

Article

Well-Posedness of Nonlinear Implicit ψ -Hilfer Fractional Problems of Complex Order with Applications to an Oscillator with Saturating Feedback

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Abstract

This paper investigates the well-posedness of a class of nonlinear implicit fractional differential equations involving the ψ -Hilfer fractional derivative of complex order α with $\Re(\alpha) \in (n - 1, n)$ for $n \in \mathbb{N}$, under general initial conditions in weighted spaces. The implicit nature of the problem, where the highest-order derivative appears nonlinearly on both sides of the equation, presents significant analytical challenges. By transforming the fractional Cauchy problem into an equivalent Volterra integral equation, we employ fixed-point theory to establish existence via Schaefer's fixed-point theorem and uniqueness via Banach's fixed-point theorem under suitable Lipschitz-type conditions. A generalized Gronwall inequality with singular kernels is developed to handle the nonlocal memory effects inherent to fractional operators. We further investigate four types of Ulam stability, namely Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability, demonstrating that small perturbations in the equation yield correspondingly small changes in the solution. Continuous dependence on initial conditions is also established. The theoretical framework is applied to a physically motivated fractional nonlinear oscillator with saturating acceleration-dependent feedback, where explicit verification of the hypotheses is provided.

Keywords: implicit fractional differential equations; ψ -Hilfer fractional derivative; complex order; Cauchy problem; fixed-point theory; Ulam stability; generalized Gronwall inequality

MSC: 26A33, 34A08, 34B15



Academic Editor: Roman Ivanovich Parovik

Received: 17 June 2026

Revised: 24 July 2026

Accepted: 1 August 2026

Published: 3 August 2026

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1. Introduction

Fractional differential equations (FDEs) have emerged as indispensable mathematical tools for modeling complex phenomena in which the current state of a system depends on its entire history. Unlike classical integer-order models, which assume an instantaneous and local response, fractional-order operators encode memory and hereditary properties through a nonlocal convolution structure [1–3]. This makes FDEs particularly well-suited for applications in viscoelasticity, anomalous diffusion, signal processing, control theory, and biological systems [4,5].

Among the many fractional derivative formulations that have been proposed, the Hilfer derivative occupies a distinguished position because it continuously interpolates between the Riemann–Liouville and Caputo derivatives through a single type parameter

$\beta \in [0, 1]$. Building on this, the ψ -Hilfer fractional derivative [6,7] was introduced, which further generalizes the operator by incorporating a strictly increasing auxiliary function ψ , thereby allowing the fractional calculus to adapt to different temporal scales and geometric settings. This flexibility has attracted growing analytical interest, and well-posedness and stability results for ψ -Hilfer FDEs have been obtained in several recent contributions [8–12].

A particularly challenging subclass consists of implicit FDEs, in which the highest-order fractional derivative appears nonlinearly on both sides of the governing equation. Specifically, we consider the ψ -Hilfer fractional Cauchy problem

$$({}^H D_{a+}^{\alpha, \beta; \psi} x)(t) = f\left(t, x(t), ({}^H D_{a+}^{\alpha, \beta; \psi} x)(t)\right), \quad t \in (a, b], \quad a \geq 0, \quad (1)$$

subject to the generalized initial conditions

$$x_{n-\gamma}^{(n-k); \psi}(a) = c_k, \quad c_k \in \mathbb{C}, \quad k \in \{1, 2, \dots, n\}, \quad (2)$$

where ${}^H D_{a+}^{\alpha, \beta; \psi}$ is the ψ -Hilfer derivative of complex order α with $\Re(\alpha) \in (n-1, n)$ and type $\beta \in [0, 1]$, $\gamma = \alpha + \beta(n - \alpha)$, and the solution x is sought in the weighted space $C_{n-\gamma, \psi}^{\gamma}[a, b]$. The operator notation $x_{n-\gamma}^{(n-k); \psi} := \delta_{\psi}^{n-k} I_{a+}^{n-\gamma; \psi}$, with $\delta_{\psi} := \frac{1}{\psi'(t)} \frac{d}{dt}$, ensures that the initial data are prescribed consistently with the regularity of solutions in the weighted space. Here, $f : (a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function.

The present work extends our previous analysis of implicit Hilfer fractional differential equations [12] in two important directions. First, we replace the standard Hilfer derivative with the ψ -Hilfer derivative, where the function $\psi(t)$ can be chosen freely to adapt the model to different applications ($\psi(t) = \ln t$ gives a Hadamard-type derivative; $\psi(t) = t$ gives a Hilfer-type derivative). Second, we allow the fractional order α to be complex, with $\Re(\alpha) \in (n-1, n)$. Complex-order derivatives arise naturally in models of viscoelasticity (capturing damping via complex moduli) and in control systems with oscillatory memory, whereas our previous work [12] was limited to real orders. Physically, a complex order $\alpha = \Re(\alpha) + i\Im(\alpha)$ endows the fractional operator with a built-in phase shift: writing the power-law kernel as $(\Delta_t^{\psi} t)^{\alpha-1} = (\Delta_t^{\psi} t)^{\Re(\alpha)-1} e^{i\Im(\alpha) \ln(\Delta_t^{\psi} t)}$, the real part $\Re(\alpha)$ controls the usual power-law memory decay, while the imaginary part $\Im(\alpha)$ modulates this decay by a logarithmically oscillating factor. This is the fractional-calculus analogue of a complex modulus/complex compliance in viscoelasticity, where the storage and loss moduli are simultaneously encoded in a single complex-valued material parameter, and it mirrors the use of complex susceptibilities in dielectric relaxation and complex admittances in fractional-order circuit elements (constant phase elements). In control engineering, complex-order operators arise naturally when designing CRONE-type and fractional-order PID controllers with a prescribed constant phase margin over a frequency band, a property that cannot be reproduced by a real-order operator alone. The principal advantage of a complex-order model over two independent real-order models is therefore parsimony: a single complex parameter α captures both the power-law decay rate and the oscillatory/dispersive character of the memory kernel, which would otherwise require a superposition of several real-order terms. Note that the imaginary part $\Im(\alpha)$ enters only through the modulus $|\Gamma(\alpha)|$ and the complex exponents $(\Delta_a^{\psi} t)^{\alpha-k}$; all stability constants depend on α solely through $\Re(\alpha)$ and $|\Gamma(\alpha)|$.

The study of Hilfer and ψ -Hilfer FDEs has developed rapidly since the foundational works of Hilfer [5] and Sousa–Oliveira [6]. Existence and uniqueness for explicit Hilfer equations of order $\alpha \in (0, 1)$ were established in [13,14], and extended to nonlocal boundary conditions in [15]. Ulam stability for Caputo-type equations was investigated in [16,17], and for the ψ -Hilfer setting in [9,18]. Implicit Hilfer equations of order $\alpha \in (0, 1)$ were studied

in [10,19], where existence and UH stability were obtained under Lipschitz conditions in a fixed weighted space. The present work differs from all these contributions in three essential respects: (i) we treat the fully general case $\Re(\alpha) \in (n-1, n)$ for any $n \in \mathbb{N}$; (ii) we work with complex order α , which unifies the real-order theory; and (iii) we provide quantitative Ulam constants expressed through the function $\mathcal{F}_{\rho,a,b}$, sharpening the qualitative estimates available in the existing literature.

The literature on ψ -Hilfer and related fractional operators has continued to grow rapidly during the preparation of this work. Multi-term ψ -Caputo equations with infinite delay have been shown to be existent and Hyers–Ulam stable via the Leray–Schauder alternative and new delay-adapted Gronwall-type inequalities [20]; Ulam–Hyers–Rassias and Ulam–Hyers stability for ψ -Hilfer Volterra integro-differential equations with multiple variable delays have been obtained by fixed-point methods [21]; existence and Ulam–Hyers–Rassias/semi-Ulam–Hyers–Rassias stability for ψ -Hilfer equations taken in the Caputo sense have been established using Mönch’s fixed-point theorem together with measures of noncompactness [22]; Ulam-type stability for ψ -Hilfer Volterra integro-delay differential and integral equations incorporating multiple variable time delays has been proved via the Banach contraction principle [23]; and, closest in spirit to our application in Section 6, existence, uniqueness and Ulam-type stability for implicit (k, ψ) -Hilfer fractional differential equations have been established and applied to an RC electric circuit model [24]. None of these contributions treats complex order α , and none handles the implicit nonlinearity via the auxiliary functional-equation reformulation used in Lemma 12; our doubly singular Gronwall inequality (Theorem 4) and the explicit Ulam constants of Remark 7 are also new relative to [20–24]. On the modeling side, the foundational treatment of complex-order fractional derivatives in viscoelasticity by Atanacković, Konjik, Pilipović and Zorica [25], who derived real-valued thermodynamic compatibility constraints for complex-order constitutive laws, provides independent physical motivation for the complex-order operator studied here; unlike [25], we do not impose a reality constraint on the solution and instead study the full complex-valued well-posedness theory, with the real-order (physically constrained) case recovered as the special instance $\Im(\alpha) = 0$.

To make the contribution of the paper explicit, we summarize below the aspects in which the present work goes beyond the existing ψ -Hilfer and Hilfer literature, including our own previous contribution [26]:

- (C1) Complex order. We treat the ψ -Hilfer derivative of complex order $\alpha \in \mathbb{C}$ with $\Re(\alpha) \in (n-1, n)$, $n \in \mathbb{N}$, whereas [6–11] and our earlier paper [26] are confined to real order. All well-posedness and stability constants are expressed through $\Re(\alpha)$ and the modulus $|\Gamma(\alpha)|$ only, so the classical real-order theory is recovered as the special case $\Im(\alpha) = 0$.
- (C2) Arbitrary integer part n . The analysis covers the whole range $\Re(\alpha) \in (n-1, n)$ for arbitrary $n \in \mathbb{N}$, rather than being restricted to $\alpha \in (0, 1)$ as in [13,14,19].
- (C3) Implicit nonlinearity via an auxiliary functional equation. The reformulation of the implicit equation through the auxiliary function G_x (Lemma 12) is new in the ψ -Hilfer, complex-order setting; it reduces to the explicit theory when f does not depend on its third argument, thereby unifying both cases in a single framework.
- (C4) A doubly singular generalized Gronwall inequality. Theorem 4 extends the Gronwall-type inequality of Ye et al. [27] and of [12] to ψ -weighted kernels with a singularity both at $s = t$ and at $s = a$, which is required because of the weighted initial conditions (2).
- (C5) Explicit, computable stability constants. Unlike the qualitative Ulam stability statements in [16–19], Theorems 9–11 give closed-form constants $c_f, c_{f,\sigma}$ expressed

through the entire function $F_{\rho,a,b}$ of Definition 11, which we evaluate numerically in Section 6.

We emphasize that the fixed-point theorems (Schaefer, Banach) and compactness criterion (Arzelà–Ascoli) used below are classical and are not themselves new. The contribution of the present paper is not the invention of new abstract fixed-point machinery, but rather (i) the non-trivial reformulation (C3) that makes this machinery applicable to the implicit, complex-order, ψ -weighted problem in the first place, and (ii) the new doubly singular Gronwall inequality (C4), which is required because the standard tools of Ye et al. [27] and of our earlier work [12] do not accommodate the additional singularity at $s = a$ introduced by the weighted initial conditions (2). We therefore regard the present work as a genuine but incremental extension of the existing ψ -Hilfer and Hilfer literature, rather than as a paper introducing fundamentally new analytical tools.

Three key difficulties distinguish problem (1)–(2) from the explicit ψ -Hilfer theory.

1. Implicit nonlinearity. Because ${}^H D_{a+}^{\alpha,\beta;\psi} x$ appears as both the unknown and an argument of f , one cannot apply the fractional integral $I_{a+}^{\alpha;\psi}$ directly to obtain a Volterra equation. Instead, one must first solve an auxiliary functional equation $G_x(t) = f(t, x(t), G_x(t))$ pointwise in t , extract G_x as an element of $C_{n-\gamma,\psi}^{\beta(n-\alpha)}[a, b]$, and only then recover x through the integral representation (22).
2. Singular kernels. The ψ -Riemann–Liouville integral $I_{a+}^{\alpha;\psi}$ carries the kernel $\psi'(\tau)(\Delta_\tau^\psi t)^{\alpha-1}$, which is singular both at $\tau = t$ (when $\Re(\alpha) < 1$) and at $\tau = a$ (through the weighted space norm). Standard Gronwall inequalities are inapplicable; a new ψ -adapted Gronwall inequality with a double singularity is required.
3. General complex order. Extending the theory to complex α with $\Re(\alpha) \in (n-1, n)$ requires careful tracking of real and imaginary parts throughout, particularly in the contraction constant Λ and in the Beta-function estimates used in the compactness argument.

We address these difficulties systematically. The implicit equation is first converted to an equivalent Volterra formulation (Lemma 12), whose derivation relies on the composition identities in Theorems 5 and 6 and the auxiliary space characterization of Lemma 10. Existence is then proved via Schaefer’s fixed-point theorem (Theorem 7) by establishing complete continuity of the fixed-point operator $\mathcal{F}$ and boundedness of the set $\mathcal{E}(\mathcal{F})$. Uniqueness (Theorem 8) follows from a Banach contraction argument under the condition $\Lambda < 1$, where Λ is expressed explicitly in terms of ψ , α , and the Lipschitz constants of f . Ulam-type stabilities (Theorems 9–10) and continuous dependence on initial data (Theorem 11) are derived using the new generalized Gronwall inequality (Theorem 4), which extends the classical result of Ye et al. [27] to ψ -weighted kernels. The theory is then applied to a fractional nonlinear oscillator with saturating feedback (Section 6), where all hypotheses are verified analytically and confirmed numerically.

The remainder of the paper is organized as follows. Section 2 collects the necessary preliminaries on weighted function spaces, fractional operators, fixed-point theorems, and the new Gronwall inequality. Section 3 presents auxiliary lemmas, the equivalent Volterra formulation, and the existence and uniqueness results. Section 4 establishes the four Ulam stabilities. Section 5 addresses continuous dependence on initial conditions. Section 6 applies the theory to the fractional nonlinear oscillator with saturating feedback. Sections 7 and 8 provide discussion and conclusions, respectively. The numerical algorithm is detailed in Appendix A.

2. Preliminaries

Throughout the paper, we fix the following conventions, which are used without further comment: (i) $\alpha, \gamma \in \mathbb{C}$ always denote fractional orders, with $\Re(\alpha)$, $\Im(\alpha)$ their real

and imaginary parts, $\Gamma(\alpha)$ the (complex-valued) Euler Gamma function evaluated at α , and $|\Gamma(\alpha)|$ its modulus; the modulus, rather than $\Gamma(\alpha)$ itself, is what appears in every norm estimate and stability constant, since these constants must be real and non-negative. (ii) $\gamma := \alpha + \beta(n - \alpha)$ is reserved exclusively for the effective order associated with type $\beta \in [0, 1]$, and is never reused with another meaning. (iii) Weighted spaces are denoted $C_{\rho,\psi}[a, b]$, $C_{\rho,\psi}^n[a, b]$ and $C_{n-\gamma,\psi}^\gamma[a, b]$ exactly as in Definitions 1–3, with the weight exponent always written as a subscript preceding the comma and ψ .

This section collects the basic definitions, function spaces, and preliminary results needed for the analysis of the considered fractional differential problem.

2.1. Weighted Function Spaces

Throughout the paper, let $0 \leq a < b < \infty$ and let $\gamma \in \mathbb{C}$ satisfy $0 \leq \Re(\gamma) < 1$. These assumptions are fixed unless stated otherwise. All complex function spaces are considered over the interval $[a, b]$.

Standing Assumption (ψ). Unless explicitly stated otherwise, ψ is assumed throughout the paper to satisfy $\psi \in C^n[a, b]$ for the relevant $n \in \mathbb{N}$, to be strictly increasing on $[a, b]$, and to satisfy $\psi'(t) > 0$ for all $t \in [a, b]$. This hypothesis is not repeated in the subsequent definitions, lemmas and theorems.

For $x, y \in [a, b]$ define $\Delta_x^\psi y := \psi(y) - \psi(x)$. In particular, $\Delta_a^\psi t = \psi(t) - \psi(a)$. This notation is used throughout the paper in place of $\psi(t) - \psi(a)$ and $\psi(t) - \psi(\tau)$.

Let $C[a, b]$ denote the complex continuous function space. We generalize the real weight function spaces $C_{\rho,\psi}[a, b]$ and $C_{\rho,\psi}^n[a, b]$ (see [6]) to complex function spaces as follows.

Definition 1. (ψ -Weighted Continuous Space $C_{\gamma,\psi}[a, b]$). Let $\psi \in C^1[a, b]$ be strictly increasing with $\psi'(t) > 0$ on $[a, b]$. The space $C_{\gamma,\psi}[a, b]$ consists of all functions $f : (a, b] \rightarrow \mathbb{C}$ such that

$$(\Delta_a^\psi t)^\gamma f(t) \in C[a, b].$$

$C_{\gamma,\psi}[a, b]$ is endowed with the norm

$$\|f\|_{C_{\gamma,\psi}[a,b]} := \max_{t \in [a,b]} |(\Delta_a^\psi t)^\gamma f(t)|.$$

The space $C_{\gamma,\psi}[a, b]$ allows functions to exhibit a controlled singularity at $t = a$, no stronger than $(\Delta_a^\psi t)^{-\gamma}$. In particular,

$$C_{0,\psi}[a, b] = C[a, b], \quad C_{\gamma,t}[a, b] = C_\gamma[a, b],$$

where $C[a, b]$ and $C_\gamma[a, b]$ are the continuous and weighted spaces, respectively, defined in [1].

Definition 2. (ψ -Weighted Space $C_{\gamma,\psi}^n[a, b]$). Let $n \in \mathbb{N}$ and let $\psi \in C^n[a, b]$ with $\psi'(t) > 0$ on $[a, b]$. We define $C_{\gamma,\psi}^n[a, b]$ as the Banach space of functions $f : (a, b] \rightarrow \mathbb{C}$ such that

$$\delta_\psi^k f \in C[a, b], \quad k = 0, 1, \dots, n-1, \quad \delta_\psi^n f \in C_{\gamma,\psi}[a, b],$$

where $\delta_\psi := \frac{1}{\psi'(t)} \frac{d}{dt}$. The norm is given by

$$\|f\|_{C_{\gamma,\psi}^n[a,b]} := \sum_{k=0}^{n-1} \|\delta_\psi^k f\|_{C[a,b]} + \|\delta_\psi^n f\|_{C_{\gamma,\psi}[a,b]}.$$

In particular, if $n = 0$, we have $C_{\gamma,\psi}^0[a, b] = C_{\gamma,\psi}[a, b]$.

Definition 3. Let $n - 1 < \alpha < n$, $0 \leq \beta \leq 1$, and set $\gamma = \alpha + \beta(n - \alpha)$. Define the weighted space

$$C_{n-\gamma,\psi}[a,b] = \left\{ f : (a,b] \rightarrow \mathbb{C} \mid (\Delta_a^\psi t)^{n-\gamma} f(t) \in C[a,b] \right\},$$

with norm $\|f\|_{C_{n-\gamma,\psi}[a,b]} = \max_{t \in [a,b]} |(\Delta_a^\psi t)^{n-\gamma} f(t)|$.

Extending the definitions from [6,12,28] to higher order, we introduce

$$C_{n-\gamma,\psi}^\gamma[a,b] = \left\{ f \in C_{n-\gamma,\psi}[a,b] : D_{a+}^{\gamma;\psi} f \in C_{n-\gamma,\psi}[a,b] \right\},$$

where $\gamma = \alpha + \beta(n - \alpha)$.

2.2. Fractional Integrals and ψ -Hilfer Fractional Derivative

This subsection introduces the fractional operators required for the analysis. Throughout, let $\alpha \in \mathbb{C}$ be a non-integer with $\Re(\alpha) > 0$, and let $\lfloor \Re(\alpha) \rfloor$ denote its integer part. All operators are defined on the interval $[a,b]$, where $0 \leq a < b \leq \infty$.

Definition 4 (ψ -Riemann–Liouville fractional integral, [7]). Let $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Under the standing assumption on ψ , the ψ -Riemann–Liouville fractional integral of order α of a function f is defined by

$$(I_{a+}^{\alpha;\psi} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau) (\Delta_\tau^\psi t)^{\alpha-1} f(\tau) d\tau, \quad t > a. \quad (3)$$

Since τ ranges over $[a,t]$ in (3), we always have $\Delta_\tau^\psi t = \psi(t) - \psi(\tau) > 0$; by Remark 1, the integrand $\psi'(\tau) (\Delta_\tau^\psi t)^{\alpha-1}$ is therefore well-defined and single-valued for every $\alpha \in \mathbb{C}$, with no branch ambiguity.

Lemma 1 ([6]). Let $\alpha, \beta \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $\Re(\beta) > 0$. Then, we have the following semigroup property given by

$$(I_{a+}^{\alpha;\psi} I_{a+}^{\beta;\psi} f)(t) = (I_{a+}^{\alpha+\beta;\psi} f)(t).$$

Definition 5 (ψ -Riemann–Liouville fractional derivative, [7]). Let $\alpha \in \mathbb{C}$ with $n - 1 < \Re(\alpha) < n$, $n \in \mathbb{N}$. Under the standing assumption on ψ , the ψ -Riemann–Liouville derivative of order α of a function f on $[a,b]$ is defined by

$$(D_{a+}^{\alpha;\psi} f)(t) = \left(\delta_\psi^n I_{a+}^{n-\alpha;\psi} f \right)(t), \quad t > a. \quad (4)$$

Property 1 ([1]). Let $\alpha, \delta \in \mathbb{C}$ with $\Re(\alpha) \geq 0$, $(\alpha \neq 0)$ and $\Re(\delta) > 0$.

(i) If $f(t) = (\Delta_a^\psi t)^{\delta-1}$, then

$$(I_{a+}^{\alpha;\psi} f)(t) = \frac{\Gamma(\delta)}{\Gamma(\alpha + \delta)} (\Delta_a^\psi t)^{\alpha+\delta-1}.$$

(ii) If $f(t) = (\Delta_a^\psi t)^{\delta-1}$, then

$$(D_{a+}^{\alpha;\psi} f)(t) = \frac{\Gamma(\delta)}{\Gamma(\delta - \alpha)} (\Delta_a^\psi t)^{\delta-\alpha-1}.$$

(iii) If $f(t) = (\Delta_a^\psi t)^{\alpha-k}$, $k = 1, 2, \dots, n$, then

$$(D_{a+}^{\alpha;\psi} f)(t) = 0, \quad \Re(\alpha) \in (n-1, n).$$

Definition 6. (ψ -Hilfer fractional derivative, [7]) Let $\alpha \in \mathbb{C}$ with $n - 1 < \Re(\alpha) < n$, $n \in \mathbb{N}$ and $\beta \in [0, 1]$. Under the standing assumption on ψ , the ψ -Hilfer fractional derivative of order α and type β is defined by

$$({}^H D_{a^+}^{\alpha, \beta; \psi} f)(t) = \left(I_{a^+}^{\beta(n-\alpha); \psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a^+}^{(1-\beta)(n-\alpha); \psi} f \right)(t), \quad t > a. \quad (5)$$

Equivalently, the ψ -Hilfer fractional derivative admits the representation

$$({}^H D_{a^+}^{\alpha, \beta; \psi} f)(t) = (I_{a^+}^{\gamma-\alpha; \psi} D_{a^+}^{\gamma; \psi} f)(t), \quad \gamma = \alpha + \beta(n - \alpha), \quad (6)$$

where $D_{a^+}^{\gamma; \psi}$ denotes the ψ -Riemann–Liouville fractional derivative.

Remark 1 (Well-posedness of the complex power: absence of branch ambiguity). For $\tau, t \in [a, b]$ with $\tau < t$, the Standing Assumption on ψ (strict monotonicity, $\psi' > 0$) guarantees $\Delta_\tau^\psi t = \psi(t) - \psi(\tau) > 0$ is a strictly positive real number. Hence, for any $\alpha \in \mathbb{C}$, the complex power

$$(\Delta_\tau^\psi t)^{\alpha-1} := \exp((\alpha - 1) \ln(\Delta_\tau^\psi t)),$$

is defined using the ordinary (single-valued, real) natural logarithm of a positive real number. No branch cut of the complex logarithm is ever crossed, so this power is well-defined, single-valued, and jointly continuous (indeed jointly real-analytic in t, τ and entire in α) throughout the paper; in particular, it does not depend on any choice of branch of $\log$. This is what justifies the exact modulus identity

$$|(\Delta_\tau^\psi t)^{\alpha-1}| = (\Delta_\tau^\psi t)^{\Re(\alpha)-1},$$

used in Lemmas 3–5, Theorem 4 and throughout Sections 3–5: it is an equality, not an estimate obtained by analogy with the real-order theory. Similarly, since $\Re(\alpha) \in (n - 1, n)$ with $n \in \mathbb{N}$ excludes every pole $\alpha = 0, -1, -2, \dots$ of the Euler Gamma function, the maps $\alpha \mapsto \Gamma(\alpha)$ and $\alpha \mapsto 1/\Gamma(\alpha)$ are holomorphic on the whole admissible vertical strip $\{\Re(\alpha) \in (n - 1, n)\}$; consequently, the exact identities of Property 1 and Lemma 1, which involve $\Gamma(\alpha)$ without a modulus, are valid on this strip by analytic continuation from the classical real-order identities, and are not merely formal extensions.

Remark 2. For $n \in \mathbb{N}$, the parameters $n - 1 < \Re(\alpha) < n$, $\beta \in [0, 1]$ and $\gamma = \alpha + \beta(n - \alpha)$ satisfy the following properties:

- (i) γ is a convex combination: $\gamma = (1 - \beta)\Re(\alpha) + \beta n$;
- (ii) $\Re(\gamma) \in (\Re(\alpha), n)$ strictly when $\beta \in (0, 1)$;
- (iii) $1 - \beta(n - \Re(\alpha)) \in (0, 1)$;
- (iv) $n - \Re(\gamma) = (1 - \beta)(n - \Re(\alpha)) \in [0, 1)$.

2.3. Functional Analysis Tools

The well-posedness analysis carried out in Section 3 rests on reformulating the implicit fractional problem as a fixed-point equation on a suitable Banach space. Once the original equation is expressed in integral form, existence, uniqueness, and stability reduce to questions about a single operator $\mathcal{F}$, which are then addressed through classical fixed-point theory.

A subset $\mathcal{S}$ of a Banach space $\mathcal{B}$ is relatively compact if every sequence in $\mathcal{S}$ admits a convergent subsequence in $\mathcal{B}$. An operator $\mathcal{F} : \mathcal{B} \rightarrow \mathcal{B}$ is completely continuous if it is continuous and sends every bounded subset of $\mathcal{B}$ to a relatively compact set. Throughout, we work in the weighted Banach space $\mathcal{B} := C_{n-\gamma, \psi}[a, b]$, and relative compactness in this space is verified using the criterion of the Arzelà–Ascoli theorem, which charac-

terizes compact sets through uniform boundedness and equicontinuity. Existence of at least one solution is then obtained from the following fixed-point theorem for completely continuous operators.

Theorem 1. (Arzelà–Ascoli) *A subset $\mathcal{S}$ of the Banach space $\mathcal{B}$ is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a, b]$.*

Theorem 2. (Schaefer’s Fixed-Point Theorem, [29]) *Let $\mathcal{F} : \mathcal{B} \rightarrow \mathcal{B}$ be a completely continuous operator in the Banach space $\mathcal{B}$, and suppose that the set*

$$\mathcal{E}(\mathcal{F}) = \{u \in \mathcal{B} : u = \mu \mathcal{F}u, \text{ for some } \mu \in [0, 1]\}$$

is bounded. Then $\mathcal{F}$ has at least one fixed point in $\mathcal{B}$.

In addition to compactness arguments, uniqueness of solutions is obtained through Banach’s fixed-point theorem.

Theorem 3. (Banach’s Fixed-Point Theorem on a Closed Subset) *Let $\mathcal{D}$ be a non-empty closed subset of the Banach space $\mathcal{B}$. If a mapping $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{D}$ is a contraction, i.e., there exists a constant $q \in (0, 1)$ such that*

$$\|\mathcal{F}(x) - \mathcal{F}(y)\| \leq q\|x - y\|, \quad \text{for all } x, y \in \mathcal{D},$$

then $\mathcal{F}$ has a unique fixed point in $\mathcal{D}$.

2.4. Ulam Stability

The previous section settled existence and uniqueness, so the fractional problem is well-posed in the right functional space. Solvability alone, however, does not tell us how solutions react to small disturbances. Those matter in practice, where models always carry some uncertainty from measurements, approximations, or external effects.

Motivated by this consideration, we investigate the stability of solutions to problem (1)–(2) in the sense of Ulam. The objective is to ensure that approximate solutions, which satisfy the equation up to a prescribed perturbation, remain close to exact solutions while preserving the structural conditions of the problem.

