

Article

Fractional Geometry of Zeros for Terminal-Anchored Riemann–Liouville and Caputo Derivatives: A Fractional Gauss–Lucas Theory

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Abstract

We develop a terminal-explicit geometric theory for the algebraic zero sets associated with the Riemann–Liouville and Caputo fractional derivatives of a complex polynomial. Expanding the polynomial about an arbitrary complex terminal a reduces both operators to gamma-weighted polynomial transforms. The coordinate shift $w = z - a$ is algebraically equivalent to a zero-terminal formulation; the terminal dependence re-enters through the shifted coefficients and through the pullback of the resulting geometry to the original z -plane. We derive exact barycenter identities, terminal-centered disk bounds, multiset convergence at the endpoint orders, first-order deformation formulas for simple roots, and solvable examples. The disk bounds provide a star-shaped localization framework rather than a convex-hull theorem. A regular-polygon family yields an exact homothety for the Riemann–Liouville roots, while numerical examples show curved generic trajectories and a nondegenerate Caputo evolution. The terminal therefore acts as a distinguished geometric anchor, with radial attraction occurring under additional algebraic symmetry.

Keywords: Gauss–Lucas theorem; fractional derivative; Riemann–Liouville derivative; Caputo derivative; star-shaped containment; complex fractional calculus

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1. Introduction

The classical Gauss–Lucas theorem, originating with Lucas and developed in modern geometric form by Marden, asserts that the critical points of a nonconstant complex polynomial lie in the convex hull of its zeros [1,2]. The theorem remains a central organizing principle in the geometry of polynomials; complementary modern perspectives appear in Sendov’s monograph and Baez’s exposition [3,4]. It also belongs to the broader study of zero-preserving and stability-preserving polynomial operators [5].

Fractional calculus introduces a distinguished terminal into differentiation. Foundational accounts may be found in the books of Oldham and Spanier [6] and Miller and Ross [7]. Systematic analytic treatments are given by Samko, Kilbas, and Marichev [8], Podlubny [9], Kilbas, Srivastava, and Trujillo [10], and Diethelm [11]; Caputo’s original paper remains a standard source for the Caputo derivative in applications [12]. For analytic functions, formulas of Osler and of Lavoie, Osler, and Tremblay display the gamma-factor

structure of fractional differentiation [13,14]. Recent work has developed complex-domain, computational, distributional, and operator-theoretic viewpoints [15–18].

Caparatta, Pauli, and Saidak provided the principal catalyst for the present study by investigating the locations and asymptotic behavior of zeros of fractional derivatives of polynomials and by obtaining Mahler-measure bounds [19]. Their analysis establishes the richness of fractional root trajectories in the standard zero-terminal setting. The present work focuses on a complementary geometric layer: terminal-explicit barycenter identities, coefficient-controlled star-shaped localization, endpoint perturbation formulas, and exact terminal-centered homotheties. This distinction is now stated explicitly because the root-trajectory problem itself originates in that earlier work.

For a fixed terminal a , the substitution $w = z - a$ converts the operator to a zero-terminal transform of the shifted polynomial $F(w) = P(a + w)$. Thus, the coordinate substitution itself carries no independent novelty. The geometric content arises because the coefficients of F depend on the position of a relative to the original zeros, and the resulting root information must be translated back to the original z -plane. This produces formulas involving a , the original barycenter, and the terminal radius $R_a(P) = \max_j |\zeta_j - a|$. The construction is covariant under a simultaneous translation of the polynomial and its terminal, while a fixed terminal selects a preferred origin for the coefficient transform.

The connection with Gauss–Lucas is therefore formulated at the level of a research program. The classical theorem supplies convex-hull containment for the ordinary derivative. The fractional transforms studied here yield terminal-centered coefficient bounds and, in special symmetric families, exact homothetic motion. The general disk bound is a star-shaped localization result; it carries a different logical status from the classical convex-hull theorem. Individual zero trajectories can curve, and their distances from the terminal can vary nonmonotonically.

Our algebraic starting point is the expansion

$$P(a + w) = \sum_{k=0}^n c_k w^k.$$

The Riemann–Liouville and Caputo derivatives are governed by ordinary polynomial factors whose coefficients are multiplied by

$$\lambda_k(\alpha) := \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}.$$

Vieta’s formulas, symmetric-function estimates, classical root bounds, Rouché’s theorem, and the analytic implicit-function theorem then give finite-dimensional information about the roots.

The results have a direct interpretation for polynomial characteristic equations and finite-dimensional spectral reductions. In a fractional evolution model, a Laplace transform, modal truncation, or spatial discretization may produce a polynomial factor whose roots encode decay rates or stability margins. The barycenter law then measures the aggregate displacement of those spectral roots, while the deformation formulas approximate individual simple modes. Caputo models of anomalous drug transport illustrate the broader role of memory and nonlocal boundary data in biological media [20]. A spatial center-of-mass interpretation requires an explicit model-dependent map from the physical state to a polynomial characteristic root. Fixed-point methods address existence, uniqueness and stability in function spaces [21]; the present zero geometry supplies spectral localization when a finite polynomial characteristic factor is also available. Delay equations generally lead to quasipolynomials, so their direct treatment requires an extension beyond the polynomial class developed here.

