

Final Developments on the Various Approximation Properties

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Abstract

In this paper, we review some important results and introduce some open problems on modern variants related to the classical approximation property in points of view such as bounded approximation properties, approximation properties via operator ideals, approximation properties via tensor norms and approximated identities.

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1. The Approximation Property and Its Bounded Versions

The approximation property was first introduced in Banach's book [1], and was systematically investigated by Grothendieck [2] from various points of view. Since then, the approximation property and its variants have been intensively studied; for example, readers may see [3] for various results and references for them. Throughout this paper, Banach spaces are denoted by X and Y over $\mathbb{R}$ or $\mathbb{C}$, with the dual spaces X^* and Y^* , and the closed unit ball of X is denoted by B_X . The ideal of finite rank operators is denoted by $\mathcal{F}$.

Definition 1. We say that X has the **approximation property** (AP) if for every $\epsilon > 0$ and every compact subset K of X , there exists an $S \in \mathcal{F}(X, X)$ such that

$$\sup_{x \in K} \|Sx - x\| \leq \epsilon.$$

For the convenience of notions, we use the *topology of uniform convergence on each compact set*, which will be denoted by τ_c . Then X has the AP if

$$I_X \in \overline{\mathcal{F}(X, X)}^{\tau_c},$$

where I_X is the identity map on X .

Definition 2. Let $\lambda \geq 1$. We say that X has the **λ -bounded approximation property** (λ -BAP) if

$$I_X \in \overline{\{S \in \mathcal{F}(X, X) : \|S\| \leq \lambda\}}^{\tau_c}.$$

We say that X has the BAP if X has the λ -BAP for some $\lambda \geq 1$.

Trivially, if X has the BAP, then X has the AP. The readers may consider the following question.

Question 1. If X has the AP, then does X have the BAP?



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Figiel and Johnson [4] gave a negative answer to Question 1.

Theorem 1. *There exists a Banach space X_{FJ} that has the AP but fails to have the BAP; moreover, X_{FJ}^* is separable.*

It is well known that if X^* is separable and has the AP, then X^* has the BAP (cf. Theorem 1.e.15 [5]). The following is a long-standing open problem.

Problem 1. *For every Banach space X , if X^* has the AP, then does X^* have the BAP?*

In fact, it is not known whether X^* has the BAP when X is separable and X^* has the AP.

Definition 3. *Let $\lambda \geq 1$. We say that X has the **weak λ -bounded approximation property** (weak λ -BAP) if for every Banach space Y and every $T \in \mathcal{W}(X, Y)$,*

$$I_X \in \overline{\{S \in \mathcal{F}(X, X) : \|TS\| \leq \lambda \|T\|\}}^{\tau_c},$$

where $\mathcal{W}$ is the ideal of weakly compact operators. We say that X has the weak BAP if X has the weak λ -BAP for some $\lambda \geq 1$.

Obviously, if X has the λ -BAP, then X has the weak λ -BAP, and trivially, if X has the weak BAP, then X has the AP. The weak BAP was introduced and studied in [6–9]. Lima and Oja [8] showed that the AP does not imply the weak BAP in general (Proposition 2.2 [8]). However, for every Banach space X , if X^* has the AP, then X^* has the weak BAP (Corollary 3.4 [8]). They conjectured that the following question is false.

Problem 2. *For every Banach space X , if X has the weak BAP, then does X have the BAP?*

According to (Corollary 1 [9]), we have

Theorem 2. *If X^* is separable and X has the weak BAP, then X has the BAP.*

The following was shown in [10].

Theorem 3. *The weak BAP and the BAP are equivalent for every Banach space if they are equivalent for every separable Banach space.*

Consequently, Problem 2 can be reformulated as follows.

Problem 3. *For every separable Banach space X , if X has the weak BAP, then does X have the BAP?*

Definition 4. *We say that X has the **strong approximation property** (strong AP) if for every separable reflexive Banach space Y and every $T \in \mathcal{K}(X, Y)$, there exists a $\lambda_T > 0$ such that*

$$T \in \overline{\{S \in \mathcal{F}(X, Y) : \|S\| \leq \lambda_T\}}^{\tau_c},$$

where $\mathcal{K}$ is the ideal of compact operators.

A simple verification shows that if X has the weak BAP, then X has the strong AP, and trivially, if X has the strong AP, then X has the AP. Oja [11] introduced the strong AP and showed that the AP does not imply the strong AP in general (Theorem 2.1 [11]), and conjectured that the following question is false.

Question 2. If X has the strong AP, then does X have the weak BAP?

However, Kim and Zheng (Theorem 22 [12]) proved that Question 2 is true.

2. The Approximation Property and Operator Ideals

In this section, we consider the approximation property from the perspective of operator ideals. Grothendieck [2] proved that X has the AP if and only if for every Banach space Y ,

$$\mathcal{K}(Y, X) = \overline{\mathcal{F}(Y, X)}^{\|\cdot\|} \quad (1)$$

(cf. Theorem 1.e.4 [5]). The readers may consider any Banach operator ideal instead of the ideal $\mathcal{K}$ in (2).

Definition 5 (cf. [13,14]). Let $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ be a Banach operator ideal. We say that X has the $\mathcal{I}$ -approximation property ($\mathcal{I}$ -AP) if for every Banach space Y ,

$$\mathcal{I}(Y, X) = \overline{\mathcal{F}(Y, X)}^{\|\cdot\|_{\mathcal{I}}}.$$

Grothendieck [2] proved that a subset K of X is relatively norm-compact if and only if there exists a norm null sequence $(x_n)_n$ in X such that

$$K \subset \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_1} \right\}. \quad (2)$$

Naturally, the readers may extend the sequence space ℓ_1 in (2) to the case ℓ_p ($1 \leq p \leq \infty$). In this direction, Sinha and Karn [15] introduced the notion of p -compactness.

