

Article

An Additional Characterization of the Triple Consisting of Normal, Gamma, and Inverse Gaussian Distributions

Shaul K. Bar-Lev

Faculty of Industrial Engineering and Technology Management, HIT—Holon Institute of Technology,
Holon 5810201, Israel; shaulb@hit.ac.il

Abstract

Two characterizations of the triple consisting of the normal, gamma, and inverse Gaussian families are already known: one through the saddlepoint method, and another through the coincidence of the bilateral UMPU and generalized likelihood ratio tests. Bar-Lev and Reiser considered a steep two-parameter exponential family with canonical parameters (θ_1, θ_2) , canonical statistics $u_1(X)$ and $u_2(X)$, and mean parameters (η_1, η_2) , where $\eta_i = E_\theta[u_i(X)]$, $i = 1, 2$. They showed that the relation $\theta_2 = -\theta_1\phi'(\eta_2)$ provides a sufficient condition for obtaining the normal, gamma, and inverse Gaussian families. In the present paper, we impose an additional mild local smoothness and positivity condition on the transformed kernel on the u_2 -scale and prove the converse. Thus, within this locally regular framework, the above relation is not only sufficient but also necessary for obtaining the same triple. Accordingly, this gives a third characterization of the triple consisting of the normal, gamma, and inverse Gaussian families. This points to an intrinsic structural phenomenon: three different and apparently unrelated characterizations single out precisely the same classical triple.

Keywords: two-parameter exponential families; mixed parameterization; normal distribution; gamma distribution; inverse Gaussian distribution; variance functions; BLR characterization

MSC: 62E10; 62F03; 62H05

1. Introduction

The normal, gamma, and inverse Gaussian families form a remarkable triple in the theory of natural exponential families. In chronological order, two characterizations of this triple are already known. Ref. [1], using exact saddlepoint formulas for reproductive exponential families, obtained one characterization (see also [2]). Ref. [3] obtained another, based on the coincidence between the bilateral UMPU test and the generalized likelihood ratio test.

Between these two contributions stands the paper of [4], hereafter abbreviated as BLR. BLR established a sufficient condition within a steep two-parameter exponential family and raised the question of whether this condition is also necessary. The purpose of the present note is to revisit this converse problem. Under an additional local smoothness and positivity hypothesis imposed only on a nonempty open subinterval of H_2 , we prove the converse and hence obtain an additional characterization of the same classical triple. Thus, within the stated framework, we arrive at a third characterization of the natural exponential families generated, on the transformed scale, by the normal, gamma, and inverse Gaussian



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laws. It is striking that this same triple is singled out by three different and apparently unrelated approaches.

The sense in which the present result gives a third characterization should be understood as follows. Ref. [1] single out the triple through the exact saddlepoint/reproductive structure, while Ref. [3] single it out through the coincidence of two testing procedures. The present paper instead starts from the structural BLR mixed parameterization condition and proves, under the stated local kernel regularity hypothesis, that this condition is not merely sufficient for the three BLR examples but also necessary. Thus the characterization obtained here is not a restatement of either of the two earlier criteria; it is the converse completion of the BLR structural route.

BLR study a steep two-parameter exponential family dominated by the Lebesgue measure, written as

$$f(x; \theta) = a(x) \exp\{\theta_1 u_1(x) + \theta_2 u_2(x) + c(\theta)\}, \quad \theta = (\theta_1, \theta_2) \in \Theta \subset \mathbb{R}^2. \quad (1)$$

Here, $a(x) dx$ is the carrier measure, and u_1, u_2 are the canonical statistics. The canonical parameter set is

$$D = \left\{ \theta \in \mathbb{R}^2 : \int \exp\{\theta_1 u_1(x) + \theta_2 u_2(x)\} a(x) dx < \infty \right\}, \quad (2)$$

and $\Theta = \text{int } D$ is assumed to be nonempty. The function c is the negative log-Laplace transform, or equivalently the log-normalizing function in this sign convention:

$$c(\theta) = -\log \int \exp\{\theta_1 u_1(x) + \theta_2 u_2(x)\} a(x) dx. \quad (3)$$

Consequently the mean parameters are

$$\eta_i = E_\theta[u_i(X)] = -\frac{\partial c(\theta)}{\partial \theta_i}, \quad i = 1, 2, \quad (4)$$

or, equivalently,

$$\eta(\theta) = -\nabla c(\theta). \quad (5)$$

In the standard terminology, the family is regular when its natural parameter space is open. BLR assume steepness: whenever θ tends to a boundary point of Θ , the associated mean vector tends to the boundary of the mean parameter space. In the standard steepness formulation, this is expressed as

$$\|\nabla c(\theta)\| \longrightarrow \infty \quad (6)$$

as θ tends to the boundary of Θ , with the sign convention above understood.

Let H_2 denote the range of η_2 . By the general theory of mixed parameterizations (see [5]), the pair (θ_1, η_2) is variation-independent. That is, the admissible set of values of (θ_1, η_2) is the Cartesian product $\Theta_1 \times H_2$. Moreover, Θ_1 and H_2 are open intervals.

BLR then assume that there exists a differentiable function ϕ on H_2 such that

$$\theta_2 = -\theta_1 \phi'(\eta_2). \quad (7)$$

Under this condition they prove that

$$c(\theta_1, \theta_2) = \theta_1 \chi(\eta_2) - M(\theta_1), \quad (8)$$

where

$$\chi(\eta_2) = \eta_2 \phi'(\eta_2) - \phi(\eta_2), \quad (9)$$

and that

$$\eta_1 = \phi(\eta_2) + M'(\theta_1), \quad M''(\theta_1) > 0. \quad (10)$$

They also show that either $\Theta_1 \subset (-\infty, 0)$ and $\phi'' > 0$ on H_2 , or $\Theta_1 \subset (0, \infty)$ and $\phi'' < 0$ on H_2 .