The main results are an exact polynomial reduction with a precise branch and path prescription; a terminal-point classification covering $P(a) = 0$ and $n = 1$; barycenter identities for the algebraic Riemann–Liouville and Caputo root multisets; explicit terminal-centered disk bounds; endpoint convergence with multiplicities; first-order formulas for simple branches near $\alpha = 0$ and $\alpha = 1$; numerical trajectory plots; and an exact regular-polygon model. Section 2 fixes the complex-domain conventions and derives the polynomial factors. Section 3 develops the barycenter and containment results. Section 4 treats endpoint convergence and local deformations. Section 5 gives numerical illustrations, and Section 6 presents the exact symmetric model.

2. Preliminaries and Exact Reduction

We work with a polynomial

$$P(z) = \sum_{m=0}^n a_m z^m, \quad a_n \neq 0,$$

of degree $n \geq 1$, and fix a terminal $a \in \mathbb{C}$ and an order $0 < \alpha < 1$.

Choose an angle $\theta \in \mathbb{R}$ and the slit domain

$$\Omega_\theta := a + \{re^{i\phi} : r > 0, \theta < \phi < \theta + 2\pi\},$$

with the branch $\text{Log}_\theta(z - a)$. For $z = a + w \in \Omega_\theta$, the integrals below are taken on the radial segment $t = a + sw$, $0 \leq s \leq 1$, and every fractional power is evaluated with this branch. The path prescription is part of the definition. We set

$$D_a^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_a^z \frac{f(t)}{(z-t)^\alpha} dt, \quad (1)$$

and

$$C D_a^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \int_a^z \frac{f'(t)}{(z-t)^\alpha} dt. \quad (2)$$

For polynomial data, evaluation along the prescribed segment gives the finite expansions below. Replacing the logarithm branch multiplies the complete expression by a common nonzero phase, so the polynomial-factor root multisets are branch-independent.

Set

$$F(w) := P(a + w) = \sum_{k=0}^n c_k w^k, \quad c_k = \frac{P^{(k)}(a)}{k!}.$$

If $\zeta_1, \dots, \zeta_n$ are the zeros of P , counted with multiplicity, then

$$\tilde{\zeta}_j := \zeta_j - a, \quad F(w) = c_n \prod_{j=1}^n (w - \tilde{\zeta}_j).$$

The monomial formulas are

$$D_0^\alpha w^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} w^{k-\alpha} \quad (k \geq 0), \quad (3)$$

and

$$C D_0^\alpha w^k = \begin{cases} 0, & k = 0, \\ \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} w^{k-\alpha}, & k \geq 1. \end{cases} \quad (4)$$

We write

$$\lambda_k(\alpha) := \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}, \quad k \in \mathbb{N} \cup \{0\}. \quad (5)$$

For a polynomial Q , $\mathcal{Z}(Q)$ denotes its multiset of complex roots, counted with algebraic multiplicity. For a scalar b , the notation $b + \mathcal{Z}(Q)$ means the translated multiset $\{b + q : q \in \mathcal{Z}(Q)\}$.

Lemma 1 (Exact polynomial reduction). *Let $F(w) = P(a + w) = \sum_{k=0}^n c_k w^k$. Then,*

$$D_a^\alpha P(a + w) = w^{-\alpha} \mathcal{R}_\alpha(w), \quad \mathcal{R}_\alpha(w) := \sum_{k=0}^n c_k \lambda_k(\alpha) w^k, \quad (6)$$

and

$${}^C D_a^\alpha P(a + w) = w^{1-\alpha} \mathcal{C}_\alpha(w), \quad \mathcal{C}_\alpha(w) := \sum_{k=1}^n c_k \lambda_k(\alpha) w^{k-1}. \quad (7)$$

The algebraic Riemann–Liouville and Caputo root multisets are

$$\mathcal{Z}_{\text{alg}}^{\text{RL}}(P; a, \alpha) := a + \mathcal{Z}(\mathcal{R}_\alpha), \quad \mathcal{Z}_{\text{alg}}^{\text{C}}(P; a, \alpha) := a + \mathcal{Z}(\mathcal{C}_\alpha). \quad (8)$$

Their elements distinct from a are precisely the zeros of the corresponding fractional derivatives in Ω_θ .