Definition 6. Let $1 \leq p \leq \infty$. We say that a subset K of X is (absolutely) p -compact if there exists $(x_n)_n \in \ell_p(X)$ ($c_0(X)$ if $p = \infty$) such that

$$K \subset p\text{-co}(x_n)_n := \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_{p^*}} \right\},$$

where $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\ell_p(X)$ (respectively, $c_0(X)$) is the Banach space with the norm $\|\cdot\|_p$ (respectively, $\|\cdot\|_{\infty}$) of all X -valued absolutely p -summable (respectively, norm-null) sequences. A linear map $T : X \rightarrow Y$ is said to be p -compact if $T(B_X)$ is a p -compact subset of Y .

We see that the set $p\text{-co}(x_n)_n$ in Definition 6 is a convex set. Consequently, the convex hulls of X -valued absolutely p -summable sequences are p -compact subsets of X . See [15] for some properties of p -compact sets. Since introducing the p -compactness, the approximation properties of operator ideals have been intensively studied, and this is ongoing. Readers may refer to [13,14,16–28] and the references therein for related topics. Delgado, Piñeiro and Serrano [19] defined a norm on the space $\mathcal{K}_p(X, Y)$ of all p -compact operators from X to Y . For $T \in \mathcal{K}_p(X, Y)$, let

$$\|T\|_{\mathcal{K}_p} := \inf \{ \|(y_n)_n\|_p : (y_n)_n \in \ell_p(Y) \text{ and } T(B_X) \subset p\text{-co}(y_n)_n \}.$$

Then $[\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}]$ is a Banach operator ideal [20].

For $1 \leq p \leq \infty$, let $\ell_p^w(X)$ be the closed subspace of the Banach space $\ell_p^w(X)$ with the norm $\|\cdot\|_p^w$ of all X -valued weakly p -summable sequences, which consists of all sequences $(x_n)_n$ satisfying that

$$\lim_{m \rightarrow \infty} \|(0, \dots, 0, x_m, x_{m+1}, \dots)\|_p^w = 0.$$

The author called the sequences *unconditionally p -summable sequences* in [21].

Definition 7. Let $1 \leq p \leq \infty$. We say that a subset K of X is **unconditionally p -compact** if there exists $(x_n)_n \in \ell_p^u(X)$ ($c_0(X)$ if $p = \infty$) such that $K \subset p\text{-co}(x_n)_n$. A linear map $T : X \rightarrow Y$ is said to be **unconditionally p -compact** if $T(B_X)$ is an unconditionally p -compact subset of Y .

Clearly, every p -compact set is an unconditionally p -compact set. The author [21] defined a norm on the space $\mathcal{K}_{up}(X, Y)$ of all unconditionally p -compact operators from X to Y . For $T \in \mathcal{K}_{up}(X, Y)$, let

$$\|T\|_{\mathcal{K}_{up}} := \inf \{ \|(y_n)_n\|_p^w : (y_n)_n \in \ell_p^u(Y) \text{ and } T(B_X) \subset p\text{-co}(y_n)_n \}.$$

Then $[\mathcal{K}_{up}, \|\cdot\|_{\mathcal{K}_{up}}]$ is a Banach operator ideal.

Note that the ideals $[\mathcal{K}_{u\infty}, \|\cdot\|_{\mathcal{K}_{u\infty}}]$ and $[\mathcal{K}_\infty, \|\cdot\|_{\mathcal{K}_\infty}]$ are the ideal $[\mathcal{K}, \|\cdot\|_{\mathcal{K}}]$ of compact operators. Consequently, in view of (1) and Definition 5, the $\mathcal{K}_{u\infty}$ -AP and the $\mathcal{K}_\infty$ -AP are the AP. Moreover, we have

Proposition 1 (Remark 2.5 [21]). *If X has the AP, then X has the $\mathcal{K}_{up}$ -AP and the $\mathcal{K}_p$ -AP for every $1 \leq p < \infty$.*

In (Corollary 1.2 [21]) and (Corollary 3.6 [19]), it was shown that every Banach space has the $\mathcal{K}_{u2}$ -AP and the $\mathcal{K}_2$ -AP. In (Section 5 [22]) and (Proposition 3.6 and Example 3.7 [29]), for the case $1 \leq p < 2$, it was shown that the converse of Proposition 1 does not hold in general as follows.

Theorem 4. *Let $1 \leq p < 2$. Then the $\mathcal{K}_{up}$ -AP implies neither the $\mathcal{K}_p$ -AP nor the AP, and the $\mathcal{K}_p$ -AP implies neither the $\mathcal{K}_{up}$ -AP nor the AP.*

Problem 4. *Let $2 < p < \infty$. If X has the $\mathcal{K}_{up}$ -AP (respectively, $\mathcal{K}_p$ -AP), then does X have the $\mathcal{K}_p$ -AP (respectively, $\mathcal{K}_{up}$ -AP) or the AP?*

It is well known that if X^* has the AP, then X has the AP (cf. Theorem 1.e.7(a) [5]). The following theorem was shown in (Theorem 1.1 [21]), (Theorem 1.1 [22]), and (Theorem 4.7 [27]).

Theorem 5. *Let $1 \leq p < \infty$. If X^* has the $\mathcal{K}_{up}$ -AP, then X has the $\mathcal{K}_p$ -AP, and if X^* has the $\mathcal{K}_p$ -AP, then X has the $\mathcal{K}_{up}$ -AP.*

According to (Corollary 1.4 [21]) and (Corollary 1.2 [22]), the converse statements in Theorem 5 do not hold in general. Naturally, the readers may ask the following question.