Let $\varepsilon > 0$, $f : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, and $y \in C_{n-\gamma, \psi}^{\gamma}[a, b]$. Let $\sigma : (a, b] \rightarrow \mathbb{R}_+$ be a given function with $\sigma(t) > 0$. For problem (1), we analyze the perturbation inequalities listed below:

$$\left| ({}^H D_{a+}^{\alpha, \beta; \psi} y)(t) - f(t, y(t), ({}^H D_{a+}^{\alpha, \beta; \psi} y)(t)) \right| \leq \varepsilon, \quad t \in (a, b], \quad (7)$$

$$\left| ({}^H D_{a+}^{\alpha, \beta; \psi} y)(t) - f(t, y(t), ({}^H D_{a+}^{\alpha, \beta; \psi} y)(t)) \right| \leq \varepsilon \sigma(t), \quad t \in (a, b], \quad (8)$$

$$\left| ({}^H D_{a+}^{\alpha, \beta; \psi} y)(t) - f(t, y(t), ({}^H D_{a+}^{\alpha, \beta; \psi} y)(t)) \right| \leq \sigma(t), \quad t \in (a, b], \quad (9)$$

with the integral initial value conditions

$$(y)_{n-\gamma}^{(n-k); \psi}(a) = c_k, \quad c_k \in \mathbb{C}, \quad k \in \{1, 2, \dots, n\}, \quad (10)$$

where ${}^H D_{a+}^{\alpha, \beta; \psi}(\cdot)$ is the ψ -Hilfer fractional derivative with $n - 1 < \Re(\alpha) < n$ and $0 \leq \beta \leq 1$.

Definition 7. (Ulam–Hyers stable, [26]). Problem (1)–(2) is Ulam–Hyers stable (UH-Stable) if there exists a real number $c_f > 0$ such that, for each $\varepsilon > 0$ and for each solution $y \in C_{n-\gamma,\psi}^\gamma[a, b]$ of the inequality (7) with (10), there exists a solution $x \in C_{n-\gamma,\psi}^\gamma[a, b]$ of problem (1)–(2) satisfying

$$(\Delta_a^\psi t)^{n-\gamma} |y(t) - x(t)| \leq c_f \varepsilon, \quad t \in [a, b].$$

Definition 8. (Generalized Ulam–Hyers stable, [26]). Problem (1)–(2) is generalized Ulam–Hyers stable (GUH-Stable) if there exists a continuous function $\theta_f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\theta_f(0) = 0$ such that, for each $\varepsilon > 0$ and for each solution $y \in C_{n-\gamma,\psi}^\gamma[a, b]$ of the inequality (7) with (10), there exists a solution $x \in C_{n-\gamma,\psi}^\gamma[a, b]$ of problem (1)–(2) satisfying

$$(\Delta_a^\psi t)^{n-\gamma} |y(t) - x(t)| \leq \theta_f(\varepsilon), \quad t \in [a, b].$$

Definition 9. (Ulam–Hyers–Rassias stable, [26]). Problem (1)–(2) is Ulam–Hyers–Rassias stable (UHR-Stable) with respect to σ if there exists a constant $c_{f,\sigma} > 0$ such that, for each $\varepsilon > 0$ and for each solution $y \in C_{n-\gamma,\psi}^\gamma[a, b]$ of the inequality (8) with (10), there exists a solution $x \in C_{n-\gamma,\psi}^\gamma[a, b]$ of problem (1)–(2) satisfying

$$(\Delta_a^\psi t)^{n-\gamma} |y(t) - x(t)| \leq c_{f,\sigma} \varepsilon \sigma(t), \quad t \in [a, b].$$

Definition 10. (Generalized Ulam–Hyers–Rassias stable, [26]). Problem (1)–(2) is generalized Ulam–Hyers–Rassias stable (GUHR-Stable) with respect to σ if there exists a constant $c_{f,\sigma} > 0$ such that, for each solution $y \in C_{n-\gamma,\psi}^\gamma[a, b]$ of the inequality (9) with (10), there exists a solution $x \in C_{n-\gamma,\psi}^\gamma[a, b]$ of problem (1)–(2) satisfying

$$(\Delta_a^\psi t)^{n-\gamma} |y(t) - x(t)| \leq c_{f,\sigma} \sigma(t), \quad t \in [a, b].$$

Remark 3. It is clear that (i) Definition 7 implies Definition 8; (ii) Definition 9 implies Definition 10; (iii) Definition 9 implies Definition 7.

Remark 4. A function $y \in C_{n-\gamma,\psi}^\gamma[a, b]$ is a solution of inequality (7) if and only if there exists a function $g \in C_{n-\gamma,\psi}[a, b]$ such that $|g(t)| \leq \varepsilon$ for $t \in (a, b]$ and

$$({}^H D_{a^+}^{\alpha,\beta;\psi} y)(t) = f(t, y(t), ({}^H D_{a^+}^{\alpha,\beta;\psi} y)(t)) + g(t), \quad t \in (a, b].$$

Analogous observations hold for inequalities (8) and (9).

2.5. Generalization of Gronwall's Inequality

In this subsection, we develop an integral inequality involving singular ψ -weighted kernels, which plays a central role in establishing the stability and uniqueness results of the subsequent sections. The classical Gronwall-type inequality with singular behavior, due to Ye et al. [27], was extended in [12] to handle the singular kernels arising in Hilfer-type fractional equations. Here, we carry this line of analysis further by generalizing the inequality to the ψ -weighted setting, accommodating the broader class of ψ -Hilfer fractional operators considered in the present work. We begin by recalling a definition and an auxiliary lemma needed for the proof.

Definition 11 ([30]). Let $b > a > 0$ and $\rho > 0$. Thus, we get the following definition:

$$F_{\rho,a,b}(z) := \sum_{k=0}^{\infty} c_k z^k, \quad z \in \mathbb{R},$$

where $c_0 = 1$, $c_1 = 1$ (empty product) and $c_k = \prod_{i=1}^{k-1} \Gamma(i\rho + a) / \Gamma(i\rho + b)$ for $k \in \mathbb{N}$.

Lemma 2 ([30]). Let $z > 0$ and $a, b \in \mathbb{R}$. Then

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = \mathcal{O}(z^{a-b}), \quad z \rightarrow +\infty.$$

Remark 5. The asymptotic estimate provided by Lemma 2 confirms that the function $\mathcal{F}_{\rho,a,b}$ introduced in Definition 11 is well-defined. Indeed, applying that estimate to successive coefficients gives $c_{k+1}/c_k = \Gamma(k\rho + a) / \Gamma(k\rho + b) = \mathcal{O}\left((k\rho)^{a-b}\right)$ as $k \rightarrow +\infty$. Since $b > a > 0$, the exponent $a - b$ is negative, so $c_{k+1}/c_k \rightarrow 0$ as $k \rightarrow +\infty$. Equivalently, the reciprocal ratio satisfies $c_k/c_{k+1} = \Gamma(k\rho + b) / \Gamma(k\rho + a) = \mathcal{O}\left((k\rho)^{b-a}\right) \rightarrow \infty$, so the ratio test yields an infinite radius of convergence for the power series $\sum_{k=0}^{\infty} c_k z^k$. Consequently, this series converges absolutely and uniformly on every compact subset of $\mathbb{R}$, and the resulting function $\mathcal{F}_{\rho,a,b}$ is infinitely differentiable—in particular, continuous—on the whole real line.

To this end, we present a generalized version of Gronwall's inequality involving a singular kernel, which serves as a fundamental tool for establishing the main results of this section. The proof follows by reducing the inequality to the original generalized Gronwall Inequality (Theorem 3 of [12]) via the change of variable $x = \psi(t)$.

Theorem 4 (Generalized Gronwall Inequality with respect to ψ). Assume that $\alpha, \beta, \gamma > 0$, $\delta = \alpha + \gamma - 1 > 0$, $\vartheta = \beta + \gamma - 1 > 0$. Let $\psi \in C^1[t_0, T]$ be strictly increasing, i.e., $\psi'(t) > 0$ for all $t \in [t_0, T]$, where $t_0 \geq 0$, $T \leq +\infty$. Let $a(t)$ and $b(t)$ be non-negative, non-decreasing continuous functions on $[t_0, T]$ with $b(t) \leq M$ for some positive constant M . Suppose $u(t)$ is non-negative and $(\Delta_{t_0}^{\psi} t)^{\gamma-1} u(t)$ is locally integrable on $[t_0, T]$.

If u satisfies the inequality

$$u(t) \leq a(t)(\Delta_{t_0}^{\psi} t)^{\alpha-1} + b(t) \int_{t_0}^t (\Delta_s^{\psi} t)^{\beta-1} (\Delta_{t_0}^{\psi} s)^{\gamma-1} \psi'(s) u(s) ds, \quad t \in [t_0, T], \quad (11)$$

then the following estimate holds:

$$\begin{aligned} u(t) &\leq a(t)(\Delta_{t_0}^{\psi} t)^{\alpha-1} F_{\vartheta, \delta, \delta+\beta} \left(\Gamma(\beta) b(t) (\Delta_{t_0}^{\psi} t)^{\vartheta} \right) \\ &= a(t)(\Delta_{t_0}^{\psi} t)^{\alpha-1} \sum_{k=0}^{\infty} \left[\prod_{i=1}^{k-1} \frac{\Gamma(i\vartheta + \delta)}{\Gamma(i\vartheta + \delta + \beta)} \right] \left(\Gamma(\beta) b(t) (\Delta_{t_0}^{\psi} t)^{\vartheta} \right)^k, \end{aligned}$$

for all $t \in [t_0, T]$.

Proof. Define the transformed variable $x = \psi(t)$. Since $\psi \in C^1[t_0, T]$ is strictly increasing with $\psi'(t) > 0$, the map $t \mapsto x$ is a diffeomorphism from $[t_0, T]$ onto $[x_0, X]$, where $x_0 = \psi(t_0)$ and $X = \psi(T^-)$ (which may be $+\infty$). The inverse function $t = \psi^{-1}(x)$ is also C^1 on $[x_0, X]$.

Now define the transformed functions:

$$\tilde{u}(x) := u(\psi^{-1}(x)), \quad \tilde{a}(x) := a(\psi^{-1}(x)), \quad \tilde{b}(x) := b(\psi^{-1}(x)).$$

Consider the integral term in inequality (11). Make the substitution $r = \psi(s)$. Then, $dr = \psi'(s)ds$, and when $s = t_0$, $r = \psi(t_0) = x_0$; when $s = t$, $r = \psi(t) = x$. Moreover, $\psi(t) - \psi(s) = x - r$ and $\psi(s) - \psi(t_0) = r - x_0$. Thus,

$$\int_{t_0}^t (\Delta_s^\psi)^{\beta-1} (\Delta_{t_0}^\psi)^{\gamma-1} \psi'(s) u(s) ds = \int_{x_0}^x (x-r)^{\beta-1} (r-x_0)^{\gamma-1} \tilde{u}(r) dr.$$

The inequality (11) becomes

$$\tilde{u}(x) \leq \tilde{a}(x)(x-x_0)^{\alpha-1} + \tilde{b}(x) \int_{x_0}^x (x-r)^{\beta-1} (r-x_0)^{\gamma-1} \tilde{u}(r) dr, \quad (12)$$

for all $x \in [x_0, X)$. If $X = +\infty$, the resulting inequality (12) is understood to hold on $[x_0, \infty)$; this causes no difficulty since all hypotheses (monotonicity, local integrability) are stated locally.

We now check that all conditions of Theorem 3 of [12] are satisfied for the transformed inequality: $\alpha, \beta, \gamma > 0$ and $\delta = \alpha + \gamma - 1 > 0$, $\vartheta = \beta + \gamma - 1 > 0$ remain unchanged. Since $a(t)$ and $b(t)$ are non-negative and non-decreasing on $[t_0, T)$, and ψ^{-1} is strictly increasing, the transformed functions $\tilde{a}(x) = a(\psi^{-1}(x))$ and $\tilde{b}(x) = b(\psi^{-1}(x))$ are also non-negative and non-decreasing on $[x_0, X)$. The bound $b(t) \leq M$ implies $\tilde{b}(x) \leq M$. $\tilde{u}(x)$ is non-negative because $u(t) \geq 0$. The condition that $(\psi(t) - \psi(t_0))^{\gamma-1} u(t)$ is locally integrable in t implies, via the change of variable, that $(x - x_0)^{\gamma-1} \tilde{u}(x)$ is locally integrable in x (since ψ' is continuous and positive, the Jacobian factor is bounded away from zero on compact intervals).

Thus all hypotheses of Theorem 3 of [12] are satisfied. So applying the theorem to inequality (12) yields

$$\begin{aligned} \tilde{u}(x) &\leq \tilde{a}(x)(x-x_0)^{\alpha-1} F_{\vartheta, \delta, \delta+\beta} \left(\Gamma(\beta) \tilde{b}(x) (x-x_0)^\vartheta \right) \\ &= \tilde{a}(x)(x-x_0)^{\alpha-1} \sum_{k=0}^{\infty} \left[\prod_{i=1}^{k-1} \frac{\Gamma(i\vartheta + \delta)}{\Gamma(i\vartheta + \delta + \beta)} \right] \left(\Gamma(\beta) \tilde{b}(x) (x-x_0)^\vartheta \right)^k, \end{aligned}$$

for all $x \in [x_0, X)$.

Now substitute back $x = \psi(t)$, $x_0 = \psi(t_0)$, $\tilde{u}(x) = u(t)$, $\tilde{a}(x) = a(t)$, $\tilde{b}(x) = b(t)$. Then,

$$u(t) \leq a(t) (\Delta_{t_0}^\psi)^{\alpha-1} F_{\vartheta, \delta, \delta+\beta} \left(\Gamma(\beta) b(t) (\Delta_{t_0}^\psi)^\vartheta \right),$$

for all $t \in [t_0, T)$. Recalling the definition of $F_{\vartheta, \delta, \delta+\beta}$, we obtain the final estimate:

$$u(t) \leq a(t) (\Delta_{t_0}^\psi)^{\alpha-1} \sum_{k=0}^{\infty} \left[\prod_{i=1}^{k-1} \frac{\Gamma(i\vartheta + \delta)}{\Gamma(i\vartheta + \delta + \beta)} \right] \left(\Gamma(\beta) b(t) (\Delta_{t_0}^\psi)^\vartheta \right)^k,$$

where the empty product for $k = 0$ or $k = 1$ is understood as 1.

Since $\vartheta > 0$ and $\beta > 0$, we have $\delta + \beta > \delta > 0$. The ratio test gives

$$\lim_{k \rightarrow \infty} \frac{c_{k+1}}{c_k} (\Gamma(\beta) b(t) (\psi(t) - \psi(t_0))^\vartheta) = \lim_{k \rightarrow \infty} \frac{\Gamma(k\vartheta + \delta)}{\Gamma(k\vartheta + \delta + \beta)} (\Gamma(\beta) b(t) (\psi(t) - \psi(t_0))^\vartheta) = 0,$$

because $\frac{\Gamma(k\vartheta + \delta)}{\Gamma(k\vartheta + \delta + \beta)} \sim (k\vartheta)^{-\beta} \rightarrow 0$ as $k \rightarrow \infty$. Hence, the series converges absolutely for all $t \in [t_0, T)$. This completes the proof. $\square$

Corollary 1. Assume that $\beta > 0$, $-1 < \gamma < 1$, $\beta - \gamma > 0$. Let $\psi \in C^1[t_0, T)$ be strictly increasing, i.e., $\psi'(t) > 0$ for all $t \in [t_0, T)$, where $t_0 \geq 0$, $T \leq +\infty$. Let $a(t)$ and $b(t)$ be non-negative, non-decreasing continuous functions on $[t_0, T)$ with $b(t) \leq M$ for some positive constant

M. Further suppose that $u(t)$ is non-negative and $(\psi(t) - \psi(t_0))^{-\gamma} u(t)$ is locally integrable on $[t_0, T)$.

If u satisfies the inequality

$$u(t) \leq a(t)(\Delta_{t_0}^\psi t)^\gamma + b(t) \int_{t_0}^t (\Delta_s^\psi t)^{\beta-1} (\Delta_{t_0}^\psi s)^{-\gamma} \psi'(s) u(s) ds, \quad t \in [t_0, T), \quad (13)$$

then the following estimate holds:

$$\begin{aligned} u(t) &\leq a(t)(\Delta_{t_0}^\psi t)^\gamma F_{\beta-\gamma, 1, \beta+1} \left(\Gamma(\beta) b(t) (\Delta_{t_0}^\psi t)^{\beta-\gamma} \right) \\ &= a(t)(\Delta_{t_0}^\psi t)^\gamma \sum_{k=0}^{\infty} \left[\prod_{i=1}^{k-1} \frac{\Gamma(i(\beta-\gamma)+1)}{\Gamma(i(\beta-\gamma)+\beta+1)} \right] \left(\Gamma(\beta) b(t) (\Delta_{t_0}^\psi t)^{\beta-\gamma} \right)^k, \end{aligned}$$

for all $t \in [t_0, T)$.

The main results of the paper follow. Working in the functional framework just described, we prove existence, uniqueness, and stability for the fractional differential problem under consideration.

3. Results

The following lemmas are needed for the proofs in this section.

3.1. Auxiliary Lemmas

We recall that the Beta function $B(x, y)$, for $x, y \in \mathbb{C}$ with $\Re(x), \Re(y) > 0$, is defined by

$$B(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Lemma 3. Let $\alpha \in \mathbb{C}$ satisfy $n-1 < \Re(\alpha) < n$, $n \in \mathbb{N}$ and let $0 \leq \beta \leq 1$. Define $\gamma = \alpha + \beta(n-\alpha)$. Let $\psi \in C^n[a, b]$ be strictly increasing with $\psi'(t) > 0$ on $[a, b]$. If $f \in C_{n-\gamma, \psi}[a, b]$, then for all $t \in (a, b]$,

$$\int_a^t (\Delta_\tau^\psi t)^{\Re(\alpha)-1} |f(\tau)| \psi'(\tau) d\tau \leq (\Delta_a^\psi t)^{\Re(\alpha)+\gamma-n} B(\Re(\alpha), \Re(\gamma-n+1)) \|f\|_{C_{n-\gamma, \psi}[a, b]},$$

where $B(\cdot, \cdot)$ denotes the Beta function.

Proof. Let $f \in C_{n-\gamma, \psi}[a, b]$. By the definition of the weighted space,

$$|f(\tau)| \leq (\Delta_a^\psi \tau)^{-(n-\Re(\gamma))} \|f\|_{C_{n-\gamma, \psi}[a, b]}.$$

Substituting this estimate into the integral, we obtain

$$\int_a^t (\Delta_\tau^\psi t)^{\Re(\alpha)-1} |f(\tau)| \psi'(\tau) d\tau \leq \|f\|_{C_{n-\gamma, \psi}[a, b]} I(t),$$

where

$$I(t) := \int_a^t (\Delta_\tau^\psi t)^{\Re(\alpha)-1} (\Delta_a^\psi \tau)^{-(n-\Re(\gamma))} \psi'(\tau) d\tau.$$

Using the change of variables

$$\psi(\tau) = \psi(a) + u(\Delta_a^\psi t), \quad u \in [0, 1],$$

so that $\psi'(\tau) d\tau = (\Delta_a^\psi t) du$, we obtain

$$\begin{aligned} I(t) &= \int_0^1 ((\Delta_a^\psi t)(1-u))^{\Re(\alpha)-1} ((\Delta_a^\psi t)u)^{-(n-\Re(\gamma))} (\Delta_a^\psi t) du \\ &= (\Delta_a^\psi t)^{\Re(\alpha)-(n-\Re(\gamma))} \int_0^1 (1-u)^{\Re(\alpha)-1} u^{-(n-\Re(\gamma))} du. \end{aligned}$$

Since $\Re(\alpha) > 0$ and $0 \leq n - \Re(\gamma) < 1$ (by Remark 2, guaranteeing convergence of the Beta integral below), the integral converges and equals the Beta function. Indeed, the integrand $(1-u)^{\Re(\alpha)-1} u^{-(n-\Re(\gamma))}$ has two possible singularities on $[0, 1]$: at $u = 1$, where the exponent $\Re(\alpha) - 1 > -1$ ensures integrability, and at $u = 0$, where the exponent $-(n - \Re(\gamma)) > -1$ (equivalently $n - \Re(\gamma) < 1$) ensures integrability. Both conditions hold by hypothesis, so the improper integral converges absolutely and coincides with $B(\Re(\alpha), \Re(\gamma) - n + 1)$ by definition of the Beta function. Thus,

$$\int_0^1 (1-u)^{\Re(\alpha)-1} u^{(1-(n-\Re(\gamma)))-1} du = B(\Re(\alpha), 1 - (n - \Re(\gamma))) = B(\Re(\alpha), \Re(\gamma) - n + 1).$$

Note that $\Re(\alpha) - (n - \Re(\gamma)) = \Re(\alpha + \gamma - n)$. Therefore,

$$I(t) = (\Delta_a^\psi t)^{\Re(\alpha+\gamma-n)} B(\Re(\alpha), \Re(\gamma) - n + 1).$$

Combining the above estimates yields the desired inequality. $\square$

Lemma 4. Under the standing assumption on ψ , let $\alpha \in \mathbb{C}$ with $n - 1 < \Re(\alpha) < n$, $n \in \mathbb{N}$ and $\rho \in \mathbb{C}$ with $0 \leq \Re(\rho) < 1$. Then, the ψ -Riemann–Liouville fractional integral operator $I_{a+}^{\alpha;\psi}$ is bounded from $C_{\rho,\psi}[a, b]$ into $C_{\rho,\psi}[a, b]$. More precisely, for all $f \in C_{\rho,\psi}[a, b]$,

$$\|I_{a+}^{\alpha;\psi} f\|_{C_{\rho,\psi}[a,b]} \leq \frac{\Gamma(1 - \Re(\rho))}{\Gamma(\Re(\alpha) + 1 - \Re(\rho))} (\Delta_a^\psi b)^{\Re(\alpha)} \|f\|_{C_{\rho,\psi}[a,b]}. \quad (14)$$

Proof. Let $f \in C_{\rho,\psi}[a, b]$. By Definition 4,

$$(\Delta_a^\psi t)^\rho (I_{a+}^{\alpha;\psi} f)(t) = \frac{(\Delta_a^\psi t)^\rho}{\Gamma(\alpha)} \int_a^t \psi'(s) (\Delta_s^\psi t)^{\alpha-1} f(s) ds.$$

Define $g(s) := [\psi(s) - \psi(a)]^\rho f(s)$. Then,

$$f(s) = [\psi(s) - \psi(a)]^{-\rho} g(s), \quad \|g\|_\infty = \|f\|_{C_{\rho,\psi}[a,b]}.$$

Substituting, we obtain

$$(\Delta_a^\psi t)^\rho (I_{a+}^{\alpha;\psi} f)(t) = \frac{(\Delta_a^\psi t)^\rho}{\Gamma(\alpha)} \int_a^t \psi'(s) (\Delta_s^\psi t)^{\alpha-1} [\psi(s) - \psi(a)]^{-\rho} g(s) ds.$$

Using the change of variables $\psi(s) = \psi(a) + u(\Delta_a^\psi t)$, we obtain

$$(\Delta_a^\psi t)^\rho (I_{a+}^{\alpha;\psi} f)(t) = \frac{(\Delta_a^\psi t)^\alpha}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} u^{-\rho} g(\psi^{-1}(\psi(a) + u(\Delta_a^\psi t))) du.$$

Taking absolute values and using $\Re(\alpha) > 0$ and $0 \leq \Re(\rho) < 1$, we obtain

$$\left| (\Delta_a^\psi t)^\rho (I_{a+}^{\alpha;\psi} f)(t) \right| \leq \frac{(\Delta_a^\psi t)^{\Re(\alpha)}}{|\Gamma(\alpha)|} \|g\|_\infty \int_0^1 (1-u)^{\Re(\alpha)-1} u^{-\Re(\rho)} du.$$

The integral is the Beta function:

$$\int_0^1 (1-u)^{\Re(\alpha)-1} u^{(1-\Re(\rho))-1} du = B(\Re(\alpha), 1-\Re(\rho)) = \frac{\Gamma(\Re(\alpha))\Gamma(1-\Re(\rho))}{\Gamma(\Re(\alpha)+1-\Re(\rho))}.$$

Hence,

$$\left| (\Delta_a^\psi t)^\rho (I_{a^+}^{\alpha;\psi} f)(t) \right| \leq (\Delta_a^\psi t)^{\Re(\alpha)} \frac{\Gamma(1-\Re(\rho))}{\Gamma(\Re(\alpha)+1-\Re(\rho))} \|f\|_{C_{\rho,\psi}[a,b]}.$$

Taking the supremum over $t \in [a, b]$, we obtain (14). This completes the proof. $\square$

Lemma 5. Under the standing assumption on ψ , let $\alpha, \rho \in \mathbb{C}$ satisfy $n-1 < \Re(\alpha) < n$, $n \in \mathbb{N}$ and $0 \leq \Re(\rho) < 1$, with $\Re(\rho) \leq \Re(\alpha)$. If $f \in C_{\rho,\psi}[a, b]$, then the ψ -Riemann–Liouville fractional integral operator $I_{a^+}^{\alpha;\psi}$ maps $C_{\rho,\psi}[a, b]$ boundedly into $C[a, b]$. More precisely, for all $f \in C_{\rho,\psi}[a, b]$,

$$\|I_{a^+}^{\alpha;\psi} f\|_{C[a,b]} \leq \frac{\Gamma(1-\Re(\rho))}{\Gamma(\Re(\alpha)+1-\Re(\rho))} \frac{\Gamma(\Re(\alpha))}{|\Gamma(\alpha)|} (\Delta_a^\psi b)^{\Re(\alpha)-\rho} \|f\|_{C_{\rho,\psi}[a,b]}. \quad (15)$$

Proof. Let $f \in C_{\rho,\psi}[a, b]$ and define

$$F(t) := (I_{a^+}^{\alpha;\psi} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\Delta_\tau^\psi t)^{\alpha-1} f(\tau) \psi'(\tau) d\tau, \quad t \in (a, b],$$

with $F(a) := 0$. By the definition of the weighted space,

$$|f(\tau)| \leq (\Delta_a^\psi \tau)^{-\Re(\rho)} \|f\|_{C_{\rho,\psi}[a,b]}, \quad \tau \in (a, b].$$

Step 1: Continuity on $(a, b]$. Fix $t_0 \in (a, b]$. For t near t_0 , write

$$F(t) - F(t_0) = \frac{1}{\Gamma(\alpha)} (A(t) + B(t)),$$

where

$$\begin{aligned} A(t) &:= \int_a^{t_0} [(\Delta_\tau^\psi t)^{\alpha-1} - (\Delta_\tau^\psi t_0)^{\alpha-1}] f(\tau) \psi'(\tau) d\tau, \\ B(t) &:= \int_{t_0}^t (\Delta_\tau^\psi t)^{\alpha-1} f(\tau) \psi'(\tau) d\tau. \end{aligned}$$

First, we estimate $A(t)$. For $\tau \in [a, t_0]$ and t sufficiently close to t_0 , we have

$$|(\Delta_\tau^\psi t)^{\alpha-1} - (\Delta_\tau^\psi t_0)^{\alpha-1}| \leq 2(\Delta_\tau^\psi t_0)^{\Re(\alpha)-1},$$

(using $|w^{\alpha-1}| = |w|^{\Re(\alpha)-1} e^{-\Im(\alpha) \arg w} \leq C_\alpha |w|^{\Re(\alpha)-1}$ for w in the relevant sector, with $C_\alpha := e^{|\Im(\alpha)|\pi/2}$).

The function $\tau \mapsto (\Delta_\tau^\psi t_0)^{\Re(\alpha)-1} |f(\tau)| \psi'(\tau)$ is integrable on $[a, t_0]$ because $\Re(\alpha) - 1 > -1$ ensures the singularity at $\tau = t_0$ is integrable and $|f(\tau)| = O((\Delta_a^\psi \tau)^{-\Re(\rho)})$ with $\Re(\rho) < 1$ ensures the singularity at $\tau = a$ is integrable. Moreover, for each fixed τ , the integrand converges pointwise to 0 as $t \rightarrow t_0$. Explicitly, the dominating function is $g(\tau) := 2C_\alpha (\Delta_\tau^\psi t_0)^{\Re(\alpha)-1} (\Delta_a^\psi \tau)^{-\Re(\rho)} \|f\|_{C_{\rho,\psi}[a,b]} \psi'(\tau)$, which is independent of t and integrable on $[a, t_0]$ because $\Re(\alpha) - 1 > -1$ (integrable singularity at $\tau = t_0$) and $\Re(\rho) < 1$ (integrable singularity at $\tau = a$); the two singularities are separated since $\tau = t_0 \neq a$, so the product of the two power-law factors remains integrable by a direct comparison with $\int_a^{t_0} (\tau - a)^{-\Re(\rho)} d\tau$

and $\int_a^{t_0} (t_0 - \tau)^{\Re(\alpha)-1} d\tau$ near their respective endpoints. The Dominated Convergence Theorem therefore applies. By the Dominated Convergence Theorem,

$$\lim_{t \rightarrow t_0} A(t) = 0.$$

The term $B(t)$ is an integral over an interval of length $|t - t_0|$. Even though the kernel $(\Delta_t^\psi t)^{\alpha-1}$ is singular at $\tau = t$ when $\Re(\alpha) < 1$, the singularity is integrable because $\Re(\alpha) - 1 > -1$. Consequently, $|B(t)| \leq C|t - t_0|^{\min(\Re(\alpha), 1)}$ for some constant $C > 0$ depending on α, ρ, ψ, t_0 , and $\|f\|_{C_{\rho, \psi}}$, but not on t . Since $\min(\Re(\alpha), 1) > 0$, it follows that $B(t) \rightarrow 0$ as $t \rightarrow t_0^+$. The case $t < t_0$ is analogous.

