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A Conditional Structural Closure Framework Under Admissible Low-Mach Realizability for Persistent Concentration Exclusion in Three-Dimensional Incompressible Flow

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Abstract

A persistent concentration of scale-critical quantities is widely regarded as a necessary precursor to finite-time singularity formation in the three-dimensional incompressible Navier–Stokes equations. This paper develops a conditional structural closure framework for excluding a dynamically sustained persistent concentration within an admissible low-Mach realizable class. A localized amplitude-modulation decomposition of High–High transfer activity is introduced together with a localized Littlewood–Paley bridge linking transfer-amplitude concentration to mechanical concentration. A concentration-representation principle and a low-Mach inheritance mechanism then show that a persistent mechanical concentration necessarily generates a nontrivial thermo-acoustic trace. A central result is a localized structural necessity principle establishing that a persistent concentration necessarily generates a localized transfer-amplitude concentration. Combining the localized Littlewood–Paley bridge, the concentration-representation mechanism, the low-Mach inheritance principle, and a thermo-acoustic ε -regularity framework, we show that the resulting thermo-acoustic concentration scenario is excluded within the admissible low-Mach realizable class. Consequently, a dynamically sustained persistent concentration cannot occur in this class. The result is a conditional structural closure theorem: it excludes a persistent concentration only within the admissible low-Mach realizable class and does not constitute an unconditional regularity theorem for arbitrary solutions of the three-dimensional incompressible Navier–Stokes equations. The sole non-universal assumption is admissible low-Mach realizability, which is shown to follow whenever an admissible entropy-coercive, weakly compressible approximation exists. The principal remaining open problem is the universality of admissible low-Mach realizability.

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1. Introduction

1.1. The Persistent Concentration Problem

The three-dimensional incompressible Navier–Stokes equations constitute one of the central open problems in mathematical fluid mechanics [1–3]. While the existence of global weak solutions was established by Leray and Hopf [1,2], the global regularity of smooth solutions remains unresolved and is recognized as one of the Clay Millennium Problems [3].

A common viewpoint underlying modern regularity theory is that any hypothetical finite-time singularity must be accompanied by a concentration of scale-critical quantities [4–8]. In particular, the ε -regularity framework of Caffarelli, Kohn, and Nirenberg implies that singular behavior can occur only if a suitable localized concentration persists across sufficiently small scales [8].

Consequently, a fundamental question is whether a concentration sequence can remain dynamically sustained as the characteristic scale approaches zero. Let $z_0 = (x_0, t_0)$ be a space–time point and let $r_n \rightarrow 0$ be a sequence of shrinking scales. The present work focuses on concentration sequences associated with such scales and investigates whether a persistent concentration can be maintained under the combined constraints of nonlinear transfer dynamics and thermodynamic admissibility.

From a physical viewpoint, a persistent concentration sequence represents a hypothetical scenario in which localized mechanical activity repeatedly accumulates within progressively smaller space–time regions without dissipating. Such behavior would be expected near a potential singularity, where concentration remains dynamically active across a cascade of shrinking scales rather than being dispersed by viscosity or nonlinear transfer mechanisms. Persistent concentration sequences therefore provide a natural framework for investigating whether a singularity-forming concentration can remain sustained in a small-scale limit.

Specifically, a persistent concentration sequence is defined as a sequence of localized states associated with (z_n, r_n) , where $r_n \rightarrow 0$, for which the localized mechanical concentration functional $E(z, r)$, introduced later in Section 3, remains non-vanishing in the scale limit: $\limsup_{n \rightarrow \infty} E(z_n, r_n) > 0$. The present work investigates whether such concentration sequences can remain dynamically sustained under the combined constraints of nonlinear transfer dynamics, concentration representation, and thermo-acoustic admissibility.

1.2. Existing Approaches

The existing approaches to the Navier–Stokes regularity problem may be broadly classified into analytical and structural frameworks.

Analytical approaches establish regularity through a priori estimates, integrability criteria, or localized energy inequalities. Representative examples include the Prodi–Serrin criteria [4,5], the Beale–Kato–Majda criterion [6], backward uniqueness and blow-up analysis [7,9], and the ε -regularity theory of Caffarelli, Kohn, and Nirenberg [8]. Additional developments include the well-posedness theory of Koch and Tataru [10], blow-up scenarios for averaged models [11], and modern regularity frameworks summarized by Seregin and Lemarié-Rieusset [9,12,13].