Question 3. *Let $2 < p < \infty$. If X^* has the $\mathcal{K}_{up}$ -AP (respectively, the $\mathcal{K}_p$ -AP), then does X have the $\mathcal{K}_{up}$ -AP (respectively, the $\mathcal{K}_p$ -AP)?*

In (Section 5 [22]) and (Proposition 3.6 [29]), for the case $1 \leq p < 2$, it was shown that Question 3 is false.

Problem 5. *Let $2 < p < \infty$. If X^* has the $\mathcal{K}_{up}$ -AP (respectively, the $\mathcal{K}_p$ -AP), then does X have the $\mathcal{K}_{up}$ -AP (respectively, the $\mathcal{K}_p$ -AP)?*

Additionally, we review an approximation property related to the p -compactness.

Definition 8. Let $1 \leq p < \infty$. We say that X has the *p -approximation property* (p -AP) if

$$I_X \in \overline{\mathcal{F}(X, X)}^{\tau_p},$$

where τ_p is the topology of uniform convergence on each p -compact set.

The p -AP was introduced and studied in [15,18], and if X has the AP, then X has the p -AP because every p -compact set is norm-compact. We remark that

$$I_X \in \overline{\mathcal{F}(X, X)}^{\tau_{up}}$$

if and only if X has the $\mathcal{K}_{up}$ -AP, where τ_{up} is the topology of uniform convergence on each unconditionally p -compact set (Theorem 2.3 [21]). It was shown that if X has the q -AP, then X has the p -AP for $1 \leq p < q < \infty$ and every Banach space has the 2-AP, and for $p > 2$, there exists a Banach space failing to have the p -AP (Section 6 [15]).

Theorem 6 (Theorem 28 [12]). For $2 < p < \infty$, if X^* has the p -AP, then X has the p -AP.

If X has the $\mathcal{K}_{up}$ -AP or the $\mathcal{K}_p$ -AP, then X has the p -AP (Remark 2.5 [21]). However, we do not know whether the converse is true or false.

Problem 6. Let $2 < p < \infty$. If X has the p -AP, then does X have the $\mathcal{K}_{up}$ -AP or the $\mathcal{K}_p$ -AP?

A notion of compactness determined by operator ideals was introduced in [30].

Definition 9. Let $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ be a Banach operator ideal. We say that a subset K of X is *$\mathcal{I}$ -compact* if there exists a Banach space Z , $U \in \mathcal{I}(Z, X)$ and a relatively compact subset C of Z such that $K \subset U(C)$. A linear map $T : X \rightarrow Y$ is said to be *$\mathcal{I}$ -compact* if $T(B_X)$ is an $\mathcal{I}$ -compact subset of Y .

Let $\mathcal{K}_{\mathcal{I}}(X, Y)$ be the space of all $\mathcal{I}$ -compact operators from X to Y . A method to measure the size of $\mathcal{I}$ -compact sets was introduced in [26]. For an $\mathcal{I}$ -compact subset K of X , let

$$m_{\mathcal{I}}(K; X) := \inf\{\|U\|_{\mathcal{I}} : U \in \mathcal{I}(Z, X), \text{ relatively compact } C \subset B_Z, K \subset U(C)\}$$

and let $\|T\|_{\mathcal{K}_{\mathcal{I}}} := m_{\mathcal{I}}(T(B_X); Y)$ for $T \in \mathcal{K}_{\mathcal{I}}(X, Y)$. Then $[\mathcal{K}_{\mathcal{I}}, \|\cdot\|_{\mathcal{K}_{\mathcal{I}}}]$ is a Banach operator ideal (Section 2 [26]). According to (Remarks 1.3 and 1.7 [26]), a subset K of X is relatively norm-compact if and only if K is $\mathcal{K}$ -compact. In this case, $m_{\mathcal{K}}(K; X) = \sup_{x \in K} \|x\|$. Consequently, $[\mathcal{K}_{\mathcal{K}}, \|\cdot\|_{\mathcal{K}_{\mathcal{K}}}] = [\mathcal{K}, \|\cdot\|]$.

The following theorem was shown in (Proposition 3.3 [26]).

Theorem 7. If X has the BAP, then X has the $\mathcal{K}_{\mathcal{I}}$ -AP for every Banach operator ideal $\mathcal{I}$.

However, we do not know whether the following question is true or false.

Problem 7. If X has the AP, then does X have the $\mathcal{K}_{\mathcal{I}}$ -AP for every Banach operator ideal $\mathcal{I}$?

A partial result of Problem 7 was obtained in (Proposition 3.5 [26]).

Proposition 2. If X has the AP, then X has the $\mathcal{K}_{\mathcal{I}}$ -AP for every right-accessible Banach operator ideal $\mathcal{I}$.

According to Theorem 4, the converse of Proposition 2 does not hold in general because the ideal $\mathcal{K}_p$ is isometrically equal to the ideal $\mathcal{K}_{\mathcal{I}}$ for some right-accessible Banach operator ideal $\mathcal{I}$ (cf. Remark 1.7 [26]).

3. The Approximation Property and Tensor Norms

Grothendieck [2] used the injective tensor norm and the projective tensor norm to reformulate the AP.