Proof. Insert the finite expansion of F into the radial integrals and apply (3) and (4) termwise. Factoring the common powers $w^{-\alpha}$ and $w^{1-\alpha}$ gives (6) and (7). These powers are nonzero on Ω_θ , which proves the final assertion. $\square$

Remark 1 (The terminal point and the degree-one case). *Let $q := \min\{k : c_k \neq 0\}$. For the Riemann–Liouville transform,*

$$\mathcal{R}_\alpha(w) = w^q \tilde{\mathcal{R}}_\alpha(w), \quad \tilde{\mathcal{R}}_\alpha(0) = c_q \lambda_q(\alpha) \neq 0.$$

When $P(a) \neq 0$, one has $q = 0$ and $D_a^\alpha P(a + w) \sim c_0 \lambda_0(\alpha) w^{-\alpha}$, so a is singular. When P has a zero of multiplicity $q \geq 1$ at a , the polynomial factor has an algebraic root of multiplicity q at a , and the fractional expression behaves as $w^{q-\alpha} \tilde{\mathcal{R}}_\alpha(w)$ and extends continuously to the value zero at a .

For the Caputo transform, let $r := \min\{k \geq 1 : c_k \neq 0\}$. Then,

$$\mathcal{C}_\alpha(w) = w^{r-1} \tilde{\mathcal{C}}_\alpha(w), \quad {}^C D_a^\alpha P(a + w) = w^{r-\alpha} \tilde{\mathcal{C}}_\alpha(w).$$

Thus, the polynomial factor has an algebraic terminal root of multiplicity $r - 1$, and the full Caputo expression extends continuously to zero at a . Since a is a branch point, multiplicities in this paper refer to the polynomial factors $\mathcal{R}_\alpha$ and $\mathcal{C}_\alpha$. For $n = 1$, $\mathcal{C}_\alpha$ is a nonzero constant; the Caputo algebraic root multiset is empty, while the continuous terminal value remains zero.

Lemma 2 (Gamma-weight identities). *For every $k \geq 0$,*

$$\frac{\lambda_{k+1}(\alpha)}{\lambda_k(\alpha)} = \frac{k+1}{k+1-\alpha} > 1. \quad (9)$$

For $1 \leq m \leq n$,

$$\frac{\lambda_{n-m}(\alpha)}{\lambda_n(\alpha)} = \prod_{j=0}^{m-1} \frac{n-j-\alpha}{n-j}, \quad (10)$$

and this ratio is strictly decreasing in $\alpha \in (0, 1)$.

Proof. The gamma recurrence gives (9). Iteration from n to $n - m$ gives (10). Every factor in the product is positive and strictly decreasing in α . $\square$

3. Terminal-Centered Geometry of Fractional Zeros

Theorem 1 (Exact barycenter law). Let $\mathfrak{c}(P) := n^{-1} \sum_{j=1}^n \zeta_j$. Write

$$\mathcal{Z}_{\text{alg}}^{\text{RL}}(P; a, \alpha) = \{\hat{z}_1^{\text{RL}}(\alpha), \dots, \hat{z}_n^{\text{RL}}(\alpha)\}.$$

Then,

$$\frac{1}{n} \sum_{j=1}^n \hat{z}_j^{\text{RL}}(\alpha) = a + \frac{n - \alpha}{n} (\mathfrak{c}(P) - a). \quad (11)$$

For $n \geq 2$, write

$$\mathcal{Z}_{\text{alg}}^{\text{C}}(P; a, \alpha) = \{\hat{z}_1^{\text{C}}(\alpha), \dots, \hat{z}_{n-1}^{\text{C}}(\alpha)\}.$$

Then,

$$\frac{1}{n-1} \sum_{j=1}^{n-1} \hat{z}_j^{\text{C}}(\alpha) = a + \frac{n - \alpha}{n-1} (\mathfrak{c}(P) - a). \quad (12)$$

Proof. The coefficient of w^{n-1} in F is

$$c_{n-1} = -c_n \sum_{j=1}^n \zeta_j. \quad (13)$$

Vieta's formula applied to $\mathcal{R}_\alpha$ gives

$$\sum_{j=1}^n (\hat{z}_j^{\text{RL}}(\alpha) - a) = -\frac{c_{n-1} \lambda_{n-1}(\alpha)}{c_n \lambda_n(\alpha)} = \frac{n - \alpha}{n} \sum_{j=1}^n (\zeta_j - a),$$

which yields (11). The leading two coefficients of $\mathcal{C}_\alpha$ are the same pair, while its degree is $n - 1$. Hence,

$$\sum_{j=1}^{n-1} (\hat{z}_j^{\text{C}}(\alpha) - a) = \frac{n - \alpha}{n} \sum_{j=1}^n (\zeta_j - a),$$

and division by $n - 1$ proves (12). $\square$

Remark 2 (Geometric meaning of the coefficients). The Riemann–Liouville factor

$$s_{\text{RL}}(\alpha) = \frac{n - \alpha}{n}$$

belongs to $((n - 1)/n, 1)$, so its algebraic-root barycenter lies on the segment from a to $\mathfrak{c}(P)$. At $\alpha = 0$, it equals $\mathfrak{c}(P)$; as $\alpha \rightarrow 1^-$, it tends to $a + (n - 1)(\mathfrak{c}(P) - a)/n$, consistent with the limiting polynomial factor $wF'(w)$.