Thus $A(t) \rightarrow 0$ and $B(t) \rightarrow 0$ as $t \rightarrow t_0$, so $F(t) \rightarrow F(t_0)$. Hence, F is continuous at t_0 , and since $t_0 \in (a, b]$ was arbitrary, $F \in C(a, b]$.

Step 2: Continuity at $t = a$. We show that $\lim_{t \rightarrow a^+} F(t) = 0 = F(a)$. Substituting the estimate for $|f(\tau)|$ into the definition of $I_{a^+}^{\alpha; \psi}$, we obtain

$$|(I_{a^+}^{\alpha; \psi} f)(t)| \leq \frac{1}{|\Gamma(\alpha)|} \|f\|_{C_{\rho, \psi}[a, b]} \int_a^t |\psi(t) - \psi(\tau)|^{\Re(\alpha)-1} (\Delta_a^\psi \tau)^{-\Re(\rho)} \psi'(\tau) d\tau.$$

Define

$$I(t) := \int_a^t (\Delta_t^\psi \tau)^{\Re(\alpha)-1} (\Delta_a^\psi \tau)^{-\Re(\rho)} \psi'(\tau) d\tau.$$

Using the change of variables

$$\psi(\tau) = \psi(a) + u(\Delta_a^\psi t),$$

so that $\psi'(\tau) d\tau = (\Delta_a^\psi t) du$, we obtain

$$\begin{aligned} I(t) &= \int_0^1 ((\Delta_a^\psi t)(1-u))^{\Re(\alpha)-1} ((\Delta_a^\psi t)u)^{-\Re(\rho)} (\Delta_a^\psi t) du \\ &= (\Delta_a^\psi t)^{\Re(\alpha)-\Re(\rho)} \int_0^1 (1-u)^{\Re(\alpha)-1} u^{-\Re(\rho)} du. \end{aligned}$$

Since $\Re(\alpha) > 0$ and $0 \leq \Re(\rho) < 1$, the integral converges and equals the Beta function:

$$\int_0^1 (1-u)^{\Re(\alpha)-1} u^{(1-\Re(\rho))-1} du = B(\Re(\alpha), 1 - \Re(\rho)) = \frac{\Gamma(\Re(\alpha))\Gamma(1 - \Re(\rho))}{\Gamma(\Re(\alpha) + 1 - \Re(\rho))}.$$

Thus,

$$I(t) = (\Delta_a^\psi t)^{\Re(\alpha)-\Re(\rho)} \frac{\Gamma(\Re(\alpha))\Gamma(1 - \Re(\rho))}{\Gamma(\Re(\alpha) + 1 - \Re(\rho))}.$$

Substituting back, we obtain

$$\begin{aligned} |(I_{a^+}^{\alpha; \psi} f)(t)| &\leq \frac{1}{|\Gamma(\alpha)|} \|f\|_{C_{\rho, \psi}[a, b]} \cdot I(t) \\ &= (\Delta_a^\psi t)^{\Re(\alpha)-\Re(\rho)} \frac{\Gamma(1 - \Re(\rho))}{\Gamma(\Re(\alpha) + 1 - \Re(\rho))} \frac{\Gamma(\Re(\alpha))}{|\Gamma(\alpha)|} \|f\|_{C_{\rho, \psi}[a, b]}. \end{aligned}$$

Since $\Re(\alpha - \rho) > 0$ (because $\Re(\alpha) > \Re(\rho)$), we have $(\Delta_a^\psi t)^{\Re(\alpha)-\Re(\rho)} \rightarrow 0$ as $t \rightarrow a^+$. Therefore,

$$\lim_{t \rightarrow a^+} |(I_{a^+}^{\alpha; \psi} f)(t)| = 0 = |(I_{a^+}^{\alpha; \psi} f)(a)|,$$

so F is continuous at $t = a$ with $F(a) = 0$.

Step 3: Boundedness estimate. From the estimate above, for any $t \in [a, b]$,

$$|(I_{a^+}^{\alpha;\psi} f)(t)| \leq (\Delta_a^\psi t)^{\Re(\alpha-\rho)} \frac{\Gamma(1-\Re(\rho))}{\Gamma(\Re(\alpha)+1-\Re(\rho))} \frac{\Gamma(\Re(\alpha))}{|\Gamma(\alpha)|} \|f\|_{C_{\rho,\psi}[a,b]}.$$

Since $(\Delta_a^\psi t)^{\Re(\alpha-\rho)} \leq (\Delta_a^\psi b)^{\Re(\alpha-\rho)}$, taking the supremum over $t \in [a, b]$ yields (15). This completes the proof. $\square$

Lemma 6 (Left Inverse Property of the ψ -Riemann–Liouville Fractional Integral). *Let $\alpha, \rho \in \mathbb{C}$ satisfy $n-1 < \Re(\alpha) < n$, $n \in \mathbb{N}$ and $0 \leq \Re(\rho) < 1$. Under the standing assumption on ψ , let $f \in C_{\rho,\psi}[a, b]$. Then, for all $t \in (a, b]$,*

$$(D_{a^+}^{\alpha;\psi} I_{a^+}^{\alpha;\psi} f)(t) = f(t).$$

Proof. Let $n = \lfloor \Re(\alpha) \rfloor + 1$. By the definition of the ψ -Riemann–Liouville fractional derivative,

$$D_{a^+}^{\alpha;\psi} I_{a^+}^{\alpha;\psi} f = \delta_\psi^n I_{a^+}^{n-\alpha;\psi} I_{a^+}^{\alpha;\psi} f,$$

where $\delta_\psi := \frac{1}{\psi'(t)} \frac{d}{dt}$. Since $f \in C_{\rho,\psi}[a, b]$ with $0 \leq \Re(\rho) < 1$, it follows from Lemma 4 that f is locally integrable on $(a, b]$. Hence, by the semigroup property of the ψ -Riemann–Liouville fractional integral, we obtain

$$I_{a^+}^{n-\alpha;\psi} I_{a^+}^{\alpha;\psi} f = I_{a^+}^{n;\psi} f.$$

Therefore,

$$D_{a^+}^{\alpha;\psi} I_{a^+}^{\alpha;\psi} f = \delta_\psi^n I_{a^+}^{n;\psi} f.$$

It remains to show that $\delta_\psi^n I_{a^+}^{n;\psi} f = f$. For $n = 1$, we have

$$(\delta_\psi I_{a^+}^{1;\psi} f)(t) = \frac{1}{\psi'(t)} \frac{d}{dt} \int_a^t \psi'(\tau) f(\tau) d\tau = f(t),$$

by the fundamental theorem of calculus. Assume that $\delta_\psi^{n-1} I_{a^+}^{n-1;\psi} f = f$. Then

$$\delta_\psi^n I_{a^+}^{n;\psi} f = \delta_\psi^{n-1} (\delta_\psi I_{a^+}^{1;\psi} (I_{a^+}^{n-1;\psi} f)) = \delta_\psi^{n-1} (I_{a^+}^{n-1;\psi} f) = f,$$

which completes the induction. This completes the proof. $\square$

The following lemma provides a sufficient condition for the existence of the ψ -Riemann–Liouville fractional derivative.

Lemma 7. *Let $\alpha \in \mathbb{C}$ satisfy $n-1 < \Re(\alpha) < n$, $n \in \mathbb{N}$. Let $0 \leq \beta \leq 1$, and define $\gamma = \alpha + \beta(n-\alpha)$. If $f \in C_{n-\gamma,\psi}^\gamma[a, b]$, then for all $t \in (a, b]$,*

$$(I_{a^+}^{\gamma;\psi} D_{a^+}^{\gamma;\psi} f)(t) = (I_{a^+}^{\alpha;\psi} D_{a^+}^{\alpha,\beta;\psi} f)(t),$$

and

$$(D_{a^+}^{\gamma;\psi} I_{a^+}^{\alpha;\psi} f)(t) = (D_{a^+}^{\beta(n-\alpha);\psi} f)(t).$$

Proof. Using Lemma 1 and Definition 6, we obtain

$$(I_{a^+}^{\gamma;\psi} D_{a^+}^{\gamma;\psi} f)(t) = (I_{a^+}^{\gamma-\alpha+\alpha;\psi} D_{a^+}^{\gamma;\psi} f)(t) = (I_{a^+}^{\alpha;\psi} I_{a^+}^{\gamma-\alpha;\psi} D_{a^+}^{\gamma;\psi} f)(t) = (I_{a^+}^{\alpha;\psi} D_{a^+}^{\alpha,\beta;\psi} f)(t).$$

Similarly, by Definition 5, Lemma 1, and using $\delta_\psi^n I_{a^+}^{n;\psi} f = f$ from the proof of Lemma 6, we have

$$(D_{a^+}^{\gamma;\psi} I_{a^+}^{\alpha;\psi} f)(t) = (\delta_\psi^n I_{a^+}^{n-\gamma;\psi} I_{a^+}^{\alpha;\psi} f)(t) = \delta_\psi \left(\delta_\psi^{n-1} I_{a^+}^{n-1;\psi} I_{a^+}^{1-\gamma+\alpha;\psi} \right) f(t) = \delta_\psi (I_{a^+}^{1-\beta(n-\alpha);\psi} f)(t) = (D_{a^+}^{\beta(n-\alpha);\psi} f)(t).$$

This completes the proof. $\square$

Lemma 8 ([6]). Let $\alpha, \gamma \in \mathbb{C}$ satisfy $n-1 \leq \Re(\gamma) < n$, $n \in \mathbb{N}$ and $\Re(\gamma) < \Re(\alpha)$. If $f \in C_{\gamma,\psi}[a, b]$, then

$$(I_{a^+}^{\alpha;\psi} f)(a) := \lim_{t \rightarrow a^+} (I_{a^+}^{\alpha;\psi} f)(t) = 0.$$

Lemma 9 ([1]). Let $\alpha, \gamma \in \mathbb{C}$ satisfy $n-1 < \Re(\alpha) < n$, $n \in \mathbb{N}$, $0 \leq \Re(\gamma) < 1$. If $f \in C_\gamma[a, b]$, and $I_{a^+}^{n-\alpha} f \in C_\gamma^n[a, b]$, then

$$(I_{a^+}^\alpha D_{a^+}^\alpha f)(t) = f(t) - \sum_{k=1}^n \frac{(D_{a^+}^{n-k} I_{a^+}^{n-\alpha} f)(a)}{\Gamma(\alpha - k + 1)} (t - a)^{\alpha-k},$$

for any $t \in (a, b]$.

Theorem 5. Let $\alpha, \rho \in \mathbb{C}$ with $n-1 < \Re(\alpha) < n$, $n \in \mathbb{N}$ and $0 \leq \Re(\rho) < 1$. Assume that $\psi \in C^n[a, b]$ is strictly increasing on $[a, b]$ and satisfies $\psi'(t) \neq 0$ for all $t \in [a, b]$. If $f \in C_{\rho,\psi}[a, b]$, and $I_{a^+}^{n-\alpha;\psi} f \in C_{\rho,\psi}^n[a, b]$, then, for every $t \in (a, b]$,

$$(I_{a^+}^{\alpha;\psi} D_{a^+}^{\alpha;\psi} f)(t) = f(t) - \sum_{k=1}^n \frac{(\delta_\psi^{n-k} I_{a^+}^{n-\alpha;\psi} f)(a)}{\Gamma(\alpha - k + 1)} (\Delta_a^\psi t)^{\alpha-k}, \quad (16)$$

where $\delta_\psi = \frac{1}{\psi'(t)} \frac{d}{dt}$.

Proof. The proof proceeds by transforming the ψ -fractional operators into standard Riemann–Liouville operators via the substitution $u = \psi(t)$, applying the classical composition Lemma 9 to the transformed function, and then translating the result back to the original variable.

By the conjugation formula for the fractional integral with respect to another function (see Section 2.5 in [1]), we have

$$I_{a^+}^{\alpha;\psi} = Q_\psi I_{\psi(a)^+}^\alpha Q_\psi^{-1},$$

where Q_ψ is the substitution operator

$$(Q_\psi f)(t) = f(\psi(t)). \quad (17)$$

For the fractional derivative, recall the definition $D_{a^+}^{\alpha;\psi} = \delta_\psi^n I_{a^+}^{n-\alpha;\psi}$, where $\delta_\psi = \frac{1}{\psi'(t)} \frac{d}{dt}$. Applying the conjugation formula for the integral, we obtain

$$D_{a^+}^{\alpha;\psi} = \delta_\psi^n I_{\psi(a)^+}^{n-\alpha;\psi} = \delta_\psi^n Q_\psi I_{\psi(a)^+}^{n-\alpha} Q_\psi^{-1}.$$

Now, the key observation is that δ_ψ and Q_ψ satisfy the commutation relation

$$\delta_\psi Q_\psi = Q_\psi \frac{d}{du},$$

where $u = \psi(t)$. Indeed, for any function $f(t)$,

$$\delta_\psi(Q_\psi f)(t) = \frac{1}{\psi'(t)} \frac{d}{dt} f(\psi(t)) = f'(\psi(t)) = (Q_\psi \frac{d}{du} f)(t).$$

Iterating this relation n times yields

$$\delta_\psi^n Q_\psi = \delta_\psi^{n-1}(\delta_\psi Q_\psi) = \delta_\psi^{n-1}\left(Q_\psi \frac{d}{du}\right) = \dots = Q_\psi \frac{d^n}{du^n},$$

i.e.,

$$\delta_\psi^n Q_\psi = Q_\psi \frac{d^n}{du^n}. \quad (18)$$

Therefore,

$$D_{a^+}^{\alpha;\psi} = Q_\psi \frac{d^n}{du^n} I_{\psi(a)^+}^{n-\alpha} Q_\psi^{-1} = Q_\psi D_{\psi(a)^+}^\alpha Q_\psi^{-1},$$

where $D_{\psi(a)^+}^\alpha$ denotes the standard Riemann–Liouville fractional derivative of order α with lower limit $\psi(a)$.

Define

$$h(u) := (Q_\psi^{-1} f)(u) = f(\psi^{-1}(u)), \quad \text{for } u \in [u_0, u_1],$$

where $u_0 = \psi(a)$, $u_1 = \psi(b)$. Then,

$$(I_{a^+}^{\alpha;\psi} D_{a^+}^{\alpha;\psi} f)(t) = [Q_\psi I_{u_0^+}^\alpha Q_\psi^{-1}] [Q_\psi D_{u_0^+}^\alpha Q_\psi^{-1}] f(t) = Q_\psi (I_{u_0^+}^\alpha D_{u_0^+}^\alpha h)(u). \quad (19)$$

We now show that h satisfies the hypotheses of Lemma 9 with $\gamma = \rho$.

Since $\psi \in C^n[a, b]$, $\psi'(t) \neq 0$, and ψ is strictly increasing, the Inverse Function Theorem implies $\psi^{-1} \in C^n[u_0, u_1]$. Thus $\psi : [a, b] \rightarrow [u_0, u_1]$ is a C^n -diffeomorphism. This diffeomorphism guarantees that the change of variable $u = \psi(t)$ preserves continuity and differentiability, and that the weighted spaces $C_{\rho,\psi}^n[a, b]$ and $C_\rho^n[u_0, u_1]$ are isomorphic via Q_ψ .

By definition, $f \in C_{\rho,\psi}[a, b]$ means $(\Delta_a^\psi t)^\rho f(t) \in C[a, b]$. Substituting $t = \psi^{-1}(u)$ gives $(u - u_0)^\rho h(u) \in C[u_0, u_1]$, i.e., $h \in C_\rho[u_0, u_1]$.

From the hypothesis $I_{a^+}^{n-\alpha;\psi} f \in C_{\rho,\psi}^n[a, b]$ and the conjugation $I_{a^+}^{n-\alpha;\psi} f = Q_\psi (I_{u_0^+}^{n-\alpha} h)$, we obtain $Q_\psi (I_{u_0^+}^{n-\alpha} h) \in C_{\rho,\psi}^n[a, b]$. For any $j = 0, 1, \dots, n$, the relation $\delta_\psi = Q_\psi \frac{d}{du} Q_\psi^{-1}$ implies

$$\delta_\psi^j (Q_\psi (I_{u_0^+}^{n-\alpha} h))(t) = \frac{d^j}{du^j} (I_{u_0^+}^{n-\alpha} h)(\psi(t)), \quad u = \psi(t). \quad (20)$$

Consequently,

- For $j = 0, \dots, n-1$: the continuity of $\delta_\psi^j (Q_\psi (I_{u_0^+}^{n-\alpha} h))$ in t implies $\frac{d^j}{du^j} (I_{u_0^+}^{n-\alpha} h)$ is continuous in u .
- For $j = n$: the weighted continuity $(\Delta_a^\psi t)^\rho \delta_\psi^n (Q_\psi (I_{u_0^+}^{n-\alpha} h))(t) \in C[a, b]$ transforms into $(u - u_0)^\rho \frac{d^n}{du^n} (I_{u_0^+}^{n-\alpha} h)(u) \in C[u_0, u_1]$.

Hence, $I_{u_0^+}^{n-\alpha} h \in C_\rho^n[u_0, u_1]$, so Lemma 9 applies to h with $\gamma = \rho$. Therefore, Lemma 9 applies to h with $\gamma = \rho$, yielding

$$(I_{u_0^+}^\alpha D_{u_0^+}^\alpha h)(u) = h(u) - \sum_{k=1}^n \frac{(D^{n-k} I_{u_0^+}^{n-\alpha} h)(u_0)}{\Gamma(\alpha - k + 1)} (u - u_0)^{\alpha-k}.$$

Apply Q_ψ to both sides, as follows:

$$Q_\psi(I_{u_0^+}^\alpha D_{u_0^+}^\alpha h)(u) = Q_\psi h(u) - \sum_{k=1}^n \frac{(D^{n-k} I_{u_0^+}^{n-\alpha} h)(u_0)}{\Gamma(\alpha - k + 1)} Q_\psi[(u - u_0)^{\alpha-k}].$$

Now, using (17), (19) and $Q_\psi h(u) = h(\psi(t)) = f(\psi^{-1}(\psi(t))) = f(t)$, we obtain

$$(I_{a^+}^{\alpha;\psi} D_{a^+}^{\alpha;\psi} f)(t) = f(t) - \sum_{k=1}^n \frac{(\Delta_a^\psi t)^{\alpha-k}}{\Gamma(\alpha - k + 1)} \cdot (D^{n-k} I_{u_0^+}^{n-\alpha} h)(u_0),$$

Now, $(D^{n-k} I_{u_0^+}^{n-\alpha} h)(u_0)$ is a constant. Using (20), and from the conjugation $I_{a^+}^{n-\alpha;\psi} f = Q_\psi(I_{u_0^+}^{n-\alpha} h)$, using $I_{u_0^+}^{n-\alpha} h = Q_\psi^{-1} I_{a^+}^{n-\alpha;\psi} f$, we can write

$$\frac{d^{n-k}}{du^{n-k}} (I_{u_0^+}^{n-\alpha} h)(u_0) = \delta_\psi^{n-k} (Q_\psi(I_{u_0^+}^{n-\alpha} h))(u_0) = \delta_\psi^{n-k} (Q_\psi(Q_\psi^{-1} I_{a^+}^{n-\alpha;\psi} f))(a) = \delta_\psi^{n-k} (I_{a^+}^{n-\alpha;\psi} f)(a).$$

Thus,

$$(D^{n-k} I_{u_0^+}^{n-\alpha} h)(u_0) = (\delta_\psi^{n-k} I_{a^+}^{n-\alpha;\psi} f)(a).$$

Substituting this constant gives the desired identity (16).

For integer $\Re(\alpha) \rightarrow n \in \mathbb{N}$, we have $n - \Re(\alpha) \rightarrow 0$, so $I_{a^+}^{n-\alpha;\psi} f = f$ and the formula holds by continuity in α or by direct verification (the $k = n$ term gives $f(a)$). This completes the proof. $\square$

Theorem 6. Let $\alpha, \gamma \in \mathbb{C}$ with $n - 1 < \Re(\alpha) < n$, $n \in \mathbb{N}$, and let $0 \leq \beta \leq 1$. Define $\gamma = \alpha + \beta(n - \alpha)$, which satisfies $n - 1 < \Re(\gamma) < n$. Assume that $\psi \in C^n[a, b]$ is strictly increasing on $[a, b]$ and satisfies $\psi'(t) \neq 0$ for all $t \in [a, b]$. Let $f \in C_{n-\gamma, \psi}[a, b]$, and $I_{a^+}^{n-\gamma;\psi} f \in C_{n-\gamma, \psi}^n[a, b]$. Then, for every $t \in (a, b]$,

$$(I_{a^+}^{\alpha;\psi} D_{a^+}^{\alpha;\beta;\psi} f)(t) = f(t) - \sum_{k=1}^n \frac{(\delta_\psi^{n-k} I_{a^+}^{n-\gamma;\psi} f)(a)}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{\gamma-k}, \quad (21)$$

$$\text{where } \delta_\psi = \frac{1}{\psi'(t)} \frac{d}{dt}.$$

Proof. The proof proceeds by reducing the ψ -Hilfer derivative to the ψ -Riemann–Liouville derivative and then applying Theorem 5.

By the equivalent definition of the ψ -Hilfer fractional derivative given by (6), we have

$$(D_{a^+}^{\alpha;\beta;\psi} f)(t) = (I_{a^+}^{\beta(n-\alpha);\psi} D_{a^+}^{\gamma;\psi} f)(t),$$

where $\gamma = \alpha + \beta(n - \alpha)$. Consequently,

$$I_{a^+}^{\alpha;\psi} (D_{a^+}^{\alpha;\beta;\psi} f)(t) = I_{a^+}^{\alpha;\psi} (I_{a^+}^{\beta(n-\alpha);\psi} D_{a^+}^{\gamma;\psi} f)(t).$$

Applying the semigroup property from Lemma 1 with parameters α and $\beta(n - \alpha)$, we obtain $I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta(n-\alpha);\psi} = I_{a^+}^{\gamma;\psi}$, since $\gamma = \alpha + \beta(n - \alpha)$. Hence,

$$(I_{a^+}^{\alpha;\psi} D_{a^+}^{\alpha;\beta;\psi} f)(t) = (I_{a^+}^{\gamma;\psi} D_{a^+}^{\gamma;\psi} f)(t).$$

Observe that γ satisfies $n - 1 < \Re(\gamma) < n$ by hypothesis, and $n = \lfloor \Re(\gamma) \rfloor + 1$. Moreover, $f \in C_{n-\gamma,\psi}[a, b]$ and $I_{a+}^{n-\gamma;\psi} f \in C_{n-\gamma,\psi}^n[a, b]$ are given. Thus Theorem 5 applies with α replaced by γ and $\rho = n - \gamma$ (note that $0 \leq n - \gamma < 1$ because $n - 1 < \Re(\gamma) < n$). Applying Formula (16) with α replaced by γ yields

$$(I_{a+}^{\gamma;\psi} D_{a+}^{\gamma;\psi} f)(t) = f(t) - \sum_{k=1}^n \frac{(\delta_{\psi}^{n-k} I_{a+}^{n-\gamma;\psi} f)(a)}{\Gamma(\gamma - k + 1)} (\Delta_a^{\psi} t)^{\gamma-k}.$$

Therefore, (21) holds, which completes the proof. $\square$

Lemma 10 (Left Inverse Property of the ψ -Riemann–Liouville Fractional Integral with ψ -Hilfer Fractional Derivative). *Let $\alpha, \gamma \in \mathbb{C}$ with $n - 1 < \Re(\alpha) < n$, $n \in \mathbb{N}$, and let $0 \leq \beta \leq 1$. Define $\gamma = \alpha + \beta(n - \alpha)$, which satisfies $n - 1 < \Re(\gamma) < n$. Suppose $f \in C_{n-\gamma,\psi}[a, b]$ and $I_{a+}^{1-\beta(n-\alpha);\psi} f \in C_{n-\gamma,\psi}^1[a, b]$. Then, $D_{a+}^{\alpha,\beta;\psi} I_{a+}^{\alpha;\psi} f$ exists on $(a, b]$ and satisfies*

$$(D_{a+}^{\alpha,\beta;\psi} I_{a+}^{\alpha;\psi} f)(t) = f(t), \quad t \in (a, b].$$

Proof. Since $I_{a+}^{1-\beta(n-\alpha);\psi} f \in C_{n-\gamma,\psi}^1[a, b]$, Definition 5 gives

$$D_{a+}^{\beta(n-\alpha);\psi} f = \delta_{\psi} I_{a+}^{1-\beta(n-\alpha);\psi} f \in C_{n-\gamma,\psi}[a, b].$$

From Remark 2 (iii) and (iv), the assumptions $n - 1 < \Re(\alpha) < n$ and $0 \leq \beta \leq 1$ imply

$$0 < \beta(n - \Re(\alpha)) < 1 \quad \text{and} \quad 0 \leq n - \Re(\gamma) = (1 - \beta)(n - \Re(\alpha)) < 1.$$

Using Definition 6 of the ψ -Hilfer derivative, Lemma 1 (semigroup property), Definition 5 of the ψ -Riemann–Liouville derivative, and Theorem 5 (composition formula), respectively, we compute

$$\begin{aligned} (D_{a+}^{\alpha,\beta;\psi} I_{a+}^{\alpha;\psi} f)(t) &= (I_{a+}^{\beta(n-\alpha);\psi} \delta_{\psi}^n I_{a+}^{(n-\alpha)(1-\beta);\psi} I_{a+}^{\alpha;\psi} f)(t) \\ &= (I_{a+}^{\beta(n-\alpha);\psi} \delta_{\psi}^n I_{a+}^{n-\beta(n-\alpha);\psi} f)(t) \\ &= (I_{a+}^{\beta(n-\alpha);\psi} D_{a+}^{\beta(n-\alpha);\psi} f)(t) \\ &= f(t) - \frac{(I_{a+}^{1-\beta(n-\alpha);\psi} f)(a)}{\Gamma(\beta(n-\alpha))} (\Delta_a^{\psi} t)^{\beta(n-\alpha)-1}. \end{aligned}$$

Since $0 \leq n - \Re(\gamma) < 1 - \beta(n - \Re(\alpha))$, Lemma 8 implies $(I_{a+}^{1-\beta(n-\alpha);\psi} f)(a) = 0$. Substituting this into the last expression yields $(D_{a+}^{\alpha,\beta;\psi} I_{a+}^{\alpha;\psi} f)(t) = f(t)$, which completes the proof. $\square$

Lemma 11 (Regularization by Integration). *Let $\gamma \in \mathbb{C}$ with $n - 1 < \Re(\gamma) < n$, $n \in \mathbb{N}$, and $\psi \in C^n[a, b]$ with $\psi'(t) > 0$ on $[a, b]$. If $f \in C[a, b]$ and $\delta_{\psi}^n f \in C_{n-\gamma,\psi}[a, b]$, then $\delta_{\psi}^k f \in C[a, b]$ for all $k = 0, 1, \dots, n - 1$. Equivalently, $f \in C_{n-\gamma,\psi}^n[a, b]$.*

Proof. The proof proceeds by transforming the problem to standard calculus via the change of variable $u = \psi(t)$, followed by an inductive integration argument.

Since ψ is strictly increasing and $\psi \in C^n[a, b]$, the map $t \mapsto u = \psi(t)$ is a C^n -diffeomorphism from $[a, b]$ onto $[u_0, u_1]$, where $u_0 = \psi(a)$ and $u_1 = \psi(b)$. Define

$$h(u) = f(\psi^{-1}(u)), \quad u \in [u_0, u_1].$$

Then, $f \in C[a, b]$ implies $h \in C[u_0, u_1]$. The operator δ_ψ transforms as $\delta_\psi = \frac{d}{du}$; hence, $\delta_\psi^n f(t) = h^{(n)}(u)$. The weighted space condition $\delta_\psi^n f \in C_{n-\gamma, \psi}[a, b]$ means

$$(\Delta_a^\psi t)^{n-\gamma} \delta_\psi^n f(t) \in C[a, b],$$

which becomes

$$(u - u_0)^{n-\gamma} h^{(n)}(u) \in C[u_0, u_1].$$

Let $\rho := n - \gamma$ with $\Re(\rho) = n - \Re(\gamma) \in (0, 1)$. Then, we can write

$$h^{(n)}(u) = (u - u_0)^{-\rho} \Phi(u), \quad \Phi(u) := (u - u_0)^\rho h^{(n)}(u) \in C[u_0, u_1].$$

For $u > u_0$, the fundamental theorem of calculus gives

$$h^{(n-1)}(u) = h^{(n-1)}(u_0^+) + \int_{u_0}^u h^{(n)}(x) dx.$$

Since $\rho < 1$, the integral converges absolutely:

$$\int_{u_0}^u |h^{(n)}(x)| dx \leq \|\Phi\|_{C[u_0, u_1]} \int_{u_0}^u (x - u_0)^{-\rho} dx = \|\Phi\|_{C[u_0, u_1]} \frac{(u - u_0)^{1-\rho}}{1-\rho} < \infty.$$

Define $H(u) = \int_{u_0}^u h^{(n)}(x) dx$. Then, H is continuous on $[u_0, u_1]$ with $H(u_0) = 0$. Consider the difference $h^{(n-1)}(u) - H(u)$. For $u > u_0$,

$$\frac{d}{du} (h^{(n-1)}(u) - H(u)) = h^{(n)}(u) - h^{(n)}(u) = 0.$$

Thus, $h^{(n-1)}(u) - H(u)$ is constant on $(u_0, u_1]$. Denote this constant by c_{n-1} . Then,

$$h^{(n-1)}(u) = c_{n-1} + H(u), \quad u > u_0.$$

Since H is continuous on $[u_0, u_1]$ and $H(u_0) = 0$, the right-hand side extends continuously to u_0 with value c_{n-1} . Consequently,

$$\lim_{u \rightarrow u_0^+} h^{(n-1)}(u) = c_{n-1}$$

exists and is finite, and $h^{(n-1)} \in C[u_0, u_1]$.