Definition 10. Let $X \otimes Y$ be the algebraic tensor product of X and Y and let $u \in X \otimes Y$. The *injective tensor norm* ε is defined by

$$\varepsilon(u; X, Y) = \sup \left\{ \left| \sum_{n=1}^m x^*(x_n) y^*(y_n) \right| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\},$$

where $\sum_{n=1}^m x_n \otimes y_n$ is any representation of u , and the *projective tensor norm* π is defined by

$$\pi(u; X, Y) = \inf \left\{ \sum_{n=1}^m \|x_n\| \|y_n\| : u = \sum_{n=1}^m x_n \otimes y_n, m \in \mathbb{N} \right\}.$$

The normed space $X \otimes Y$ equipped with a norm α will be denoted by $X \otimes_{\alpha} Y$ and its completion is $X \hat{\otimes}_{\alpha} Y$. Grothendieck [2] proved that X has the AP if and only if for every Banach space Y , and the canonical inclusion map

$$J_{\pi} : Y \hat{\otimes}_{\pi} X \hookrightarrow Y \hat{\otimes}_{\varepsilon} X \quad (3)$$

is injective (cf. Theorem 5.6 [31]). In view of (3), the readers naturally may consider the following notion (cf. Section 21.7 [31]).

Definition 11. Let α be a tensor norm. Then we say that X has the *α -approximation property* (α -AP) if for every Banach space Y , the canonical inclusion map

$$J_{\alpha} : Y \hat{\otimes}_{\alpha} X \hookrightarrow Y \hat{\otimes}_{\varepsilon} X$$

is injective.

For every tensor norm α , it is well known that X has the α -AP if X has the AP (cf. Proposition 21.7(1) [31]).

Definition 12. For $\sum_{n=1}^m x_n \otimes y_n \in X \otimes Y$, let

$$\left| \sum_{n=1}^m x_n \otimes y_n \right|_{\sigma} = \sup \left\{ \sum_{n=1}^m |x^*(x_n) y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\}.$$

For $u \in X \otimes Y$, the *σ -tensor norm* α_{σ} is defined by

$$\alpha_{\sigma}(u; X, Y) = \inf \left\{ \left| \sum_{n=1}^m x_n \otimes y_n \right|_{\sigma} : u = \sum_{n=1}^m x_n \otimes y_n, m \in \mathbb{N} \right\}.$$

The fundamental problems on the α_{σ} -AP include the following questions.

Problem 8. Does every Banach space have the α_{σ} -AP? If X has the α_{σ} -AP, then does X have the AP?

An operator $T : X \rightarrow Y$ is called **nuclear** if there exist sequences $(x_n^*)_n$ in X^* and $(y_n)_n$ in Y with $\sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| < \infty$ such that

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n,$$

where $x_n^* \otimes y_n$ is an operator from X to Y defined by $(x_n^* \otimes y_n)(x) = x_n^*(x)y_n$. The operator $R : c_0 \rightarrow c_0$ defined by $R = \sum_{n=1}^{\infty} \lambda_n e_n^* \otimes e_n$ is a simple example of a nuclear operator, where each e_n is the standard unit vector in c_0 and each e_n^* is the coordinate functional and $(\lambda_n)_n \in \ell_1$. The readers may refer to [32] for some basic backgrounds of nuclear operators. Let $\mathcal{N}(X, Y)$ be the space of all nuclear operators from X to Y and for $T \in \mathcal{N}(X, Y)$, and let

$$\|T\|_{\mathcal{N}} := \inf \left\{ \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| : T = \sum_{n=1}^{\infty} x_n^* \otimes y_n, \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| < \infty \right\}.$$

It is well known that $[\mathcal{N}, \|\cdot\|_{\mathcal{N}}]$ is a Banach operator ideal. The σ -tensor norm is closely related to the following operator (Section 23.2 [33]).

Definition 13. We say that an operator $T : X \rightarrow Y$ is **σ -nuclear** if there exist sequences $(x_n^*)_n$ in X^* and $(y_n)_n$ in Y such that

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$$

unconditionally converges in the Banach space $(\mathcal{L}(X, Y), \|\cdot\|)$ of all operators from X to Y .

Clearly, every nuclear operator is σ -nuclear. Let $\mathcal{N}_{\sigma}(X, Y)$ be the space of all σ -nuclear operators from X to Y and for $T \in \mathcal{N}_{\sigma}(X, Y)$, and let

$$\|T\|_{\mathcal{N}_{\sigma}} := \inf \left\{ \left| \sum_{n=1}^{\infty} x_n^* \otimes y_n \right|_{\sigma} : T = \sum_{n=1}^{\infty} x_n^* \otimes y_n \text{ unconditionally converges} \right\},$$

where $|\sum_{n=1}^{\infty} x_n^* \otimes y_n|_{\sigma} := \sup \{ \sum_{n=1}^{\infty} |x_n^*(x)y_n^*(y)| : x \in B_X, y^* \in B_{Y^*} \}$. Then $[\mathcal{N}_{\sigma}, \|\cdot\|_{\mathcal{N}_{\sigma}}]$ is a Banach operator ideal (Theorem 23.2.2 [33]). It is well known that for $T \in \mathcal{L}(X, Y)$, if the adjoint operator $T^* : Y^* \rightarrow X^*$ is nuclear, then T is nuclear whenever X^* or Y^{***} has the AP [2]. The following theorem was shown in (Corollary 1 [34]).

Theorem 8. For $T \in \mathcal{L}(X, Y)$, if $T^* \in \mathcal{N}_{\sigma}(Y^*, X^*)$, then $T \in \mathcal{N}_{\sigma}(X, Y)$ whenever X^* or Y^{***} has the AP.

The following operator was introduced in (Section 23.1 [33]).

Definition 14. We say that an operator $T : X \rightarrow Y$ is **absolutely τ -summing** if there exists a $C > 0$ such that

$$\sum_{n=1}^m |y_n^*(Tx_n)| \leq C \sup \left\{ \sum_{n=1}^m |x^*(x_n)y_n^*(y)| : x^* \in B_{X^*}, y \in B_Y \right\}$$

for every finite sequences $(x_n)_{n=1}^m$ in X and $(y_n^*)_{n=1}^m$ in Y^* .