If, in addition, u_2 is one-to-one, BLR write

$$y = u_2(x) = g(x), \quad x = g^{-1}(y), \quad (11)$$

and introduce the transformed kernel

$$r(y) = a(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|. \quad (12)$$

We call r the transformed kernel, since it is the density kernel of the transformed variable $Y = u_2(X)$ with respect to the Lebesgue measure on the u_2 -scale. BLR then show that, after centering u_1 by an appropriate constant if necessary,

$$u_1(g^{-1}(y)) = \phi(y). \quad (13)$$

Combining Equations (7)–(9), and (11)–(13), the density of $Y = u_2(X)$ in the mixed parameters (θ_1, η_2) is

$$f_Y(y; \theta_1, \eta_2) = r(y) \exp\{\theta_1[\phi(y) - \phi(\eta_2) - \phi'(\eta_2)(y - \eta_2)] - M(\theta_1)\}. \quad (14)$$

For a sample $X_1, \dots, X_n$, BLR define

$$T_i = \sum_{j=1}^n u_i(X_j), \quad i = 1, 2, \quad (15)$$

and prove that, for every $n \geq 2$,

$$Z_n = T_1 - n\phi(T_2/n) \quad (16)$$

is independent of T_2 . Furthermore, BLR prove the following reproducibility property: T_2/n has the same distribution as the transformed statistic $u_2(X)$, with parameter vector $(n\theta_1, n\theta_2)$ in place of (θ_1, θ_2) . Equivalently, in the transformed density (14), the same kernel r is used and the parameters θ_1, θ_2 are replaced by $n\theta_1, n\theta_2$. BLR established that their condition is satisfied by the normal, gamma, and inverse Gaussian examples, and left open the converse question.

The present paper has two aims. First, we explain in detail why a Jensen-type saddle-point argument does not, by itself, provide a rigorous converse proof in the BLR setting. Second, under a local regularity assumption on the transformed kernel, imposed only on a nonempty open interval, we prove the converse directly from BLR's exact sample-size-two factorization. The main ingredients are the analyticity of the mixed inverse map, the resulting analyticity of ϕ , the exact form of the transformed density on the u_2 -scale, and a local analysis of the statistic Z_2 . Finally, we recover the variance-function powers 0, 2, and 3, thereby linking the result with [6] For comprehensive accounts of exponential families and their associated variance functions, see [5,7] respectively.

The paper is organized as follows. Section 2 discusses the mathematical gaps in the Jensen-type route. Section 3 states the converse theorem and gives its proof. A brief concluding section closes the paper.

2. Why the Jensen-Type Route Does Not Settle the BLR Converse

In this section we explain, in mathematical terms, why a Jensen-type saddlepoint development does not by itself provide a proof of the BLR converse.

This criticism is not meant to deny that one may formally write preliminary cumulant expressions before invoking BLR Lemma 4.1. The point is that such expressions do not, by themselves, prove the BLR converse. Once the BLR setting is imposed, Assumption 3.2 and Lemma 4.1 give the exact identity used below, and the corresponding one-observation cumulants become degenerate. Without that identity, on the other hand, the cumulant expressions are not yet tied to the BLR condition strongly enough to identify the three families. The first genuinely informative exact object is therefore the sample-size-two statistic.

The central point is that the Jensen-type note starts from a one-observation quantity

$$Z = u_1(X) - \phi(u_2(X)), \quad U = u_2(X),$$

and then tries to extract differential information on ϕ from mixed cumulants of (Z, U) . This is not the BLR statistic. BLR work with

$$Z_n = T_1 - n\phi(T_2/n), \quad T_i = \sum_{j=1}^n u_i(X_j), \quad i = 1, 2,$$

and their theorem states that Z_n is independent of T_2 for every $n \geq 2$. Consequently, the first nontrivial case is $n = 2$, not $n = 1$.

Indeed, BLR's Assumption 3.2 together with their Lemma 4.1 implies that, after an inessential centering of u_1 , one has

$$u_1(g^{-1}(y)) = \phi(y)$$

on H_2 . Substituting $y = u_2(X)$ gives the exact identity

$$u_1(X) - \phi(u_2(X)) = 0 \quad \text{almost surely.}$$

Thus the one-observation variable Z introduced above is actually degenerate. In particular, for every $r \geq 1$ and $s \geq 0$,

$$\kappa_{r+s}(Z, \dots, Z, U, \dots, U) = 0,$$

where Z occurs r times and U occurs s times. This holds for the trivial reason that one of the arguments is constant. Hence, the one-observation mixed cumulants carry no nontrivial information relevant to the classification.

It is useful to see exactly where a truncated cumulant argument breaks down. Let

$$\eta_2 = E(U), \quad \xi = U - \eta_2, \quad W = u_1(X) - E\{u_1(X)\}.$$

Since $u_1(X) = \phi(U)$ almost surely, we have the exact identity

$$W = \phi(U) - E\{\phi(U)\}.$$

Because ϕ is real analytic on H_2 , we may expand at η_2 and obtain

$$W = \phi'(\eta_2)\xi + \frac{1}{2}\phi''(\eta_2)(\xi^2 - E[\xi^2]) + \frac{1}{6}\phi'''(\eta_2)(\xi^3 - E[\xi^3]) + \frac{1}{24}\phi^{(4)}(\eta_2)(\xi^4 - E[\xi^4]) + \dots$$

Therefore,

$$Z = u_1(X) - \phi(U) = E\{u_1(X) - \phi(U)\} + W - [\phi(U) - E\{\phi(U)\}] = 0,$$

and the exact cancelation of all orders is built into the BLR model.