Although these approaches provide powerful restrictions on possible singularities, they do not explicitly identify the nonlinear transfer mechanisms responsible for sustaining a scale-critical concentration.

A complementary viewpoint investigates the internal structure of a turbulent transfer. Beginning with the cascade theories of Kolmogorov, Onsager, and Kraichnan [14–16], subsequent developments have emphasized the role of triadic interactions, transfer locality, and coherent interaction structures [17–21]. These studies suggest that only a highly

restricted subset of nonlinear interactions remains dynamically relevant near a potential concentration event.

Understanding the structure of this subset is therefore essential for analyzing the persistence of a scale-critical concentration.

The overall logical hierarchy of the conditional structural closure framework is summarized schematically in Figure 1. The argument begins by assuming the existence of a persistent concentration sequence of scale-critical mechanical activity. The localized structural necessity principle established in Sections 5 and 6 then shows that a persistent concentration necessarily generates a localized transfer-amplitude concentration.

The localized Littlewood–Paley bridge developed in Section 7 and Appendix C converts this transfer-amplitude concentration into a localized mechanical concentration. Under the admissible low-Mach realizability assumption, the low-Mach inheritance principle established in Section 3 implies that a persistent mechanical concentration necessarily generates a nontrivial thermodynamic trace. This trace, in turn, activates the thermo-acoustic concentration functional.

The thermo-acoustic ε -regularity and exclusion framework developed in Section 2 then excludes the resulting thermo-acoustic concentration branch. Consequently, the thermodynamic activity generated by a persistent concentration becomes incompatible with thermo-acoustic admissibility. As illustrated in Figure 1, this contradiction rules out the original persistence hypothesis.

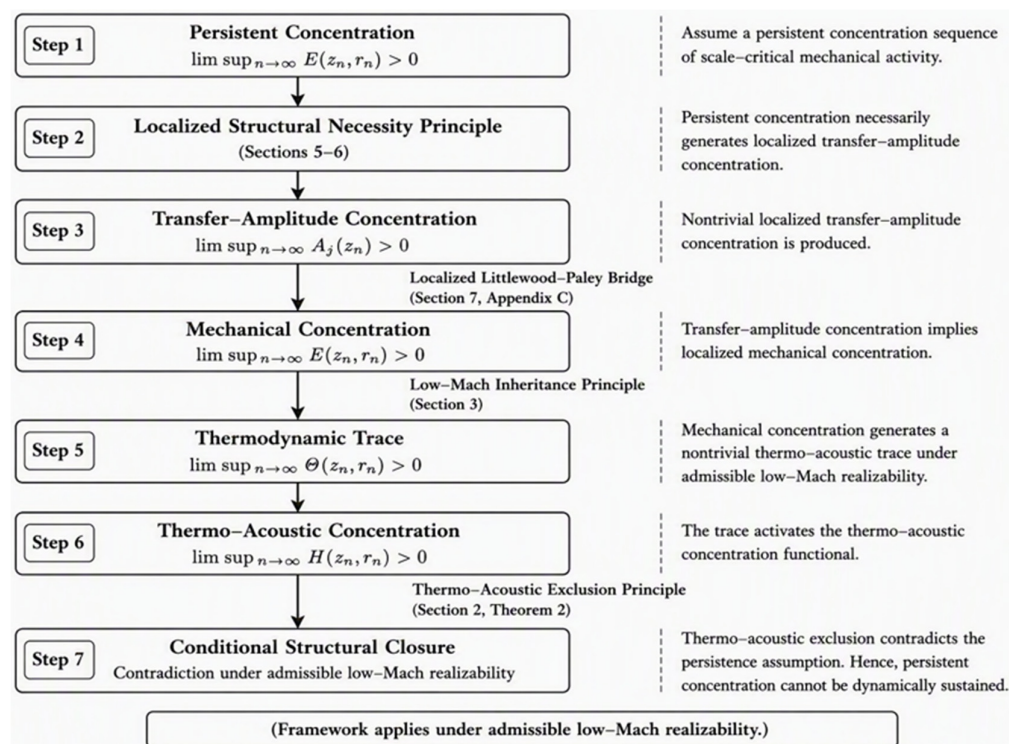


Figure 1. Schematic flow of the conditional structural closure framework.