The Caputo factor

$$s_{\text{C}}(\alpha) = \frac{n - \alpha}{n - 1}$$

belongs to $(1/n, (n - 1))$. Its barycenter lies on the ray from a through $\mathfrak{c}(P)$, beyond $\mathfrak{c}(P)$ for $0 < \alpha < 1$, and approaches $\mathfrak{c}(P)$ as $\alpha \rightarrow 1^-$. At $\alpha = 0$, it equals $a + n(\mathfrak{c}(P) - a)/(n - 1)$. The two coefficients share the numerator because Vieta's sum uses the same ratio $\lambda_{n-1}/\lambda_n = (n - \alpha)/n$. Their denominators differ because $\mathcal{R}_\alpha$ has n algebraic roots and $\mathcal{C}_\alpha$ has $n - 1$. Dependence on lower coefficients disappears because a root sum is determined solely by the leading and subleading coefficients.

The barycenter is consequently a robust first-moment statistic of the transformed spectral roots. In applications where a polynomial characteristic equation represents modes of a discretized fractional system, it gives an exact aggregate shift under variation of α . Higher moments and individual trajectories retain dependence on the remaining coefficients, which is why the barycenter law alone supplies no pointwise radial-motion principle.

Theorem 2 (Terminal-centered star-domain containment). *Let*

$$R_a(P) := \max_{1 \leq j \leq n} |\zeta_j - a|.$$

Define

$$M_{n,\alpha}^{\text{RL}} := \max_{1 \leq m \leq n} \left\{ \binom{n}{m} \prod_{j=0}^{m-1} \frac{n-j-\alpha}{n-j} \right\}^{1/m}, \quad (14)$$

and, for $n \geq 2$,

$$M_{n,\alpha}^{\text{C}} := \max_{1 \leq m \leq n-1} \left\{ \binom{n}{m} \prod_{j=0}^{m-1} \frac{n-j-\alpha}{n-j} \right\}^{1/m}. \quad (15)$$

Every $z \in \mathcal{Z}_{\text{alg}}^{\text{RL}}(P; a, \alpha)$ satisfies

$$|z - a| \leq 2R_a(P)M_{n,\alpha}^{\text{RL}}. \quad (16)$$

For $n \geq 2$, every $z \in \mathcal{Z}_{\text{alg}}^{\text{C}}(P; a, \alpha)$ satisfies

$$|z - a| \leq 2R_a(P)M_{n,\alpha}^{\text{C}}. \quad (17)$$

The factors $M_{n,\alpha}^{\text{RL}}$ and $M_{n,\alpha}^{\text{C}}$ are strictly decreasing in $\alpha \in (0, 1)$. The quoted radii are strictly decreasing when $R_a(P) > 0$ and are identically zero when $R_a(P) = 0$. The containing disks are star-shaped with respect to a .

Proof. The factorization of F gives

$$F(w) = c_n \sum_{m=0}^n (-1)^m e_m(\zeta_1, \dots, \zeta_n) w^{n-m},$$

where

$$e_0 := 1, \quad e_m(\zeta_1, \dots, \zeta_n) := \sum_{1 \leq j_1 < \dots < j_m \leq n} \zeta_{j_1} \cdots \zeta_{j_m} \quad (1 \leq m \leq n). \quad (18)$$

Thus,

$$\frac{c_{n-m}}{c_n} = (-1)^m e_m(\zeta_1, \dots, \zeta_n), \quad (19)$$

and

$$|e_m(\zeta_1, \dots, \zeta_n)| \leq \binom{n}{m} R_a(P)^m. \quad (20)$$

After division by $c_n \lambda_n(\alpha)$, the coefficient of w^{n-m} in the monic polynomial $\mathcal{R}_\alpha / (c_n \lambda_n)$ has modulus at most

$$\binom{n}{m} R_a(P)^m \prod_{j=0}^{m-1} \frac{n-j-\alpha}{n-j}.$$

Fujiwara's bound [22] states that every root of a monic polynomial $w^N + b_1 w^{N-1} + \dots + b_N$ satisfies

$$|w| \leq 2 \max_{1 \leq m \leq N} |b_m|^{1/m}.$$

This proves (16). Applying the same argument to the degree- $(n-1)$ polynomial $C_\alpha / (c_n \lambda_n)$ proves (17). Strict decrease of the M -factors follows from Lemma 2 and the maximum over finitely many positive strictly decreasing functions. Multiplication by $R_a(P)$ gives the stated radius behavior. $\square$

Remark 3 (Scope of the localization theorem). *Theorem 2 is a coefficient-based root localization theorem. The phrase fractional Gauss–Lucas refers here to the program of relating fractional roots to the original zero geometry and to the terminal. Monotonicity belongs to the enclosing radius. Individual trajectories may curve and may have nonmonotone distance from a ; Section 5 provides an explicit numerical instance.*