This should be compared with the Jensen-type calculation, which replaces the exact identity by the first-order truncation

$$Z \simeq W - \phi'(\eta_2)\xi$$

and then substitutes this approximation into an exact cumulant relation, for example,

$$\kappa_3(Z, Z, U) = 0.$$

This step is unjustified unless one proves that the omitted terms make no contribution to the exact cumulant. In the present setting, however, there is neither a small parameter nor an asymptotic limiting regime. The neglected remainder

$$R_2 = \frac{1}{2}\phi''(\eta_2)(\xi^2 - E[\xi^2]) + \frac{1}{6}\phi'''(\eta_2)(\xi^3 - E[\xi^3]) + \dots$$

is itself a legitimate random variable of the same order of magnitude as the terms retained in the displayed approximation. By the multilinearity of cumulants,

$$\begin{aligned} \kappa_3(Z, Z, U) &= \kappa_3(W - \phi'(\eta_2)\xi, W - \phi'(\eta_2)\xi, U) \\ &\quad + 2\kappa_3(W - \phi'(\eta_2)\xi, R_2, U) + \kappa_3(R_2, R_2, U). \end{aligned}$$

The last two terms do not vanish automatically, and the Jensen-type argument provides no proof that they are negligible or that they cancel. Since the exact value of $\kappa_3(Z, Z, U)$ is zero only because Z itself is constant, the omitted terms must in fact cancel the displayed terms exactly. Thus any differential equation derived before establishing this cancelation is unsupported.

The same objection arises at the next order. The Jensen-type argument then invokes the second-order truncation

$$Z \simeq W - \phi'(\eta_2)\xi - \frac{1}{2}\phi''(\eta_2)(\xi^2 - E[\xi^2])$$

inside the exact fourth-order relation

$$\kappa_4(Z, Z, Z, U) = 0.$$

Again, however, the exact variable Z is constant, so every term omitted from the Taylor expansion must participate in the cancelation. Since this is not proved, the passage from the exact cumulant identity to a polynomial condition on the exponent is incomplete.

There is also a second and independent mathematical problem. An argument that imposes from the start the nonvanishing of ϕ''' effectively excludes the normal case. But BLR Example 5.1 is precisely the normal family, for which

$$\phi''' \equiv 0.$$

A derivation that excludes the normal case a priori cannot prove the full BLR converse.

Finally, it is useful to compare the one-observation quantity $Z = u_1(X) - \phi(u_2(X))$ with the first genuinely informative BLR statistic. After the change in variable $Y = u_2(X)$, the sample-size-two statistic becomes

$$\begin{aligned} Z_2 &= u_1(X_1) + u_1(X_2) - 2\phi\left(\frac{u_2(X_1) + u_2(X_2)}{2}\right) \\ &= \phi(Y_1) + \phi(Y_2) - 2\phi\left(\frac{Y_1 + Y_2}{2}\right). \end{aligned}$$

This quantity is nondegenerate, and BLR prove that it is independent of

$$T_2 = Y_1 + Y_2.$$

The exact change in variables

$$M = \frac{Y_1 + Y_2}{2}, \quad D = \frac{Y_1 - Y_2}{2}$$

then yields an exact local factorization. This is precisely where genuine information enters the argument.

For these reasons, the saddlepoint-inspired route does not, by itself, establish the BLR converse. The difficulty is not merely one of presentation. The one-observation quantity used in that approach is degenerate in the BLR model, and the cumulant calculations insert truncated expansions into exact identities without proving that the omitted terms vanish or cancel. The proof below therefore begins instead from BLR's exact sample-size-two independence statement.

3. The Converse Theorem

We keep throughout this section the notation and numbered formulas introduced in Section 1. In particular, Equation (1) is the original two-parameter family, Equation (4) defines the mean parameters, Equation (7) is the BLR condition, Equations (8)–(10) are the corresponding BLR identities, Equation (12) defines the transformed kernel r , Equation (13) is BLR Lemma 4.1 after the inessential centering of u_1 , Equation (14) is the transformed density on the u_2 -scale, and Equations (15) and (16) are the sample-size statistics. We shall also use the BLR reproducibility property stated immediately after Equation (16).

Before presenting the main theorem, we record three preliminary lemmas needed for the proof.

Lemma 1. *The map*

$$F : (\theta_1, \theta_2) \mapsto (\theta_1, \eta_2)$$

is real analytic on Θ and has a real analytic inverse from $\Theta_1 \times H_2$ onto Θ .

Proof of Lemma 1. BLR recall that $c(\theta_1, \theta_2)$ is real analytic on the open set Θ . Therefore,

$$\eta_2(\theta_1, \theta_2) = -\frac{\partial c(\theta_1, \theta_2)}{\partial \theta_2}$$

is real analytic on Θ , and hence, the map F is real analytic. Its Jacobian matrix is

$$J_F(\theta) = \begin{pmatrix} 1 & 0 \\ \partial \eta_2 / \partial \theta_1 & \partial \eta_2 / \partial \theta_2 \end{pmatrix}.$$

Therefore,

$$\det J_F(\theta) = \frac{\partial \eta_2}{\partial \theta_2}.$$

By exponential-family theory,

$$\frac{\partial \eta_2}{\partial \theta_2} = \text{Var}_\theta(u_2(X)) > 0, \quad \forall \theta \in \Theta.$$

Thus the Jacobian determinant never vanishes. By the real-analytic inverse function theorem, F has a real-analytic local inverse near every point of Θ . BLR also prove that (θ_1, η_2) is

a global mixed parameterization; that is, F is one-to-one from Θ onto $\Theta_1 \times H_2$. Therefore the local inverses patch together and define a global real-analytic inverse from $\Theta_1 \times H_2$ onto Θ . This proves the lemma. $\square$

Lemma 2. *The functions $\theta_2(\theta_1, \eta_2)$ and*

$$\tilde{c}(\theta_1, \eta_2) = c(\theta_1, \theta_2(\theta_1, \eta_2))$$

are real analytic on $\Theta_1 \times H_2$.