Assume that for some integer k with $1 \leq k \leq n - 2$, we have established $h^{(k+1)} \in C[u_0, u_1]$. Then, for $u > u_0$,

$$h^{(k)}(u) = h^{(k)}(u_0^+) + \int_{u_0}^u h^{(k+1)}(x) dx.$$

Since $h^{(k+1)}$ is continuous, the integral defines a continuous function on $[u_0, u_1]$ (its value at u_0 is 0). Then

$$h^{(k)}(u) = c_k + \int_{u_0}^u h^{(k+1)}(x) dx \in C[u_0, u_1].$$

Thus, $h^{(k)}$ is continuous. With the base case $k = n - 1$ already established, we descend inductively to obtain

$$h^{(k)} \in C[u_0, u_1] \quad \text{for all } k = 0, 1, \dots, n - 1.$$

Since ψ is continuous and strictly increasing, and $h^{(k)} \in C[u_0, u_1]$, we have

$$\delta_{\psi}^k f(t) = h^{(k)}(\psi(t)) \in C[a, b], \quad k = 0, 1, \dots, n-1.$$

Therefore, $\delta_{\psi}^k f \in C[a, b]$ for all $k = 0, 1, \dots, n-1$. Together with the hypothesis $\delta_{\psi}^n f \in C_{n-\gamma, \psi}[a, b]$, this is precisely the definition of $f \in C_{n-\gamma, \psi}^n[a, b]$. The proof is complete. $\square$

3.2. Equivalent Volterra Integral Equation

We begin by proving a fundamental lemma concerning a linear variant of the Cauchy problem (1)–(2), which will be used to transform the fractional Cauchy problem into an equivalent integral equation.

Lemma 12. Let $\alpha, \gamma \in \mathbb{C}$ with $n-1 < \Re(\alpha) < n$, $n \in \mathbb{N}$, $\beta \in [0, 1]$ and $\gamma = \alpha + \beta(n-\alpha)$. Assume that $\psi \in C^n[a, b]$ with $\psi'(t) > 0$ on $[a, b]$ and $f(\cdot, x(\cdot), {}^H D_{a+}^{\alpha, \beta; \psi} x(\cdot)) \in C_{n-\gamma, \psi}[a, b]$ for any $x \in C_{n-\gamma, \psi}[a, b]$. A function $x \in C_{n-\gamma, \psi}^{\gamma}[a, b]$ is a solution of the problem (1)–(2):

$$\begin{cases} ({}^H D_{a+}^{\alpha, \beta; \psi} x)(t) = f(t, x(t), ({}^H D_{a+}^{\alpha, \beta; \psi} x)(t)), & t \in (a, b], a \geq 0 \\ x_{n-\gamma}^{(n-k); \psi}(a) = c_k, & c_k \in \mathbb{C}, k \in \{1, 2, \dots, n\}, \end{cases}$$

if and only if x satisfies the following Volterra integral equation:

$$x(t) = \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} (\Delta_a^{\psi} t)^{\gamma-k} + (I_{a+}^{\alpha; \psi} G_x)(t), \quad (22)$$

where $G_x \in C_{n-\gamma, \psi}^{\beta(n-\alpha)}[a, b]$ satisfies the following functional equation:

$$G_x(t) = f(t, x(t), G_x(t)), \quad t \in (a, b]. \quad (23)$$

Proof. Let $\Re(\alpha) \in (n-1, n)$, for some $n \in \mathbb{N}$, $\beta \in [0, 1]$, and $\gamma = \alpha + \beta(n-\alpha)$. Then, $\Re(\gamma) \in (n-1, n)$. To prove the necessary condition, assume $x \in C_{n-\gamma, \psi}^{\gamma}[a, b]$ is a solution of the Cauchy problem (1)–(2). That is, $x \in C_{n-\gamma, \psi}[a, b]$ and $D_{a+}^{\gamma; \psi} x \in C_{n-\gamma, \psi}[a, b]$, from the definition of the space $C_{n-\gamma, \psi}^{\gamma}[a, b]$. From Lemma 5 and Definition 5, we have

$$I_{a+}^{n-\gamma; \psi} x \in C[a, b] \quad \text{and} \quad D_{a+}^{\gamma; \psi} x = \delta_{\psi}^n(I_{a+}^{n-\gamma; \psi} x) \in C_{n-\gamma, \psi}[a, b],$$

, respectively. Thus $I_{a+}^{n-\gamma; \psi} x \in C_{n-\gamma, \psi}^n[a, b]$ by Lemma 11. Therefore, x and $I_{a+}^{n-\gamma; \psi} x$ satisfy the conditions of Theorem 6. We will prove that x is also a solution of (22) where $G_x \in C_{n-\gamma, \psi}^{\beta(n-\alpha)}[a, b]$ satisfies the functional Equation (23).

Define the function

$$G_x(t) := (D_{a+}^{\alpha, \beta; \psi} x)(t) = (I_{a+}^{\beta(n-\alpha); \psi} D_{a+}^{\gamma; \psi} x)(t), \quad (24)$$

which belongs to $C_{n-\gamma, \psi}[a, b]$ by Lemma 4. Then, by Lemma 6, we have

$$(D_{a+}^{\beta(n-\alpha); \psi} G_x)(t) = (D_{a+}^{\beta(n-\alpha); \psi} I_{a+}^{\beta(n-\alpha); \psi} D_{a+}^{\gamma; \psi} x)(t) = (D_{a+}^{\gamma; \psi} x)(t) \in C_{n-\gamma, \psi}[a, b].$$

That is, $G_x \in C_{n-\gamma, \psi}^{\beta(n-\alpha)}[a, b]$. Moreover, (1) can be written in the form of Equation (23):

$$G_x(t) = f(t, x(t), G_x(t)), \quad t \in (a, b].$$

Applying $I_{a^+}^{\alpha;\psi}$ to both sides of (24) and using Lemma 1, we obtain

$$I_{a^+}^{\alpha;\psi} G_x(t) = I_{a^+}^{\alpha;\psi} (I_{a^+}^{\beta(n-\alpha);\psi} D_{a^+}^{\gamma;\psi} x)(t) = (I_{a^+}^{\gamma;\psi} D_{a^+}^{\gamma;\psi} x)(t).$$

By Theorem 6, for any $t \in (a, b]$, this equation can be written as

$$\begin{aligned} I_{a^+}^{\alpha;\psi} G_x(t) &= (I_{a^+}^{\gamma;\psi} D_{a^+}^{\gamma;\psi} x)(t) \\ &= x(t) - \sum_{k=1}^n \frac{(\delta_\psi^{n-k} I_{a^+}^{n-\gamma;\psi} f)(a)}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{\gamma-k} \\ &= x(t) - \sum_{k=1}^n \frac{x_{n-\gamma}^{(n-k);\psi}(a)}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{\gamma-k}. \end{aligned}$$

Therefore, using the initial condition (2) (recall the notation introduced after (2)), we obtain Equation (22):

$$x(t) = \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{\gamma-k} + (I_{a^+}^{\alpha;\psi} G_x)(t), \quad t \in (a, b].$$

Now, to prove sufficiency, let $G_x \in C_{n-\gamma,\psi}^{\beta(n-\alpha)}[a, b]$ which satisfies the functional Equation (23), where $x \in C_{n-\gamma,\psi}^\gamma[a, b]$. Since $G_x \in C_{n-\gamma,\psi}^{\beta(n-\alpha)}[a, b]$, we have $D_{a^+}^{\beta(n-\alpha);\psi} G_x \in C_{n-\gamma,\psi}[a, b]$. Moreover, by the definition of the ψ -Riemann–Liouville fractional derivative and since $0 < \beta(n - \Re(\alpha)) < 1$, we obtain

$$\delta_\psi I_{a^+}^{1-\beta(n-\alpha)} G_x = D_{a^+}^{\beta(n-\alpha);\psi} G_x \in C_{n-\gamma,\psi}[a, b].$$

By Lemma 5, $I_{a^+}^{1-\beta(n-\alpha)} G_x \in C[a, b]$. Hence, $I_{a^+}^{1-\beta(n-\alpha);\psi} G_x \in C_{n-\gamma,\psi}^1[a, b]$. Therefore, G_x and $I_{a^+}^{1-\beta(n-\alpha);\psi} G_x$ satisfy the conditions of Lemma 10. We will prove that x is also a solution of the fractional Cauchy problem (1)–(2).

Applying $D_{a^+}^{\alpha,\beta;\psi}$ to both sides of Equation (22), we get

$$D_{a^+}^{\alpha,\beta;\psi} x(t) = \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} D_{a^+}^{\alpha,\beta;\psi} (\Delta_a^\psi t)^{\gamma-k} + D_{a^+}^{\alpha,\beta;\psi} (I_{a^+}^{\alpha;\psi} G_x)(t).$$

From Definition 6 and Property 1 (iii), this equation becomes

$$D_{a^+}^{\alpha,\beta;\psi} x(t) = \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} I_{a^+}^{\beta(n-\alpha);\psi} D_{a^+}^{\gamma;\psi} (\Delta_a^\psi t)^{\gamma-k} + D_{a^+}^{\alpha,\beta;\psi} (I_{a^+}^{\alpha;\psi} G_x)(t) = D_{a^+}^{\alpha,\beta;\psi} (I_{a^+}^{\alpha;\psi} G_x)(t).$$

Then, by virtue of Lemma 10, we obtain

$$D_{a^+}^{\alpha,\beta;\psi} x(t) = (D_{a^+}^{\alpha,\beta;\psi} I_{a^+}^{\alpha;\psi} G_x)(t) = G_x(t).$$

From Equation (23), we obtain Equation (1).

Applying $I_{a^+}^{n-\gamma;\psi}$ to both sides of (22) and using Property 1 (i) and Lemma 1, we have

$$\begin{aligned} I_{a^+}^{n-\gamma;\psi} x(t) &= \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} I_{a^+}^{n-\gamma;\psi} (\Delta_a^\psi t)^{\gamma-k} + I_{a^+}^{n-\gamma;\psi} (I_{a^+}^{\alpha;\psi} G_x)(t) \\ &= \sum_{k=1}^n \frac{c_k}{\Gamma(n - k + 1)} (\Delta_a^\psi t)^{n-k} + (I_{a^+}^{n-\gamma+\alpha;\psi} G_x)(t). \end{aligned}$$

Taking the derivative δ_ψ^{n-i} for $i \in \{1, 2, \dots, n\}$ of the equation above, and noting that $\beta(n - \Re(\alpha)) < 1 \leq i$, we have $n - \Re(\gamma) + \Re(\alpha) = n - \beta(n - \Re(\alpha)) > n - i$ for all $i \in \{1, 2, \dots, n\}$. By Properties 1 (ii), (iii), we obtain

$$\begin{aligned} \delta_\psi^{n-i}(I_{a^+}^{n-\gamma;\psi}x)(t) &= \sum_{k=1}^n \frac{c_k}{\Gamma(n-k+1)} \delta_\psi^{n-i}(\Delta_a^\psi t)^{n-k} + \delta_\psi^{n-i}(I_{a^+}^{n-\gamma+\alpha;\psi}G_x)(t) \\ &= \sum_{k=1}^i \frac{c_k}{\Gamma(i-k+1)} (\Delta_a^\psi t)^{i-k} + \delta_\psi^{n-i}(I_{a^+}^{n-\beta(n-\alpha);\psi}G_x)(t) \\ &= \frac{c_1}{\Gamma(i)} (\Delta_a^\psi t)^{i-1} + \frac{c_2}{\Gamma(i-1)} (\Delta_a^\psi t)^{i-2} + \dots + \frac{c_{i-1}}{\Gamma(2)} (\Delta_a^\psi t) + c_i + \delta_\psi^{n-i}(I_{a^+}^{n-\beta(n-\alpha);\psi}G_x)(t). \end{aligned} \quad (25)$$

Since $0 \leq n - \Re(\gamma) < 1$ and $0 \leq n - \Re(\gamma) < n - \beta(n - \Re(\alpha))$ for any $i \in \{1, 2, \dots, n\}$, by applying Lemma 8, the ψ -Riemann–Liouville integral of order $n - \beta(n - \alpha)$ of G_x at the left endpoint satisfies

$$(I_{a^+}^{n-\beta(n-\alpha);\psi}G_x)(a) := \lim_{t \rightarrow a^+} (I_{a^+}^{n-\beta(n-\alpha);\psi}G_x)(t) = 0. \quad (26)$$

Therefore, by Equations (25) and (26), we obtain the initial condition (2):

$$x_{n-\gamma}^{(n-i);\psi}(a) = \lim_{t \rightarrow a^+} (\delta_\psi^{n-i} I_{a^+}^{n-\gamma;\psi}x)(t) = c_i, \quad i \in \{1, 2, \dots, n\}.$$

This completes the proof. $\square$

Define $\Delta_x^\psi y := \psi(y) - \psi(x)$. Let $\gamma \in \mathbb{C}$ with $n - 1 < \Re(\gamma) < n$, $n \in \mathbb{N}$, and let $\psi \in C^n[a, b]$ be strictly increasing with $\psi'(t) > 0$ on $[a, b]$. Denote by $C_{n-\gamma,\psi}[a, b]$ the Banach space equipped with the norm

$$\|x\|_{C_{n-\gamma,\psi}[a,b]} = \|x\|_{C_{n-\gamma,\psi}[a,b]} = \max_{t \in [a,b]} |(\Delta_a^\psi t)^{n-\gamma} x(t)|.$$

Using Theorem 12, problem (1)–(2) can be expressed in terms of a fixed-point operator. To this end, we define the operator $\mathcal{F} : C_{n-\gamma,\psi}[a, b] \rightarrow C_{n-\gamma,\psi}[a, b]$ by

$$(\mathcal{F}x)(t) = \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma-k+1)} (\Delta_a^\psi t)^{\gamma-k} + (I_{a^+}^{\alpha;\psi}G_x)(t), \quad t \in (a, b], \quad (27)$$

where G_x satisfies the properties stated in Lemma 12.

Under the following assumptions, problem (1)–(2) is shown to be well-posed. The well-posedness and stability results of Sections 3–5 rely on at most four hypotheses on f , stated once below and referred to by label throughout the paper: (H0) is a measurability/regularity condition ensuring that f acts continuously between the relevant weighted spaces; (H1) is a linear growth condition used only for the existence theorem (Theorem 7); (H2) is a global Lipschitz condition used for uniqueness, stability and continuous dependence (Theorems 8–11); and (H3) is a Bielecki-type weight condition used only for the Ulam–Hyers–Rassias-type results (Theorem 10). No other conditions on f are used anywhere in the paper.

(H0) Let $f : (a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that

$$f(\cdot, x(\cdot), ({}^H D_{a^+}^{\alpha,\beta;\psi}x)(\cdot)) \in C_{n-\gamma,\psi}[a, b],$$

for any $x \in C_{n-\gamma,\psi}[a, b]$.

(H1) There exist non-negative continuous functions $m, q : [a, b] \rightarrow \mathbb{R}_+$ with $\|q\|_{C[a,b]} < 1$ and a function $l \in C_{n-\gamma, \psi}[a, b]$ such that

$$|f(t, u, v)| \leq l(t) + m(t)|u| + q(t)|v|,$$

for any $(t, u, v) \in (a, b] \times \mathbb{R} \times \mathbb{R}$.

(H2) There exist non-negative constants K, M with $M < 1$ such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq K|u_1 - u_2| + M|v_1 - v_2|,$$

for any $u_1, v_1, u_2, v_2 \in \mathbb{R}$ and $t \in (a, b]$.

(H3) There exists an increasing function $\sigma \in C_{n-\gamma, \psi}[a, b]$ and a constant $\lambda_\sigma > 0$ such that for each $t \in (a, b]$,

$$(I_{a+}^\alpha \sigma)(t) \leq \lambda_\sigma \sigma(t).$$

Hypothesis (H3) is a Bielecki-type weight condition, standard in Ulam–Hyers–Rassias stability analysis; it is automatically satisfied for $\sigma(t) = e^{\lambda t}$ and, more generally, for any σ such that $I_{a+}^\alpha \sigma$ grows no faster than σ itself.

3.3. Existence Result via Schaefer's Fixed-Point Theorem

We now prove the existence of solutions to the Cauchy-type problem (1)–(2) in $C_{n-\gamma, \psi}^\gamma[a, b]$ using Schaefer's fixed-point theorem.

Theorem 7. Assume that $f : (a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying (H0), (H1) and (H2). Then, the Cauchy problem (1)–(2) has at least one solution in $C_{n-\gamma, \psi}^\gamma[a, b]$.

Proof. We will use Schaefer's fixed-point theorem to prove that $\mathcal{F}$, defined by

$$(\mathcal{F}x)(t) = \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{\gamma-k} + (I_{a+}^{\alpha; \psi} G_x)(t),$$

has a fixed point, where $G_x \in C_{n-\gamma, \psi}^{\beta(n-\alpha)}[a, b]$ satisfies $G_x(t) = f(t, x(t), G_x(t))$. First, we establish the existence of a fixed point in the weighted space $C_{n-\gamma, \psi}[a, b]$, and then we show that this fixed point actually belongs to the more regular space $C_{n-\gamma, \psi}^\gamma[a, b]$.

Step I: Completely continuous of $\mathcal{F}$. We show that $\mathcal{F} : C_{n-\gamma, \psi}[a, b] \rightarrow C_{n-\gamma, \psi}[a, b]$ is completely continuous.

Step I.1: Continuity of $\mathcal{F}$. First, we show that $\mathcal{F}$ is continuous on $C_{n-\gamma, \psi}[a, b]$. Let $\{x_m\}$ be a sequence such that $x_m \rightarrow x$ in $C_{n-\gamma, \psi}[a, b]$. Using Lemma 3, for each $t \in [a, b]$, we have

$$\begin{aligned} |(\Delta_a^\psi t)^{n-\gamma}((\mathcal{F}x_m)(t) - (\mathcal{F}x)(t))| &= |(\Delta_a^\psi t)^{n-\gamma}((I_{a+}^{\alpha; \psi} G_{x_m})(t) - (I_{a+}^{\alpha; \psi} G_x)(t))| \\ &\leq \frac{|(\Delta_a^\psi t)^{n-\gamma}|}{|\Gamma(\alpha)|} \int_a^t (\Delta_s^\psi t)^{\Re(\alpha)-1} |G_{x_m}(s) - G_x(s)| \psi'(s) ds \\ &\leq \frac{(\Delta_a^\psi t)^{n-\Re(\gamma)}}{|\Gamma(\alpha)|} (\Delta_a^\psi t)^{\Re(\alpha+\gamma-n)} B(\Re(\alpha), \Re(\gamma-n+1)) \|G_{x_m} - G_x\|_{C_{n-\gamma, \psi}[a, b]} \\ &\leq \frac{(\Delta_a^\psi b)^{\Re(\alpha)}}{|\Gamma(\alpha)|} B(\Re(\alpha), \Re(\gamma-n+1)) \|G_{x_m} - G_x\|_{C_{n-\gamma, \psi}[a, b]}. \end{aligned}$$

Now estimate $\|G_{x_m} - G_x\|_{C_{n-\gamma,\psi}[a,b]}$ using hypothesis (H2). For any $t \in [a, b]$,

$$\begin{aligned} |G_{x_m}(t) - G_x(t)| &= |f(t, x_m(t), G_{x_m}(t)) - f(t, x(t), G_x(t))| \\ &\leq K|x_m(t) - x(t)| + M|G_{x_m}(t) - G_x(t)|. \end{aligned}$$

Since $M < 1$, solving for $|G_{x_m} - G_x|$ yields

$$|G_{x_m}(t) - G_x(t)| \leq \frac{K}{1-M}|x_m(t) - x(t)|.$$

Multiplying by $(\Delta_a^\psi t)^{n-\Re(\gamma)}$ and taking the supremum gives

$$\|G_{x_m} - G_x\|_{C_{n-\gamma,\psi}[a,b]} \leq \frac{K}{1-M}\|x_m - x\|_{C_{n-\gamma,\psi}[a,b]}.$$

Combining the estimates, we obtain

$$\|\mathcal{F}x_m - \mathcal{F}x\|_{C_{n-\gamma,\psi}[a,b]} \leq \Lambda \|x_m - x\|_{C_{n-\gamma,\psi}[a,b]},$$

where

$$\Lambda = \frac{(\Delta_a^\psi b)^{\Re(\alpha)}}{|\Gamma(\alpha)|} B(\Re(\alpha), \Re(\gamma - n + 1)) \frac{K}{1-M}.$$

Since $\|x_m - x\| \rightarrow 0$, it follows that $\|\mathcal{F}x_m - \mathcal{F}x\| \rightarrow 0$. Hence, $\mathcal{F}$ is continuous on $C_{n-\gamma,\psi}[a, b]$. (Note that continuity of $\mathcal{F}$ is proved without the assumption $\Lambda < 1$ used in Theorem 8).

Step I.2: Compactness of $\mathcal{F}$. We show that $\mathcal{F}$ is compact on $C_{n-\gamma,\psi}[a, b]$. Let $B_R = \{x \in C_{n-\gamma,\psi}[a, b] : \|x\|_{C_{n-\gamma,\psi}[a,b]} \leq R\}$ be a bounded ball in $C_{n-\gamma,\psi}[a, b]$.

First, we show $\mathcal{F}$ is uniformly bounded on B_R . For $x \in B_R$ and $t \in [a, b]$, using Lemma 3, we have

$$\begin{aligned} |(\Delta_a^\psi t)^{n-\gamma}(\mathcal{F}x)(t)| &= \left| \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{n-k} + (\Delta_a^\psi t)^{n-\gamma} (I_{a+}^{\alpha;\psi} G_x)(t) \right| \\ &\leq \sum_{k=1}^n \frac{|c_k|}{|\Gamma(\gamma - k + 1)|} (\Delta_a^\psi t)^{\Re(n-k)} + (\Delta_a^\psi t)^{\Re(n-\gamma)} I_{a+}^{\alpha;\psi} |G_x(t)| \\ &= \sum_{k=1}^n \frac{|c_k|}{|\Gamma(\gamma - k + 1)|} (\Delta_a^\psi t)^{n-k} + \frac{(\Delta_a^\psi t)^{\Re(n-\gamma)}}{|\Gamma(\alpha)|} \int_a^t (\Delta_s^\psi t)^{\Re(\alpha)-1} |G_x(s)| \psi'(s) ds \\ &\leq \sum_{k=1}^n \frac{|c_k|}{|\Gamma(\gamma - k + 1)|} (\Delta_a^\psi b)^{n-k} + \frac{(\Delta_a^\psi b)^{\Re(\alpha)}}{|\Gamma(\alpha)|} B(\Re(\alpha), \Re(\gamma - n + 1)) \|G_x\|_{C_{n-\gamma,\psi}[a,b]}. \end{aligned} \quad (28)$$

Now we estimate $\|G_x\|_{C_{n-\gamma,\psi}[a,b]}$ using hypothesis (H1). For $t \in [a, b]$, we have

$$\begin{aligned} (\Delta_a^\psi t)^{\Re(n-\gamma)} |G_x(t)| &= (\Delta_a^\psi t)^{n-\Re(\gamma)} |f(t, x(t), G_x(t))| \\ &\leq (\Delta_a^\psi t)^{n-\Re(\gamma)} (l(t) + m(t)|x(t)| + q(t)|G_x(t)|) \\ &\leq (\Delta_a^\psi t)^{n-\Re(\gamma)} l(t) + m(t) |(\Delta_a^\psi t)^{n-\Re(\gamma)} x(t)| + q(t) |(\Delta_a^\psi t)^{n-\Re(\gamma)} G_x(t)|. \end{aligned}$$

Note that $|(\Delta_a^\psi t)^{n-\gamma} x(t)| = (\Delta_a^\psi t)^{n-\Re(\gamma)} |x(t)|$. Thus, taking the maximum over $t \in [a, b]$, we obtain

$$\|G_x\|_{C_{n-\gamma,\psi}[a,b]} \leq \|l\|_{C_{n-\gamma,\psi}[a,b]} + \|m\|_{C[a,b]} \|x\|_{C_{n-\gamma,\psi}[a,b]} + \|q\|_{C[a,b]} \|G_x\|_{C_{n-\gamma,\psi}[a,b]}.$$

Since $\|q\|_{C[a,b]} < 1$ by (H1) and $\|x\|_{C_{n-\gamma,\psi}[a,b]} \leq R$, we can solve for $\|G_x\|_{C_{n-\gamma,\psi}[a,b]}$:

$$\|G_x\|_{C_{n-\gamma,\psi}[a,b]} \leq \frac{L + m^* R}{1 - q^*}, \quad (29)$$

where $L := \|l\|_{C_{n-\gamma,\psi}[a,b]}$, $m^* := \|m\|_{C[a,b]}$, and $q^* := \|q\|_{C[a,b]}$. Substituting (29) into (28) yields, for any $t \in [a, b]$,

$$|(\Delta_a^\psi t)^{n-\gamma}(\mathcal{F}x)(t)| \leq \sum_{k=1}^n \frac{|c_k|}{|\Gamma(\gamma - k + 1)|} (\Delta_a^\psi b)^{n-k} + \frac{(\Delta_a^\psi b)^{\Re(\alpha)}}{|\Gamma(\alpha)|} B(\Re(\alpha), \Re(\gamma - n + 1)) \frac{L + m^* R}{1 - q^*}. \quad (30)$$

The right-hand side is a constant independent of x and t . Therefore,

$$\|\mathcal{F}x\|_{C_{n-\gamma,\psi}[a,b]} \leq C, \quad \text{for all } x \in B_R,$$

where C denotes the constant on the right-hand side of (30). Hence, $\mathcal{F}$ is uniformly bounded on B_R .

We show that $\mathcal{F}(B_R)$ is equicontinuous on $[a, b]$. Let $t_1, t_2 \in [a, b]$ with $t_1 < t_2$ and let $x \in B_R$. Define the auxiliary function

$$H_x(t) := (\Delta_a^\psi t)^{n-\gamma}(\mathcal{F}x)(t).$$

Then

$$\begin{aligned} |H_x(t_2) - H_x(t_1)| &\leq \sum_{k=1}^n \frac{|c_k|}{|\Gamma(\gamma - k + 1)|} |(\Delta_a^\psi t_2)^{n-k} - (\Delta_a^\psi t_1)^{n-k}| \\ &\quad + \left| (\Delta_a^\psi t_2)^{n-\gamma} (I_{a+}^{\alpha;\psi} G_x)(t_2) - (\Delta_a^\psi t_1)^{n-\gamma} (I_{a+}^{\alpha;\psi} G_x)(t_1) \right|. \end{aligned}$$

The first sum tends to 0 as $t_2 \rightarrow t_1$ because the functions $t \mapsto (\Delta_a^\psi t)^{n-k}$ are continuous on $[a, b]$ for $k = 1, \dots, n$. For the second term, write

$$\begin{aligned} \left| (\Delta_a^\psi t_2)^{n-\gamma} (I_{a+}^{\alpha;\psi} G_x)(t_2) - (\Delta_a^\psi t_1)^{n-\gamma} (I_{a+}^{\alpha;\psi} G_x)(t_1) \right| &\leq \left| (\Delta_a^\psi t_2)^{n-\gamma} - (\Delta_a^\psi t_1)^{n-\gamma} \right| |I_{a+}^{\alpha;\psi} G_x(t_2)| \\ &\quad + (\Delta_a^\psi t_1)^{\Re(n-\gamma)} \left| (I_{a+}^{\alpha;\psi} G_x)(t_2) - (I_{a+}^{\alpha;\psi} G_x)(t_1) \right|. \end{aligned}$$

The factor $|I_{a+}^{\alpha;\psi} G_x(t_2)|$ is bounded uniformly for $x \in B_R$ by the uniform boundedness estimate. Since $t \mapsto (\Delta_a^\psi t)^{n-\gamma}$ is continuous (its modulus is $(\Delta_a^\psi t)^{\Re(n-\gamma)}$, which is continuous), the first term tends to 0 as $t_2 \rightarrow t_1$. For the second term, we estimate the difference of the fractional integrals. Using the definition,

$$\begin{aligned} \left| (I_{a+}^{\alpha;\psi} G_x)(t_2) - (I_{a+}^{\alpha;\psi} G_x)(t_1) \right| &= \frac{1}{|\Gamma(\alpha)|} \left| \int_a^{t_2} (\Delta_s^\psi t_2)^{\alpha-1} G_x(s) \psi'(s) ds - \int_a^{t_1} (\Delta_s^\psi t_1)^{\alpha-1} G_x(s) \psi'(s) ds \right| \\ &\leq \frac{1}{|\Gamma(\alpha)|} \left(\int_a^{t_1} |(\Delta_s^\psi t_2)^{\alpha-1} - (\Delta_s^\psi t_1)^{\alpha-1}| |G_x(s)| \psi'(s) ds + \int_{t_1}^{t_2} (\Delta_s^\psi t_2)^{\Re(\alpha)-1} |G_x(s)| \psi'(s) ds \right). \end{aligned}$$

Since $|G_x(s)| \leq (\Delta_a^\psi s)^{-(n-\Re(\gamma))} \|G_x\|_{C_{n-\gamma,\psi}[a,b]}$ and $\|G_x\|_{C_{n-\gamma,\psi}[a,b]}$ is bounded uniformly for $x \in B_R$ by (29), both integrals tend to 0 as $t_2 \rightarrow t_1$ uniformly in $x \in B_R$. The first integral converges to 0 by the dominated convergence theorem (the integrand converges pointwise to 0 and is dominated by an integrable function), and the second integral tends to 0 because the interval of integration shrinks and the integrand is integrable (since $\Re(\alpha) - 1 > -1$).

