

Article

The Two-Maxian Problem on Block Graphs with Distance Constraint

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Abstract

This paper investigates the constrained two-maxian problem defined on block graphs, a special class of composite graphs containing trees and cacti as subclasses. The problem requires placing two facilities on a block graph such that the Euclidean/topological distance between the two facilities is bounded by a given upper limit, while maximizing the total weighted maximum distance from all client vertices to the farthest facility. We first analyze the structural properties of block graphs and the vertex optimality of the constrained two-maxian solution, proving that at least one facility in any optimal solution must be located at a vertex of the block graph. Based on this key property, we design an $O(n^4)$ time algorithm for the constrained two-maxian problem on block graphs. We further compare the proposed method with existing algorithms for trees and cycles and discuss a possible extension to cactus graphs.

Keywords: facility location problem; maxian problem; block graph; distance constraint; polynomial-time algorithm

MSC: 90B10; 90B80; 90C27

1. Introduction

Facility location theory is a core branch of operations research and combinatorial optimization, widely applied in logistics planning, public service layout, and undesirable facility deployment. Classical location models mainly include median problems and center problems: the median model minimizes the total weighted distance from clients to facilities, while the center model minimizes the maximum weighted distance from clients to facilities. However, in practical scenarios such as the layout of garbage landfills, chemical plants, and noisy equipment, decision-makers need to place facilities as far as possible from client nodes. This type of problem is collectively called the obnoxious facility location problem, among which the p -maxian problem is the most representative one.

The p -maxian problem aims to select p facilities in a graph to maximize the sum of weighted maximum distances from all vertices to the farthest facility. Early research on maxian problems focused on classic graph structures. Burkard et al. [1] first studied the p -maxian problem on trees and proposed a linear-time algorithm. Kang and Cheng [2] extended the research to block graphs and designed a linear-time algorithm for the unconstrained p -maxian problem on block graphs. Cheng and Kang [3] solved the p -maxian problem on interval graphs with linear complexity, and Kang et al. [4] proposed a quadratic-time algorithm for the two-maxian problem on cactus graphs.



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In real engineering applications, the placement of obnoxious facilities is often subject to distance constraints between facilities: for example, monitoring equipment and auxiliary facilities need to be kept within a certain distance for interconnection and collaborative work; scattered waste disposal points also require limited spacing to facilitate centralized management. Nguyen et al. [5] pioneered the constrained p-maxian problem with facility distance constraints and gave a linear-time algorithm for the constrained two-maxian problem on trees. Bai and Du [6] studied the constrained two-maxian problem on cycles, proved the vertex-related optimality property of optimal solutions, and constructed a linear-time algorithm for cyclic graphs.

Block graphs are a superclass of trees and cactus graphs, where every biconnected component (block) is a clique. Due to its flexible structure, block graphs can describe more complex network systems than trees and cycles. Nevertheless, the constrained two-maxian problem on block graphs has not been systematically studied so far. Motivated by the above research background, this paper takes block graphs as the research object, explores the theoretical properties and solving algorithms of the constrained two-maxian problem, and fills the research gap of constrained maxian problems on block graphs.

The rest of this paper is organized as follows: Section 2 gives the basic definitions of block graphs, formalizes the constrained two-maxian problem, and deduces fundamental structural properties. Section 3 designs a polynomial-time algorithm and analyzes its time complexity. Section 4 discusses the correlation between constrained and unconstrained problems. Section 5 summarizes the full work and prospects future research directions.

2. Problem Definition and Basic Properties

2.1. Basic Concepts of Block Graphs

Let $G = (V, E)$ be a connected undirected graph, where V is the vertex set and E is the edge set. A block (biconnected component) of G refers to a maximal subgraph without cut vertices. A graph G is defined as a block graph if every block of G is a complete graph (clique) (see Figure 1). Trees are special block graphs where each block is a two-vertex clique (single edge); cactus graphs are not block graphs in general, but block graphs share many structural characteristics with trees and cycles.