Let $\mathcal{P}_{\tau}(X, Y)$ be the space of all absolutely τ -summing operators from X to Y and for $T \in \mathcal{P}_{\tau}(X, Y)$, and let $\|T\|_{\mathcal{P}_{\tau}} := \inf C$, where the infimum is taken over all such inequalities in Definition 14. Then $[\mathcal{P}_{\tau}, \|\cdot\|_{\mathcal{P}_{\tau}}]$ is a Banach operator ideal (Theorem 23.1.2 [33]). Let $T \in \mathcal{P}_{\tau}(X, Y)$. If $\sum_{n=1}^{\infty} y_n^* \otimes x_n \in \mathcal{N}_{\sigma}(Y, X)$, then we see that

$$\lim_{l,m \rightarrow \infty} \sum_{n=l}^m |y_n^*(Tx_n)| \leq \|T\|_{\mathcal{P}_\tau} \lim_{l,m \rightarrow \infty} \sup \left\{ \sum_{n=l}^m |y_n^*(y)x^*(x_n)| : y \in B_Y, x^* \in B_{X^*} \right\} = 0.$$

Consequently, $(y_n^*(Tx_n))_n \in \ell_1$. Thus, we can define a locally convex topology, which will be denoted by τ_σ , on $\mathcal{P}_\tau(X, Y)$ generated by the seminorms

$$\|(y_n^*(Tx_n))_n\|_{\ell_1}$$

for every $\sum_{n=1}^\infty y_n^* \otimes x_n \in \mathcal{N}_\sigma(Y, X)$. It was shown in (Theorem [35]) that X has the AP if and only if for every Banach space Y ,

$$\mathcal{K}(X, Y) \subset \overline{\mathcal{F}(X, Y)}^{\tau_c}.$$

The following was shown in (Theorem 1 [34]).

Theorem 9. *Let X be a Banach space. Then X has the α_σ -AP if and only if for every Banach space Y ,*

$$\mathcal{P}_\tau(X, Y) = \overline{\mathcal{F}(X, Y)}^{\tau_\sigma}.$$

It is well known that X has the AP if and only if for every Banach space Y ,

$$\mathcal{K}(Y, X) \subset \overline{\mathcal{F}(Y, X)}^{\tau_c}$$

(cf. [2]). In this direction, we have the following questions.

Problem 9. *If X has the α_σ -AP, then for every Banach space Y ,*

$$\mathcal{P}_\tau(Y, X) = \overline{\mathcal{F}(Y, X)}^{\tau_\sigma}?$$

Or, if for every Banach space Y ,

$$\mathcal{P}_\tau(Y, X) = \overline{\mathcal{F}(Y, X)}^{\tau_\sigma},$$

then does X have the α_σ -AP?

As previously mentioned, if X^* has the AP, then X has the AP, but the converse statement does not hold in general (cf. Theorem 1.e.7 [5]). So we ask

Problem 10. *If X^* has the α_σ -AP, then does X have the α_σ -AP? Or, if X has the α_σ -AP, then does X^* have the α_σ -AP?*

4. The Approximation Property and Approximate Identities

The theory of approximate identities is a fundamental tool in the study of the theory of Banach algebras. Readers may refer to [36] for basic information and results on approximate identities. In this section, we consider some relationships between approximation properties and approximate identities. Throughout this section, we will denote by $\mathcal{A}$ a normed algebra with the norm $\|\cdot\|_{\mathcal{A}}$.

Definition 15. *We say that $\mathcal{A}$ has a **left approximate identity** (LAI) (respectively, **right approximate identity** (RAI)) if for every $A_1, \dots, A_n \in \mathcal{A}$ and every $\epsilon > 0$, there exists an $A \in \mathcal{A}$ such that*

$$\|AA_i - A_i\|_{\mathcal{A}} \leq \epsilon \quad (1 \leq i \leq n) \quad (\text{respectively, } \|A_iA - A_i\|_{\mathcal{A}} \leq \epsilon \quad (1 \leq i \leq n)).$$

The following is a bounded version of LAI and RAI.

Definition 16. Let $\lambda \geq 1$. We say that $\mathcal{A}$ has a λ -bounded left approximate identity (λ -BLAI) (respectively, λ -bounded right approximate identity (λ -BRAI)) if for every $A_1, \dots, A_n \in \mathcal{A}$ and every $\epsilon > 0$, there exists an $A \in \mathcal{A}$ with $\|A\|_{\mathcal{A}} \leq \lambda$ such that

$$\|AA_i - A_i\|_{\mathcal{A}} \leq \epsilon \quad (1 \leq i \leq n) \quad (\text{respectively, } \|A_iA - A_i\|_{\mathcal{A}} \leq \epsilon \quad (1 \leq i \leq n)).$$

We say that X has a BLAI (respectively, BRAI) if X has a λ -BLAI (respectively, λ -BRAI) for some $\lambda \geq 1$.

In [37–40], the approximate identities were studied in general normed algebras and in [41–43], they were studied in algebras of compact operators. We need to introduce another classical approximation property.

Definition 17. Let $\lambda \geq 1$. We say that X has the compact approximation property (CAP) (respectively, λ -bounded compact approximation property (λ -BCAP)) if

$$I_X \in \overline{\mathcal{K}(X, X)}^{\tau_c} \quad (\text{respectively, } I_X \in \overline{\{S \in \mathcal{K}(X, X) : \|S\| \leq \lambda\}}^{\tau_c}).$$

We say that X has the BCAP if X has the λ -BCAP for some $\lambda \geq 1$.