Proof of Lemma 2. By Lemma 1 the inverse map

$$(\theta_1, \eta_2) \mapsto (\theta_1, \theta_2)$$

is real analytic. Therefore $\theta_2(\theta_1, \eta_2)$ is real analytic on $\Theta_1 \times H_2$. Since c is real analytic on Θ , the composition

$$(\theta_1, \eta_2) \mapsto c(\theta_1, \theta_2(\theta_1, \eta_2))$$

is also real analytic. This proves the lemma. $\square$

Lemma 3. *The function ϕ is real analytic on H_2 .*

Proof of Lemma 3. Under the BLR condition (7) one has

$$\theta_2(\theta_1, \eta_2) = -\theta_1 \phi'(\eta_2).$$

BLR also show that either $\Theta_1 \subset (-\infty, 0)$ or $\Theta_1 \subset (0, \infty)$, so in particular, $\theta_1 \neq 0$ for every $\theta_1 \in \Theta_1$. Fix one value $\theta_1^0 \in \Theta_1$. Then

$$\phi'(\eta_2) = -\frac{\theta_2(\theta_1^0, \eta_2)}{\theta_1^0}.$$

By Lemma 2, the right-hand side is real analytic in η_2 on H_2 . Hence ϕ' is real analytic on H_2 , and therefore, ϕ is real analytic on H_2 , up to an additive constant. This proves the lemma. $\square$

Theorem 1. *Assume the BLR setting summarized in Equations (1)–(16), including steepness, the one-to-one assumption on u_2 , and the BLR condition Equation (7). Let r be the transformed kernel defined in Equation (12). Suppose that r has a representative which is strictly positive and of class C^4 on some nonempty open interval $I \subset H_2$. Then, on the transformed u_2 -scale, the only possible forms are, up to an affine change in the u_2 -scale and the addition of an affine term to ϕ ,*

$$\phi(\eta_2) = A\eta_2^2 + B\eta_2 + C,$$

$$\phi(\eta_2) = A \log(a\eta_2 + b) + B\eta_2 + C,$$

or

$$\phi(\eta_2) = \frac{A}{a\eta_2 + b} + B\eta_2 + C,$$

where in the last two cases, $a\eta_2 + b > 0$ on H_2 . Then, for each fixed $\theta_1 \in \Theta_1$, the variance function of the one-parameter transformed family generated by $Y = u_2(X)$, written in the mean parameter

η_2 , has, after affine normalization of the u_2 -scale and with the normalized mean, again denoted by η_2 , one of the three forms

$$V_{\theta_1}(\eta_2) = \alpha(\theta_1), \quad V_{\theta_1}(\eta_2) = \alpha(\theta_1)\eta_2^2, \quad V_{\theta_1}(\eta_2) = \alpha(\theta_1)\eta_2^3,$$

where $\alpha(\theta_1) > 0$. Moreover, the corresponding transformed kernel and support are those of the normal, gamma, and inverse Gaussian BLR examples, respectively; the normalized supports are $\mathbb{R}$, $(0, \infty)$, and $(0, \infty)$.

Proof of Theorem 1. The proof is divided into six steps.

Step 1. Analyticity of the mixed inverse map and of ϕ . The only external result required at this stage is the real-analytic inverse function theorem; see [8], Chapter 2. The desired conclusion therefore follows from the preceding three lemmas.

Step 2. The transformed family and the exact variance identity. Set

$$Y = u_2(X) = g(X).$$

By Equations (7)–(9) and (11)–(13), the density of Y in the mixed parameters (θ_1, η_2) is precisely Equation (14). We now compute the variance for the transformed variable. Since

$$\eta_2 = E_\theta(Y),$$

we have

$$\frac{\partial \eta_2}{\partial \theta_2} = \text{Var}_\theta(Y).$$

Differentiate

$$\theta_2 = -\theta_1 \phi'(\eta_2)$$

with respect to θ_2 , keeping θ_1 fixed. One obtains

$$1 = -\theta_1 \phi''(\eta_2) \frac{\partial \eta_2}{\partial \theta_2}.$$

Thus, for each fixed value of θ_1 , the variance function of the one-parameter subfamily indexed by η_2 is

$$V_{\theta_1}(\eta_2) = \text{Var}_{\theta_1, \eta_2}(Y) = -\frac{1}{\theta_1 \phi''(\eta_2)}. \quad (17)$$

Equation (17) should be read in the mixed-parametric sense. We are not fixing θ_1 once and for all in the original two-parameter family. Rather, for each admissible value of θ_1 , the mixed parametrization gives a one-parameter natural exponential family

$$\{P_{\theta_1, \eta_2} : \eta_2 \in H_2\}$$

in the mean parameter η_2 . Along this one-parameter subfamily, θ_1 is held fixed, and therefore, $V_{\theta_1}(\eta_2)$ is a function of the mean variable η_2 . The role of θ_1 is only to multiply this variance function by a positive scale factor. The shape of the variance function as a function of η_2 is therefore determined entirely by ϕ'' and is the same for all values of θ_1 ; only the multiplicative constant changes with θ_1 . This identity will later place the BLR converse in the power-variance-function framework of [6] up to this fixed scale factor.

Step 3. The exact factorization at sample size two. Because BLR already prove the independence of T_2 and Z_n for every $n \geq 2$, it is logical enough to work with one convenient

sample size. We choose $n = 2$. Let Y_1 and Y_2 be independent copies of Y . Introduce the mean-difference coordinates

$$M = \frac{Y_1 + Y_2}{2}, \quad D = \frac{Y_1 - Y_2}{2}.$$

Then

$$Y_1 = M + D, \quad Y_2 = M - D,$$

The Jacobian of the transformation $(Y_1, Y_2) \mapsto (M, D)$ has absolute value $1/2$; equivalently, the inverse transformation $(M, D) \mapsto (Y_1, Y_2)$ has absolute Jacobian 2. Thus the density of (M, D) contains the factor 2 appearing below. Define

$$q_m(d) = \phi(m + d) + \phi(m - d) - 2\phi(m).$$

Then

$$Z_2 = \phi(Y_1) + \phi(Y_2) - 2\phi(M) = q_M(D).$$

Using Equation (14), the joint density of (M, D) is

$$f_{M,D}(m, d) = 2r(m + d)r(m - d) \exp\{\theta_1 q_m(d) + 2\theta_1[\phi(m) - \phi(\eta_2) - \phi'(\eta_2)(m - \eta_2)] - 2M(\theta_1)\}.$$

By the explicit BLR reproducibility statement just recalled, $M = T_2/2$ has the transformed density with the same kernel r and with parameter pair $(2\theta_1, 2\theta_2)$. Hence,

$$f_M(m) = r(m) \exp\{2\theta_1[\phi(m) - \phi(\eta_2) - \phi'(\eta_2)(m - \eta_2)] - M(2\theta_1)\}.$$

This is the only point at which the full BLR reproducibility assertion for T_2/n is used, beyond the independence statement. No additional assumption is introduced here: this is precisely the BLR reproducibility statement recalled after Equation (16), specialized to $n = 2$. BLR also show that Z_2 is independent of $T_2 = 2M$. Since M is a one-to-one function of T_2 , it follows that Z_2 is independent of M .