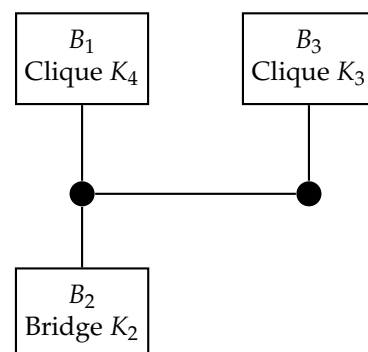


Figure 1. A standard block graph containing clique blocks. The block graph consists of cut vertices (black dots) and blocks, where B_1 represents a clique K_4 , B_3 denotes a clique K_3 , and B_2 denotes a bridge block K_2 .

For any two points x, y on graph G , let $d(x, y)$ denote the shortest path distance between x and y . Each vertex $v_i \in V$ is assigned a non-negative weight w_i , representing the client demand or weight of the vertex. Let $X(G)$ be the set of all points on G (including vertices and interior points on edges).

2.2. Definition of Constrained Two-Maxian Problem

For a block graph $G = (V, E)$ with $|V| = n$, the unconstrained two-maxian problem is to select two points $X = \{x_1, x_2\} \subset X(G)$ to maximize the objective function:

$$F(X) := \sum_{i=1}^n w_i \cdot \max\{d(v_i, x_1), d(v_i, x_2)\} \quad (1)$$

In practical scenarios, we add a distance constraint between the two facilities: given an upper bound $\mathcal{U} > 0$, require $d(x_1, x_2) \leq \mathcal{U}$. The constrained two-maxian problem on block graphs is formulated as the following optimization model:

$$\max F(x_1, x_2) = \sum_{i=1}^n w_i \cdot \max\{d(v_i, x_1), d(v_i, x_2)\} \quad (2)$$

$$\text{s.t. } d(x_1, x_2) \leq \mathcal{U}, x_1, x_2 \in X(G) \quad (3)$$

A set of points $\{x_1^*, x_2^*\}$ satisfying the constraints and maximizing the objective function is called an optimal solution of the constrained two-maxian problem.

2.3. Auxiliary Set Division

For any feasible solution $\{x_1, x_2\}$, divide the vertex set V into three disjoint subsets:

$$V_1 = \{v_i \in V \mid d(v_i, x_1) \geq d(v_i, x_2)\}$$

and

$$V_2 = \{v_i \in V \mid d(v_i, x_1) < d(v_i, x_2)\}.$$

Vertices in V_1 take x_1 as the reference facility, and vertices in V_2 take x_2 as the reference facility. The objective function of the constrained two-maxian problem can be rewritten as

$$F(x_1, x_2) = \sum_{v_i \in V_1} w_i d(v_i, x_1) + \sum_{v_i \in V_2} w_i d(v_i, x_2). \quad (4)$$

This decomposition lays a foundation for subsequent property proof and algorithm design.

2.4. Optimality Property of Optimal Solutions

For the unconstrained two-maxian problems on cactus graphs and block graphs, existing studies have proved that the optimal solution can be located entirely at vertices [2,4]. However, after adding the distance constraint $d(x_1, x_2) \leq \mathcal{U}$, this full vertex optimality no longer holds, which is similar to the conclusion on cycle graphs. We hereby prove the core property of the constrained two-maxian problem on block graphs:

Theorem 1. *For the constrained two-maxian problem defined on a block graph G , there exists an optimal solution $X^* = \{x_1^*, x_2^*\}$ satisfying two properties: the distance between the two facilities equals the upper bound $\mathcal{U}$, i.e., $d(x_1^*, x_2^*) = \mathcal{U}$, and at least one of x_1^* and x_2^* is a vertex of G .*

Proof. We split the proof into two parts corresponding to the two statements in the theorem.

Any optimal solution satisfies $d(x_1^*, x_2^*) = \mathcal{U}$. Suppose for contradiction that there exists a feasible optimal pair $\{x_1, x_2\}$ with $d_0 = d(x_1, x_2) < \mathcal{U}$. We move x_1 and x_2 simultaneously in opposite directions, correspondingly obtaining two new points $\tilde{x}_1$ and $\tilde{x}_2$ such that $d(\tilde{x}_1, \tilde{x}_2) = \mathcal{U}$. Since the partition of vertex subsets V_1, V_2 remains unchanged and each service distance becomes larger, it is straightforward to verify that the objective function value strictly rises. This contradicts the optimality of $\{x_1, x_2\}$. Therefore, every optimal feasible pair must satisfy $d(x_1^*, x_2^*) = \mathcal{U}$.