For a Banach space X , let us consider the operator normed algebras $\mathcal{F}(X, X)$, $\overline{\mathcal{F}(X, X)}^{\|\cdot\|}$ and $\mathcal{K}(X, X)$. The following results were shown in (Theorems 2.5 and 2.6 [42]).

Theorem 10. Let $\lambda \geq 1$ and let X be a Banach space. Then we have the following statements.

- (a) X has the λ -BAP if and only if the algebra $\mathcal{F}(X, X)$ has a λ -BLAI.
- (b) X has the λ -BCAP if and only if the algebra $\mathcal{K}(X, X)$ has a λ -BLAI.
- (c) If X has the AP, then the algebra $\overline{\mathcal{F}(X, X)}^{\|\cdot\|}$ has an LAI.
- (d) If X has the CAP, then the algebra $\mathcal{K}(X, X)$ has an LAI.

The natural questions are whether the converse statements in (c) and (d) in Theorem 10 are true.

Problem 11. If $\overline{\mathcal{F}(X, X)}^{\|\cdot\|}$ has an LAI, then does X have the AP? Or, if $\mathcal{K}(X, X)$ has an LAI, then does X have the CAP?

The following right version of Theorem 10 is due to [43].

Theorem 11. Let $\lambda \geq 1$ and let X be a Banach space. Then we have the following statements.

- (a) X^* has the λ -BAP if and only if the algebra $\mathcal{F}(X, X)$ has a λ -BRAI.
- (b) $I_{X^*} \in \overline{\{S^* : S \in \mathcal{K}(X, X), \|S\| \leq \lambda\}}^{\tau_c}$ if and only if the algebra $\mathcal{K}(X, X)$ has a λ -BRAI.
- (c) If X^* has the AP, then the algebra $\overline{\mathcal{F}(X, X)}^{\|\cdot\|}$ has an RAI.
- (d) If $I_{X^*} \in \overline{\{S^* : S \in \mathcal{K}(X, X)\}}^{\tau_c}$, then the algebra $\mathcal{K}(X, X)$ has an RAI.

As in Problem 11, we ask

Problem 12. If $\overline{\mathcal{F}(X, X)}^{\|\cdot\|}$ has an RAI, then does X^* have the AP? Or, if $\mathcal{K}(X, X)$ has an RAI, then does X^* have the CAP?

Definition 18. We say that $\mathcal{A}$ has a **left approximate unit** (LAU) (respectively, **right approximate unit** (RAU)) if for every $A \in \mathcal{A}$ and every $\epsilon > 0$, there exists a $B \in \mathcal{A}$ such that

$$\|BA - A\|_{\mathcal{A}} \leq \epsilon \text{ (respectively, } \|AB - A\|_{\mathcal{A}} \leq \epsilon).$$

The following is a bounded version of LAU and RAU.

Definition 19. Let $\lambda \geq 1$. We say that $\mathcal{A}$ has a **λ -bounded left approximate unit** (λ -BLAU) (respectively, **λ -bounded right approximate unit** (λ -BRAU)) if for every $A \in \mathcal{A}$ and every $\epsilon > 0$, there exists a $B \in \mathcal{A}$ with $\|B\|_{\mathcal{A}} \leq \lambda$ such that

$$\|BA - A\|_{\mathcal{A}} \leq \epsilon \text{ (respectively, } \|AB - A\|_{\mathcal{A}} \leq \epsilon).$$

We say that X has a BLAU (respectively, BRAU) if X has a λ -BLAU (respectively, λ -BRAU) for some $\lambda \geq 1$.

Trivially, if a normed algebra has a λ -BLAI (respectively, λ -BRAI), then it has a λ -BLAU (respectively, λ -BRAU). A question was whether the converse is true in general. Wichmann (Theorem 1 [39]) gave an affirmative answer to this question.

Theorem 12. Let $\lambda \geq 1$. For every normed algebra $\mathcal{A}$, if $\mathcal{A}$ has a λ -BLAU, then $\mathcal{A}$ has a λ -BLAI.

Using the same argument of their proof, we have

Theorem 13. Let $\lambda \geq 1$. For every normed algebra $\mathcal{A}$, if $\mathcal{A}$ has a λ -BRAU, then $\mathcal{A}$ has a λ -BRAI.

Naturally, the readers are may led to the following question.

Problem 13. For every normed algebra $\mathcal{A}$, if $\mathcal{A}$ has an LAU (respectively, RAU), then does $\mathcal{A}$ have an LAI (respectively, RAI)?

Moreover, we do not know whether the following questions are true or false.

Problem 14. For every Banach space X , if the algebra $\mathcal{K}(X, X)$ has an LAU (respectively, RAU), then does $\mathcal{K}(X, X)$ have an LAI (respectively, RAI)? Or, if the algebra $\overline{\mathcal{F}(X, X)}^{\|\cdot\|}$ has an LAU (respectively, RAU), then does $\overline{\mathcal{F}(X, X)}^{\|\cdot\|}$ have an LAI (respectively, RAI)?

Definition 20. We say that X has the property $(\mathcal{K})$ (respectively, $(\overline{\mathcal{F}})$) if for every compact subset K of X , there exists a $T \in \mathcal{K}(X, X)$ (respectively, $\overline{\mathcal{F}(X, X)}^{\|\cdot\|}$) such that

$$K \subset \overline{T(B_X)}.$$

Dixon (Theorem 2.7 [42]) obtained some partial results for Problem 14.

Theorem 14. Assume that X has the property $(\mathcal{K})$ (respectively, $(\overline{\mathcal{F}})$). Then the following statements are equivalent.