Now fix $m \in H_2$. Since H_2 is an open interval, there exists $\delta > 0$ such that $m + d$ and $m - d$ belong to H_2 whenever $|d| < \delta$. Because ϕ'' has a fixed sign on H_2 , the map

$$d \mapsto q_m(d)$$

is strictly monotone on each of the two branches $(0, \delta)$ and $(-\delta, 0)$. Indeed,

$$\frac{d}{dd} q_m(d) = \phi'(m + d) - \phi'(m - d),$$

and this derivative has a fixed nonzero sign on each branch for sufficiently small $|d|$.

We work first on the branch $0 < d < \delta$. The opposite branch gives the same local information, and its contribution may be absorbed into the one-variable density of Z_2 . On this branch the change in variables $(m, d) \mapsto (m, z)$, with $z = q_m(d)$, has Jacobian

$$\left| \frac{dz}{dd} \right| = |\phi'(m + d) - \phi'(m - d)|.$$

There is no singularity problem on this branch. For $0 < d < \delta$ the denominator is nonzero, because ϕ'' has a fixed sign; moreover it is asymptotic to $2\phi''(m)d$ as $d \downarrow 0$. The endpoint $d = 0$ is not part of the branch. The resulting $z^{-1/2}$ behavior of the branch density near $z = 0$ is the ordinary integrable square-root singularity coming from the change from d to $z = q_m(d) \sim \phi''(m)d^2$. Let $p_{\theta_1}^+(z)$ denote the contribution to the density of Z_2 with respect

to z obtained from the branch $d > 0$. Equivalently, it is the push-forward density of the restriction of the joint density of (M, D) to that branch. Independence gives

$$\frac{2r(m+d)r(m-d)}{|\phi'(m+d) - \phi'(m-d)|} \exp\{\theta_1 q_m(d) - 2M(\theta_1) + M(2\theta_1)\} = r(m)p_{\theta_1}^+(q_m(d)).$$

Equivalently, if we define the one-variable factor

$$d_{\theta_1}^+(z) = p_{\theta_1}^+(z) \exp\{-\theta_1 z + 2M(\theta_1) - M(2\theta_1)\},$$

then the exact local factorization is

$$\frac{2r(m+d)r(m-d)}{|\phi'(m+d) - \phi'(m-d)|} = r(m)d_{\theta_1}^+(q_m(d)). \quad (18)$$

Thus, all factors depending on $z = q_m(d)$ alone, including the exponential factor involving θ_1 , have been absorbed into $d_{\theta_1}^+$. If the full density of Z_2 is used instead of the one-branch density, the two symmetric branch contributions only modify the right-hand one-variable factor. This has no effect on the differential invariants extracted below.

Step 4. The differential invariants. We first treat the convex case $\phi'' > 0$ on H_2 and set

$$h = \phi''.$$

The concave case is identical after replacing ϕ by $-\phi$, and the final forms are the same up to a change in constants. Since r is positive and of class C^4 on I , and since ϕ is real analytic by Lemma 3, all local Taylor expansions used below are legitimate for $m \in I$ and for d values sufficiently small that $m \pm d \in I$.

From the exact factorization (18), for fixed $m \in I$ and $d \downarrow 0$, one has

$$q_m(d) = hd^2 + \frac{h''}{12}d^4 + \frac{h^{(4)}}{360}d^6 + O(d^8),$$

and

$$\phi'(m+d) - \phi'(m-d) = 2hd + \frac{h''}{3}d^3 + \frac{h^{(4)}}{60}d^5 + O(d^7),$$

where all derivatives of h are evaluated at m . Also

$$r(m+d)r(m-d) = r^2 + (rr'' - (r')^2)d^2 + O(d^4).$$

Set

$$L(z) = z^{1/2}d_{\theta_1}^+(z).$$

Then (18) gives

$$L(q_m(d)) = \frac{2r(m+d)r(m-d)\sqrt{q_m(d)}}{r(m)|\phi'(m+d) - \phi'(m-d)|}.$$

Letting $d \downarrow 0$ gives

$$L(q_m(d)) \longrightarrow \frac{r(m)}{\sqrt{h(m)}}.$$

Since $q_m(d) \downarrow 0$ and L depends only on the single variable z , the limit $\lim_{z \downarrow 0} L(z)$ exists and is independent of m . Therefore, there exists a constant $C_0 > 0$ such that

$$r(m)^2 = C_0^2 h(m), \quad m \in I. \quad (19)$$

Thus, for the chosen positive C^4 representative of r , we may write

$$r(m) = C_0 \sqrt{h(m)}.$$

Substitution into (18) leaves a constant factor C_0 , which may be absorbed into the one-variable density $d_{\theta_1}^+$. After this harmless normalization, (18) becomes

$$d_{\theta_1}^+(q_m(d)) = \frac{2\sqrt{h(m+d)h(m-d)}}{\sqrt{h(m)}[\phi'(m+d) - \phi'(m-d)]}. \quad (20)$$

Here we have used $d > 0$ and $h > 0$, so the denominator is positive for sufficiently small d .