At least one facility in the optimal pair is a vertex of G . From Part 1, we only need to consider pairs $\{x_1, x_2\}$ with $d(x_1, x_2) = U$. Define the total weight of each subset as

$$W(V_1) = \sum_{v_i \in V_1} w_i$$

and

$$W(V_2) = \sum_{v_i \in V_2} w_i.$$

We carry out synchronous displacement of x_1 and x_2 with their mutual distance maintained at U all the time: if $W(V_1) \geq W(V_2)$, shift the two points simultaneously in the direction opposite to V_1 ; otherwise, move them synchronously toward the direction opposite to V_2 . Terminate the movement after a displacement distance $\delta > 0$, where x_1 arrives at a vertex denoted by v_1 , and the new position of x_2 is written as x'_2 . Let V_3 stand for the set of vertices originally served by x_2 that are now served by v_1 after the movement. The objective function value after moving is expressed as

$$\begin{aligned} F(v_1, x'_2) &= \sum_{v_i \in V_1 \cup V_3} w_i(d(v_i, x_1) + \delta) + \sum_{v_i \in V_2 \setminus V_3} w_i(d(v_i, x_2) - \delta) \\ &= \sum_{v_i \in V_1} w_i d(v_i, x_1) + \sum_{v_i \in V_2} w_i d(v_i, x_2) + (W(V_1) - W(V_2))\delta \\ &\quad + \sum_{v_i \in V_3} w_i \{ (d(v_i, x_1) + \delta) - (d(v_i, x_2) - \delta) \} \end{aligned} \quad (5)$$

then we have

$$\begin{aligned} F(v_1, x'_2) - F(x_1, x_2) &= (W(V_1) - W(V_2))\delta + \sum_{v_i \in V_3} w_i(d(v_i, x_1) - d(v_i, x_2) + 2\delta) \\ &\geq 0, \end{aligned} \quad (6)$$

since $W(V_1) \geq W(V_2)$ and $(d(v_i, x_1) + \delta) \geq (d(v_i, x_2) - \delta)$, $v_i \in V_3$.

The constructed feasible pair $\{v_1, x'_2\}$ satisfies $d(v_1, x'_2) = U$, and v_1 is a vertex of G . Its objective value is no less than that of the original interior point pair $\{x_1, x_2\}$. Therefore, there must exist an optimal solution containing at least one vertex of G . The proof is complete. $\square$

It is not hard to see that to solve the constrained two-maxian problem on a block graph, we only need to traverse all vertices and the edge interior points with distance exactly U from each vertex, which greatly reduces the solution search space.

3. Algorithm Design and Complexity Analysis

Based on Theorem 1, we design a polynomial-time traversal algorithm for the constrained two-maxian problem on block graphs, combined with the structural characteristics of block graphs.

3.1. Preprocessing Steps

1. Graph initialization: Input the block graph $G = (V, E)$, vertex weights w_i , edge length set, and distance upper bound U . Let $n = |V|$.
2. Mark critical interior points: For each vertex $u \in V$, find all interior points p on edges of G such that $d(u, p) = U$, and record these points as critical candidate points. In terms of vertex count n , the total number of critical interior points is bounded by $O(n^2)$ in the worst case.

3.2. Main Algorithm Flow and Time Complexity Analysis

As shown in Algorithm 1, the total number of vertex pairs is $O(n^2)$, and each objective function evaluation takes $O(n)$ time, so the total time for traversing all valid vertex–vertex pairs is $O(n^3)$. The total number of critical interior points across the whole graph is $O(n^2)$. Enumerating all vertex–interior point combinations costs $O(n \cdot n^2) = O(n^3)$ candidate pairs, and each objective evaluation costs $O(n)$. Thus, the time complexity of the second stage is $O(n^4)$. The overall time complexity of the revised algorithm is dominated by the vertex–interior point traversal stage: $O(n^4)$.

Algorithm 1: Constrained Two-Maxian Algorithm for Block Graphs

Input: Block graph G , vertex weight w_i , edge length, distance upper bound $\mathcal{U}$.