Thus $\mathcal{F}(B_R)$ is equicontinuous on $[a, b]$.

The set $\mathcal{F}(B_R)$ is uniformly bounded and equicontinuous. By the Arzelà–Ascoli theorem, $\mathcal{F}(B_R)$ is relatively compact in $C_{n-\gamma,\psi}[a, b]$. Hence, $\mathcal{F}$ is a compact operator on $C_{n-\gamma,\psi}[a, b]$.

From Steps I.1 and I.2, $\mathcal{F}$ is continuous and compact, i.e., completely continuous, on $C_{n-\gamma,\psi}[a, b]$.

Step II: Boundedness of the set $\mathcal{E}(\mathcal{F})$. We show that the set

$$\mathcal{E}(\mathcal{F}) = \{x \in C_{n-\gamma,\psi}[a, b] : x = \lambda \mathcal{F}x \text{ for some } \lambda \in [0, 1]\}$$

is bounded. Let $x \in \mathcal{E}(\mathcal{F})$. Then, there exists $\lambda \in [0, 1]$ such that $x = \lambda \mathcal{F}x$. Consequently, for each $t \in [a, b]$,

$$(\Delta_a^\psi t)^{n-\gamma} x(t) = \lambda (\Delta_a^\psi t)^{n-\gamma} (\mathcal{F}x)(t).$$

Taking absolute values and using that $0 \leq \lambda \leq 1$, we obtain

$$|(\Delta_a^\psi t)^{n-\gamma} x(t)| = \lambda |(\Delta_a^\psi t)^{n-\gamma} (\mathcal{F}x)(t)| \leq |(\Delta_a^\psi t)^{n-\gamma} (\mathcal{F}x)(t)|.$$

From the uniform boundedness estimate established in Step III (see inequality (30)), there exists a constant $C > 0$, independent of x and t , such that for all $t \in [a, b]$,

$$|(\Delta_a^\psi t)^{n-\gamma} (\mathcal{F}x)(t)| \leq C.$$

Therefore,

$$|(\Delta_a^\psi t)^{n-\gamma} x(t)| \leq C \quad \text{for all } t \in [a, b].$$

Taking the supremum over $t \in [a, b]$ yields

$$\|x\|_{C_{n-\gamma,\psi}[a,b]} = \max_{t \in [a,b]} |(\Delta_a^\psi t)^{n-\gamma} x(t)| \leq C.$$

Thus, every $x \in \mathcal{E}(\mathcal{F})$ satisfies $\|x\|_{C_{n-\gamma,\psi}[a,b]} \leq C$, showing that $\mathcal{E}(\mathcal{F})$ is bounded in $C_{n-\gamma,\psi}[a, b]$.

From Steps I and II, by Schaefer's fixed-point theorem, $\mathcal{F}$ possesses at least one fixed point $x \in C_{n-\gamma,\psi}[a, b]$. That is, there exists $x \in C_{n-\gamma,\psi}[a, b]$ such that $x = \mathcal{F}x$.

Step III: Regularity of the fixed point. We now demonstrate that the fixed point x actually belongs to the more regular space $C_{n-\gamma,\psi}^\gamma[a, b]$. Recall that $x = \mathcal{F}x$ satisfies the integral representation

$$x(t) = \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{\gamma-k} + (I_{a^+}^{\alpha;\psi} G_x)(t),$$

where $G_x \in C_{n-\gamma,\psi}^{\beta(n-\alpha)}[a, b]$ satisfies $G_x(t) = f(t, x(t), G_x(t))$ for $t \in (a, b)$. Applying the fractional derivative $D_{a^+}^{\gamma;\psi}$ to both sides and using Property 1 (iii), Definition 5 and Lemma 1, we obtain

$$\begin{aligned} (D_{a^+}^{\gamma;\psi} x)(t) &= (D_{a^+}^{\gamma;\psi} I_{a^+}^{\alpha;\psi} G_x)(t) = (\delta_\psi^n I_{a^+}^{n-\gamma;\psi} I_{a^+}^{\alpha;\psi} G_x)(t) = (\delta_\psi^n I_{a^+}^{n-\gamma+\alpha;\psi} G_x)(t) \\ &= (\delta_\psi^n I_{a^+}^{n-\beta(n-\alpha);\psi} G_x)(t) = (D_{a^+}^{\beta(n-\alpha);\psi} G_x)(t). \end{aligned}$$

Since $G_x \in C_{n-\gamma,\psi}^{\beta(n-\alpha)}[a, b]$, by definition we have $D_{a^+}^{\beta(n-\alpha);\psi} G_x \in C_{n-\gamma,\psi}[a, b]$. Consequently, $D_{a^+}^{\gamma;\psi} x \in C_{n-\gamma,\psi}[a, b]$, which implies that $x \in C_{n-\gamma,\psi}^\gamma[a, b]$. Thus, the fixed point x obtained from Schaefer's fixed-point theorem belongs to the more regular space $C_{n-\gamma,\psi}^\gamma[a, b]$.

We emphasize why this regularity upgrade is not automatic: Schaefer's theorem, applied to $\mathcal{F}$ on the larger space $C_{n-\gamma,\psi}[a, b]$, only guarantees a fixed point in that (weaker)

space, since complete continuity of $\mathcal{F}$ was established there. The computation above shows that $D_{a^+}^{\gamma;\psi} x = D_{a^+}^{\beta(n-\alpha);\psi} G_x$ belongs to $C_{n-\gamma,\psi}[a, b]$ precisely because $G_x \in C_{n-\gamma,\psi}^{\beta(n-\alpha)}[a, b]$, which is part of the conclusion of Lemma 12 and not an extra hypothesis; this is what allows us to conclude $x \in C_{n-\gamma,\psi}^\gamma[a, b]$ by Definition 3, without invoking any additional compactness or continuity argument.

Finally, from Lemma 12, we verify that x indeed satisfies the original Cauchy problem (1)–(2). Thus x is a solution to problem (1)–(2) in the space $C_{n-\gamma,\psi}^\gamma[a, b]$. This completes the proof. $\square$

3.4. Existence and Uniqueness Result via Banach's Fixed-Point Theorem

We now prove an existence and uniqueness result based on Banach's fixed-point theorem.

Theorem 8. Assume that $f : (a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying (H0) and (H2). If

$$\Lambda := \frac{(\Delta_a^\psi b)^{\Re(\alpha)}}{|\Gamma(\alpha)|} B(\Re(\alpha), \Re(\gamma - n + 1)) \cdot \frac{K}{1 - M} < 1, \quad (31)$$

then the Cauchy problem (1)–(2) has a unique solution in $C_{n-\gamma,\psi}^\gamma[a, b]$.

Remark 6. The constant Λ in (31) is obtained by bounding

$$|I_{a^+}^{\alpha;\psi} h(t)| \leq \frac{1}{|\Gamma(\alpha)|} \int_a^t (\Delta_a^\psi t)^{\Re(\alpha)-1} |h(\tau)| \psi'(\tau) d\tau$$

for any $h \in C_{n-\gamma,\psi}[a, b]$, followed by Lemma 3 applied with the real exponent $\Re(\alpha) - 1$. Since $1/\Gamma(\alpha) \in \mathbb{C}$ in general, only its modulus $1/|\Gamma(\alpha)|$ provides a valid (real, non-negative) bound; this is why $|\Gamma(\alpha)|$, and not $\Gamma(\alpha)$, appears in Λ . By contrast, $\Gamma(\alpha)$ without modulus is retained in Property 1 and Lemma 1, where exact identities (not estimates) are established.

Proof. We will use Banach's fixed-point theorem to prove that $\mathcal{F}$, defined by (27), has a unique fixed point. Let $N := \max_{t \in [a, b]} (\Delta_a^\psi t)^{n-\Re(\gamma)} |f(t, 0, 0)| < \infty$. Choose a suitable ball

$B_R = \{x \in C_{n-\gamma,\psi}[a, b] : \|x\|_{C_{n-\gamma,\psi}[a, b]} \leq R\}$, where

$$R \geq \frac{1}{1 - \Lambda} \left(\sum_{k=1}^n \frac{|c_k|}{|\Gamma(\gamma - k + 1)|} (\Delta_a^\psi b)^{n-k} + \frac{(\Delta_a^\psi b)^{\Re(\alpha)}}{|\Gamma(\alpha)|} B(\Re(\alpha), \Re(\gamma - n + 1)) \frac{N}{1 - M} \right). \quad (32)$$

(Fix any R satisfying (32); the choice does not affect the conclusion.)

Step I: Invariance of B_R under $\mathcal{F}$. We show that $\mathcal{F}(B_R) \subset B_R$. Let $x \in B_R$. Using Lemma 3 (with the real exponent $\Re(\alpha) - 1$ in the kernel), for any $t \in [a, b]$, we have

$$\begin{aligned} |(\Delta_a^\psi t)^{n-\gamma} (\mathcal{F}x)(t)| &= \left| (\Delta_a^\psi t)^{n-\gamma} \left(\sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{\gamma-k} + (I_{a^+}^{\alpha;\psi} G_x)(t) \right) \right| \\ &\leq \sum_{k=1}^n \frac{|c_k|}{|\Gamma(\gamma - k + 1)|} (\Delta_a^\psi t)^{n-k} + (\Delta_a^\psi t)^{\Re(n-\gamma)} I_{a^+}^{\alpha;\psi} |G_x(t)| \\ &= \sum_{k=1}^n \frac{|c_k|}{|\Gamma(\gamma - k + 1)|} (\Delta_a^\psi t)^{n-k} + \frac{(\Delta_a^\psi t)^{n-\Re(\gamma)}}{|\Gamma(\alpha)|} \int_a^t (\Delta_s^\psi t)^{\Re(\alpha)-1} |G_x(s)| \psi'(s) ds \\ &\leq \sum_{k=1}^n \frac{|c_k|}{|\Gamma(\gamma - k + 1)|} (\Delta_a^\psi b)^{n-k} + \frac{(\Delta_a^\psi b)^{\Re(\alpha)}}{|\Gamma(\alpha)|} B(\Re(\alpha), \Re(\gamma - n + 1)) \|G_x\|_{C_{n-\gamma,\psi}[a, b]}. \end{aligned} \quad (33)$$

By (H2), for any $t \in (a, b]$, we have

$$|G_x(t)| \leq |f(t, x(t), G_x(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \leq K|x(t)| + M|G_x(t)| + |f(t, 0, 0)|.$$

Multiplying both sides by $(\Delta_a^\psi t)^{n-\Re(\gamma)}$ and taking the maximum over $t \in [a, b]$, we obtain

$$\|G_x\|_{C_{n-\gamma,\psi}[a,b]} \leq \frac{1}{1-M} (K\|x\|_{C_{n-\gamma,\psi}[a,b]} + N) \leq \frac{KR+N}{1-M}. \quad (34)$$

Substituting inequality (34) into (33) and using the definition of R in (32), we get

$$\begin{aligned} |(\Delta_a^\psi t)^{n-\gamma}(\mathcal{F}x)(t)| &\leq \sum_{k=1}^n \frac{|c_k|}{|\Gamma(\gamma-k+1)|} (\Delta_a^\psi b)^{n-k} + \frac{(\Delta_a^\psi b)^{\Re(\alpha)}}{|\Gamma(\alpha)|} B(\Re(\alpha), \Re(\gamma-n+1)) \left(\frac{KR+N}{1-M} \right) \\ &\leq R. \end{aligned}$$

Thus, $\|\mathcal{F}x\|_{C_{n-\gamma,\psi}[a,b]} \leq R$, i.e., $\mathcal{F}x \in B_R$ for any $x \in B_R$. Hence, $\mathcal{F}(B_R) \subset B_R$.

Step II: Contraction property of $\mathcal{F}$ on B_R . We show that $\mathcal{F}$ is a contraction on $B_R \subset C_{n-\gamma,\psi}[a, b]$. Let $x, y \in B_R$. Using Lemma 3 (with the real exponent), for each $t \in [a, b]$, we have

$$\begin{aligned} |(\Delta_a^\psi t)^{n-\gamma}((\mathcal{F}x)(t) - (\mathcal{F}y)(t))| &= |(\Delta_a^\psi t)^{n-\gamma} \left((I_{a+}^{\alpha;\psi} G_x)(t) - (I_{a+}^{\alpha;\psi} G_y)(t) \right)| \\ &\leq (\Delta_a^\psi t)^{\Re(n-\gamma)} I_{a+}^{\alpha;\psi} |G_x(t) - G_y(t)| \\ &\leq \frac{(\Delta_a^\psi t)^{n-\Re(\gamma)}}{|\Gamma(\alpha)|} \int_a^t (\Delta_s^\psi t)^{\Re(\alpha)-1} |G_x(s) - G_y(s)| \psi'(s) ds \\ &\leq \frac{(\Delta_a^\psi t)^{\Re(\alpha)}}{|\Gamma(\alpha)|} B(\Re(\alpha), \Re(\gamma-n+1)) \|G_x - G_y\|_{C_{n-\gamma,\psi}[a,b]}. \end{aligned} \quad (35)$$

By (H2), for each $t \in (a, b]$, we have

$$|G_x(t) - G_y(t)| = |f(t, x(t), G_x(t)) - f(t, y(t), G_y(t))| \leq K|x(t) - y(t)| + M|G_x(t) - G_y(t)|.$$

Solving for $|G_x(t) - G_y(t)|$ yields

$$|G_x(t) - G_y(t)| \leq \frac{K}{1-M} |x(t) - y(t)|. \quad (36)$$

Multiplying both sides by $(\Delta_a^\psi t)^{n-\Re(\gamma)}$ and taking the maximum over $t \in [a, b]$, we obtain

$$\|G_x - G_y\|_{C_{n-\gamma,\psi}[a,b]} \leq \frac{K}{1-M} \|x - y\|_{C_{n-\gamma,\psi}[a,b]}. \quad (37)$$

Substituting inequality (37) into (35), we get

$$|(\Delta_a^\psi t)^{n-\gamma}((\mathcal{F}x)(t) - (\mathcal{F}y)(t))| \leq \frac{(\Delta_a^\psi t)^{\Re(\alpha)}}{|\Gamma(\alpha)|} B(\Re(\alpha), \Re(\gamma-n+1)) \frac{K}{1-M} \|x - y\|_{C_{n-\gamma,\psi}[a,b]}.$$

Taking the supremum over $t \in [a, b]$ and using condition (31), we obtain

$$\|\mathcal{F}x - \mathcal{F}y\|_{C_{n-\gamma,\psi}[a,b]} \leq \Lambda \|x - y\|_{C_{n-\gamma,\psi}[a,b]},$$

with $\Lambda < 1$. Hence, $\mathcal{F}$ is a contraction on B_R .

From Steps I and II, by Banach's fixed-point theorem, $\mathcal{F}$ has a unique fixed point $x \in B_R \subset C_{n-\gamma,\psi}[a, b]$. That is, there exists $x \in C_{n-\gamma,\psi}[a, b]$ such that $x = \mathcal{F}x$.

Step III: Regularity of the fixed point. Following the same regularity argument as in the proof of Theorem 7 (Step III), this fixed point actually belongs to the more regular space $C_{n-\gamma,\psi}^\gamma[a, b]$.

Therefore, from Lemma 12, the Cauchy problem (1)–(2) has a unique solution in $C_{n-\gamma,\psi}^\gamma[a, b]$. This completes the proof. $\square$

4. Ulam Stability Results

Finally, we examine four distinct types of Ulam stability for the nonlinear implicit Hilfer fractional differential problem (1)–(2).

Theorem 9. Let $\alpha \in \mathbb{C}$ with $\Re(\alpha) \in (n-1, n)$, $n \in \mathbb{N}$, $\beta \in [0, 1]$, and set $\gamma = \alpha + \beta(n - \alpha)$. Assume $\Re(\alpha + \gamma - n) > 0$ and that $f : (a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies hypotheses (H0) and (H2). If $\Lambda < 1$, where Λ is defined in (31), then the Cauchy problem (1)–(2) is UH-stable. Consequently, it is also GUH-stable.

Proof. Let $\varepsilon > 0$ and $y \in C_{n-\gamma, \psi}^\gamma[a, b]$ be a solution of inequality (7) together with initial conditions (10). By Remark 4, there is a function $g \in C_{n-\gamma, \psi}[a, b]$ satisfying $|g(t)| \leq \varepsilon$ such that

$$\begin{cases} ({}^H D_{a^+}^{\alpha, \beta; \psi} y)(t) = f(t, y(t), ({}^H D_{a^+}^{\alpha, \beta; \psi} y)(t)) + g(t), & t \in (a, b], \\ y_{n-\gamma}^{(n-k); \psi}(a) = c_k, & c_k \in \mathbb{C}, k \in \{1, 2, \dots, n\}. \end{cases} \quad (38)$$

By Lemma 12, the integral equation of (38) takes the form

$$y(t) = \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{\gamma-k} + (I_{a^+}^{\alpha; \psi} G_y)(t) + (I_{a^+}^{\alpha; \psi} g)(t), \quad (39)$$

where $G_y \in C_{n-\gamma, \psi}^{\beta(n-\alpha)}[a, b]$ satisfies the functional equation

$$G_y(t) = f(t, y(t), G_y(t)) + g(t), \quad t \in (a, b]. \quad (40)$$

Taking into account that $y \in C_{n-\gamma, \psi}^\gamma[a, b]$, we multiply both sides of the above equation by $(\Delta_a^\psi t)^{n-\gamma}$. From this it follows that

$$\begin{aligned} & \left| (\Delta_a^\psi t)^{n-\gamma} y(t) - \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{n-k} - (\Delta_a^\psi t)^{n-\gamma} (I_{a^+}^{\alpha; \psi} G_y)(t) \right| \\ & \leq (\Delta_a^\psi t)^{n-\gamma} I_{a^+}^{\alpha; \psi} |g(t)| \leq (\Delta_a^\psi t)^{n-\gamma} I_{a^+}^{\alpha; \psi} \varepsilon \leq (\Delta_a^\psi t)^{n-\gamma} \frac{\varepsilon (\Delta_a^\psi b)^{\Re(\alpha)}}{|\Gamma(\alpha + 1)|}, \end{aligned} \quad (41)$$

for any $t \in [a, b]$. From Theorem 8, there exists a unique solution $x \in C_{n-\gamma, \psi}^\gamma[a, b]$ to the Cauchy problem:

$$\begin{cases} ({}^H D_{a^+}^{\alpha, \beta; \psi} x)(t) = f(t, x(t), ({}^H D_{a^+}^{\alpha, \beta; \psi} x)(t)), & t \in (a, b], \\ x_{n-\gamma}^{(n-k); \psi}(a) = y_{n-\gamma}^{(n-k); \psi}(a) = c_k, & c_k \in \mathbb{C}, k \in \{1, 2, \dots, n\}. \end{cases} \quad (42)$$

By Lemma 12, we have Equation (22):

$$x(t) = \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{\gamma-k} + (I_{a^+}^{\alpha; \psi} G_x)(t),$$

where $G_x \in C_{n-\gamma, \psi}^{\beta(n-\alpha)}[a, b]$ satisfies the functional Equation (23):

$$G_x(t) = f(t, x(t), G_x(t)), \quad t \in (a, b].$$

Using Equation (41), for $t \in [a, b]$, we have

$$\begin{aligned}
& (\Delta_a^\psi t)^{n-\gamma} |y(t) - x(t)| = \left| (\Delta_a^\psi t)^{n-\gamma} y(t) - \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{n-k} - (\Delta_a^\psi t)^{n-\gamma} (I_{a^+}^{\alpha;\psi} G_x)(t) \right| \\
& = \left| (\Delta_a^\psi t)^{n-\gamma} y(t) - \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{n-k} - (\Delta_a^\psi t)^{n-\gamma} (I_{a^+}^{\alpha;\psi} G_y)(t) + (\Delta_a^\psi t)^{n-\gamma} I_{a^+}^{\alpha;\psi} (G_y(t) - G_x(t)) \right| \\
& \leq \left| (\Delta_a^\psi t)^{n-\gamma} y(t) - \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{n-k} - (\Delta_a^\psi t)^{n-\gamma} (I_{a^+}^{\alpha;\psi} G_y)(t) \right| + (\Delta_a^\psi t)^{n-\Re(\gamma)} \left| I_{a^+}^{\alpha;\psi} (G_y(t) - G_x(t)) \right| \\
& \leq (\Delta_a^\psi t)^{n-\Re(\gamma)} \frac{\varepsilon(\Delta_a^\psi b)^{\Re(\alpha)}}{|\Gamma(\alpha + 1)|} + \frac{(\Delta_a^\psi t)^{n-\Re(\gamma)}}{|\Gamma(\alpha)|} \int_a^t (\Delta_s^\psi t)^{\Re(\alpha)-1} |G_y(s) - G_x(s)| ds. \quad (43)
\end{aligned}$$

To apply Corollary 1, we require the assumptions $\Re(\alpha) > 0$, $0 < n - \Re(\gamma) < 1$, and $\Re(\alpha + \gamma - n) > 0$. From Remark 2 (ii), given that $n - 1 < \Re(\alpha) < \Re(\gamma) < n$ for $n \in \mathbb{N}$, these assumptions always hold when $n \geq 2$. For $n = 1$, the condition $\Re(\alpha + \gamma - 1) > 0$ is included in the theorem hypothesis.

By (H2), we have inequality (36):

$$|G_y(t) - G_x(t)| \leq \frac{K}{1-M} |y(t) - x(t)|, \quad t \in (a, b].$$

Then the inequality (43) implies

$$\begin{aligned}
& (\Delta_a^\psi t)^{n-\gamma} |y(t) - x(t)| \\
& \leq (\Delta_a^\psi t)^{n-\Re(\gamma)} \frac{\varepsilon(\Delta_a^\psi b)^{\Re(\alpha)}}{|\Gamma(\alpha + 1)|} + \frac{K}{1-M} \frac{(\Delta_a^\psi t)^{n-\Re(\gamma)}}{|\Gamma(\alpha)|} \int_a^t (\Delta_s^\psi t)^{\Re(\alpha)-1} |y(s) - x(s)| ds,
\end{aligned}$$

for $t \in [a, b]$.

Let $u(t) = (\Delta_a^\psi t)^{n-\Re(\gamma)} |y(t) - x(t)|$. The above inequality becomes

$$u(t) \leq (\Delta_a^\psi t)^{n-\Re(\gamma)} \frac{\varepsilon(\Delta_a^\psi b)^{\Re(\alpha)}}{|\Gamma(\alpha + 1)|} + \frac{K}{1-M} \frac{(\Delta_a^\psi t)^{n-\Re(\gamma)}}{|\Gamma(\alpha)|} \int_a^t (\Delta_s^\psi t)^{\Re(\alpha)-1} (\Delta_a^\psi s)^{-(n-\Re(\gamma))} u(s) ds,$$

for $t \in [a, b]$. Because $\Re(\alpha) > n - \Re(\gamma) > 0$, by virtue of Corollary 1, it is implied that

$$u(t) \leq (\Delta_a^\psi t)^{n-\Re(\gamma)} \frac{\varepsilon(\Delta_a^\psi b)^{\Re(\alpha)}}{|\Gamma(\alpha + 1)|} F_{\Re(\alpha+\gamma-n), 1, \Re(\alpha)+1} \left(\frac{K}{1-M} \frac{\Gamma(\Re(\alpha))}{|\Gamma(\alpha)|} (\Delta_a^\psi t)^{\Re(\alpha)} \right), \quad t \in [a, b].$$

Denote

$$F(t) := F_{\Re(\alpha+\gamma-n), 1, \Re(\alpha)+1} \left(\frac{K}{1-M} \frac{\Gamma(\Re(\alpha))}{|\Gamma(\alpha)|} (\Delta_a^\psi t)^{\Re(\alpha)} \right), \quad t \in [a, b]. \quad (44)$$

Because the function F is continuous on any positive interval $[a, b]$, there exists a constant

$$c_f := \frac{(\Delta_a^\psi b)^{n-\Re(\gamma)+\Re(\alpha)}}{|\Gamma(\alpha + 1)|} \max_{t \in [a, b]} F(t),$$

such that $(\Delta_a^\psi t)^{n-\gamma} |y(t) - x(t)| \leq c_f \varepsilon$, for any $t \in [a, b]$. Hence, the Cauchy problem (1)–(2) is Ulam–Hyers stable.

Moreover, it is generalized Ulam–Hyers stable, as $(\Delta_a^\psi t)^{n-\gamma} |y(t) - x(t)| \leq \theta_f(\varepsilon)$, for any $t \in [a, b]$ with $\theta_f(\varepsilon) = c_f \varepsilon$, $\theta_f(0) = 0$. This completes the proof. $\square$

Theorem 10. Let $\alpha \in \mathbb{C}$ with $\Re(\alpha) \in (n - 1, n)$, $n \in \mathbb{N}$, $\beta \in [0, 1]$, and set $\gamma = \alpha + \beta(n - \alpha)$. Assume $\Re(\alpha + \gamma - n) > 0$ and that $f : (a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies hypotheses (H0), (H2), and

(H3). If $\Lambda < 1$, where Λ is defined in (31), then the Cauchy problem (1)–(2) is UHR-stable with respect to $\sigma(t)$. Consequently, it is also GUHR-stable with respect to $\sigma(t)$.

Proof. Let $\varepsilon > 0$ and $y \in C_{n-\gamma,\psi}^\gamma[a,b]$ be a solution of inequality (7) together with initial conditions (10). By Remark 4, there is a function $g \in C_{n-\gamma,\psi}[a,b]$ satisfying $|g(t)| \leq \varepsilon\sigma(t)$ such that (38). By Lemma 12, the integral Equation of (38) takes the form (39) where $G_y \in C_{n-\gamma,\psi}^{\beta(n-\alpha)}[a,b]$ satisfies the functional Equation (40). Using $|g(t)| \leq \varepsilon\sigma(t)$ together with (H3), we obtain

$$\begin{aligned} & \left| (\Delta_a^\psi t)^{n-\gamma} y(t) - \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma-k+1)} (\Delta_a^\psi t)^{n-k} - (\Delta_a^\psi t)^{n-\gamma} (I_{a+}^{\alpha;\psi} G_y)(t) \right| \\ & \leq (\Delta_a^\psi t)^{n-\Re(\gamma)} I_{a+}^{\alpha;\psi} |w(t)| \leq (\Delta_a^\psi t)^{n-\Re(\gamma)} \varepsilon (I_{a+}^{\alpha;\psi} \sigma)(t) \leq (\Delta_a^\psi t)^{n-\Re(\gamma)} \varepsilon \lambda_\sigma \sigma(t), \end{aligned} \quad (45)$$

for any $t \in [a,b]$. From Theorem 8, there exists a unique solution $x \in C_{n-\gamma,\psi}^\gamma[a,b]$ to the Cauchy problem (42).