Therefore, a dynamically sustained persistent concentration cannot occur within the admissible low-Mach realizable class. This yields the conditional structural closure theorem established later in the paper.

1.3. Motivation

Two recent developments motivate the present work.

First, a structural reduction framework for the incompressible Navier–Stokes equations was established in [22]. Using Fourier–helical localization, triadic decomposition, coherent-family analysis, and residence-time compression estimates, that work showed that most nonlinear transfer pathways become asymptotically perturbative at sufficiently small scales. As a consequence, coherent same-scale High–High interactions were identified as the only remaining non-perturbative candidates capable of supporting dynamically significant scale-critical transfers [22].

Second, a thermo-acoustic admissibility framework for compressible Navier–Stokes–Fourier dynamics was developed in [23]. Within an admissible thermodynamic class, a persistent concentration was shown to generate thermo-acoustic activity controlled by entropy-coercive concentration functionals. The resulting thermo-acoustic exclusion mechanism ruled out dynamically sustained concentration sequences in that admissible framework [23].

These two results provide complementary information. The first identifies the only remaining nonlinear transfer mechanism capable of sustaining concentration, whereas the second provides an exclusion mechanism once concentration generates sufficient thermo-acoustic activity.

The present work connects these two viewpoints through a concentration-representation framework. A localized transfer concentration is first represented as a localized mechanical concentration and subsequently as a thermo-acoustic trace through an admissible low-Mach realization. This representation hierarchy provides the missing link between transfer dynamics and thermo-acoustic concentration exclusion and leads to a unified structural closure theory for persistent concentration sequences.

From the viewpoint of incompressible fluid dynamics, low-Mach limits provide a classical mechanism through which compressible flows approach incompressible behavior while retaining thermodynamic information. In physically relevant regimes, acoustic fluctuations become asymptotically weak as the Mach number tends to zero, while the underlying mechanical flow structure remains essentially unchanged. This suggests that localized mechanical concentration may admit a thermodynamic representation through a suitable low-Mach realization.

The admissible low-Mach realizability condition introduced below should therefore be viewed as a concentration-representation principle rather than as an additional dynamical law. Its role is to identify concentration sequences that can be represented by an entropy-coercive, weakly compressible family whose thermodynamic traces remain compatible with incompressible concentration mechanisms.

This viewpoint is central to the present framework. The low-Mach realization serves as a bridge between localized mechanical concentration and thermo-acoustic quantities. Through this bridge, concentration information can be transferred from incompressible transfer dynamics to the thermo-acoustic admissibility framework. The resulting representation mechanism ultimately allows thermo-acoustic exclusion to be applied to persistent concentration scenarios.

Assumption 1 (Admissible Low-Mach Realizability). *A concentration sequence associated with (z_n, r_n) is said to satisfy admissible low-Mach realizability if there exists a family of thermodynamic profiles (ρ_n, u_n, T_n) such that:*

1. **(Kinematic Consistency)** *As $n \rightarrow \infty$, the velocity fields converge to the incompressible velocity field associated with the concentration sequence.*
2. **(Low-Mach Scaling)** *The local Mach number $M_n = |u_n|/c_{s,n}$.*

Satisfies $M_n \rightarrow 0$.

1. **(Thermodynamic Admissibility)** The family satisfies the admissibility conditions introduced in Section 2, including thermodynamic non-degeneracy, constitutive admissibility, entropy coercivity, and uniform local energy control.
2. **(Trace Compatibility)** The associated thermodynamic traces remain compatible with the localized mechanical concentration structure, so that concentration information is preserved under the low-Mach realization.

1.4. Main Idea

The central idea of the present work is to formulate a structural concentration-representation framework for persistent concentration sequences. A localized amplitude-modulation decomposition of coherent High–High transfers is introduced. For each shell index j and concentration center z_0 , the localized transfer activity satisfies

$$T_j^{HH}(z_0) = A_j(z_0) M_j(z_0), \quad (1)$$

Here, $T_j^{HH}(z_0)$ denotes localized High–High transfer activity, $A_j(z_0)$ denotes a localized transfer amplitude, and $M_j(z_0)$ denotes a localized modulation factor associated with phase organization.

The key observation is that a persistent concentration necessarily generates a persistent transfer-amplitude concentration. This necessity principle is established through the localized amplitude-modulation analysis developed in Sections 5 and 6.