- (a) X has the CAP (respectively, AP).
- (b) $\mathcal{K}(X, X)$ (respectively, $\overline{\mathcal{F}(X, X)}^{\|\cdot\|}$) has an LAI.
- (c) $\mathcal{K}(X, X)$ (respectively, $\overline{\mathcal{F}(X, X)}^{\|\cdot\|}$) has an LAU.

Casazza and Jarchow (Corollary 3.8 [44]) proved that there exists a Banach space failing to have the property $(\overline{\mathcal{F}})$. Dixon [42] conjectured that the following question is false.

Problem 15. *Does every Banach space have the property $(\mathcal{K})$?*

5. Conclusions

The main goal of this work is to introduce some open problems on recent various approximation properties, which are connected to the classical approximation property, to the researchers in the field of Banach space theory. Problem 1 is an old problem but it was reanalyzed in [8]. Since it was shown in [8] that X^* has the weak BAP if X^* has the AP, Problem 1 has an affirmative answer if the following question is true.

Problem 16. *If the dual space X^* has the weak BAP, does X^* have the BAP?*

The separable case is also unknown for Problem 16.

Problem 17. *If X is separable and X^* has the weak BAP, does X^* have the BAP?*

Recall that the ideal $\mathcal{K}$ is isometrically equal to the ideal $\mathcal{K}_{\mathcal{K}}$. Consequently, X has the AP if and only if X has the $\mathcal{K}_{\mathcal{K}}$ -AP. In view of Problem 7, the author suggests an additional problem related to it.

Problem 18. *Find Banach operator ideals $\mathcal{I}$ such that X has the AP if and only if X has the $\mathcal{K}_{\mathcal{I}}$ -AP.*

Since the σ -tensor norm α_{σ} was recently introduced, it would be an important work to establish fundamental properties of α_{σ} . Grothendieck [45] introduced and studied the fundamental fourteen tensor norms. The author suggests the following problem.

Problem 19. *Establish some relationships between the σ -tensor norm and Grothendieck's fourteen tensor norms.*

Dixon (Theorem 4.3 [42]) proved that if X has the BCAP, then X has the property $(\mathcal{K})$. Trivially, if X has the property $(\overline{\mathcal{F}})$, then X has the property $(\mathcal{K})$. Casazza and Jarchow (Corollary 3.8 [44]) showed that the Pisier space X_P [46] does not have the property $(\overline{\mathcal{F}})$. Then the author suggests the following problem.

Problem 20. *Does the space X_P have the property $(\mathcal{K})$?*

Concerning Problem 15, if the space X_P does not have the property $(\mathcal{K})$, then Problem 15 would have a negative answer. Moreover, since we do not know whether the property $(\mathcal{K})$ and the property $(\overline{\mathcal{F}})$ are equivalent, if the space X_P has the property $(\mathcal{K})$, we would see that the property $(\mathcal{K})$ is different from the property $(\overline{\mathcal{F}})$.

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References

1. Banach, S. *Theorie des Operations Lineaires*; Fundus Kultury Narodowej: Warszawa, Poland, 1932.
2. Grothendieck, A. *Produits Tensoriels Topologiques et Espaces Nucléaires*; Memoirs of the American Mathematical Society; American Mathematical Society: Providence, RI, USA, 1955; Volume 16.
3. Casazza, P.G. Approximation Properties. In *Handbook of the Geometry of Banach Spaces Johnson*; Johnson, W.B., Lindenstrauss, J., Eds.; Elsevier: Amsterdam, The Netherlands, 2001; Volume 1, pp. 271–316.