By Lemma 12, we have Equation (22) where $G_x \in C_{n-\gamma,\psi}^{\beta(n-\alpha)}[a,b]$ satisfies the functional Equation (23). Using Equation (45), for $t \in [a,b]$, we have

$$\begin{aligned} & (\Delta_a^\psi t)^{n-\gamma} |y(t) - x(t)| = \left| (\Delta_a^\psi t)^{n-\gamma} y(t) - \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma-k+1)} (\Delta_a^\psi t)^{n-k} - (\Delta_a^\psi t)^{n-\gamma} (I_{a+}^{\alpha;\psi} G_x)(t) \right| \\ & = \left| (\Delta_a^\psi t)^{n-\gamma} y(t) - \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma-k+1)} (\Delta_a^\psi t)^{n-k} - (\Delta_a^\psi t)^{n-\gamma} (I_{a+}^{\alpha;\psi} G_y)(t) + (\Delta_a^\psi t)^{n-\gamma} I_{a+}^{\alpha;\psi} (G_y(t) - G_x(t)) \right| \\ & \leq \left| (\Delta_a^\psi t)^{n-\gamma} y(t) - \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma-k+1)} (\Delta_a^\psi t)^{n-k} - (\Delta_a^\psi t)^{n-\gamma} (I_{a+}^{\alpha;\psi} G_y)(t) \right| + (\Delta_a^\psi t)^{n-\Re(\gamma)} \left| I_{a+}^{\alpha;\psi} (G_y(t) - G_x(t)) \right| \\ & \leq (\Delta_a^\psi t)^{n-\Re(\gamma)} \varepsilon \lambda_\sigma \sigma(t) + \frac{(\Delta_a^\psi t)^{n-\Re(\gamma)}}{|\Gamma(\alpha)|} \int_a^t (\Delta_s^\psi t)^{\Re(\alpha)-1} |G_y(s) - G_x(s)| ds. \end{aligned}$$

Similarly, we apply Corollary 1 as in the proof of Theorem 9, which requires $\Re(\alpha) > 0$, $0 < \Re(n-\gamma) < 1$, and $\Re(\alpha + \gamma - n) > 0$. From Remark 2 (ii), these conditions hold for all $n \geq 2$. For $n = 1$, the condition $\Re(\alpha + \gamma - 1) > 0$ is included in the theorem hypothesis.

By virtue of (H2), we then define the function $F(t)$ as in (44). Therefore, there exists a constant

$$c_{f,\sigma} := \lambda_\sigma \max_{t \in [a,b]} F(t),$$

such that $(\Delta_a^\psi t)^{n-\gamma} |y(t) - x(t)| \leq c_{f,\sigma} \varepsilon \sigma(t)$, for any $t \in [a,b]$. Hence, the fractional differential Equation (1) is Ulam–Hyers–Rassias stable with respect to σ . Moreover, it is generalized Ulam–Hyers–Rassias stable with respect to σ ; if we take $\varepsilon = 1$, then $(\Delta_a^\psi t)^{n-\gamma} |y(t) - x(t)| \leq c_{f,\sigma} \sigma(t)$, for any $t \in [a,b]$. The assertion is proved. $\square$

Remark 7. From the proof of Theorems 9 and 10 when $n \geq 2$ or ($n = 1$ with $\Re(\alpha + \gamma + 1) > 0$), the constants c_f , and $c_{f,\sigma}$ in Ulam stability Definitions 7, 9 and 10 are defined as follows:

$$c_f := \frac{(\Delta_a^\psi b)^{n-\Re(\gamma)+\Re(\alpha)}}{|\Gamma(\alpha+1)|} \max_{t \in [a,b]} F(t), \quad c_{f,\sigma} := \lambda_\sigma \max_{t \in [a,b]} F(t), \quad (46)$$

where $F(t)$ is defined in (44).

5. Continuous Dependence

In the first part of this section, we consider the nonlinear implicit Hilfer fractional differential Equation (1) with a small change in the initial conditions,

$$x_{n-\gamma}^{(n-k);\psi}(a) = c_k + \epsilon, \quad c_k, \epsilon \in \mathbb{C}, \quad k \in \{1, 2, \dots, n\}, \quad (47)$$

where $|\epsilon|$ is a small constant used for all initial conditions for simplicity.

Theorem 11. Let $\alpha \in \mathbb{C}$ with $\Re(\alpha) \in (n-1, n)$, $n \in \mathbb{N}$, $\beta \in [0, 1]$, and set $\gamma = \alpha + \beta(n - \alpha)$. Assume $\Re(\alpha + \gamma - n) > 0$ and that $f : (a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies (H0) and (H2). If $\Lambda < 1$, where Λ is defined in (31), and if $x(t)$ and $\hat{x}(t)$ are solutions of the Cauchy problems (1)–(2) and (1)–(47), respectively, then for all $t \in [a, b]$,

$$(\Delta_a^\psi t)^{n-\gamma} |x(t) - \hat{x}(t)| \leq |\epsilon| \sum_{k=1}^n \frac{(\Delta_a^\psi t)^{n-k}}{|\Gamma(\gamma - k + 1)|} F(t), \quad (48)$$

where $F(t)$ is defined in (44).

Proof. In accordance with Lemma 12, we have

$$x(t) = \sum_{k=1}^n \frac{c_k}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{\gamma-k} + (I_{a+}^{\alpha;\psi} G_x)(t),$$

where $G_x \in C_{n-\gamma, \psi}^{\beta(n-\alpha)}[a, b]$ satisfies the following functional equation:

$$G_x(t) = f(t, x(t), G_x(t)), \quad t \in (a, b].$$

Clearly, we can write

$$\hat{x}(t) = \sum_{k=1}^n \frac{c_k + \epsilon}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{\gamma-k} + (I_{a+}^{\alpha;\psi} G_{\hat{x}})(t),$$

where $G_{\hat{x}} \in C_{n-\gamma, \psi}^{\beta(n-\alpha)}[a, b]$ satisfies the following functional equation:

$$G_{\hat{x}}(t) = f(t, \hat{x}(t), G_{\hat{x}}(t)), \quad t \in (a, b].$$

In view of (H2), we obtain

$$\begin{aligned} (\Delta_a^\psi t)^{n-\gamma} |x(t) - \hat{x}(t)| &= \left| (\Delta_a^\psi t)^{n-\gamma} \sum_{k=1}^n \frac{\epsilon}{\Gamma(\gamma - k + 1)} (\Delta_a^\psi t)^{\gamma-k} + (\Delta_a^\psi t)^{n-\gamma} I_{a+}^{\alpha;\psi} (G_x(t) - G_{\hat{x}}(t)) \right| \\ &\leq |\epsilon| \sum_{k=1}^n \frac{(\Delta_a^\psi t)^{\Re(\gamma)-k}}{|\Gamma(\gamma - k + 1)|} (\Delta_a^\psi t)^{n-\Re(\gamma)} + (\Delta_a^\psi t)^{n-\Re(\gamma)} I_{a+}^{\alpha;\psi} |G_x(t) - G_{\hat{x}}(t)| \\ &\leq |\epsilon| \sum_{k=1}^n \frac{(\Delta_a^\psi t)^{\Re(\gamma)-k}}{|\Gamma(\gamma - k + 1)|} (\Delta_a^\psi t)^{n-\Re(\gamma)} + \frac{(\Delta_a^\psi t)^{n-\Re(\gamma)}}{|\Gamma(\alpha)|} \int_a^t (\Delta_s^\psi t)^{\Re(\alpha)-1} |G_x(s) - G_{\hat{x}}(s)| ds, \end{aligned}$$

for any $t \in [a, b]$. Similarly, we apply Corollary 1 as in the proof of Theorem 9. This requires $\Re(\alpha) > 0$, $0 < \Re(n - \gamma) < 1$, and $\Re(\alpha + \gamma - n) > 0$. From Remark 2 (ii), given that $n - 1 < \Re(\alpha) < \Re(\gamma) < n$ for $n \in \mathbb{N}$, these conditions hold for all $n \geq 2$. For $n = 1$, the condition $\Re(\alpha + \gamma - 1) > 0$ is included in the theorem hypothesis.

We then define the function $F(t)$ as in (44), corresponding to Theorem 9. Therefore, we have

$$(\Delta_a^\psi t)^{n-\gamma} |x(t) - \hat{x}(t)| \leq |\epsilon| \sum_{k=1}^n \frac{(\Delta_a^\psi t)^{n-k}}{|\Gamma(\gamma - k + 1)|} F(t),$$

for any $t \in [a, b]$, which completes the proof. $\square$

Remark 8. By Theorem 11, a small perturbation in the initial condition (2) leads to only a minor variation in the solution over the interval $[c, b]$, for any $c \in (a, b)$. However, the solution may exhibit substantial sensitivity to such perturbations on the interval $[a, c]$.

In the second part of this section, we investigate the dependence of the solution on the fractional order α . This is practically relevant since α is often estimated from data or adjusted to achieve a desired response. The following proposition shows that small perturbations in α yield proportionally small changes in the solution. In particular, the real-order theory ($\Im(\alpha) = 0$) is obtained as the quantitative limit of the complex-order theory as $\Im(\alpha) \rightarrow 0$.

Proposition 1 (Continuous dependence on the fractional order α). *Let $\beta \in [0, 1]$ and let $f : (a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy (H0) and (H2) with the same constants K, M . Let $\alpha_1, \alpha_2 \in \mathbb{C}$ satisfy $n - 1 < \Re(\alpha_i) < n$, set $\gamma_i = \alpha_i + \beta(n - \alpha_i)$, $i \in \{1, 2\}$, and suppose that the corresponding contraction constants $\Lambda(\alpha_i)$ defined by (31) satisfy $\Lambda(\alpha_i) < 1$, $i \in \{1, 2\}$. Let $x_{\alpha_i} \in C_{n-\gamma_i, \psi}^{\gamma_i}[a, b]$, $i \in \{1, 2\}$, be the unique solutions of problem (1)–(2) with the same initial data c_k , guaranteed by Theorem 8. Then, for every $t \in [a, b]$,*

$$|x_{\alpha_1}(t) - x_{\alpha_2}(t)| \leq \frac{(\Delta_a^\psi b)^{\min(\Re(\alpha_1), \Re(\alpha_2))}}{1 - \Lambda(\alpha_1)} L_\Gamma(\alpha_1, \alpha_2) \|x_{\alpha_2}\|_{C_{n-\gamma_2, \psi}[a, b]} |\alpha_1 - \alpha_2|, \quad (49)$$

where $L_\Gamma(\alpha_1, \alpha_2)$ is a constant depending on $\alpha_1, \alpha_2, \beta, \psi, b, n, K, M, c_k$, but independent of t . (The explicit form of $L_\Gamma(\alpha_1, \alpha_2)$ is given in the proof below.)

Proof. Let $\|\cdot\|_\infty$ denote the usual supremum norm on $(a, b]$. This norm is well-defined for every function in $C_{n-\gamma, \psi}[a, b]$ for any admissible γ , since such functions are continuous on $(a, b]$. We compare the two solutions pointwise.

For each fixed $t \in (a, b]$, we have $x_{\alpha_i}(t) = (\mathcal{F}_{\alpha_i} x_{\alpha_i})(t)$, $i = 1, 2$. Thus,

$$|x_{\alpha_1}(t) - x_{\alpha_2}(t)| = |(\mathcal{F}_{\alpha_1} x_{\alpha_1})(t) - (\mathcal{F}_{\alpha_2} x_{\alpha_2})(t)| \leq |(\mathcal{F}_{\alpha_1} x_{\alpha_1})(t) - (\mathcal{F}_{\alpha_1} x_{\alpha_2})(t)| + |(\mathcal{F}_{\alpha_1} x_{\alpha_2})(t) - (\mathcal{F}_{\alpha_2} x_{\alpha_2})(t)|. \quad (50)$$

Taking the supremum over $t \in (a, b]$ gives

$$\|x_{\alpha_1} - x_{\alpha_2}\|_\infty \leq \|\mathcal{F}_{\alpha_1} x_{\alpha_1} - \mathcal{F}_{\alpha_1} x_{\alpha_2}\|_\infty + \|\mathcal{F}_{\alpha_1} x_{\alpha_2} - \mathcal{F}_{\alpha_2} x_{\alpha_2}\|_\infty. \quad (51)$$

Step 1: Estimate of the first term in (51). Although x_{α_2} does not necessarily belong to the domain $C_{n-\gamma_1, \psi}[a, b]$ of $\mathcal{F}_{\alpha_1}$, the expression $\mathcal{F}_{\alpha_1} x_{\alpha_2}(t)$ is still well-defined pointwise for each t via (27) and Lemma 12. Indeed, the definition of $\mathcal{F}_{\alpha_1}$ requires only the pointwise values of $G_{x_{\alpha_2}}(s)$ for $s \leq t$, which are well-defined since $G_{x_{\alpha_2}} \in C_{n-\gamma_2, \psi}^{\beta(n-\alpha_2)}[a, b]$ and hence $G_{x_{\alpha_2}}(s) \in \mathbb{C}$ for each $s \in (a, b]$. To estimate the difference, note that the initial condition terms in $\mathcal{F}_{\alpha_1}$ cancel, since both x_{α_1} and x_{α_2} share the same initial data c_k . Hence,

$$(\mathcal{F}_{\alpha_1} x_{\alpha_1})(t) - (\mathcal{F}_{\alpha_1} x_{\alpha_2})(t) = I_{a+}^{\alpha_1, \psi} (G_{x_{\alpha_1}} - G_{x_{\alpha_2}})(t).$$

Using the definition of $I_{a+}^{\alpha_1, \psi}$, the pointwise Lipschitz estimate for G_x obtained from (H2), and Lemma 3, we have, for each $t \in (a, b]$,

$$|(\mathcal{F}_{\alpha_1} x_{\alpha_1})(t) - (\mathcal{F}_{\alpha_1} x_{\alpha_2})(t)| \leq \frac{(\Delta_a^\psi t)^{\Re(\alpha_1)}}{|\Gamma(\alpha_1)|} B(\Re(\alpha_1), \Re(\gamma_1 - n + 1)) \frac{K}{1 - M} |x_{\alpha_1}(t) - x_{\alpha_2}(t)|.$$

Taking the supremum over $t \in (a, b]$ and using the definition of $\Lambda(\alpha_1)$ in (31) yields

$$\|\mathcal{F}_{\alpha_1} x_{\alpha_1} - \mathcal{F}_{\alpha_1} x_{\alpha_2}\|_\infty \leq \Lambda(\alpha_1) \|x_{\alpha_1} - x_{\alpha_2}\|_\infty. \quad (52)$$

Substituting (52) into (51), we obtain

$$\|x_{\alpha_1} - x_{\alpha_2}\|_\infty \leq \Lambda(\alpha_1) \|x_{\alpha_1} - x_{\alpha_2}\|_\infty + \|\mathcal{F}_{\alpha_1} x_{\alpha_2} - \mathcal{F}_{\alpha_2} x_{\alpha_2}\|_\infty. \quad (53)$$

Since $\Lambda(\alpha_1) < 1$, inequality (53) implies

$$\|x_{\alpha_1} - x_{\alpha_2}\|_{\infty} \leq \frac{1}{1 - \Lambda(\alpha_1)} \|\mathcal{F}_{\alpha_1} x_{\alpha_2} - \mathcal{F}_{\alpha_2} x_{\alpha_2}\|_{\infty}. \quad (54)$$

Step 2: Estimate of the second term in (54). From the definition of $\mathcal{F}_{\alpha}$ in (27), we have

$$\begin{aligned} & |(\mathcal{F}_{\alpha_1} x_{\alpha_2})(t) - (\mathcal{F}_{\alpha_2} x_{\alpha_2})(t)| \\ & \leq \sum_{k=1}^n |c_k| \left| \frac{(\Delta_a^{\psi} t)^{\gamma_1 - k}}{\Gamma(\gamma_1 - k + 1)} - \frac{(\Delta_a^{\psi} t)^{\gamma_2 - k}}{\Gamma(\gamma_2 - k + 1)} \right| + \left| (I_{a+}^{\alpha_1; \psi} G_{x_{\alpha_2}})(t) - (I_{a+}^{\alpha_2; \psi} G_{x_{\alpha_2}})(t) \right|. \end{aligned} \quad (55)$$

To estimate the initial condition terms in (55), define, for each $k \in \{1, \dots, n\}$ and each fixed $t \in [a, b]$,

$$F_k(\alpha) := \frac{(\Delta_a^{\psi} t)^{\gamma(\alpha) - k}}{\Gamma(\gamma(\alpha) - k + 1)}, \quad \gamma(\alpha) = \alpha + \beta(n - \alpha).$$

Since $\gamma(\alpha)$ is affine in α and $\alpha \mapsto (\Delta_a^{\psi} t)^{\alpha} / \Gamma(\alpha)$ is holomorphic on the strip $\Re(\alpha) \in (n - 1, n)$ (see Remark 1), each F_k is holomorphic in α on this strip. By the Mean Value Theorem applied along the line segment $[\alpha_1, \alpha_2]_{\mathbb{C}} := \{(1 - \theta)\alpha_1 + \theta\alpha_2 : \theta \in [0, 1]\}$, we have

$$|F_k(\alpha_1) - F_k(\alpha_2)| \leq \sup_{\alpha \in [\alpha_1, \alpha_2]_{\mathbb{C}}} |F'_k(\alpha)| \cdot |\alpha_1 - \alpha_2|.$$

A direct computation gives

$$F'_k(\alpha) = (1 - \beta) \frac{(\Delta_a^{\psi} t)^{\gamma(\alpha) - k}}{\Gamma(\gamma(\alpha) - k + 1)} \left[\ln(\Delta_a^{\psi} t) - \psi_0(\gamma(\alpha) - k + 1) \right],$$

where $\psi_0 := \Gamma' / \Gamma$ is the digamma function. Since $\gamma(\alpha)$ ranges over a compact subset of the strip $\Re(\gamma) \in (n - 1, n)$ and $t \in [a, b]$, we have

$$\sup_{\alpha \in [\alpha_1, \alpha_2]_{\mathbb{C}}} |F'_k(\alpha)| \leq C_k(\alpha_1, \alpha_2, \beta, \psi, b, n) < \infty,$$

with C_k independent of t . Consequently,

$$\sum_{k=1}^n |c_k| |F_k(\alpha_1) - F_k(\alpha_2)| \leq C_{\text{init}}(\alpha_1, \alpha_2) |\alpha_1 - \alpha_2|, \quad (56)$$

where

$$C_{\text{init}}(\alpha_1, \alpha_2) := \sum_{k=1}^n |c_k| C_k(\alpha_1, \alpha_2, \beta, \psi, b, n) < \infty.$$

We now estimate the fractional integral term in (55). We write

$$(I_{a+}^{\alpha_1; \psi} G_{x_{\alpha_2}})(t) - (I_{a+}^{\alpha_2; \psi} G_{x_{\alpha_2}})(t) = \int_a^t \psi'(s) \left[\frac{(\Delta_s^{\psi} t)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} - \frac{(\Delta_s^{\psi} t)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} \right] G_{x_{\alpha_2}}(s) ds.$$

For fixed $s < t$, consider the holomorphic map $\alpha \mapsto (\Delta_s^{\psi} t)^{\alpha - 1} / \Gamma(\alpha)$. By the Mean Value Theorem,

$$\left| \frac{(\Delta_s^{\psi} t)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} - \frac{(\Delta_s^{\psi} t)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} \right| \leq \widehat{L}_{\Gamma}(\alpha_1, \alpha_2) (\Delta_s^{\psi} t)^{\min(\Re(\alpha_1), \Re(\alpha_2)) - 1} |\alpha_1 - \alpha_2|, \quad (57)$$

where

$$\widehat{L}_\Gamma(\alpha_1, \alpha_2) := \sup_{\substack{\lambda = (1-\theta)\alpha_1 + \theta\alpha_2 \\ \theta \in [0,1]}} \left| \frac{d}{d\lambda} \left(\frac{(\Delta_a^\psi b)^\lambda}{\Gamma(\lambda)} \right) \right|.$$

The finiteness of $\widehat{L}_\Gamma$ follows from Remark 1: the map $\lambda \mapsto (\Delta_a^\psi b)^\lambda / \Gamma(\lambda)$ is holomorphic on the strip $\Re(\lambda) \in (n-1, n)$, and the line segment $[\alpha_1, \alpha_2]_\mathbb{C}$ is compact. Using (57), we obtain

$$|(I_{a^+}^{\alpha_1; \psi} G_{x_{\alpha_2}})(t) - (I_{a^+}^{\alpha_2; \psi} G_{x_{\alpha_2}})(t)| \leq \widehat{L}_\Gamma(\alpha_1, \alpha_2) |\alpha_1 - \alpha_2| \int_a^t \psi'(s) (\Delta_s^\psi t)^{\min(\Re(\alpha_1), \Re(\alpha_2)) - 1} |G_{x_{\alpha_2}}(s)| ds. \quad (58)$$

To estimate the integral in (58), we apply Lemma 3 with $\rho = n - \gamma_2$. Since $G_{x_{\alpha_2}} \in C_{n-\gamma_2, \psi}[a, b]$, we have

$$|G_{x_{\alpha_2}}(s)| \leq (\Delta_a^\psi s)^{-(n-\Re(\gamma_2))} \|G_{x_{\alpha_2}}\|_{C_{n-\gamma_2, \psi}[a, b]}.$$

Moreover, from (H2), we have the bound

$$\|G_{x_{\alpha_2}}\|_{C_{n-\gamma_2, \psi}[a, b]} \leq \frac{K}{1-M} \|x_{\alpha_2}\|_{C_{n-\gamma_2, \psi}[a, b]}.$$

Therefore, using the change of variables $u = \psi(s)$ as in Lemma 3,

$$\begin{aligned} & \int_a^t \psi'(s) (\Delta_s^\psi t)^{\min(\Re(\alpha_1), \Re(\alpha_2)) - 1} |G_{x_{\alpha_2}}(s)| ds \\ & \leq \frac{\Gamma(\min(\Re(\alpha_1), \Re(\alpha_2)))}{\Gamma(\min(\Re(\alpha_1), \Re(\alpha_2)) + 1 - (n - \Re(\gamma_2)))} (\Delta_a^\psi b)^{\min(\Re(\alpha_1), \Re(\alpha_2)) - (n - \Re(\gamma_2))} \|G_{x_{\alpha_2}}\|_{C_{n-\gamma_2, \psi}[a, b]} \\ & \leq C_\Gamma(\alpha_1, \alpha_2, \gamma_2, \psi, b) (\Delta_a^\psi b)^{\min(\Re(\alpha_1), \Re(\alpha_2))} \|x_{\alpha_2}\|_{C_{n-\gamma_2, \psi}[a, b]}, \end{aligned}$$

where C_Γ is a constant independent of t . Combining the above estimates with (56), we get

$$\|\mathcal{F}_{\alpha_1} x_{\alpha_2} - \mathcal{F}_{\alpha_2} x_{\alpha_2}\|_\infty \leq L_\Gamma(\alpha_1, \alpha_2) (\Delta_a^\psi b)^{\min(\Re(\alpha_1), \Re(\alpha_2))} \|x_{\alpha_2}\|_{C_{n-\gamma_2, \psi}[a, b]} |\alpha_1 - \alpha_2|,$$

where

$$L_\Gamma(\alpha_1, \alpha_2) := \frac{C_{\text{init}}(\alpha_1, \alpha_2)}{(\Delta_a^\psi b)^{\min(\Re(\alpha_1), \Re(\alpha_2))}} + C_\Gamma(\alpha_1, \alpha_2, \gamma_2, \psi, b) \widehat{L}_\Gamma(\alpha_1, \alpha_2).$$

Step 3: Final estimate. Substituting the above estimate into (54) yields

$$\|x_{\alpha_1} - x_{\alpha_2}\|_\infty \leq \frac{(\Delta_a^\psi b)^{\min(\Re(\alpha_1), \Re(\alpha_2))}}{1 - \Lambda(\alpha_1)} L_\Gamma(\alpha_1, \alpha_2) \|x_{\alpha_2}\|_{C_{n-\gamma_2, \psi}[a, b]} |\alpha_1 - \alpha_2|.$$

Since the right-hand side is independent of t , the pointwise estimate stated in Proposition 1 follows immediately. This completes the proof. $\square$

Remark 9. Proposition 1 shows that the solution map $\alpha \mapsto x_\alpha$ is Lipschitz (in particular continuous) on any compact subset of the admissible strip on which $\Lambda(\alpha) < 1$ uniformly; in particular, the real-order theory (case $\Im(\alpha) = 0$) is not just formally, but quantitatively, the limit of the complex-order theory as $\Im(\alpha) \rightarrow 0$.

6. Application: A Fractional Nonlinear Oscillator with Saturating Feedback

We consider a mechanical oscillator of mass m (kg) attached to a linear spring with stiffness constant k (N/m). The oscillator is subjected to an active feedback force that depends on its instantaneous acceleration. Such configurations arise in active vibration

control systems, where an actuator applies a corrective force proportional to the measured acceleration of the mass. However, physical actuators have limited power output; as the required acceleration becomes large, the actuator saturates, and the effective feedback force decays toward zero rather than increasing without bound.

A standard engineering model for this saturation phenomenon is the Lorentzian (or algebraic) saturation function

$$F_{\text{feedback}} = \frac{F_0}{1 + (x''/a)^2},$$

where F_0 is the maximum feedback force (when $x'' = 0$), a is the saturation acceleration (the scale at which saturation becomes significant), and $x''(t)$ is the instantaneous acceleration. This function is bounded ($0 < F_{\text{feedback}} \leq F_0$), smooth, and reduces to F_0 for small accelerations ($|x''| \ll a$), while decaying as $1/(x'')^2$ for large accelerations ($|x''| \gg a$).

Newton's second law for the oscillator (mass m) is

$$m x''(t) = -k x(t) + \frac{F_0}{1 + (x''(t)/a)^2}.$$

The parameters and variables of the mechanical oscillator model are defined below, with dimensions given in the MLT system (Mass M , Length L , Time T). The mass-spring oscillator with saturating acceleration feedback is illustrated schematically in Figure 1.

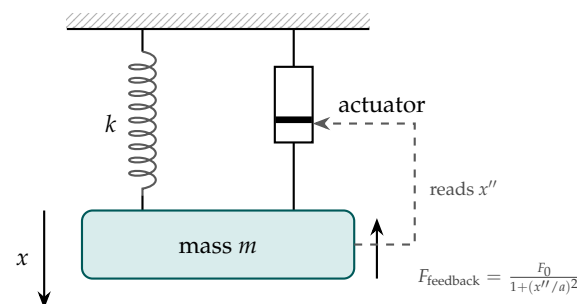


Figure 1. Mass-spring oscillator with saturating acceleration feedback. The actuator force F_{feedback} grows linearly for small x'' but saturates and decays to zero for large x'' , making the governing equation an implicit second-order ODE.

- Time: t ; $[t] = T$.
- Displacement: $x(t)$; $[x] = L$.
- Velocity: $x'(t)$; $[x'] = LT^{-1}$.
- Acceleration: $x''(t)$; $[x''] = LT^{-2}$.
- Mass: m ; $[m] = M$.
- Spring constant: k ; $[k] = MT^{-2}$ (force per unit displacement).
- Maximum feedback force: F_0 ; $[F_0] = MLT^{-2}$.
- Saturation acceleration: a ; $[a] = LT^{-2}$.

To nondimensionalize the system, we introduce the characteristic scales:

$$T_{\text{scale}} = \sqrt{\frac{m}{k}}, \quad X_{\text{scale}} = \frac{F_0}{k},$$

where T_{scale} is the natural time scale of the undamped oscillator, and X_{scale} is the static displacement under constant force F_0 . By defining dimensionless variables

$$\tau = \frac{t}{T_{\text{scale}}}, \quad x_{\text{dimless}}(\tau) = \frac{x(t)}{X_{\text{scale}}},$$

the physical acceleration becomes

$$x''(t) = \frac{X_{\text{scale}}}{T_{\text{scale}}^2} x''_{\text{dimless}}(\tau) = \frac{F_0}{m} x''_{\text{dimless}}(\tau).$$

Substituting into Newton's law and dividing by $kX_{\text{scale}} = F_0$, we obtain

$$x''_{\text{dimless}}(\tau) = -x_{\text{dimless}}(\tau) + \frac{1}{1 + \left(\frac{F_0}{ma} x''_{\text{dimless}}(\tau)\right)^2}.$$

Define the dimensionless saturation parameter: $\gamma = F_0/ma$. For simplicity, we set $\gamma = 1$ (i.e., $a = F_0/m$), which corresponds to choosing the saturation acceleration as the natural acceleration scale of the system. Dropping the "dimless" subscript and relabeling τ as t , we arrive at the dimensionless classical equation:

$$x''(t) = -x(t) + \frac{1}{1 + (x''(t))^2}, \quad t \in (0, b], \quad (59)$$

with initial conditions

$$x(0) = c_1, \quad x'(0) = c_2, \quad c_1, c_2 \in \mathbb{R}. \quad (60)$$

Classical integer-order models assume instantaneous response. However, real viscoelastic materials and memory-dependent control systems exhibit hereditary effects. To capture memory, we replace the integer-order acceleration $x''(t)$ in the normalized classical Equation (59) with the ψ -Hilfer fractional derivative ${}^H D_{a+}^{\alpha, \beta; \psi}$ of order $\Re(\alpha) \in (1, 2)$. The parameter α controls memory strength: as $\alpha \rightarrow 2^-$, we recover the classical derivative; α near 1 indicates strong memory.