Expanding the right-hand side of Equation (20), and then rewriting the result in terms of $z = q_m(d)$, gives

$$d_{\theta_1}^+(z) = z^{-1/2}\{C_1 + C_1 A_2(m)z + C_1 B_2(m)z^2 + O(z^3)\},$$

where C_1 is independent of m and

$$A_2(m) = \frac{3hh'' - 4(h')^2}{8h^3},$$

while

$$B_2(m) = \frac{40h^3h^{(4)} - 192h^2h'h^{(3)} - 85h^2(h'')^2 + 408hh'^2h'' - 144h'^4}{1152h^6}.$$

Since $d_{\theta_1}^+(z)$ depends only on z for fixed θ_1 , the coefficients $A_2(m)$ and $B_2(m)$ must be independent of m . Therefore,

$$\frac{3hh'' - 4(h')^2}{h^3} = K_1, \quad (21)$$

and

$$\frac{40h^3h^{(4)} - 192h^2h'h^{(3)} - 85h^2(h'')^2 + 408hh'^2h'' - 144h'^4}{h^6} = K_2, \quad (22)$$

where K_1 and K_2 are constants on the interval I .

Step 5. Solution of the differential system. Set

$$y = h^{-1/3}.$$

A direct differentiation shows that Equation (21) is equivalent to

$$y'' = cy^{-2} \quad (23)$$

for a constant c . We next rewrite Equation (22) in terms of y . A direct calculation gives

$$\begin{aligned} & \frac{40h^3h^{(4)} - 192h^2h'h^{(3)} - 85h^2(h'')^2 + 408hh'^2h'' - 144h'^4}{h^6} \\ &= 3y^3[-40y^2y^{(4)} + 64yy'y^{(3)} + 225y(y'')^2 + 480(y')^2y'']. \end{aligned}$$

Now substitute $y'' = cy^{-2}$. Then

$$y^{(3)} = -2cy^{-3}y'$$

and

$$y^{(4)} = 6cy^{-4}(y')^2 - 2c^2y^{-5}.$$

Substitution into the preceding display and simplification yields

$$K_2 = 3c[305c + 112y(y')^2]. \quad (24)$$

For detailed derivations of Equations (23) and (24) refer to Appendix A. Thus, there are two cases.

First suppose $c = 0$. Then $y'' = 0$, so

$$y(\eta_2) = a\eta_2 + b.$$

Consequently,

$$h(\eta_2) = y(\eta_2)^{-3} = \frac{A}{(a\eta_2 + b)^3}.$$

The constant case is included here when $a = 0$. Thus, the normal case, which corresponds to constant $h = \phi''$, is not excluded at any stage of the differential-system argument. No assumption such as $\phi''' \neq 0$ is used.

Next suppose $c \neq 0$. Multiplying Equation (23) by y' and integrating once gives

$$(y')^2 = C_1 - \frac{2c}{y}$$

for some constant C_1 . Multiplying by y gives

$$y(y')^2 = C_1 y - 2c.$$

But Equation (24) shows that $y(y')^2$ is constant. In the nonconstant case this forces $C_1 = 0$. Hence,

$$(y')^2 = -\frac{2c}{y}.$$

Now compute the second derivative of $y^{3/2}$:

$$\frac{d^2}{d\eta_2^2} y^{3/2} = \frac{3}{2} y^{1/2} y'' + \frac{3}{4} y^{-1/2} (y')^2.$$

Substituting $y'' = cy^{-2}$ and $(y')^2 = -2c/y$ gives

$$\frac{d^2}{d\eta_2^2} y^{3/2} = 0.$$

Hence,

$$y(\eta_2)^{3/2} = a\eta_2 + b,$$

that is,

$$y(\eta_2) = (a\eta_2 + b)^{2/3}.$$

Therefore,

$$h(\eta_2) = y(\eta_2)^{-3} = \frac{A}{(a\eta_2 + b)^2}.$$

We have proven that the only possibilities are

$$h(\eta_2) = \text{constant},$$

$$h(\eta_2) = \frac{A}{(a\eta_2 + b)^2},$$

and

$$h(\eta_2) = \frac{A}{(a\eta_2 + b)^3}.$$

These alternatives have been obtained first on the interval I . Since $h = \phi''$ is real analytic and does not vanish on H_2 , the identity principle for real analytic functions extends the corresponding expression for h from I to all of the connected interval H_2 . In the last two cases, H_2 is contained in one component of $a\eta_2 + b \neq 0$, and the constants may be renamed if necessary.

Integrating twice gives, up to affine terms,

$$\phi(\eta_2) = A\eta_2^2 + B\eta_2 + C,$$

$$\phi(\eta_2) = A \log(a\eta_2 + b) + B\eta_2 + C,$$

and

$$\phi(\eta_2) = \frac{A}{a\eta_2 + b} + B\eta_2 + C.$$

The proof above treated the convex case $\phi'' > 0$. If $\phi'' < 0$, replace ϕ with $-\phi$. The same argument then applies to $-\phi$, and after changing the signs of the constants, one obtains exactly the same three functional forms. In either sign case, the local kernel relation is

$$r(\eta_2)^2 = C_0^2 |\phi''(\eta_2)|, \quad \eta_2 \in I.$$

This relation is local. The next step uses the global normalizing identity to identify the kernel on the whole transformed support, up to the usual freedom to alter representatives on sets of Lebesgue measure zero.

Step 6. Identification of the three transformed families and the variance functions.

We now identify the three resulting BLR forms directly from the transformed density in Equation (14) and the variance identity in Equation (17). The calculation shows that, after an affine change in the u_2 scale, the transformed density has exactly the Gaussian, gamma, or inverse Gaussian kernel appearing in BLR Examples 5.1–5.3. This is an identification of the transformed variable $Y = u_2(X)$, not necessarily of X itself, unless $u_2(x) = x$.

We also record why the local kernel information obtained on I is enough for the global identification. Let S denote the essential support interval of the transformed variable Y . Once one of the three analytic forms of ϕ has been obtained on H_2 , the normalizing identity in Equation (14) holds for all admissible mean values and for an open set of values of θ_1 . This identity determines the kernel on the whole support, up to sets of Lebesgue measure zero, by the uniqueness of Fourier or Laplace transforms.