4. Figiel, T.; Johnson, W.B. The approximation property does not imply the bounded approximation property. *Proc. Am. Math. Soc.* **1973**, *41*, 197–200. [[CrossRef](#)]
5. Lindenstrauss, J.; Tzafriri, L. *Classical Banach Spaces I, Sequence Spaces*; Springer: Berlin/Heidelberg, Germany, 1977.
6. Lima, Å.; Lima, V. Geometry of spaces of compact operators. *Ark. Mat.* **2008**, *46*, 113–142. [[CrossRef](#)]
7. Lima, V. The weak metric approximation property and ideals of operators. *J. Math. Anal. Appl.* **2007**, *334*, 593–603. [[CrossRef](#)]
8. Lima, Å.; Oja, E. The weak metric approximation property. *Math. Ann.* **2005**, *333*, 471–484. [[CrossRef](#)]
9. Oja, E. The impact of the Radon-Nikodým property on the weak bounded approximation property. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Math.* **2006**, *100*, 325–331.
10. Kim, J.M.; Lee, K.Y. A remark on Defant-Floret-Lima-Oja's conjecture. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Math.* **2020**, *114*, 58. [[CrossRef](#)]
11. Oja, E. The strong approximation property. *J. Math. Anal. Appl.* **2008**, *338*, 407–415. [[CrossRef](#)]
12. Kim, J.M.; Zheng, B. History, Developments and Open Problems on Approximation Properties. *Mathematics* **2020**, *8*, 1117. [[CrossRef](#)]
13. Lassalle, S.; Turco, P. On p -compact mappings and the p -approximation properties. *J. Math. Anal. Appl.* **2012**, *389*, 1204–1221. [[CrossRef](#)]
14. Oja, E. A remark on the approximation of p -compact operators by finite-rank operators. *J. Math. Anal. Appl.* **2012**, *387*, 949–952. [[CrossRef](#)]
15. Sinha, D.P.; Karn, A.K. Compact operators whose adjoints factor through subspaces of l_p . *Stud. Math.* **2002**, *150*, 17–33. [[CrossRef](#)]
16. Ain, K.; Lillemets, R.; Oja, E. Compact operators which are defined by ℓ_p -spaces. *Quaest. Math.* **2012**, *35*, 145–159. [[CrossRef](#)]
17. Delgado, J.M.; Piñeiro, C. An approximation property with respect to an operator ideal. *J. Math. Anal. Appl.* **2009**, *354*, 159–164.
18. Delgado, J.M.; Oja, E.; Piñeiro, C.; Serrano, E. The p -approximation property in terms of density of finite rank operators. *J. Math. Anal. Appl.* **2009**, *354*, 159–164. [[CrossRef](#)]
19. Delgado, J.M.; Piñeiro, C.; Serrano, E. Density of finite rank operators in the Banach space of p -compact operators. *J. Math. Anal. Appl.* **2010**, *370*, 498–505. [[CrossRef](#)]
20. Delgado, J.M.; Piñeiro, C.; Serrano, E. Operators whose adjoints are quasi p -nuclear. *Stud. Math.* **2010**, *197*, 291–304. [[CrossRef](#)]
21. Kim, J.M. The $\mathcal{K}_{up}$ -approximation property and its duality. *J. Aust. Math. Soc.* **2015**, *98*, 364–374.
22. Kim, J.M. Duality between the $\mathcal{K}_1$ - and the $\mathcal{K}_{u1}$ -approximation properties. *Houst. J. Math.* **2017**, *43*, 1133–1145.
23. Piñeiro, C.; Delgado, J.M. p -Convergent sequences and Banach spaces in which p -compact sets are q -compact. *Proc. Am. Math. Soc.* **2011**, *139*, 957–967. [[CrossRef](#)]
24. Sinha, D.P.; Karn, A.K. Compact operators which factor through subspaces of l_p . *Math. Nachr.* **2008**, *281*, 412–423. [[CrossRef](#)]
25. Galicer, D.; Lassalle, S.; Turco, P. The ideal of p -compact operators: A tensor product approach. *Stud. Math.* **2012**, *211*, 269–286. [[CrossRef](#)]
26. Lassalle, S.; Turco, P. The Banach ideal of $\mathcal{A}$ -compact operators and related approximation properties. *J. Funct. Anal.* **2013**, *265*, 2452–2464. [[CrossRef](#)]
27. Lassalle, S.; Turco, P. On null sequences for Banach operator ideals, trace duality and approximation properties. *Math. Nachr.* **2017**, *290*, 2308–2321. [[CrossRef](#)]
28. Pietsch, A. The ideal of p -compact operators and its maximal hull. *Proc. Am. Math. Soc.* **2014**, *142*, 519–530. [[CrossRef](#)]
29. Wirzenius, H. On approximation properties related to unconditionally p -compact operators and Sinha-Karn p -compact operators. *Math. Nachr.* **2023**, *296*, 4835–4853. [[CrossRef](#)]
30. Carl, B.; Stephani, I. On $\mathcal{A}$ -compact operators, generalized entropy numbers and entropy ideals. *Math. Nachr.* **1984**, *199*, 77–95. [[CrossRef](#)]
31. Defant, A.; Floret, K. *Tensor Norms and Operator Ideals*; Elsevier/North-Holland: Amsterdam, The Netherlands, 1993.
32. Diestel, J.; Jarchow, H.; Tonge, A. *Absolutely Summing Operators*; Cambridge University Press: Cambridge, UK, 1995.
33. Pietsch, A. *Operator Ideals*; North-Holland: Amsterdam, The Netherlands, 1980.
34. Kim, J.M. The α_σ -approximation property and its related operator Ideals. *Mathematics* **2024**, *12*, 2006. [[CrossRef](#)]
35. Kim, J.M. New criterion of the approximation property. *J. Math. Anal. Appl.* **2008**, *345*, 889–891. [[CrossRef](#)]
36. Doran, R.S.; Wichmann, J. *Approximate Identities and Factorization in Banach Modules*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1979; Volume 768.
37. Dixon, P.G. Approximate identities in normed algebras II. *J. Lond. Math. Soc.* **1978**, *17*, 141–151. [[CrossRef](#)]
38. Dixon, P.G. Approximate identities in normed algebras. *Proc. Lond. Math. Soc.* **1973**, *26*, 485–496. [[CrossRef](#)]
39. Wichmann, J. Bounded approximate units and bounded approximate identities. *Proc. Am. Math. Soc.* **1973**, *41*, 547–550. [[CrossRef](#)]
40. Dixon, P.G. Unbounded approximate identities in normed algebras. *Glasg. Math. J.* **1992**, *34*, 189–192. [[CrossRef](#)]
41. Zhang, Y. Unbounded approximate identities in algebras of compact operators on Banach spaces. *Math. Proc. Camb. Philos. Soc.* **2003**, *134*, 187–192. [[CrossRef](#)]

42. Dixon, P.G. Left approximate identities in algebras of compact operators on Banach spaces. *Proc. R. Soc. Edinb. Sect. A Math.* **1986**, *104*, 169–175. [[CrossRef](#)]
43. Grønbæk, N.; Willis, G. Approximate identities in Banach algebras of compact operators. *Can. Math. Bull.* **1993**, *36*, 45–53. [[CrossRef](#)]
44. Casazza, P.G.; Jarchow, H. Self-induced compactness in Banach spaces. *Proc. R. Soc. Edinb. Sect. A Math.* **1996**, *126*, 355–362. [[CrossRef](#)]
45. Grothendieck, A. Résumé de la théorie métrique des produits tensoriels topologiques. *Soc. Mat. São Paulo* **1956**, *8*, 1–79.
46. Pisier, G. Counterexample to a conjecture of Grothendieck. *Acta Math.* **1983**, *151*, 180–208. [[CrossRef](#)]

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