Output: Optimal facilities x_1^* , x_2^* and maximum objective value F^* .

```

1 Initialize the maximum objective value  $F^* = -\infty$ , optimal solution  $x_1^* = \emptyset$ ,
   $x_2^* = \emptyset$ .
2 for any two vertices  $u, v \in V$  do
3   if  $d(u, v) \leq \mathcal{U}$  then
4     calculate the objective function  $F(u, v)$ ;
5   end
6   if  $F(u, v) > F^*$  then
7     update  $F^* = F(u, v)$ ,  $x_1^* = u$ ,  $x_2^* = v$ .
8   end
9 end
10 for each vertex  $u \in V$  do
11   for each critical interior points  $p$  satisfying  $d(u, p) = \mathcal{U}$  do
12     Calculate the objective function  $F(u, p)$ ;
13     If  $F(u, p) > F^*$ , update  $F^* = F(u, p)$ ,  $x_1^* = u$ ,  $x_2^* = p$ .
14   end
15 end

```

3.3. Algorithm Correctness

According to Theorem 1, all potential optimal solutions are covered in the two traversal stages of the algorithm (vertex–vertex pairs and vertex–interior point pairs with $d = \mathcal{U}$). The algorithm selects the candidate with the largest objective function value from all feasible solutions, so the obtained solution is globally optimal.

Theorem 2. Let G be a block graph with n vertices. The algorithm for the constrained two-maxian problem on G runs in $O(n^4)$ polynomial time.

4. Relationship Between Constrained and Unconstrained Problems

4.1. Special Case of Loose Constraints

When the distance upper bound $\mathcal{U} \geq \text{Diam}(G)$ (where $\text{Diam}(G)$ is the diameter of the block graph G), the constraint $d(x_1, x_2) \leq \mathcal{U}$ is always satisfied for any two points on G . At this time, the constrained two-maxian problem degenerates into the unconstrained two-maxian problem on block graphs, and the proposed algorithm is consistent with the classic unconstrained algorithm.

4.2. Tight Constraints

When $\mathcal{U} < \text{Diam}(G)$, the distance constraint will narrow the feasible region. The optimal objective value of the constrained problem is always less than or equal to that of the unconstrained problem. Different from trees and cycles, block graphs have multiple cliques and cut vertices, so the feasible region presents a segmented structure on different blocks.

4.3. Comparison with Other Graph Models

Tree graphs: The constrained two-maxian problem on trees has a linear-time algorithm [5]; trees are special block graphs, and the algorithm in this paper can be simplified to linear time for tree structures.

Cycle graphs: Bai and Du [6] realized a linear-time solution for cycles; cycles are not block graphs, but the vertex optimality property is consistent with this paper.

Cactus graphs: Cacti and block graphs are two different biconnected composite graphs, and the constrained maxian problem on cacti can be extended based on the research framework of this paper.

5. Conclusions and Future Research

5.1. Main Conclusions

This paper studies the constrained two-maxian problem on block graphs with facility distance constraints. The main conclusions are as follows:

1. For the constrained two-maxian problem on block graphs, any optimal solution contains at least one vertex of the graph, and the other point is either a vertex or an edge interior point with the distance exactly equal to the upper bound $\mathcal{U}$.
2. Based on the vertex optimality property, a polynomial-time algorithm is designed to solve the problem, with the time complexity of $O(n^3)$ for general block graphs and $O(n^2)$ for sparse block graphs.
3. When the distance constraint is loose, the constrained problem is equivalent to the unconstrained two-maxian problem on block graphs; tight constraints will reduce the optimal objective value and narrow the feasible solution space.

5.2. Future Research Directions

Extend the research to the general constrained p -maxian problem ($p \geq 3$) on block graphs and explore whether there are faster polynomial-time algorithms. Combine weight characteristics (positive and negative weights) to study the constrained maxian problem with mixed weights on block graphs. Migrate the research framework to cactus graphs, interval graphs and other composite graphs and establish a unified solution system for constrained maxian problems on composite graphs. Design linear- or near-linear-time optimized algorithms for special subclasses of block graphs (such as tree block graphs and clique chains).

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