In the quadratic case the normalizing identity has the form

$$\int_S r(t) \exp\{-\lambda(t - \tau)^2\} dt = C(\lambda),$$

for λ in a nonempty open interval and for τ in an open interval. The left-hand side is the Gaussian convolution of the measure $r(t)\mathbf{1}_S(t)dt$. Since it is constant on an open set of centers and is real analytic in the center, analytic continuation and uniqueness of the Fourier transform force $r(t)\mathbf{1}_S(t)dt$ to be a constant multiple of Lebesgue measure on $\mathbb{R}$. Hence, in the normalized quadratic case, $S = \mathbb{R}$ and r is constant, up to null sets.

In the logarithmic case set $t = ay + b$ and $\tau = a\eta_2 + b$. The normalizing identity is equivalent, for a nonempty open set of shape parameters $k > 0$ and for a nonempty open interval of $s > 0$, to

$$\int_S r(t)t^k e^{-st} dt = C(k)s^{-k}.$$

By uniqueness of the Laplace transform, the measure $r(t)t^k \mathbf{1}_S(t)dt$ must be proportional to $t^{k-1} \mathbf{1}_{(0,\infty)}(t)dt$. Thus, $S = (0, \infty)$ and $r(t)$ is proportional to t^{-1} , up to null sets.

In the reciprocal case the same argument gives, for a nonempty open interval of $\lambda > 0$ and a nonempty open interval of $a > 0$, equality between the Laplace transform over S of the measure

$$r(t) \exp\left\{-\frac{\lambda}{2t}\right\} dt$$

and the corresponding full inverse-Gaussian transform over $(0, \infty)$. Laplace-transform uniqueness therefore forces $S = (0, \infty)$ and $r(t)$ to be proportional to $t^{-3/2}$, up to null sets. Thus the argument identifies the full BLR supports and kernels, not merely the local kernels.

First suppose that

$$\phi(\eta_2) = A\eta_2^2 + B\eta_2 + C.$$

Then $|\phi''|$ is constant, and the kernel-identification argument above gives a constant kernel. Also

$$\phi(y) - \phi(\eta_2) - \phi'(\eta_2)(y - \eta_2) = A(y - \eta_2)^2.$$

Hence, (14) becomes

$$f_Y(y; \theta_1, \eta_2) \propto \exp\{\theta_1 A(y - \eta_2)^2\}.$$

Since $\theta_1 \phi'' < 0$, one has $\theta_1 A < 0$. By the support argument above, the normalized support is $\mathbb{R}$. Thus, after an affine normalization of the u_2 scale, the transformed density is the Gaussian density of BLR Example 5.1.

Next suppose that

$$\phi(\eta_2) = A \log(a\eta_2 + b) + B\eta_2 + C.$$

Set

$$t = ay + b, \quad \tau = a\eta_2 + b.$$

Then $t > 0$ and $\tau > 0$ on H_2 , and the kernel-identification argument above gives

$$r(y) \propto (ay + b)^{-1} = t^{-1}.$$

Moreover,

$$\phi(y) - \phi(\eta_2) - \phi'(\eta_2)(y - \eta_2) = A \left[\log t - \log \tau - \frac{t - \tau}{\tau} \right].$$

Therefore, the density of $T = aY + b$ is, up to normalization,

$$f_T(t; \theta_1, \tau) \propto t^{-1} \exp\left\{\theta_1 A \left[\log t - \log \tau - \frac{t - \tau}{\tau} \right]\right\}.$$

Equivalently,

$$f_T(t; \theta_1, \tau) \propto t^{A\theta_1 - 1} \exp\left\{-\frac{A\theta_1}{\tau} t\right\}, \quad t > 0.$$

In this case, $\theta_1 A > 0$. By the support argument above, the normalized support is $(0, \infty)$. This is therefore the gamma density occurring in BLR Example 5.2.

Finally, suppose that

$$\phi(\eta_2) = \frac{A}{a\eta_2 + b} + B\eta_2 + C.$$

Again put

$$t = ay + b, \quad \tau = a\eta_2 + b.$$

Then the kernel relation gives

$$r(y) \propto (ay + b)^{-3/2} = t^{-3/2}.$$

Also

$$\begin{aligned}\phi(y) - \phi(\eta_2) - \phi'(\eta_2)(y - \eta_2) &= A \left[\frac{1}{t} - \frac{1}{\tau} + \frac{t - \tau}{\tau^2} \right] \\ &= A \left[\frac{1}{t} + \frac{t}{\tau^2} - \frac{2}{\tau} \right].\end{aligned}$$

Therefore, the density of $T = aY + b$ is, up to normalization,

$$f_T(t; \theta_1, \tau) \propto t^{-3/2} \exp \left\{ \theta_1 A \left[\frac{1}{t} + \frac{t}{\tau^2} - \frac{2}{\tau} \right] \right\}, \quad t > 0.$$

Since $\theta_1 A < 0$ in this case, write

$$\lambda = -2\theta_1 A > 0.$$

Then

$$\begin{aligned}f_T(t; \theta_1, \tau) &\propto t^{-3/2} \exp \left\{ -\frac{\lambda}{2} \left[\frac{1}{t} + \frac{t}{\tau^2} - \frac{2}{\tau} \right] \right\} \\ &\propto t^{-3/2} \exp \left\{ -\frac{\lambda}{2\tau^2} \frac{(t - \tau)^2}{t} \right\}, \quad t > 0.\end{aligned}$$

By the support argument above, the normalized support is $(0, \infty)$. This is exactly the inverse Gaussian density occurring in BLR Example 5.3. Thus, after the affine normalization $t = ay + b$, the transformed density has the BLR inverse Gaussian form.