Once a persistent amplitude concentration is present, a localized Littlewood–Paley bridge converts the transfer concentration into a localized mechanical concentration. The concentration-representation principle established in Section 3 then shows that a mechanical concentration generates a nontrivial thermodynamic trace through admissible low-Mach realizability.

The resulting thermo-acoustic activity is controlled by the thermo-acoustic admissibility framework of [23]. In particular, a thermo-acoustic concentration is excluded by the thermo-acoustic exclusion principle. Consequently, a persistent amplitude concentration is incompatible with the admissible low-Mach realizable framework.

The overall strategy therefore differs from traditional regularity approaches based on global norm estimates or continuation criteria. Rather than attempting to control all nonlinear interactions simultaneously, the present framework identifies the structural mechanism necessarily associated with a persistent concentration and follows its representation through successive concentration levels until a contradiction is obtained.

Accordingly, a dynamically sustained persistent concentration cannot occur within the admissible low-Mach realizable class considered in this work. The resulting structural closure theorem is therefore conditional upon admissible low-Mach realizability, while the universality of this realizability class remains an open problem.

1.5. Main Results

The present work is motivated by, but is mathematically distinct from, the structural reduction framework developed in [22]. Since the relationship between the two works plays an important role in the interpretation of the present results, we summarize the distinction explicitly.

The principal result of [22] is a structural reduction theorem for scale-critical transfers. Using Fourier–helical localization, triadic decomposition, coherent-transfer classification, and residence-time compression estimates, that work showed that most nonlinear transfer pathways become asymptotically perturbative at sufficiently small scales. As a consequence, coherent same-scale High–High interactions were identified as the only remaining non-perturbative candidates capable of supporting dynamically significant scale-critical transfers.

The present paper does not primarily aim to further refine the transfer-reduction argument itself. Rather, it starts from the reduced transfer structure obtained in [22] and investigates the consequences of a persistent concentration within that reduced setting. The focus is therefore shifted from transfer reduction to concentration representation and structural closure.

More specifically, the principal new ingredients of the present work are:

1. **Admissible low-Mach realizability and concentration representation.**

Sections 1–3 introduce a new realization framework linking localized mechanical concentration to thermo-acoustic traces through entropy-coercive low-Mach representations.

2. **The Low-Mach Inheritance Principle.**

Mechanical concentration is shown to generate a nontrivial thermodynamic trace under admissible low-Mach realizability.

3. **A localized amplitude-modulation framework.**

Sections 5–6 introduce localized decomposition

$$|T_j^{HH,loc}| = A_j M_j,$$

together with the localized structural necessity principle showing that a persistent concentration necessarily generates a localized transfer-amplitude concentration.

4. **A localized Littlewood–Paley bridge.**

Section 7 and Appendix C establish a quantitative connection between localized transfer-amplitude concentration and localized mechanical concentration.

5. **A thermo-acoustic structural closure mechanism.**

Combining concentration representation, low-Mach inheritance, the localized LP bridge, and thermo-acoustic exclusion yields the conditional structural closure theorem.

In summary, [22] identifies the transfer mechanisms capable of sustaining concentration, whereas the present work develops a concentration-representation and thermo-acoustic closure framework for excluding the resulting concentration scenarios within the admissible low-Mach realizable class.

Accordingly, [22] provides the transfer-reduction theory and the supplementary coherent-transfer framework summarized in Appendices D and E, whereas the concentration-representation theory, low-Mach inheritance framework, localized structural necessity principle, localized LP bridge, and structural closure theorem are new contributions of the present paper.

Furthermore, the coherent-transfer exclusion mechanism inherited from [22] is not used in the proof of Theorem 1. Its role in the present manuscript is supplementary and comparative. The proof of the structural closure theorem relies instead on the concentration-representation framework and the thermo-acoustic exclusion mechanism.

The principal result of this paper is a conditional structural closure theorem for admissible concentration sequences. More precisely, the theorem applies only to concentration sequences belonging to the admissible low-Mach realizable class introduced in Section 3.

The result should therefore be interpreted as a conditional structural closure theorem within the admissible low-Mach realizable class, rather than as an unconditional regularity theorem for arbitrary solutions of the three-dimensional incompressible Navier–Stokes equations. An additional coherent-transfer exclusion result inherited from [22] is summarized in Appendix E, but it is not required for the proof of the structural closure theorem.