Remark 4 (Sharpness and alternative bounds). *Two sources of slack enter (16) and (17): the symmetric-function estimate (20) discards phase cancellation, and Fujiwara's theorem contributes its universal factor 2. For the regular-polygon family $F(w) = w^n - \beta^n$, the exact Riemann–Liouville radius is $\mu_{n,\alpha} |\beta|$ from Theorem 5, while the universal radius is $2|\beta| M_{n,\alpha}^{\text{RL}}$. Since the $m = 1$ term gives $M_{n,\alpha}^{\text{RL}} \geq n - \alpha$, the ratio of the quoted radius to the exact radius is at least $2(n - \alpha) / \mu_{n,\alpha}$ and becomes unbounded as $\alpha \rightarrow 1^-$. This family therefore gives a concrete overestimation example.*

Cauchy–Lagrange bounds, Pellet-type annular bounds, companion-matrix estimates, and bounds exploiting coefficient sparsity can be applied directly to $\mathcal{R}_\alpha$ and $\mathcal{C}_\alpha$; standard surveys appear in [2]. Their effectiveness depends on the shifted coefficient pattern. Constructing anisotropic or sectorial bounds adapted to the terminal-centered star geometry is a priority for further work.

4. Endpoint Convergence and Local Deformations

Proposition 1 (Endpoint convergence with multiplicity). *Define the degree- $(n-1)$ polynomial*

$$Q_0(w) := \sum_{k=1}^n c_k w^{k-1} = \frac{F(w) - F(0)}{w}, \quad (21)$$

where the quotient at $w = 0$ is understood by its polynomial continuation. Then, in the root-multiset sense,

$$\mathcal{Z}(\mathcal{R}_\alpha) \longrightarrow \mathcal{Z}(F), \quad \alpha \rightarrow 0^+, \quad (22)$$

$$\mathcal{Z}(\mathcal{C}_\alpha) \longrightarrow \mathcal{Z}(Q_0), \quad \alpha \rightarrow 0^+, \quad (23)$$

$$\mathcal{Z}(\mathcal{C}_\alpha) \longrightarrow \mathcal{Z}(F'), \quad \alpha \rightarrow 1^-. \quad (24)$$

More precisely, choose disjoint disks around the distinct roots of a limiting polynomial, with every boundary free of limiting roots. For α sufficiently close to the endpoint, each disk contains the same number of transformed roots as the multiplicity of the enclosed limiting root.

Proof. The coefficients satisfy $\lambda_k(\alpha) \rightarrow 1$ as $\alpha \rightarrow 0^+$ and $\lambda_k(\alpha) \rightarrow k$ for $k \geq 1$ as $\alpha \rightarrow 1^-$. Hence, $\mathcal{R}_\alpha \rightarrow F$, $\mathcal{C}_\alpha \rightarrow Q_0$, and $\mathcal{C}_\alpha \rightarrow F'$ locally uniformly. Rouché's theorem on the chosen disk boundaries preserves the number of roots counted with multiplicity. $\square$

The next formulas resolve individual branches when the limiting root is simple. Let $\psi = \Gamma' / \Gamma$ denote the digamma function [23], and define

$$G(w) := \sum_{k=0}^n c_k \psi(k+1) w^k, \quad H(w) := \sum_{k=1}^n c_k \psi(k+1) w^{k-1}. \quad (25)$$

Theorem 3 (Small-order deformation at simple roots). *If ξ is a simple root of F , there is an analytic root branch $\rho_\xi(\alpha)$ of $\mathcal{R}_\alpha$ with $\rho_\xi(0) = \xi$ and*

$$\rho_\xi(\alpha) = \xi - \alpha \frac{G(\xi)}{F'(\xi)} + O(\alpha^2). \quad (26)$$

If η is a simple root of Q_0 , there is an analytic root branch $\kappa_\eta(\alpha)$ of $\mathcal{C}_\alpha$ with $\kappa_\eta(0) = \eta$ and

$$\kappa_\eta(\alpha) = \eta - \alpha \frac{H(\eta)}{Q'_0(\eta)} + O(\alpha^2). \quad (27)$$

For $\eta \neq 0$, this becomes

$$\kappa_\eta(\alpha) = \eta - \alpha \eta \frac{H(\eta)}{F'(\eta)} + O(\alpha^2). \quad (28)$$

Proof. Differentiation of (5) gives

$$\lambda_k(\alpha) = 1 + \psi(k+1)\alpha + O(\alpha^2).$$

Consequently,

$$\mathcal{R}_\alpha(w) = F(w) + \alpha G(w) + O(\alpha^2), \quad \mathcal{C}_\alpha(w) = Q_0(w) + \alpha H(w) + O(\alpha^2),$$

locally uniformly. The analytic implicit-function theorem at a simple root gives the two branches, and differentiation of the root identities at $\alpha = 0$ yields (26) and (27). For $\eta \neq 0$, differentiation of $wQ_0(w) = F(w) - F(0)$ at a root of Q_0 gives $Q'_0(\eta) = F'(\eta)/\eta$. $\square$