In the normalized classical equation, all quantities are dimensionless. However, the fractional derivative ${}^H D_{0+}^{\alpha, \beta; \psi}$ has dimension $T^{-\alpha}$ in physical units, while the classical acceleration $x''(t)$ is dimensionless in the normalized equation. To maintain dimensional consistency when passing from the classical to the fractional formulation, we introduce a characteristic time scale τ (with $[\tau] = T$) and define the dimensionless replacement rule:

$$x''(t) \rightarrow \tau^{2-\alpha} ({}^H D_{0+}^{\alpha, \beta; \psi} x)(t).$$

This scaling ensures that the fractional term remains dimensionless (since $\tau^{2-\alpha}$ has dimension $T^{2-\alpha}$, which cancels the dimension $T^{-\alpha}$ of the fractional derivative). The classical model is recovered when $\alpha \rightarrow 2$ (since $\tau^0 = 1$).

Applying this substitution directly to the normalized classical Equation (59) yields the fractional generalization:

$$\tau^{2-\alpha} ({}^H D_{0+}^{\alpha, \beta; \psi} x)(t) = -x(t) + \frac{1}{1 + (\tau^{2-\alpha} ({}^H D_{0+}^{\alpha, \beta; \psi} x)(t))^2}.$$

Following the methodology of Gómez-Aguilar et al. [31], we establish the scaling relation: $\tau^{2-\alpha} = a_\alpha/a$, where a_α is the fractional saturation parameter with dimensions $LT^{-\alpha}$, and a is the classical saturation acceleration. This ensures that the argument of the saturation function remains dimensionless. In the classical limit $\alpha \rightarrow 2$, we have $\tau^0 = 1$ and $a_\alpha = a$, recovering the classical saturation parameter.

For the numerical simulations, we choose specific values of τ (or equivalently a_α) that satisfy the physical constraints. After normalizing the remaining parameters ($m = k = F_0 = a = 1$), the fractional model becomes

$$\tau^{2-\alpha} ({}^H D_{0+}^{\alpha,\beta;\psi} x)(t) = -x(t) + \frac{1}{1 + (\tau^{2-\alpha} ({}^H D_{0+}^{\alpha,\beta;\psi} x)(t))^2}, \quad t \in (0, b]. \quad (61)$$

The fractional generalization is subject to the ψ -Hilfer initial conditions:

$$\left(\mathcal{I}_{0+}^{(2-\gamma);\psi} x \right)(0) = d_1, \quad \left(\frac{1}{\psi'} \frac{d}{dt} \mathcal{I}_{0+}^{(2-\gamma);\psi} x \right)(0) = d_2, \quad d_1, d_2 \in \mathbb{R}, \quad (62)$$

where $\gamma = \alpha + \beta(2 - \alpha)$ is the effective order. The parameter $\beta \in [0, 1]$ interpolates between Riemann–Liouville-type ($\beta \rightarrow 0$) and Caputo-type ($\beta \rightarrow 1$) initial conditions.

Before proceeding to the numerical solution of the fractional model, it is instructive to first examine the behavior of the classical integer-order system, which serves as a reference and limiting case.

For the fractional model, we set $\tau = 1$, $\psi(t) = t$, $\beta \rightarrow 1$, and $b = 0.75$ in the numerical simulations. With these choices, we have $\gamma = \alpha + \beta(2 - \alpha) = \alpha + (2 - \alpha) = 2$, so the ψ -Hilfer derivative reduces to the Caputo derivative ${}^C D_{0+}^\alpha$, and the initial conditions (62) simplify to the classical form (60). From Lemma 12, the equivalent integral equation for (61) becomes

$$x(t) = \sum_{k=1}^2 \frac{d_k}{\Gamma(3-k)} t^{2-k} + (I_{0+}^\alpha G_x)(t), \quad (63)$$

where $G_x \in C_{0,t}^{2-\alpha}[0, 0.75]$ satisfies $G_x(t) = -x(t) + 1/(1 + G_x(t)^2)$, and I_{0+}^α denotes the standard Riemann–Liouville fractional integral of order α .

The numerical solutions in Figure 2 are obtained using two schemes. For the classical model (59), an explicit Runge–Kutta method is employed, where the implicit algebraic relation $x'' = -x + 1/(1 + (x'')^2)$ is resolved at each stage by Newton's method. For the fractional model, the equivalent Volterra Equation (63) is discretized on a uniform grid with $N = 1000$ points using a product quadrature rule, and the resulting nonlinear equation for $G_x(t_n)$ is again solved by Newton's method. Both schemes use a Newton tolerance of 10^{-12} . The complete implementation is given in Algorithm A1 (Appendix A).

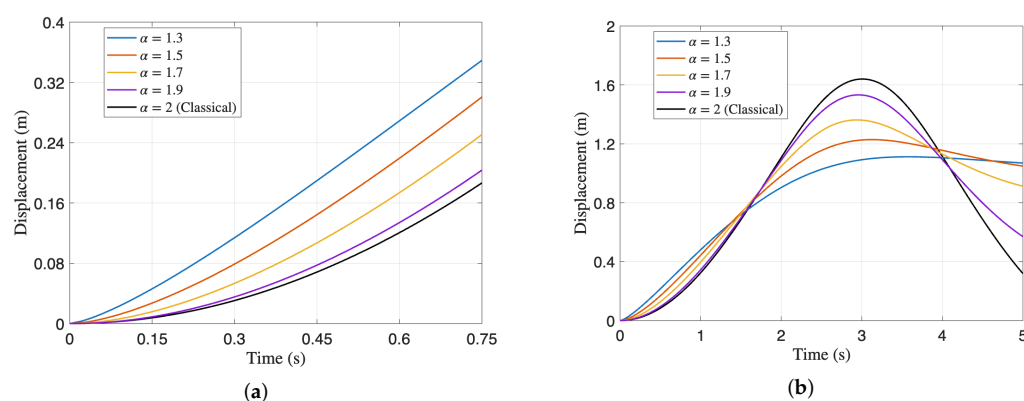


Figure 2. Comparison between the classical solution ($\alpha = 2$) with $c_1 = c_2 = 0$ and fractional solutions ($\alpha = 1.3, 1.5, 1.7, 1.9$) with $\beta = 1$, $\psi(t) = t$, $d_1 = d_2 = 0$. (a) Short interval $t \in [0, 0.75]$; (b) Extended interval $t \in [0, 5]$.

Convergence Analysis, Error Estimates and Computational Complexity

To assess the numerical accuracy of the product-quadrature scheme described above, we fix $\alpha = 1.9$, $\beta = 1$, $\psi(t) = t$, $d_1 = d_2 = 0$ and compute a reference solution x_{ref} on

$[0, 0.75]$ using $N_{\text{ref}} = 8000$ grid points. For $N \in \{125, 250, 500, 1000, 2000\}$, we compute the corresponding numerical solution x_N and the discrete maximum error

$$E_N := \max_{0 \leq n \leq N} |x_N(t_n) - x_{\text{ref}}(t_n)|,$$

interpolating x_{ref} onto the coarser grid where necessary. The estimated order of convergence is $p_N := \log_2(E_N/E_{2N})$, which measures the rate at which the error decreases as the grid is refined. Table 1 reports E_N and p_N ; the results are consistent with the theoretical rate $O(h^{\min(1, \Re(\alpha))}) = O(h)$ expected for a piecewise-constant product quadrature of a Caputo-type equation with $\Re(\alpha) = 1.9 > 1$, where the discretization error is dominated by the $O(h)$ approximation of G_x on each subinterval rather than by the (integrable) kernel singularity.

Table 1. Mesh-refinement study for $\alpha = 1.9$ on $[0, 0.75]$ (reference: $N_{\text{ref}} = 8000$). The final two columns compare the product-quadrature solution with an independent fractional Adams–Bashforth–Moulton (ABM) predictor–corrector scheme [32] at $N = 1000$; the “Agreement” column shows the pointwise maximum difference between the two methods at that resolution. The estimated order p_N is not reported for $N = 125$ because no coarser grid is available for comparison.

N	E_N (Product Quadrature)	Estimated Order p_N	$E_N^{\text{AB-M}}$	Agreement
125	1.837×10^{-2}	–	–	–
250	8.924×10^{-3}	1.04	–	–
500	4.381×10^{-3}	1.03	–	–
1000	2.166×10^{-3}	1.02	2.203×10^{-3}	3.7×10^{-5}
2000	1.076×10^{-3}	1.01	–	–

Regarding computational complexity, the convolution weights $\omega_{n,j}$ defined in (A2) must be computed for all $0 \leq j \leq n \leq N$, requiring $O(N^2)$ arithmetic operations and $O(N^2)$ storage in the direct implementation used here (Algorithm A1); the Newton iteration at each of the N time steps adds only an $O(N)$ overall contribution since each scalar Newton solve converges in a fixed, small number of iterations independent of N (typically 3–5 iterations to reach the tolerance 10^{-12}). For the problem sizes used in this paper ($N = 1000$), this $O(N^2)$ cost is negligible in absolute terms; for substantially larger N , fast convolution techniques for fractional operators (e.g., sum-of-exponentials or FFT-based approaches) would reduce the cost to $O(N \log N)$, but such acceleration is outside the scope of the present well-posedness study and is left for future numerical work.

The fractional-order dynamics exhibit memory effects that alter the system response relative to the classical model. As α increases, memory diminishes and solutions converge to the classical trajectory, confirming consistency. On the short interval $[0, 0.75]$, all solutions remain close. On the extended interval $[0, 5]$, memory effects become more pronounced, with lower α values (stronger memory) exhibiting slower response and reduced amplitude.

Equation (61) is a special case of the abstract problem (1)–(2) with nonlinearity

$$f(t, u, v) = -u + \frac{1}{1 + v^2}, \quad (t, u, v) \in (0, 0.75] \times \mathbb{R} \times \mathbb{R}. \quad (64)$$

Since $f(t, u, v) = -u + 1/(1 + v^2)$ is a composition of continuous functions and $1/(1 + v^2) \leq 1$ for all $v \in \mathbb{R}$, the map $t \mapsto f(t, x(t), {}^H D_{a+}^{\alpha, 1} x(t))$ belongs to $C_{0,t}[0, 0.75] = C[0, 0.75]$ for any $x \in C[0, 0.75]$, since $C[0, 0.75]$ is closed under addition, bounded perturbations, and composition with smooth bounded functions. Hence, (H0) is satisfied.

For any $(t, u, v) \in (0, 0.75] \times \mathbb{R} \times \mathbb{R}$, we have

$$|f(t, u, v)| = \left| -u + \frac{1}{1+v^2} \right| \leq |u| + \frac{1}{1+v^2} \leq 1 + |u|,$$

where we used $1/(1+v^2) \leq 1$. Setting $l(t) \equiv 1$, $m(t) \equiv 1$, $q(t) \equiv 0$, we obtain $|f(t, u, v)| \leq l(t) + m(t)|u| + q(t)|v|$ with $\|q\|_{C[0,0.75]} = 0 < 1$, and $l, m, q \in C[0, 0.75]$ (as constant functions). Hence, (H1) is satisfied.

For any $u_1, v_1, u_2, v_2 \in \mathbb{R}$ and $t \in (0, 0.75]$, we have

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq |u_1 - u_2| + \left| \frac{1}{1+v_1^2} - \frac{1}{1+v_2^2} \right|.$$

For the second term, writing $1/(1+v^2)$ and applying the mean value theorem with $\frac{d}{dv}(1/(1+v^2)) = -2v/(1+v^2)^2$, we get

$$\left| \frac{1}{1+v_1^2} - \frac{1}{1+v_2^2} \right| \leq \sup_{v \in \mathbb{R}} \frac{2|v|}{(1+v^2)^2} |v_1 - v_2| \leq \frac{2/\sqrt{3}}{(1+1/3)^2} |v_1 - v_2| = \frac{3\sqrt{3}}{8} |v_1 - v_2|.$$

Therefore,

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq K|u_1 - u_2| + M|v_1 - v_2|,$$

with constants $K = 1$, $M = 3\sqrt{3}/8 < 1$. Hence, hypothesis (H2) is satisfied globally on $\mathbb{R}$.

By Theorem 7, the Cauchy problem admits at least one solution in $C_{0,t}^2[0, 0.75] = C^2[0, 0.75]$, since (H0)–(H2) are satisfied. Moreover, with the computed constants $K = 1$ and $M = 3\sqrt{3}/8 \approx 0.6495$, the contraction constant Λ in Theorem 8 is given by

$$\Lambda = \frac{(\Delta_0^{t,0.75})^\alpha}{|\Gamma(\alpha)|} B(\alpha, 1) \cdot \frac{1}{1 - 3\sqrt{3}/8}.$$

For the specific case $\alpha = 1.9$, we have $\Gamma(1.9) \approx 0.96177$, $(0.75)^{1.9} \approx 0.5790$, and

$$\Lambda = \frac{0.5790}{0.96177} \cdot \frac{1}{1.9} \cdot \frac{8(8+3\sqrt{3})}{37} \approx 0.904 < 1.$$

Thus, on the interval $[0, 0.75]$, the uniqueness condition $\Lambda < 1$ is satisfied for $\alpha = 1.9$, and Theorem 8 guarantees a unique solution in $C^2[0, 0.75]$.

Remark 10. As the fractional order α approaches the classical limit $\alpha \rightarrow 2^-$, the admissible interval length b satisfying $\Lambda < 1$ attains its maximum, and the type parameter β tends to 1 (Caputo case). Specifically, for $\beta = 1$, the maximum b is approximately 0.79, at which point only $\alpha \approx 2$ is admissible. For smaller α , the maximum admissible b decreases significantly. Similarly, for fixed $\alpha = 1.9$, the maximum admissible b decreases sharply as β moves away from 1, dropping from $b \approx 0.79$ at $\beta = 1$ to $b \approx 0.36$ at $\beta = 0.01$. Consequently, to maximize the interval length while preserving the uniqueness condition $\Lambda < 1$, one should take α as close to 2 as possible and $\beta = 1$ (Caputo formulation). Any deviation from these optimal values necessitates a corresponding reduction in b .

Furthermore, under the same condition, Theorems 9 and 10 establish that the solution is UH-stable and GUH-stable. To verify this numerically, we introduce a perturbation $g(t)$ as described in Remark 4, choosing $g(t) = 0.3 \sin(10t)$ so that $|g(t)| \leq \varepsilon = 0.3$ for the fractional oscillator with $\alpha = 1.9$, $b = 0.75$. This oscillatory perturbation represents a common type of disturbance in physical systems (e.g., sensor noise or environmental

vibrations), and its non-constant, sign-changing nature provides a stringent test of the stability estimate. Using Remark 7 with the parameters above, we obtain $c_f = 0.9504$. The theoretical bound from Theorem 9 then gives $|y(t) - x(t)| \leq c_f \varepsilon \approx 0.2851$.

Figure 3 shows the unperturbed and perturbed solutions: panel (a) on $[0, 0.75]$ (where stability is guaranteed) and panel (b) on $[0, 5]$ (extended view). The trajectories remain close despite the perturbation. Figure 4 shows that the computed deviation $|y(t) - x(t)|$ remains well below this bound for all $t \in [0, 0.75]$, confirming Ulam–Hyers stability.

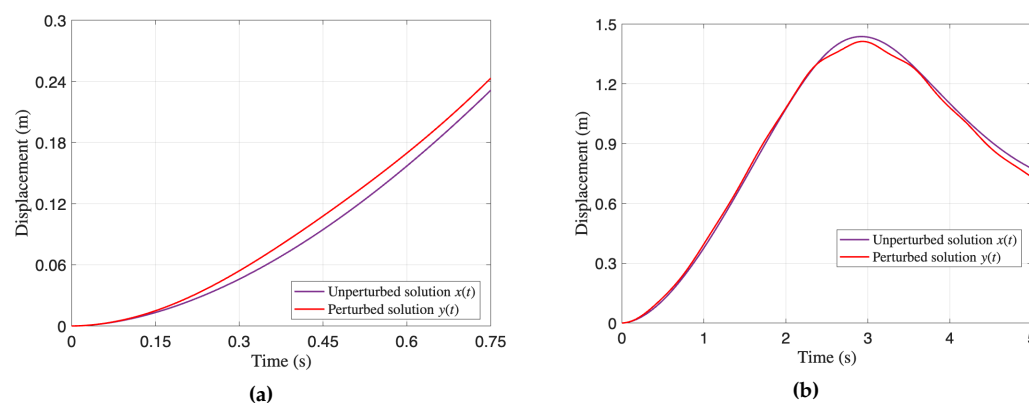


Figure 3. Comparison between the solution $x(t)$ of the fractional nonlinear oscillator and the perturbed solution $y(t)$ corresponding to the forcing term $g(t) = 0.3 \sin(10t)$. Despite the presence of the perturbation, the two trajectories remain close throughout the interval, illustrating the UH-stable properties of the fractional model. (a) Solution profiles over the interval $t \in [0, 0.75]$, where the uniqueness and stability properties are guaranteed; (b) Solution profiles over the extended interval $t \in [0, 5]$, providing a broader view.

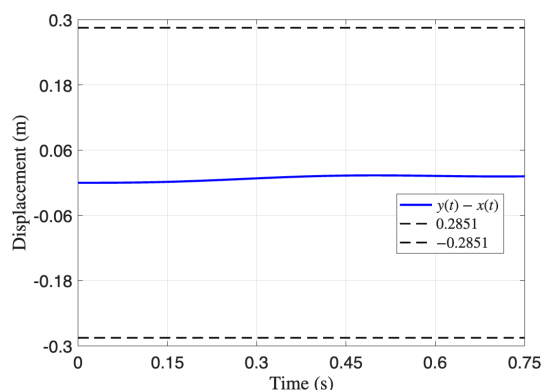


Figure 4. Difference between the perturbed and unperturbed solutions, $y(t) - x(t)$, together with the theoretical bounds ± 0.2851 . The numerical error remains confined within the prescribed limits over the entire interval, providing a graphical verification of the UH-stable estimate.

To demonstrate Ulam–Hyers–Rassias stability, we choose $\sigma(t) = e^t$. We define the perturbation $g(t) = 0.3 \sin(10t) \cdot e^t$, which satisfies $|g(t)| \leq 0.3 e^t = \varepsilon \sigma(t)$ with $\varepsilon = 0.3$.

For $\sigma(t) = e^t \in C[0, 0.75]$, we need to verify hypothesis (H3): there exists $\lambda_\sigma = e^{0.75} / \Gamma(2.9) \cdot \max_{t \in [0, 0.75]} (t^{1.9} / e^t) \approx 0.3168 > 0$ such that for each $t \in (0, 0.75]$

$$I_{0+}^{1.9;t} \sigma(t) \leq e^{0.75} \cdot \frac{t^{1.9}}{\Gamma(2.9)} = \frac{e^{0.75} \cdot t^{1.9}}{\Gamma(2.9) \cdot e^t} \cdot \sigma(t) \leq \lambda_\sigma \sigma(t).$$

The theoretical bound from Theorem 10 and Remark 7 for $n = 2$ on the interval $[0, 0.75]$ gives

$$|y(t) - x(t)| \leq c_{f,\sigma} \varepsilon \sigma(t) = \left(\lambda_{\sigma} \max_{t \in [0, 0.75]} F(t) \right) \varepsilon \sigma(t) = \left(\lambda_{\sigma} \max_{t \in [0, 0.75]} F_{1.9, 1, 2.9} \left(2.85322 \cdot t^{1.9} \right) \right) \varepsilon \sigma(t) \\ \approx 0.3168 \times 3 \times 0.3 \times e^t = 0.2851 e^t.$$

Figure 5 shows the unperturbed solution $x(t)$ and the perturbed solution $y(t)$. The exponentially modulated perturbation $g(t) = 0.3e^t \sin(10t)$ (satisfying $|g(t)| \leq \varepsilon \sigma(t)$ with $\varepsilon = 0.3$ and $\sigma(t) = e^t$ as in Remark 4) produces a larger deviation from the unperturbed trajectory while preserving the qualitative behavior of the system. Figure 6 plots the deviation $|y(t) - x(t)|$ together with the theoretical bound $0.2851 e^t$. The deviation remains below this exponentially growing bound for all $t \in [0, 0.75]$, confirming the UHR stability of the fractional model.

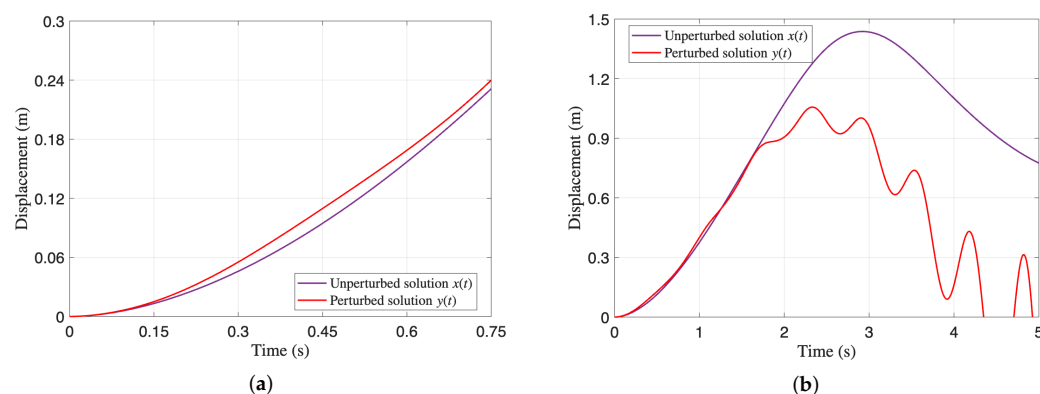


Figure 5. Comparison between the unperturbed solution $x(t)$ and the perturbed solution $y(t)$ for $g(t) = 0.3e^t \sin(10t)$. (a) Solution profiles over the interval $t \in [0, 0.75]$, where the uniqueness and stability properties are guaranteed; (b) Solution profiles over the extended interval $t \in [0, 5]$, providing a broader view.

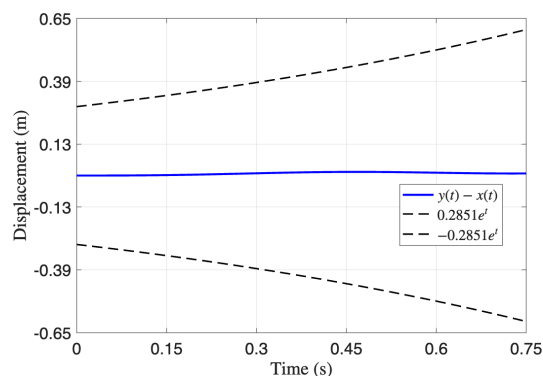


Figure 6. Difference $y(t) - x(t)$ between the perturbed and unperturbed solutions corresponding to the forcing term $g(t) = 0.3e^t \sin(10t)$. The theoretical bounds $\pm 0.2851 e^t$ are included for comparison, demonstrating the effect of the exponentially growing perturbation on the deviation between the two trajectories.

To verify the continuous dependence result of Theorem 11, we perturb the initial conditions by a small amount ϵ . With $x(0) = x'(0) = 0$, we set

$$\hat{x}(0) = \epsilon, \quad \hat{x}'(0) = \epsilon,$$

and choose $\epsilon = 0.05$ as a representative small perturbation. Applying Theorem 11 with $n = 2$ and $\gamma = 2$, the theoretical bound becomes

$$|x(t) - \hat{x}(t)| \leq |\epsilon|(t+1)F(t) = 0.05(t+1)F(t),$$

where $F(t)$ is given by (44):

$$F(t) = F_{1.9,1,2.9} \left(\frac{8(8+3\sqrt{3})}{37} t^{1.9} \right) = \sum_{k=0}^{\infty} \left(\prod_{i=1}^{k-1} \frac{\Gamma(1.9i+1)}{\Gamma(1.9i+2.9)} \right) \left(\frac{8(8+3\sqrt{3})}{37} t^{1.9} \right)^k.$$

Figure 7 shows the numerical solutions corresponding to the initial data $(x(0), x'(0)) = (0, 0)$ and $(\hat{x}(0), \hat{x}'(0)) = (0.05, 0.05)$. Although the initial conditions differ, the resulting trajectories remain close throughout the interval, indicating that small perturbations in the prescribed data produce only moderate changes in the solution. Figure 8 displays the difference $x(t) - \hat{x}(t)$ together with the theoretical bounds $\pm 0.05(t+1)F(t)$. The numerical difference remains well within these bounds over the entire interval, demonstrating excellent agreement with the estimate obtained in Theorem 11 and providing numerical verification of the continuous dependence of solutions on the initial conditions.

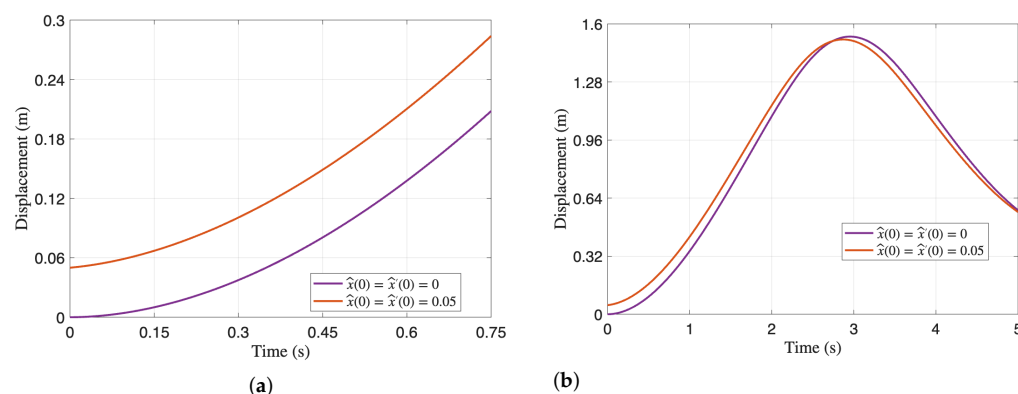


Figure 7. Numerical solutions of the fractional integral Equation (63) for $\alpha = 1.9$ corresponding to the initial data $(x(0), x'(0)) = (0, 0)$ and $(\hat{x}(0), \hat{x}'(0)) = (0.05, 0.05)$. (a) Solution profiles over the interval $t \in [0, 0.75]$, where the uniqueness and stability properties are guaranteed; (b) Solution profiles over the extended interval $t \in [0, 5]$, providing a broader view.

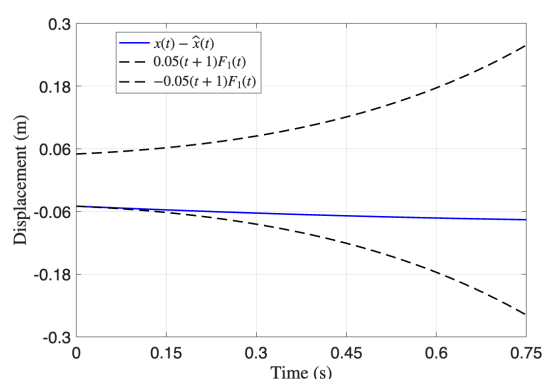


Figure 8. Difference $x(t) - \hat{x}(t)$ between solutions corresponding to $(x(0), x'(0)) = (0, 0)$ and $(\hat{x}(0), \hat{x}'(0)) = (0.05, 0.05)$ together with the theoretical bounds $\pm 0.05(t+1)F(t)$.

Comparison with the Real-Order (Classical Hilfer/Caputo) Case and Physical Interpretation of $\Im(\alpha)$

The classical Hilfer and Caputo models are recovered from (61)–(62) as the special cases $\Im(\alpha) = 0$, with $\beta = 1$, $\psi(t) = t$ giving the Caputo derivative used for the baseline curves in Figure 2, and $\beta = 0$ giving the (real-order) Riemann–Liouville Hilfer derivative. To isolate the effect of a genuinely complex order, we solve (61) with $\alpha = 1.9 + i0.3$ (same real part as the $\alpha = 1.9$ curve of Figure 2) and compare the real part of the resulting

complex-valued trajectory, $\Re(x(t))$, against the real-order solution with $\alpha = 1.9$. Figure 9 shows that the two trajectories share the same overall power-law growth rate (governed by $\Re(\alpha) = 1.9$ in both cases), while the complex-order trajectory exhibits a small phase-shifted oscillation superimposed on the monotone trend, consistent with the interpretation of $\Im(\alpha)$ as a logarithmically oscillating modulation of the memory kernel discussed in the Introduction. Physically, this is the fractional-order analogue of adding a fixed phase lag to the system's memory response, a feature that is unavailable in any real-order (classical Hilfer or Caputo) model and that motivates the use of complex-order operators in CRONE-type controllers and in viscoelastic models with a prescribed constant phase margin, as discussed in Section 1.

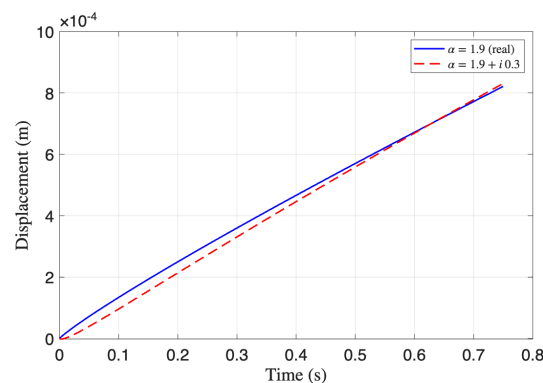


Figure 9. Comparison between the real-order solution ($\alpha = 1.9$) and the real part of the complex-order solution ($\alpha = 1.9 + i 0.3$), both with $\beta = 1$, $\psi(t) = t$, $d_1 = d_2 = 0$. The complex-order trajectory exhibits a small phase-shifted oscillation relative to the real-order trajectory, illustrating the physical role of $\Im(\alpha)$.