We now return to the variance functions. From Equation (17) one has, for each fixed θ_1 ,

$$V_{\theta_1}(\eta_2) = -\frac{1}{\theta_1 \phi''(\eta_2)}.$$

Thus, in the three cases above, before normalization,

$$V_{\theta_1}(\eta_2) = \alpha(\theta_1),$$

$$V_{\theta_1}(\eta_2) = \alpha(\theta_1)(a\eta_2 + b)^2,$$

and

$$V_{\theta_1}(\eta_2) = \alpha(\theta_1)(a\eta_2 + b)^3,$$

where $\alpha(\theta_1) > 0$. After the affine normalization of the u_2 -scale we again denote the normalized mean by η_2 . The three variance functions then become

$$V_{\theta_1}(\eta_2) = \alpha(\theta_1), \quad V_{\theta_1}(\eta_2) = \alpha(\theta_1)\eta_2^2, \quad V_{\theta_1}(\eta_2) = \alpha(\theta_1)\eta_2^3.$$

We emphasize that the preceding conclusion is made within the mixed parameterization. For each fixed $\theta_1 \in \Theta_1$, the transformed statistic $Y = u_2(X)$ generates a one-parameter natural exponential family in the natural parameter θ_2 , or equivalently in the mean parameter η_2 . The factor θ_1 therefore affects only the positive multiplicative scale of the variance function. The power of the variance function as a function of η_2 is determined solely by ϕ'' . Hence the three possible powers are 0, 2, and 3. Together with the preceding support and kernel identification, these three cases are precisely the normal, gamma, and inverse Gaussian BLR examples, after the harmless affine normalization of the u_2 -scale. This completes the proof of Theorem 1. $\square$

4. Conclusions

The purpose of this paper is to revisit the converse problem left open in BLR. The argument developed here is proven under the local smoothness and positivity hypothesis imposed in Theorem 1, namely on a nonempty open subinterval of H_2 . Within this framework, however, the proof is exact throughout, both at the level of the BLR functional forms and at the level of the resulting variance functions. It does not rely on saddlepoint approximation. Instead, it rests on four ingredients: the BLR mixed parameterization, the analyticity of the inverse mixed map, the exact transformed density on the u_2 -scale, and the exact factorization induced by the sample-size-two independence statement.

The result should therefore be read with this qualification: under the local hypothesis of Theorem 1, the BLR condition, together with the BLR sample-size-two independence mechanism, leads only to the three analytic forms corresponding to the normal, gamma, and inverse Gaussian examples. In variance-function terms, these correspond exactly to the power exponents 0, 2, and 3.

At present, it remains unclear whether the local smoothness and positivity assumption on r is intrinsic to the converse theorem itself or only to the proof given here. Thus the problem of removing this hypothesis under the original BLR assumptions remains open. All standard BLR examples, and the usual smooth exponential-family models to which the theorem is naturally applied, satisfy this local condition. At the same time, no counterexample is known to the author in which the BLR condition holds, the local smoothness of the transformed kernel fails, and a fourth family arises. Removing the hypothesis would require a separate automatic-regularity theorem for measurable solutions of the exact factorization equation, including control of the square-root singularity at $d = 0$; this lies beyond the scope of the present paper.

Although the result is theoretical, its relevance is not confined to the formal classification of distributions. Normal, gamma, and inverse Gaussian NEFs are basic building blocks in likelihood theory, generalized linear modeling, reliability, signal and noise modeling, queuing and first-passage problems, and information-processing models. This connection is especially visible in contemporary statistical learning and computer science, where exponential-family models provide analytically tractable likelihoods, conjugate structures, sufficient statistics, and variational approximations; see, for example [9–11]. In information geometry, exponential families also form the canonical examples of statistical manifolds with dual affine structures; see [12]. The present result therefore has a useful modeling interpretation: under the BLR structural condition and the stated local kernel regularity, the choice among the normal, gamma, and inverse Gaussian forms is not merely a matter of convenience, but is forced by an exact independence and reproducibility mechanism. This perspective is consistent with the broader role of natural exponential families in statistical decision theory and model construction; see [13].

We thereby obtain a third characterization of the natural exponential families generated, on the transformed scale, by the normal, gamma, and inverse Gaussian laws. It is noteworthy that this same triple is singled out by three distinct, and apparently unrelated, approaches.

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factorization, and an expanded discussion of the relevance of the result for statistical modeling, information geometry, and contemporary data-science applications.

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Appendix A. Verification of the Differential Reduction

For completeness, and in response to the request for more detail in the calculation following Equations (21) and (22), we record the elementary algebra leading from the differential invariants to the three possible forms of $h = \phi''$. Set

$$y = h^{-1/3}, \quad h = y^{-3}.$$

Then

$$\begin{aligned} h' &= -3y^{-4}y', \\ h'' &= 12y^{-5}(y')^2 - 3y^{-4}y''. \end{aligned}$$

Consequently,

$$3hh'' - 4(h')^2 = 3y^{-3}\{12y^{-5}(y')^2 - 3y^{-4}y''\} - 4\{9y^{-8}(y')^2\} = -9y^{-7}y''.$$

Since $h^3 = y^{-9}$, Equation (21) becomes

$$-9y^2y'' = K_1,$$

or equivalently

$$y'' = cy^{-2}, \quad c = -K_1/9.$$

This proves Equation (23).

For the fourth-order invariant, one differentiates further:

$$\begin{aligned} h^{(3)} &= -60y^{-6}(y')^3 + 36y^{-5}y'y'' - 3y^{-4}y^{(3)}, \\ h^{(4)} &= 360y^{-7}(y')^4 - 360y^{-6}(y')^2y'' + 36y^{-5}(y'')^2 + 48y^{-5}y'y^{(3)} - 3y^{-4}y^{(4)}. \end{aligned}$$

Substitution of these formulas into the left-hand side of Equation (22), followed by collection of powers of y , gives

$$\begin{aligned} &\frac{40h^3h^{(4)} - 192h^2h'h^{(3)} - 85h^2(h'')^2 + 408h(h')^2h'' - 144(h')^4}{h^6} \\ &= 3y^3\{-40y^2y^{(4)} + 64yy'y^{(3)} + 225y(y'')^2 + 480(y')^2y''\}. \end{aligned}$$

Using $y'' = cy^{-2}$ gives

$$y^{(3)} = -2cy^{-3}y',$$

and

$$y^{(4)} = 6cy^{-4}(y')^2 - 2c^2y^{-5}.$$

Substitution into the preceding display yields

$$3y^3\{112cy^{-2}(y')^2 + 305c^2y^{-3}\} = 3c\{305c + 112y(y')^2\},$$

which is precisely (24).

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