Define

$$J(w) := \sum_{k=1}^n c_k k \psi(k) w^{k-1}. \quad (29)$$

Theorem 4 (Classical-order deformation for simple Caputo roots). *Let ω be a simple root of F' . There is an analytic root branch $\kappa_\omega(\alpha)$ of $\mathcal{C}_\alpha$ with $\kappa_\omega(1) = \omega$ and*

$$\kappa_\omega(\alpha) = \omega + (1 - \alpha) \frac{J(\omega)}{F''(\omega)} + O((1 - \alpha)^2). \quad (30)$$

Proof. With $\varepsilon = 1 - \alpha$,

$$\lambda_k(1 - \varepsilon) = k(1 - \varepsilon \psi(k) + O(\varepsilon^2)),$$

so

$$\mathcal{C}_{1-\varepsilon}(w) = F'(w) - \varepsilon J(w) + O(\varepsilon^2)$$

locally uniformly. The implicit-function theorem at $F'(\omega) = 0$ and $F''(\omega) \neq 0$ gives the branch. Substitution of $\kappa_\omega(1 - \varepsilon) = \omega + \delta(\varepsilon)$ gives

$$0 = \delta F''(\omega) - \varepsilon J(\omega) + O(\varepsilon^2 + \varepsilon \delta + \delta^2),$$

which yields (30). $\square$

Proposition 1 supplies convergence for multiple roots as clusters with preserved multiplicity. Theorems 3 and 4 add a first-order velocity only when a simple root selects a unique analytic branch. At $\alpha = 1$, the Caputo limiting roots are the classical critical points governed by Gauss–Lucas. At $\alpha = 0$, the Riemann–Liouville limiting roots are those of P , while the Caputo limiting roots solve the anchored level-set equation $P(z) = P(a)$ after removal of the trivial factor $z - a$.

5. Numerical Illustrations

The figures were generated from the polynomial factors (6) and (7) on a grid of 400 equally spaced orders in $[0.001, 0.999]$. Consecutive root sets were matched by minimum total Euclidean displacement. The accompanying script `generate_zero_trajectory_figures.py` reproduces both plots.

For a generic Riemann–Liouville example, take $a = 0$ and

$$F(w) = \prod_{\xi \in \Xi} (w - \xi), \quad \Xi = \{-1.96 + 0.27i, 2.08 - 0.65i, -1.35 + 1.65i, -2.31 + 1.54i\}. \quad (31)$$

Figure 1 displays curved trajectories. The branch issuing from $-1.96 + 0.27i$ has distance approximately 1.978 from the terminal at $\alpha = 0.001$, reaches approximately 1.902 near $\alpha = 0.114$, and has distance approximately 2.074 at $\alpha = 0.999$. This computation separates the monotone enclosing-radius statement from pointwise radial motion.