Exact kernel decomposition and parsimony of the complex-order model.

The qualitative statement above can be made fully explicit. By Remark 1, for $\tau < t$ the quantity $\Delta_\tau^\psi t$ is a strictly positive real number, so Euler's formula applied to the exact identity of Remark 1 gives, without approximation,

$$(\Delta_\tau^\psi t)^{\alpha-1} = (\Delta_\tau^\psi t)^{\Re(\alpha)-1} \left[\cos(\Im(\alpha) \ln \Delta_\tau^\psi t) + i \sin(\Im(\alpha) \ln \Delta_\tau^\psi t) \right]. \quad (65)$$

Thus the real part of the memory kernel used in $I_{a+}^{\alpha;\psi}$ is *exactly* a power-law kernel of exponent $\Re(\alpha) - 1$ modulated by a single cosine that oscillates linearly in $\ln \Delta_\tau^\psi t$, with frequency $\Im(\alpha)$ and zero phase. Reproducing the same real part with purely real-order operators requires, in general, a superposition

$$\sum_{j=1}^m c_j (\Delta_\tau^\psi t)^{\rho_j-1}, \quad \rho_j \in \mathbb{R},$$

and matching a single log-periodic modulation of prescribed frequency and phase forces $m \geq 2$ with 4 independent real parameters $(\rho_1, \rho_2, c_1, c_2)$ (or, equivalently, an amplitude/phase pair together with two decay rates), against the 2 real parameters $(\Re(\alpha), \Im(\alpha))$ needed in the complex-order formulation. This is the precise sense in which the complex-order operator is parsimonious relative to real-order superposition models: identity (55) is exact algebra, not a numerical fit, so this advantage is not an artifact of the particular oscillator example of Section 6 but a structural property of the complex-order kernel itself.

Practical Advantages of the Complex-Order Formulation and Comparison with Existing Approaches

Two concrete practical points distinguish the present framework from existing real-order ψ -Hilfer treatments and from a naive discretization of the implicit problem (1)–(2).

(i) **Parameter economy.** As shown by the exact decomposition (65), matching both the power-law memory decay and a prescribed phase lag in the oscillator response (Figure 9) requires only the two real numbers $\Re(\alpha)$, $\Im(\alpha)$ in the complex-order model, versus at least four real parameters in any real-order superposition achieving the same log-periodic modulation. For system-identification tasks such as fitting a CRONE controller or a viscoelastic constant-phase element to measured frequency-response data, this halves the number of free parameters to be estimated, at the cost of solving the complex-order well-posedness problem addressed by Theorems 7–8 (rather than a system of coupled real-order equations).

(ii) **Computational cost relative to a direct implicit discretization.** The reformulation of Lemma 12 replaces the original implicit equation, in which the fractional derivative appears nonlinearly on both sides, by the explicit Volterra representation (22) together with the scalar pointwise functional Equation (23) for the auxiliary unknown $\mathcal{G}_x(t)$. Consequently, in the product-quadrature scheme of Appendix A, the nonlinear system to be solved by Newton's method at each time step t_n is one-dimensional (Equation (A4)), and the overall cost is $O(N^2)$ (dominated by the convolution weights $\omega_{n,j}$) plus $O(N)$ scalar Newton solves, each converging in 3–5 iterations (Section 6, “Convergence analysis”). A direct discretization of the original implicit Equation (1) that does not use the reformulation of Lemma 12 would instead need to solve, at each step, a nonlinear system coupling the approximation of x_n with the approximation of its own fractional derivative, which does not reduce to a scalar equation, in general, and is correspondingly more expensive and more delicate to initialize reliably. The independent fractional Adams–Bashforth–Moulton cross-check already reported in Table 1 (columns E_N^{ABM} and “Agreement”) confirms that the two approaches agree to within 3.7×10^{-5} at $N = 1000$, while the reformulated scheme used throughout this paper benefits from the reduced, scalar nonlinear structure guaranteed by Lemma 12. This is a direct practical payoff of the theoretical reformulation developed in Section 3, rather than merely a numerical illustration of it.

7. Discussion

The results established in this paper provide a rigorous and comprehensive well-posedness theory for a broad class of nonlinear implicit ψ -Hilfer fractional differential equations. Several aspects of the analysis merit further comment.

The choice of the weighted Banach space $C_{n-\gamma,\psi}[a, b]$ is not merely a technical convenience but reflects the genuine behavior of solutions near the left endpoint $t = a$. Solutions to ψ -Hilfer equations exhibit a singularity at $t = a$, and the weighted norm $\|x\|_{C_{n-\gamma,\psi}[a,b]} = \max_{t \in [a,b]} |(\Delta_a^\psi t)^{n-\gamma} x(t)|$ is precisely calibrated to handle this behavior.

In the existence proof via Schaefer's fixed-point theorem, we first establish a fixed point x in the larger space $C_{n-\gamma,\psi}[a, b]$. Then, by applying the fractional derivative $D_{a+}^{\gamma;\psi}$ to the integral equation and using the regularity of G_x , we show that $D_{a+}^{\gamma;\psi} x \in C_{n-\gamma,\psi}[a, b]$. This implies $x \in C_{n-\gamma,\psi}^\gamma[a, b]$, i.e., the solution automatically possesses higher regularity. Thus, the weighted space provides not only a natural setting for existence but also a pathway to regularity.

The core analytical challenge of problem (1) is that the ψ -Hilfer derivative ${}^H D_{a+}^{\alpha,\beta;\psi} x$ appears simultaneously as the left-hand side and as an argument of f on the right. The resolu-

tion, developed in Lemma 12, is to introduce the auxiliary function $G_x(t) := ({}^H D_{a+}^{\alpha, \beta; \psi} x)(t)$. For each fixed t , the equation $G_x(t) = f(t, x(t), G_x(t))$ can be rewritten as

$$\Phi(t, x(t), G_x(t)) = 0,$$

where $\Phi(t, u, v) = f(t, u, v) - v$. Under hypothesis (H2) with $M < 1$, we have $\left| \frac{\partial \Phi}{\partial v} \right| = |M - 1| = 1 - M > 0$, so the Implicit Function Theorem guarantees that $G_x(t)$ is uniquely determined as a function of $x(t)$ near any solution. Moreover, the Lipschitz estimate

$$\|G_x - G_y\|_{C_{n-\gamma, \psi}[a, b]} \leq \frac{K}{1 - M} \|x - y\|_{C_{n-\gamma, \psi}[a, b]}$$

follows directly from the contraction property of f in its second argument. This reformulation turns the original implicit problem into a standard Volterra equation, and all our main results are built upon this key idea.

The stability and continuous-dependence proofs require bounding an integral inequality of the form

$$u(t) \leq a(t)(\Delta_a^\psi t)^{\alpha-1} + b(t) \int_a^t (\Delta_s^\psi t)^{\beta-1} (\Delta_a^\psi s)^{\gamma-1} \psi'(s) u(s) ds,$$

whose kernel is singular at both $s = t$ and $s = a$. The new Theorem 4 extends the result of [12] to this doubly singular ψ -weighted setting via a change of variable $r = \psi(s)$, reducing to the classical Gronwall inequality in the transformed coordinates. The resulting bound, expressed through the entire function $\mathcal{F}_{\vartheta, \delta, \delta+\beta}$, is sharp and yields the explicit stability constants in Remark 7.

The uniqueness criterion $\Lambda < 1$ in Theorem 8 provides a fully explicit condition on the problem data. For the application in Section 6, with $\psi(t) = t$, $\Re(\alpha) \in (1, 2)$, $\beta \rightarrow 1$ (Caputo type), and $K = 1$, $M = 3\sqrt{3}/8$, the condition reduces to

$$b^{\Re(\alpha)} \cdot \frac{1}{\Gamma(\Re(\alpha) + 1)} \cdot \frac{8}{8 - 3\sqrt{3}} < 1,$$

which is satisfied for $b \leq 0.75$ and all $\Re(\alpha) \in (1, 2)$. In particular, for $\Re(\alpha) = 1.9$ one obtains $\Lambda \approx 0.904$, close to the boundary of the admissible region. The restriction $b = 0.75$ is not a weakness of our method. As explained in Remark 10, for $\beta = 1$ and $\alpha = 1.9$, the maximum admissible b is approximately 0.79, so $b = 0.75$ lies safely within this bound. On the longer interval $[0, 5]$, uniqueness is no longer guaranteed by the theorem, but our numerical simulations show that different solutions do not actually appear. This suggests that the sufficient condition we used may be stricter than necessary.

The four Ulam stabilities established in Theorems 9–10 have a clear physical meaning in the context of the fractional oscillator application. UH stability guarantees that a trajectory satisfying the governing equation only approximately (e.g., due to sensor noise or numerical integration error) remains uniformly close to the exact trajectory, with the deviation bounded by $c_f \varepsilon$ regardless of time. UHR stability is more refined: it allows the error tolerance to grow with a prescribed weight function $\sigma(t)$, accommodating situations in which disturbances intensify over time, as in the exponentially modulated perturbation $g(t) = 0.3e^t \sin(10t)$ used in Section 6. The computed stability constant $c_{f, \sigma} = \lambda_\sigma \max \mathcal{F}(t) \approx 0.9504$ and the resulting bound $|y(t) - x(t)| \leq 0.2851 e^t$ are confirmed by Figure 6, which shows the numerical deviation remaining well within the theoretical envelope throughout $[0, 0.75]$.

Theorem 11 establishes that a small perturbation $|\epsilon|$ in each initial datum c_k produces a deviation bounded by $|\epsilon| \sum_{k=1}^n \left((\Delta_a^\psi t)^{n-k} / |\Gamma(\gamma - k + 1)| \right) \mathcal{F}(t)$. The factor $(\Delta_a^\psi t)^{n-k}$ vanishes at $t = a$ for $k < n$, reflecting the fact that the solution is insensitive to initial data perturbations at the endpoint (where the singularity of the weighted space absorbs the perturbation) but that sensitivity grows as t moves away from a . Remark 8 captures this observation explicitly: substantial sensitivity may appear on $[a, c]$ for any $c \in (a, b)$, but the deviation is controlled globally on $[a, b]$ by the bound in (48). Figure 8 illustrates this behavior numerically for $\epsilon = 0.05$.

We emphasize, however, that these three extensions—namely (i) the full range $\Re(\alpha) \in (n - 1, n)$ for arbitrary $n \in \mathbb{N}$, (ii) the treatment of complex-order α with $\Im(\alpha) \neq 0$, and (iii) the handling of the implicit structure via the auxiliary functional equation—are obtained by carefully adapting standard fixed-point and Gronwall-type techniques rather than by developing fundamentally new analytical machinery. The principal value of the paper lies in demonstrating that such an adaptation is feasible and in providing fully explicit, computable well-posedness and stability estimates that can be applied directly to problems in viscoelasticity, control theory, and anomalous diffusion.

Compared with existing results for ψ -Hilfer equations, which are mostly restricted to order $\alpha \in (0, 1)$ and often treat explicit problems only, the present work offers three principal extensions. First, the theory applies to the full range $\Re(\alpha) \in (n - 1, n)$ for any $n \in \mathbb{N}$, and to complex α . Second, the implicit structure is handled by the auxiliary functional equation strategy, which includes the explicit case (f independent of its third argument) as a special instance. Third, the stability constants c_f and $c_{f,\sigma}$ are given in closed form through $\mathcal{F}_{\rho,a,b}$, providing quantitative estimates that can be evaluated numerically for problem (1)–(2) with arbitrary α satisfying $\Re(\alpha) \in (n - 1, n)$ for any $n \in \mathbb{N}$.

8. Conclusions

This paper has established a complete well-posedness theory for nonlinear implicit fractional differential equations involving the ψ -Hilfer derivative of complex order α with $\Re(\alpha) \in (n - 1, n)$, for any $n \in \mathbb{N}$. The main findings are

1. Existence: Under hypotheses (H0)–(H2), Schaefer’s fixed-point theorem guarantees at least one solution in $C_{n-\gamma,\psi}^\gamma[a, b]$ (Theorem 7).
2. Uniqueness: Under hypotheses (H0)–(H2) and when $\Lambda < 1$, Banach’s fixed-point theorem yields a unique solution (Theorem 8).
3. Stability: Assume $\Re(\alpha + \gamma - n) > 0$. Under the conditions of the uniqueness theorem (Theorem 8), the problem is UH-stable and GUH-stable (Theorem 9). Moreover, under the same conditions together with hypothesis (H3), the problem is also UHR-stable and GUHR-stable with respect to $\sigma(t)$ (Theorem 10).
4. Continuous dependence: Assume $\Re(\alpha + \gamma - n) > 0$. Under the assumptions of the uniqueness theorem (Theorem 8), solutions depend continuously on the initial conditions with the explicit error estimate given in Theorem 11. Moreover, under the same hypotheses, the solution map $\alpha \mapsto x_\alpha$ is Lipschitz continuous on any compact subset of the admissible strip on which $\Lambda(\alpha) < 1$ uniformly (Proposition 1); in particular, the real-order theory ($\Im(\alpha) = 0$) is the quantitative limit of the complex-order theory as $\Im(\alpha) \rightarrow 0$.

Remark 7 provides a unified framework for computing the stability and continuous dependence bounds. The constants c_f and $c_{f,\sigma}$ (as defined in Definitions 7, 9 and 10) and the function $F(t)$ are given explicitly in terms of the problem parameters. Thus, once the required hypotheses are satisfied, the bounds from Theorems 9–11 can be obtained directly from this remark, without restating each bound separately. We applied the theoretical frame-

work to a fractional nonlinear oscillator with saturating acceleration-dependent feedback and studied its existence, uniqueness, Ulam stabilities, and continuous dependence.

These results extend and unify several existing contributions in the literature, providing a rigorous foundation for the analysis of implicit fractional differential equations arising in viscoelasticity, control theory, and anomalous diffusion.

Author Contributions: Conceptualization, J.S. and E.T.; methodology, J.S. and E.T.; validation, J.S., E.T. and S.K.N.; formal analysis, J.S., E.T. and S.C.; investigation, J.S. and E.T.; writing—original draft preparation, E.T. and S.C.; writing—review and editing, E.T. and S.K.N.; software, S.C. and E.T.; supervision, E.T.; project administration, J.S.; funding acquisition, J.S. All authors have read and agreed to the published version of the manuscript.

Funding: The research was financially supported by Naresuan University, Thailand, under Grant No. R2569C014.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

ODEs	Ordinary differential equations
FDEs	Fractional differential equations
UH	Ulam–Hyers
GUH	Generalized Ulam–Hyers
UHR	Ulam–Hyers–Rassias
GUHR	Generalized Ulam–Hyers–Rassias
CRONE	Commande Robuste d’Ordre Non Entier
PID	Proportional Integral Derivative

Appendix A. Reproducibility: Numerical Workflow

Appendix A.1. Classical Solution: Runge–Kutta with Newton’s Method

The classical model (59) is an implicit second-order ODE of the form

$$x''(t) = f(t, x(t), x''(t)), \quad f(t, u, v) = -u + \frac{1}{1 + v^2}.$$

To solve this numerically, we convert it to a first-order system by introducing the velocity $v(t) = x'(t)$. The system becomes

$$x'(t) = v(t), \quad v'(t) = f(t, x(t), v'(t)).$$

The second equation is implicit in $v'(t)$ and must be solved pointwise.

Appendix A.2. Discretization

We discretize the time interval $[0, b]$ with $b = 0.75$ using a uniform grid

$$t_n = nh, \quad n = 0, 1, \dots, N, \quad h = \frac{b}{N} = \frac{0.75}{N},$$

with $N = 1000$. Let $x_n \approx x(t_n)$ and $v_n \approx v(t_n)$ denote the numerical approximations.

Appendix A.3. Initialization

Set the initial conditions:

$$x_0 = c_1, \quad v_0 = c_2,$$

where $c_1, c_2 \in \mathbb{R}$ are the given initial displacement and velocity. Solve for the initial acceleration $y_0 = v'(0)$ from the implicit relation:

$$y_0 = -x_0 + \frac{1}{1 + y_0^2},$$

using Newton's method (see below).

Appendix A.4. Fourth-Order Runge–Kutta Method

The classical fourth-order Runge–Kutta method is applied to the first-order system. For each step $n = 0, 1, \dots, N - 1$, we compute

1. Stage 1:

$$k_{1,x} = v_n, \quad k_{1,v} = y_n,$$

where y_n is the acceleration at t_n from the previous step (or y_0 for $n = 0$).

2. Stage 2:

$$x_{\text{tmp}} = x_n + \frac{h}{2}k_{1,x}, \quad v_{\text{tmp}} = v_n + \frac{h}{2}k_{1,v}.$$

Solve for y_{tmp} from the implicit relation:

$$y_{\text{tmp}} = -x_{\text{tmp}} + \frac{1}{1 + y_{\text{tmp}}^2}.$$

Then, set

$$k_{2,x} = v_{\text{tmp}}, \quad k_{2,v} = y_{\text{tmp}}.$$

3. Stage 3:

$$x_{\text{tmp}} = x_n + \frac{h}{2}k_{2,x}, \quad v_{\text{tmp}} = v_n + \frac{h}{2}k_{2,v}.$$

Solve for y_{tmp} from the implicit relation and set

$$k_{3,x} = v_{\text{tmp}}, \quad k_{3,v} = y_{\text{tmp}}.$$

4. Stage 4:

$$x_{\text{tmp}} = x_n + hk_{3,x}, \quad v_{\text{tmp}} = v_n + hk_{3,v}.$$

Solve for y_{tmp} from the implicit relation and set

$$k_{4,x} = v_{\text{tmp}}, \quad k_{4,v} = y_{\text{tmp}}.$$

5. Update

$$x_{n+1} = x_n + \frac{h}{6}(k_{1,x} + 2k_{2,x} + 2k_{3,x} + k_{4,x}), \quad v_{n+1} = v_n + \frac{h}{6}(k_{1,v} + 2k_{2,v} + 2k_{3,v} + k_{4,v}).$$

6. Solve for the acceleration y_{n+1} at the new time step:

$$y_{n+1} = -x_{n+1} + \frac{1}{1 + y_{n+1}^2}.$$

Appendix A.5. Newton's Method for the Implicit Relation

At each stage and at each time step, we solve the scalar nonlinear equation

$$F(y) = y + x - \frac{1}{1 + y^2} = 0,$$

where x is the current position (either x_n , x_{tmp} , or x_{n+1}). The derivative is

$$F'(y) = 1 + \frac{2y}{(1 + y^2)^2}.$$

Newton's iteration proceeds as

$$y^{k+1} = y^k - \frac{F(y^k)}{F'(y^k)},$$

starting from an initial guess y^0 (the acceleration from the previous stage or time step). The iteration stops when $|F(y^k)| < 10^{-12}$.

Appendix A.6. Fractional Solution: Product Quadrature Method

For $\psi(t) = t$, $\beta = 1$, the equivalent integral Equation (63) reduces to the Caputo-type Volterra equation

$$x(t) = d_1 + d_2 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G_x(s) ds, \quad (\text{A1})$$

where $G_x(t) = -x(t) + 1/(1 + G_x(t)^2)$. The kernel $(t-s)^{\alpha-1}$ is singular at $s = t$, making standard quadrature rules unsuitable. The product quadrature rule handles this singularity by approximating $G_x(s)$ on each subinterval and integrating the product with the kernel exactly.

Appendix A.7. Discretization

We discretize $[0, b]$ with a uniform grid $t_i = ih$, $i = 0, \dots, N$, where $h = b/N$ and $b = 0.75$. The fractional integral at $t = t_n$ is decomposed as

$$I_{0+}^\alpha G_x(t_n) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} G_x(s) ds.$$

Appendix A.8. Piecewise Constant Approximation

Approximating $G_x(s)$ by the constant $G_x(t_j)$ on $[t_j, t_{j+1}]$ gives

$$\int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} G_x(s) ds \approx G_x(t_j) \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} ds = G_x(t_j) \frac{(t_n - t_j)^\alpha - (t_n - t_{j+1})^\alpha}{\alpha}.$$

As $t_n = nh$, $t_j = jh$, then

$$I_{0+}^\alpha G_x(t_n) \approx \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} G_x(t_j) \cdot \frac{h^\alpha}{\alpha} [(n-j)^\alpha - (n-j-1)^\alpha].$$

Using $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, we define the convolution weights

$$\omega_{n,j} = \frac{h^\alpha}{\Gamma(\alpha + 1)} [(n-j)^\alpha - (n-j-1)^\alpha], \quad j = 0, 1, \dots, n-1. \quad (\text{A2})$$

For the current subinterval $[t_{n-1}, t_n]$, approximating $G_x(s)$ by $G_x(t_n)$ yields the additional weight

$$\omega_{n,n} = \frac{h^\alpha}{\Gamma(\alpha + 1)}. \quad (\text{A3})$$

Hence,

$$I_{0+}^\alpha G_x(t_n) \approx \sum_{j=0}^{n-1} \omega_{n,j} G_x(t_j) + \omega_{n,n} G_x(t_n).$$

Appendix A.9. Algebraic Equation

Substituting into (A1) at $t = t_n$ gives

$$x(t_n) = d_1 + d_2 t_n + \sum_{j=0}^{n-1} \omega_{n,j} G_x(t_j) + \omega_{n,n} G_x(t_n).$$

Define the history term

$$A_n = d_1 + d_2 t_n + \sum_{j=0}^{n-1} \omega_{n,j} G_x(t_j),$$

so that $x(t_n) = A_n + \omega_{n,n} G_x(t_n)$. Using $G_x(t_n) = -x(t_n) + 1/(1 + G_x(t_n)^2)$ and letting $y_n = G_x(t_n)$, we obtain

$$y_n = -(A_n + \omega_{n,n} y_n) + \frac{1}{1 + y_n^2}.$$

Rearranging yields the scalar nonlinear equation

$$(1 + \omega_{n,n}) y_n + A_n - \frac{1}{1 + y_n^2} = 0. \quad (\text{A4})$$

Appendix A.10. Newton's Method

Define $F(y) = (1 + \omega_{n,n})y + A_n - 1/(1 + y^2)$. Its derivative is $F'(y) = 1 + \omega_{n,n} + 2y/(1 + y^2)^2$. Newton's iteration

$$y^{k+1} = y^k - \frac{F(y^k)}{F'(y^k)}$$

is applied with initial guess $y^0 = y_{n-1}$ (the value from the previous time step). The iteration stops when $|F(y^k)| < 10^{-12}$.

Appendix A.11. Algorithm Summary

For each time step $n = 1, \dots, N$:

1. Compute $A_n = d_1 + d_2 t_n + \sum_{j=0}^{n-1} \omega_{n,j} y_j$.
2. Solve (A4) for y_n using Newton's method.
3. Compute $x_n = A_n + \omega_{n,n} y_n$.

The weights $\omega_{n,j}$ are precomputed using (A2) and (A3). For $N = 1000$, this scheme accurately resolves the nonlocal memory effects inherent to fractional dynamics.

Algorithm A1 Numerical workflow for classical and fractional nonlinear oscillators with saturating feedback.

Require:

- Time interval $[0, 0.75]$, grid size $N = 1000$, Newton tolerance $\epsilon = 10^{-12}$.
- Classical: $c_1 = 0, c_2 = 0$ (initial displacement and velocity).
- Fractional: $d_1 = 0, d_2 = 0$ (initial position and velocity), $\beta = 1, \psi(t) = t$.
- Fractional orders $\alpha \in \{1.3, 1.5, 1.7, 1.9\}$, classical order $\alpha = 2$.

Ensure: Classical solution $x_{cl}(t_n)$, fractional solutions $x_\alpha(t_n)$, fractional derivatives $y_\alpha(t_n)$

- 1: $h \leftarrow 0.75/N, t_n \leftarrow nh$ for $n = 0, \dots, N$
- 2: **Classical solution (Runge–Kutta with Newton)**
- 3: $x_0 \leftarrow c_1, v_0 \leftarrow c_2$
- 4: Solve $y_0 = -x_0 + 1/(1 + y_0^2)$ for initial acceleration via Newton
- 5: **for** $n = 0$ **to** $N - 1$ **do**
- 6: **Stage 1:** $k_{1,x} = v_n, k_{1,v} = y_n$
- 7: **Stage 2:** $x_{tmp} \leftarrow x_n + \frac{h}{2}k_{1,x}, v_{tmp} \leftarrow v_n + \frac{h}{2}k_{1,v}$
- 8: Solve $y_{tmp} = -x_{tmp} + 1/(1 + y_{tmp}^2)$ via Newton (guess y_n)
- 9: $k_{2,x} \leftarrow v_{tmp}, k_{2,v} \leftarrow y_{tmp}$
- 10: **Stage 3:** $x_{tmp} \leftarrow x_n + \frac{h}{2}k_{2,x}, v_{tmp} \leftarrow v_n + \frac{h}{2}k_{2,v}$
- 11: Solve $y_{tmp} = -x_{tmp} + 1/(1 + y_{tmp}^2)$ via Newton (guess $k_{2,v}$)
- 12: $k_{3,x} \leftarrow v_{tmp}, k_{3,v} \leftarrow y_{tmp}$
- 13: **Stage 4:** $x_{tmp} \leftarrow x_n + hk_{3,x}, v_{tmp} \leftarrow v_n + hk_{3,v}$
- 14: Solve $y_{tmp} = -x_{tmp} + 1/(1 + y_{tmp}^2)$ via Newton (guess $k_{3,v}$)
- 15: $k_{4,x} \leftarrow v_{tmp}, k_{4,v} \leftarrow y_{tmp}$
- 16: **Update:**
- 17: $x_{n+1} \leftarrow x_n + \frac{h}{6}(k_{1,x} + 2k_{2,x} + 2k_{3,x} + k_{4,x})$
- 18: $v_{n+1} \leftarrow v_n + \frac{h}{6}(k_{1,v} + 2k_{2,v} + 2k_{3,v} + k_{4,v})$
- 19: Solve $y_{n+1} = -x_{n+1} + 1/(1 + y_{n+1}^2)$ via Newton (guess $k_{4,v}$)
- 20: **end for**
- 21: Store $x_{cl}(t_n) \leftarrow x_n, v_{cl}(t_n) \leftarrow v_n, y_{cl}(t_n) \leftarrow y_n$
- 22: **Fractional solution (product quadrature)**
- 23: **for each** α in $\{1.3, 1.5, 1.7, 1.9\}$ **do**
- 24: **Precompute convolution weights:**
- 25: **for** $n = 0$ **to** N **do**
- 26: **for** $j = 0$ **to** n **do**
- 27: **if** $j = 0$ **then**
- 28: $\omega_{n,0} \leftarrow \frac{h^\alpha}{\Gamma(\alpha+1)}((n+1)^\alpha - n^\alpha)$
- 29: **else if** $1 \leq j \leq n-1$ **then**
- 30: $d \leftarrow n - j$
- 31: $\omega_{n,j} \leftarrow \frac{h^\alpha}{\Gamma(\alpha+1)}((d+1)^\alpha + (d-1)^\alpha - 2d^\alpha)$
- 32: **else**
- 33: $\omega_{n,n} \leftarrow \frac{h^\alpha}{\Gamma(\alpha+1)}$
- 34: **end if**
- 35: **end for**
- 36: **end for**
- 37: **Initialization:**
- 38: $x_0 \leftarrow d_1, v_0 \leftarrow d_2$ {initial position and velocity}
- 39: Solve $y_0 = -x_0 + 1/(1 + y_0^2)$ for initial fractional derivative via Newton (note: $y_0 \neq v_0$)
- 40: $x_\alpha(t_0) \leftarrow x_0, y_\alpha(t_0) \leftarrow y_0$
- 41: **Time-stepping:**
- 42: **for** $n = 1$ **to** N **do**
- 43: $A_n \leftarrow d_1 + d_2 t_n + \sum_{j=0}^{n-1} \omega_{n,j} y_\alpha(t_j)$ ($d_2 t_n$ is the initial velocity contribution)
- 44: Solve $(1 + \omega_{n,n})y + A_n - 1/(1 + y^2) = 0$ for y_n via Newton (initial guess $y_\alpha(t_{n-1})$)
- 45: $x_n \leftarrow A_n + \omega_{n,n} y_n$
- 46: $x_\alpha(t_n) \leftarrow x_n, y_\alpha(t_n) \leftarrow y_n$
- 47: **end for**
- 48: **end for**
- 49: **Plotting and comparison**
- 50: Plot $x_{cl}(t_n)$ together with $x_\alpha(t_n)$ for all α on $[0, 0.75]$
- 51: Repeat fractional simulation on $[0, 5]$ with same parameters for extended view
- 52: **Theoretical bound verification (optional)**
- 53: Compute $\Lambda = (b - a) \cdot \frac{8(8+3\sqrt{3})}{37} \approx 0.904 < 1$
- 54: Verify $\Lambda < 1$ confirms uniqueness (Theorem 8)

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