Furthermore, admissible low-Mach realizability is not introduced as a purely abstract assumption. As shown in Section 3 and Appendix B, every concentration sequence

admitting an admissible entropy-coercive, weakly compressible approximation automatically belongs to the admissible low-Mach realizable class. In this sense, the realizability problem is reduced to verifiable approximation properties rather than to an independent admissibility postulate.

Theorem 1 (Preview of the Main Result). *Let (z_n, r_n) be a concentration sequence belonging to the admissible low-Mach realizable class introduced in Section 3. Then, a dynamically sustained persistent concentration cannot occur. Equivalently,*

$$\text{Persistent Concentration} \Rightarrow \perp. \quad (2)$$

The proof combines five structural ingredients:

- The structural reduction of a nonlinear transfer [22];
- Concentration representation through admissible low-Mach realizability;
- Thermo-acoustic concentration exclusion [23];
- A localized amplitude concentration generated by persistent transfer activity;
- A localized amplitude-modulation structural framework.

The role of [22] is to identify coherent same-scale High–High transfer as the only remaining non-perturbative concentration mechanism and to provide the coherent-transfer analysis summarized in Appendix E. These coherent-transfer results are supplementary and are not used in the proof of the structural closure theorem. Historically, the coherent-transfer framework motivated the localized amplitude-modulation analysis developed in the present work. However, the structural closure theorem established here does not rely on the coherent-transfer exclusion mechanism itself. Instead, closure is obtained through the localized structural necessity principle together with the thermo-acoustic exclusion framework. Consequently, admissible low-Mach realizability is the sole non-universal assumption entering the proof of Theorem 1, whereas Assumption A1 appears only in the supplementary coherent-transfer results recorded in Appendix E.

A key consequence of the present reformulation is that the proof of Theorem 1 does not require an independent exclusion of the non-dispersive modulation branch. The localized structural necessity principle established in Section 6 shows that a persistent concentration necessarily generates an amplitude concentration. Since this necessary branch is excluded in Section 7, structural closure follows directly. Consequently, the coherent-transfer analysis recorded in Appendix E is supplementary and is not logically required for the proof of the structural closure theorem.

The role of [23] is to provide the thermo-acoustic admissibility framework, entropy-coercive concentration control, and thermo-acoustic exclusion mechanism used in Sections 2, 3, and 7.

The first new contribution of the present work is the introduction of an admissible low-Mach realizability framework together with a concentration-representation principle. Lemma 1 establishes a conditional concentration-representation mechanism linking localized mechanical concentration to thermodynamic traces. Section 3.3 further shows that admissible low-Mach realizability should be viewed as a realizability class rather than a universality assumption. Moreover, Proposition 1 and Proposition A1 provide constructive realizability criteria showing that every concentration sequence admitting an admissible entropy-coercive, weakly compressible approximation automatically belongs to the admissible low-Mach realizable class.

The second new contribution is the construction of a unified closure mechanism connecting transfer concentration, localized mechanical concentration, thermodynamic traces, and thermo-acoustic concentration control through a single representation framework.

The third new contribution is the localized amplitude-modulation analysis of coherent High–High transfers, which yields the localized structural necessity principle showing that a persistent concentration necessarily generates an amplitude concentration. This reduces structural closure to the exclusion of a single thermo-acoustic concentration branch. For completeness, Appendix E records an additional coherent-transfer exclusion mechanism inherited from [22] under the averaged phase-drift non-degeneracy assumption.

Together, these ingredients provide a unified local closure principle linking transfer reduction, concentration representation, thermo-acoustic admissibility, and concentration-sequence dynamics within a single framework.

The theorem should nevertheless be interpreted as a conditional structural closure theorem. The inheritance mechanism underlying the concentration-representation principle is established in the present paper. However, the universality of admissible entropy-coercive low-Mach realizations remains open. Consequently, the sole non-universal assumption underlying the structural closure theorem is admissible low-Mach realizability. No coherent-transfer assumption enters the proof of Theorem 1. Consequently, the scope of the theorem is restricted to the admissible low-Mach realizable class defined in Section 3.