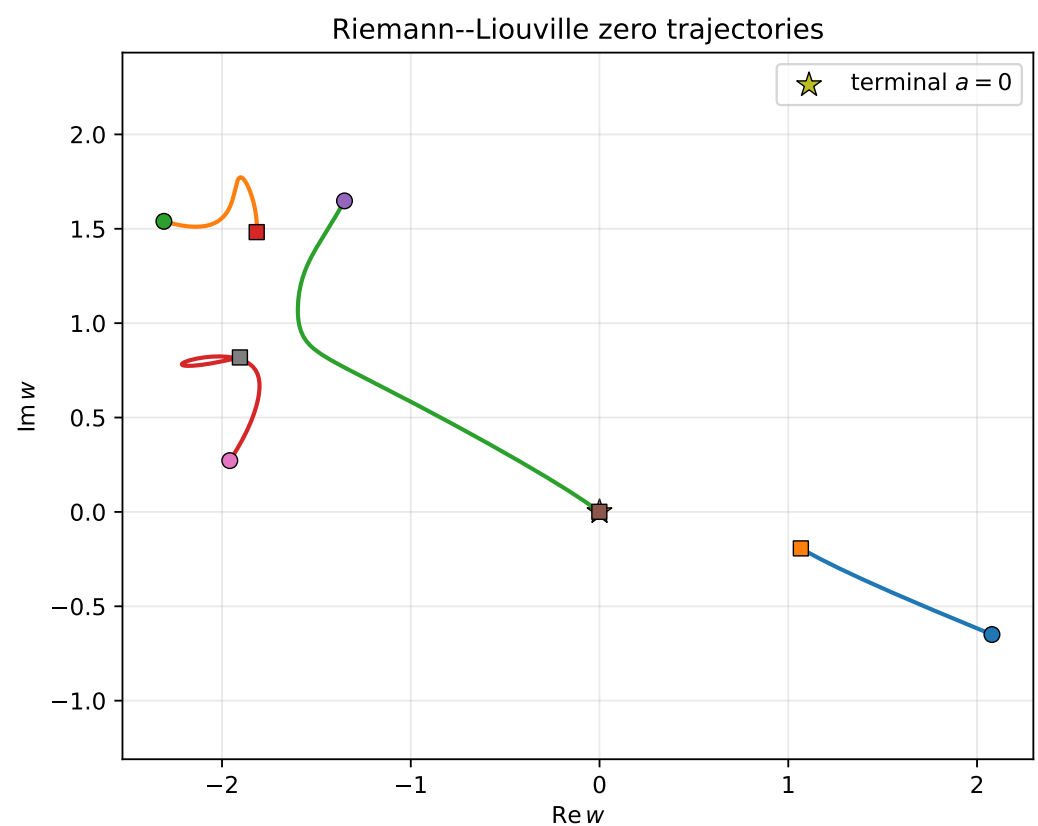


Figure 1. Riemann–Liouville algebraic-root trajectories for (31). Circles mark $\alpha = 0.001$, squares mark $\alpha = 0.999$, and the star marks the terminal. Curvature and local loops occur in the generic coefficient-coupled case.

A nondegenerate Caputo example is obtained from

$$F(w) = (w + 2)(w + \frac{1}{2})(w - 1)(w - 3) = w^4 - \frac{3}{2}w^3 - 6w^2 + \frac{7}{2}w + 3. \quad (32)$$

Its Caputo polynomial factor is

$$\mathcal{C}_\alpha(w) = \lambda_4(\alpha)w^3 - \frac{3}{2}\lambda_3(\alpha)w^2 - 6\lambda_2(\alpha)w + \frac{7}{2}\lambda_1(\alpha). \quad (33)$$

At $\alpha = 0.5$, the three roots are approximately -1.7701 , 0.4037 , and 2.6789 . As $\alpha \rightarrow 0^+$, they approach the three roots of Q_0 , approximately -2.1164 , 0.5370 , and 3.0794 ; as $\alpha \rightarrow 1^-$,

they approach the critical points of F , approximately -1.4205 , 0.2708 , and 2.2747 . Figure 2 illustrates the full interpolation.

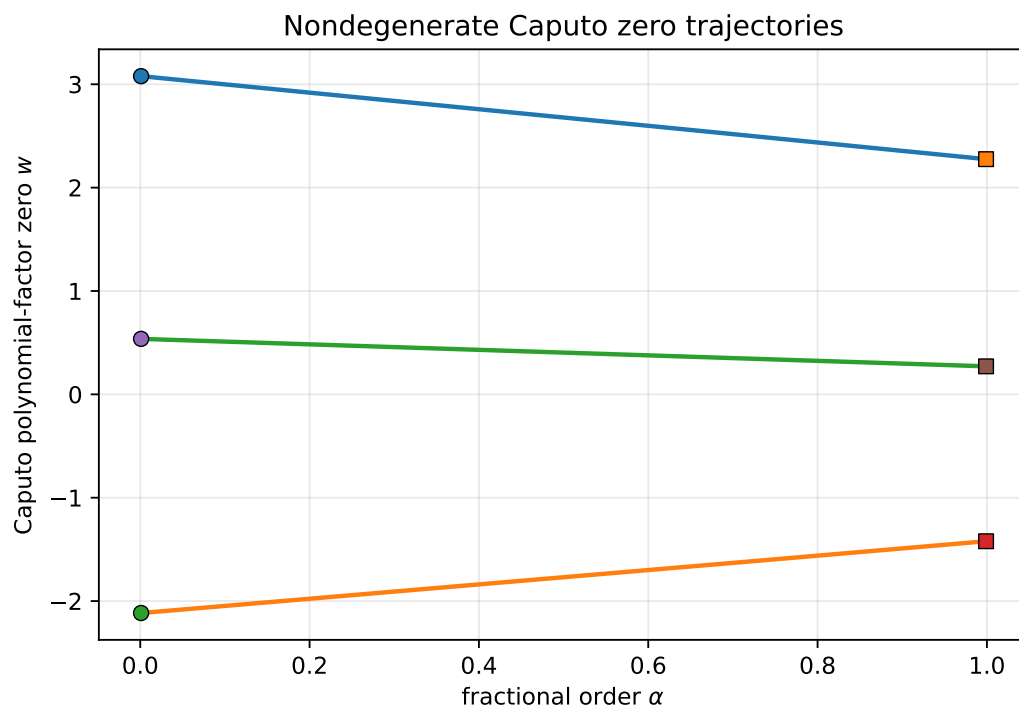


Figure 2. Three nonterminal Caputo algebraic-root trajectories for (32). Circles mark the small-order end and squares mark the near-classical end.

6. An Exact Model: Regular Polygons and Radial Motion

The preceding example shows that radial motion is a structural property requiring additional symmetry. The binomial family below has only two nonzero coefficients in terminal coordinates, so the transformed root equation remains binomial for every order.

