gómez, T.L.; Muñoz, V. Automorphisms of moduli spaces of vector bundles over a curve. *Expo. Math.* **2013**, *31*, 73–86. <https://doi.org/10.1016/j.exmath.2012.08.002>.
10. Serman, O. Moduli spaces of orthogonal and symplectic bundles over an algebraic curve. *Compos. Math.* **2008**, *144*, 721–733. <https://doi.org/10.1112/S0010437X07003247>.
11. Fringuelli, R. Automorphisms of moduli spaces of principal bundles over a smooth curve. *Internat. J. Math.* **2024**, *35*, 2450036. <https://doi.org/10.1142/S0129167X24500368>.
12. García-Prada, O.; Ramanan, S. Involutions and higher order automorphisms of Higgs bundle moduli spaces. *Proc. Lond. Math. Soc.* **2019**, *119*, 681–732. <https://doi.org/10.1112/plms.12242>.
13. Antón-Sancho, Á. Galois E_6 -bundles over a hyperelliptic algebraic curve. *Bull. Iran. Math. Soc.* **2023**, *49*, 46. <https://doi.org/10.1007/s41980-023-00785-5>.
14. Antón-Sancho, Á. The moduli space of E_6 -Higgs bundles over a curve and subvarieties. *J. Geom. Anal.* **2025**, *35*, 280. <https://doi.org/10.1007/s12220-025-02103-3>.
15. Antón-Sancho, Á. Dimensions and symmetry actions in the deformation theory of principal bundles over Riemann surfaces. *Mediterr. J. Math.* **2026**, *23*, 139. <https://doi.org/10.1007/s00009-026-03139-7>.
16. Biswas, I.; Parameswaran, A.J. On the equivariant reduction of structure group of a principal bundle to a Levi subgroup. *J. Math. Pures Appl.* **2006**, *85*, 54–70. <https://doi.org/10.1016/j.matpur.2005.10.007>.
17. Oxbury, W.; Ramanan, S. Trigonal curves and Galois $\text{Spin}(8)$ -bundles. *arXiv* **1999**, arXiv:math/9907114.
18. Antón-Sancho, Á. F_4 and $\text{PSp}(8, \mathbb{C})$ -Higgs pairs understood as fixed points of the moduli space of E_6 -Higgs bundles over a compact Riemann surface. *Open Math.* **2022**, *20*, 1723–1733. <https://doi.org/10.1515/math-2022-0543>.
19. Adams, J.F. *Lectures on Exceptional Lie Groups*; University of Chicago Press: Chicago, IL, USA, 1996.
20. Helgason, S. *Differential Geometry, Lie Groups, and Symmetric Spaces*; Graduate Studies in Mathematics; American Mathematical Society: Providence, RI, USA, 2001; Volume 34.
21. Bernardoni, F.; Cacciatori, S.L.; Cerchiai, B.L.; Scotti, A. Mapping the geometry of the F_4 group. *Adv. Theor. Math. Phys.* **2008**, *12*, 889–994. <https://dx.doi.org/10.4310/ATMP.2008.v12.n4.a6>.
22. Franco, E.; García-Prada, O.; Newstead, P.E. Higgs bundles over elliptic curves for real groups. *Q. J. Math.* **2019**, *70*, 105–146. <https://doi.org/10.1093/qmath/hay037>.
23. Baraglia, D.; Schaposnick, L. Cayley and Langlands type correspondences for orthogonal Higgs bundles. *Trans. Am. Math. Soc.* **2019**, *371*, 7451–7492. <https://doi.org/10.1090/tran/7587>.
24. Biswas, I.; García-Prada, O.; Hurtubise, J. Pseudo-real principal G -bundles over a real curve. *J. Lond. Math. Soc.* **2016**, *93*, 47–64. <https://doi.org/10.1112/jlms/jdv055>.
25. Onishchik, A.L.; Vinberg, È.B. *Lie Groups and Algebraic Groups*; Springer Series in Soviet Mathematics; Springer: Berlin/Heidelberg, Germany, 1990.
26. Besse, A.L. *Einstein Manifolds*; Classics in Mathematics; Springer: Berlin/Heidelberg, Germany, 1987. <https://doi.org/10.1007/978-3-540-74311-8>.
27. Springer, T.A.; Veldkamp, F.D. *Octonions, Jordan Algebras and Exceptional Groups*; Springer Monographs in Mathematics; Springer: Berlin/Heidelberg, Germany, 2000. <https://doi.org/10.1007/978-3-662-12622-6>.
28. Antón-Sancho, Á. Automorphisms of the moduli space of principal G -bundles induced by outer automorphisms of G . *Math. Scand.* **2018**, *122*, 53–83. <https://doi.org/10.7146/math.scand.a-26348>.

29. Demazure, M.; Grothendieck, A. *Schémas en Groupes I: Propriétés Générales des Schémas en Groupes (SGA 3, Tome I)*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1970; Volume 151.
30. Serre, J.-P. *Galois Cohomology*; Springer Monographs in Mathematics; Springer: Berlin/Heidelberg, Germany, 1997. <https://doi.org/10.1007/978-3-642-59141-9>.
31. Bourbaki, N. *Lie Groups and Lie Algebras. Chapters 4–6*; Elements of Mathematics; Springer: Berlin/Heidelberg, Germany, 2002.
32. Humphreys, J.E. *Linear Algebraic Groups*; Graduate Texts in Mathematics; Springer, New York, NY, USA, 1975; Volume 21. <https://doi.org/10.1007/978-1-4684-9443-3>.
33. Steinberg, R. *Endomorphisms of Algebraic Groups*; Memoirs of the American Mathematical Society; American Mathematical Society: Providence, RI, USA, 1968; Volume 80.
34. Humphreys, J.E. *Introduction to Lie Algebras and Representation Theory*; Graduate Texts in Mathematics; Springer: New York, NY, USA, 1972; Volume 9. <https://doi.org/10.1007/978-1-4612-6398-2>.
35. Balaji, V.; Seshadri, C.S. Semistable principal bundles—I (characteristic zero). *J. Algebra* **2002**, *258*, 321–347. [https://doi.org/10.1016/S0021-8693\(02\)00502-1](https://doi.org/10.1016/S0021-8693(02)00502-1).
36. Günaydin, M.; Sierra, G.; Townsend, P.K. Exceptional supergravity theories and the magic square. *Phys. Lett. B* **1983**, *133*, 72–76. [https://doi.org/10.1016/0370-2693\(83\)90108-9](https://doi.org/10.1016/0370-2693(83)90108-9).
37. Günaydin, M.; Sierra, G.; Townsend, P.K. The geometry of $N = 2$ Maxwell-Einstein supergravity and Jordan algebras. *Nucl. Phys. B* **1984**, *242*, 244–268. [https://doi.org/10.1016/0550-3213\(84\)90142-1](https://doi.org/10.1016/0550-3213(84)90142-1).
38. Donagi, R.; Pantev, T. Langlands duality for Hitchin systems. *Invent. Math.* **2012**, *189*, 653–735. <https://doi.org/10.1007/s00222-012-0373-8>.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.