The remainder of the paper is organized as follows. Section 2 introduces the thermo-acoustic admissibility framework. Section 3 develops the compactness theory, admissible low-Mach realizability framework, concentration-representation principle, realizability classes, and inheritance mechanism. Section 4 summarizes the structural reduction of scale-critical transfers established in [22]. Sections 5 and 6 introduce the localized amplitude-modulation framework and establish the localized structural necessity principle. Section 7 excludes the thermo-acoustic concentration branch. Section 8 discusses the coherent-transfer interpretation of persistent modulation. Section 9 proves the conditional structural closure theorem. Sections 10 and 11 discuss the scope, conditionality, limitations, and implications of the theory, with particular emphasis on admissible low-Mach realizability.

For completeness, the principal technical ingredients are collected in the appendices. Appendix A develops the thermo-acoustic ε -regularity and exclusion framework inherited from [23]. Appendix B records the compactness theory, constructive realizability criteria, concentration-representation mechanism, and low-Mach inheritance proofs. Appendix C establishes the localized Littlewood–Paley estimates and shell-uniform localization bounds required for the LP bridge. Appendix D summarizes the High–High transfer reduction theory inherited from [22]. Appendix E summarizes the coherent-transfer theory inherited from [22], including the modulation–coherence link, residence-time compression, cumulative transfer decay, and a supplementary conditional coherent-transfer exclusion result.

2. Thermo-Acoustic Admissibility Framework

2.1. Admissible Thermodynamic Class

The structural closure theory developed in this paper relies on a thermo-acoustic admissibility framework originating from compressible Navier–Stokes–Fourier dynamics [23–30]. The purpose of this framework is to identify a class of flows for which concentration phenomena remain compatible with thermodynamic consistency, constitutive non-degeneracy, and entropy production.

Let $\rho(x, t)$ denote the density, $u(x, t)$ the velocity field, $T(x, t)$ the absolute temperature, and $p(x, t)$ the pressure. These quantities are considered on a space–time cylinder Q . An admissible thermodynamic class is defined by the following conditions. (A1) Thermodynamic non-degeneracy.

There exist positive constants c_ρ , C_ρ , c_T , and C_T such that

$$c_\rho \leq \rho(x, t) \leq C_\rho, \quad (3)$$

and

$$c_T \leq T(x, t) \leq C_T. \quad (4)$$

Here, the constants are independent of the concentration scale. (A2) Constitutive admissibility. The viscosity coefficients μ and λ , and the thermal conductivity κ , satisfy

$$\mu > 0, \quad (5)$$

$$2\mu + 3\lambda \geq 0, \quad (6)$$

and

$$\kappa > 0. \quad (7)$$

These assumptions exclude degeneracy of the viscous and thermal diffusion mechanisms. (A3) Uniform local entropy-production bound. The entropy-production density D_{ent} is locally integrable and satisfies a uniform bound on every admissible cylinder Q . (A4) Entropy coercivity. There exists a positive constant c_{ent} such that

$$D_{\text{ent}} \geq c_{\text{ent}}(|D(u)|^2 + |\nabla \log T|^2 + |\nabla \log \rho|^2). \quad (8)$$

Here, $D(u)$ denotes the symmetric strain-rate tensor associated with the velocity field u . (A5) Uniform local energy control. The mechanical energy of the velocity field is locally bounded uniformly on admissible cylinders. These assumptions are consistent with the standard compressible Navier–Stokes–Fourier framework and with entropy-based compactness methods for compressible flow [23–33].

2.2. Entropy Production Structure

The entropy production structure provides the principal dissipative mechanism of the thermo-acoustic framework. For an admissible compressible flow, the local entropy balance is written as

$$\partial_t(\rho s) + \nabla \cdot (\rho s u) = D_{\text{ent}}, \quad (9)$$

where s denotes the specific entropy. The entropy-production density D_{ent} is non-negative and records irreversible thermodynamic dissipation. In the admissible class, condition (8) implies that D_{ent} controls the mechanical strain, the temperature-gradient contribution, and the density-gradient contribution. Thus, if a concentration sequence produces nontrivial mechanical or thermodynamic gradients, then the entropy-production density cannot remain asymptotically negligible. This observation is the first link between mechanical concentration and thermo-acoustic concentration.