Theorem 5 (Exact regular-polygon model). Fix $n \geq 2$, $a \in \mathbb{C}$ and $\beta \in \mathbb{C} \setminus \{0\}$, and let

$$P(z) = (z - a)^n - \beta^n. \quad (34)$$

The zeros of P are $\zeta_j = a + \beta e^{2\pi i j/n}$, $0 \leq j \leq n-1$. The Riemann–Liouville zeros in Ω_θ are

$$z_j^{\text{RL}}(\alpha) = a + \mu_{n,\alpha} \beta e^{2\pi i j/n}, \quad 0 \leq j \leq n-1, \quad (35)$$

where

$$\mu_{n,\alpha} := \left(\frac{\Gamma(n+1-\alpha)}{\Gamma(n+1)\Gamma(1-\alpha)} \right)^{1/n}. \quad (36)$$

The function $\mu_{n,\alpha}$ is strictly decreasing, takes values in $(0, 1)$, and satisfies

$$\lim_{\alpha \rightarrow 0^+} \mu_{n,\alpha} = 1, \quad \lim_{\alpha \rightarrow 1^-} \mu_{n,\alpha} = 0. \quad (37)$$

For the Caputo derivative,

$${}^{\text{C}}\mathcal{D}_a^\alpha P(a+w) = \lambda_n(\alpha) w^{n-\alpha}. \quad (38)$$

Hence, $\mathcal{C}_\alpha(w) = \lambda_n(\alpha) w^{n-1}$ has an algebraic terminal root of multiplicity $n-1$, the slit-domain derivative has no nonterminal root, and its continuous terminal value is zero.

Proof. Here, $F(w) = w^n - \beta^n$, so

$$\mathcal{R}_\alpha(w) = \lambda_n(\alpha)w^n - \lambda_0(\alpha)\beta^n.$$

Its roots satisfy

$$w^n = \frac{\lambda_0(\alpha)}{\lambda_n(\alpha)}\beta^n,$$

which gives (35). Moreover,

$$\mu_{n,\alpha}^n = \frac{\lambda_0(\alpha)}{\lambda_n(\alpha)} = \prod_{j=0}^{n-1} \frac{n-j-\alpha}{n-j}.$$

Every factor is positive and strictly decreasing, which proves the monotonicity and limits. The Caputo formula follows directly from $\mathcal{C}_\alpha(w) = \lambda_n(\alpha)w^{n-1}$. $\square$

Corollary 1 (Exact terminal-centered homothety). For (34),

$$\mathcal{Z}(\mathcal{D}_a^\alpha P) = a + \mu_{n,\alpha}(\mathcal{Z}(P) - a).$$

Thus, every Riemann–Liouville root follows the radial segment from its original position to the terminal.

Remark 5 (Extent of the radial mechanism). The exact matching in Corollary 1 comes from the two-term shifted expansion. A broader centered rotational class has the form $F(w) = w^r Q(w^q)$; gamma reweighting preserves this exponent pattern and therefore preserves the associated q -fold rotational symmetry. Multi-term coupling generally changes the radial coordinate and angle simultaneously.

When the original zeros lie on a line through a , a rotation can make the shifted coefficients real, and the transformed root multiset retains conjugation symmetry. Real-rootedness and monotone radial matching require additional hypotheses. A circle whose center differs from a produces a full shifted expansion, so the binomial homothety mechanism disappears. These observations suggest symmetry-preserving subclasses as a tractable route toward generalization.

Example 1. For $n = 2$ and $P(z) = (z - a)^2 - \beta^2$,

$$\mathcal{Z}(\mathcal{D}_a^\alpha P) = \left\{ a \pm \beta \sqrt{\frac{(1-\alpha)(2-\alpha)}{2}} \right\}.$$

The two roots remain on the original line and move monotonically toward a .

7. Conclusions

We have reduced terminal-anchored Riemann–Liouville and Caputo derivatives of a polynomial to gamma-weighted polynomial factors and developed geometric information from those factors.

The arbitrary-terminal formulation is algebraically conjugate to a zero-terminal formulation after $w = z - a$. Its value lies in terminal-explicit pullback formulas: the shifted coefficients encode the placement of a relative to the original zero set. The resulting disk theorem supplies star-shaped coefficient localization, while convex-hull containment remains the defining content of the classical Gauss–Lucas theorem. Numerical trajectories confirm that a decreasing universal radius can coexist with curved and locally outward individual motion.

The barycenter laws isolate a universal first moment controlled only by n , α , a , and $\mathfrak{c}(P)$. Their spectral interpretation becomes useful when a fractional model yields a polynomial characteristic factor through modal reduction or discretization. Fixed-point analysis and

zero localization can then support complementary stages of existence theory and spectral diagnosis. The terminal-point classification covers $P(a) = 0$, the degree-one Caputo case, and the distinction between branch-point behavior and algebraic multiplicity.

Endpoint convergence now holds for arbitrary multiplicities in the multiset sense, while the first-order expansions retain the simple-root assumption required for a unique analytic branch. The regular-polygon family identifies the additional coefficient symmetry responsible for exact radial motion, and the nondegenerate Caputo example displays genuine three-root evolution.

Future work includes sharper coefficient-sensitive bounds, anisotropic star regions, symmetry classes such as $w^r Q(w^q)$, and extensions to analytic characteristic functions and quasipolynomials arising in delayed fractional systems.

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