2.3. Thermo-Acoustic Concentration Functional

Let $z = (x, t)$ denote a space–time point and let $Q_r(z)$ denote the parabolic cylinder centered at z with radius r . Let F_{rel} denote the relative free-energy density with respect to a fixed admissible reference state. The localized thermo-acoustic concentration functional is defined by

$$H(z, r) = r^{-3} \int_{Q_r(z)} F_{\text{rel}} \, dx \, dt + r^{-1} \int_{Q_r(z)} D_{\text{ent}} \, dx \, dt. \quad (10)$$

The first term measures localized thermodynamic free-energy concentration. The second term measures localized entropy production. We also introduce the thermodynamic trace functional

$$\Theta(z, r) = r^{-1} \int_{Q_r(z)} D_{\text{ent}} \, dx \, dt. \quad (11)$$

The quantity $\Theta(z, r)$ measures the entropy-production component of the thermo-acoustic activity. By definition,

$$\Theta(z, r) \leq H(z, r). \quad (12)$$

The functional $H(z, r)$ will be used as the thermo-acoustic concentration-control quantity throughout the paper.

2.4. Entropy-Coercive Control

The entropy-coercive inequality (8) implies that $H(z, r)$ controls the principal concentration-producing gradients. Indeed, combining (8), (10), and (11), we obtain

$$H(z, r) \geq c r^{-1} \int_{Q_r(z)} (|D(u)|^2 + |\nabla \log T|^2 + |\nabla \log \rho|^2) dx dt, \quad (13)$$

for some positive constant c independent of r . Thus, the smallness of $H(z, r)$ forces simultaneous smallness of the mechanical strain, the temperature-gradient contribution, and the density-gradient contribution on $Q_r(z)$. Conversely, a persistent concentration in any of these entropy-coercive channels forces nonvanishing thermo-acoustic activity.

In particular, if a sequence of scales $r_n \rightarrow 0$ satisfies

$$\limsup_{n \rightarrow \infty} r_n^{-1} \int_{Q_{r_n}(z_0)} |D(u)|^2 dx dt > 0, \quad (14)$$

then (13) implies

$$\limsup_{n \rightarrow \infty} H(z_0, r_n) > 0. \quad (15)$$

This implication is the basic mechanism by which mechanical concentration activates thermo-acoustic concentration. The compactness tools underlying this mechanism are based on entropy bounds, weak lower semi continuity, and Aubin–Lions–Simon compactness [31–33].

2.5. Thermo-Acoustic Exclusion Principle

The thermo-acoustic admissibility theory developed in [23] yields an exclusion principle for concentration sequences with sufficiently small thermo-acoustic activity.

Theorem 2 (Thermo-Acoustic Exclusion Principle). *There exists a positive threshold ε_* such that if*

$$H(z_0, 2r) < \varepsilon_*, \quad (16)$$

then no admissible persistent concentration sequence can occur inside $Q_r(z_0)$.

Proof. By (13), the smallness condition (16) controls the entropy-coercive gradient contributions on $Q_r(z_0)$. Standard localization, compactness, and blow-up normalization arguments then exclude the formation of a nontrivial concentration profile in $Q_r(z_0)$. This is precisely the thermo-acoustic exclusion mechanism established in [23]. Detailed ε -regularity, blow-up normalization, compactness, Campanato decay, and exclusion arguments are summarized in Appendix A1–A11. Therefore, an admissible persistent concentration cannot occur under (16). $\square$

Theorem 2 will be used later to exclude the amplitude-concentration branch of the localized structural alternative.

Remark 1 (Relation to Classical ε -Regularity Criteria). *The thermo-acoustic ε -regularity framework developed here should be viewed as complementary to classical ε -regularity theories for the Navier–Stokes equations rather than as a replacement for them.*

The classical ε -regularity theory of Caffarelli, Kohn, and Nirenberg is formulated in terms of scale-invariant velocity and gradient quantities and establishes regularity through the smallness of localized mechanical concentration measures. By contrast, the thermo-acoustic ε -regularity criterion is formulated through the thermo-acoustic concentration functional $H(z, r)$, which combines localized free-energy concentration and entropy production.

The two frameworks therefore share a common structural philosophy: both exclude concentration through the smallness of a scale-localized control quantity. However, the quantities being controlled are fundamentally different. Classical ε -regularity theories act directly on mechanical variables, whereas the present framework operates through a thermodynamic representation of concentration.

From this perspective, the thermo-acoustic concentration functional may be interpreted as a thermodynamic counterpart of classical concentration-control quantities. The present approach therefore supplements existing ε -regularity theories by providing a concentration-representation mechanism linking mechanical concentration to thermodynamic traces and entropy production.

3. Low-Mach Inheritance Framework

3.1. Admissible Low-Mach Families

The thermo-acoustic admissibility framework introduced in Section 2 provides a concentration-control mechanism for compressible Navier–Stokes–Fourier dynamics. To connect this mechanism with the incompressible concentration problem, we introduce a class of admissible low-Mach families.

Definition 1 (Admissible Low-Mach Family). *Let ρ_n, u_n, T_n be a sequence of admissible compressible flows satisfying conditions (A1)–(A5) of Section 2. The sequence is called an admissible low-Mach family if the associated local Mach numbers satisfy*

$$M_n \rightarrow 0, \quad (17)$$

while the admissibility conditions remain uniform with respect to n . The objective of the present framework is not merely to study the incompressible limit itself [24–30], but rather to determine which concentration quantities remain stable under the low-Mach transition.

Remark 2 (Scope of the Definition). *The admissible low-Mach realizability condition should be interpreted as a representation condition rather than a universality assumption. The condition identifies the class of concentration sequences admitting an entropy-coercive weakly compressible low-Mach representation. Examples and realization criteria are discussed in Section 3.3 and Appendix B.4. The present paper does not assume that every incompressible concentration sequence automatically belongs to this class.*

3.2. Conditional Low-Mach Realizability and Concentration Representation

Lemma 1 (Conditional Low-Mach Realizability and Concentration Representation). *Let u be an incompressible velocity field satisfying*

$$\nabla \cdot u = 0. \quad (18)$$

Assume that there exists a family of weakly compressible lifts $(\rho_\varepsilon, u_\varepsilon, T_\varepsilon)$ such that

$$u_\varepsilon \rightarrow u \text{ strongly in } L^2_{\text{loc}}(Q), \quad (19)$$

$$D(u_\varepsilon) \rightarrow D(u) \text{ strongly in } L^2_{\text{loc}}(Q), \quad (20)$$

$$\rho_\varepsilon \rightarrow \rho_0 > 0, \quad T_\varepsilon \rightarrow T_0 > 0, \quad (21)$$

and

$$M_{\text{loc}}^{(\varepsilon)}(z, r) \rightarrow 0 \quad (22)$$

uniformly on admissible cylinders. Suppose further that

$$0 < c_\rho \leq \rho_\varepsilon \leq C_\rho, \quad 0 < c_T \leq T_\varepsilon \leq C_T, \quad (23)$$

$$\mu > 0, 2\mu + 3\lambda \geq 0, \quad \kappa > 0, \quad (24)$$

and

$$D_{\text{ent}}^{(\varepsilon)} \geq c_{\text{ent}}(|D(u_\varepsilon)|^2 + |\nabla \log T_\varepsilon|^2 + |\nabla \log \rho_\varepsilon|^2), \quad (25)$$

with a constant $c_{\text{ent}} > 0$ independent of ε . Let (z_n, r_n) be a concentration sequence associated with u . Then, every concentration sequence represented by the lifted family belongs to the admissible low-Mach realizable class. Moreover,

$$\limsup_{n \rightarrow \infty} E(z_n, r_n) > 0 \quad (26)$$

implies

$$\limsup_{n \rightarrow \infty} \Theta(z_n, r_n) > 0. \quad (27)$$

Proof. Conditions (23)–(25) imply thermodynamic non-degeneracy, constitutive admissibility, and entropy coercivity uniformly with respect to ε . Condition (22) yields

$$M_{\text{loc}}^{(\varepsilon)}(z, r) \rightarrow 0, \quad (28)$$

and therefore, the lifted family lies in the admissible low-Mach regime. Consequently, the family $(\rho_\varepsilon, u_\varepsilon, T_\varepsilon)$ forms an admissible low-Mach family. By (20), for every admissible cylinder $Q_r(z)$,

$$\int_{Q_r(z)} |D(u_\varepsilon)|^2 \, dx \, dt \rightarrow \int_{Q_r(z)} |D(u)|^2 \, dx \, dt. \quad (29)$$

Since the mechanical concentration functional is defined by

$$E(z, r) = r^{-1} \int_{Q_r(z)} |D(u)|^2 \, dx \, dt, \quad (30)$$

it follows that

$$E_\varepsilon(z, r) \rightarrow E(z, r). \quad (31)$$

Hence,

$$\limsup_{n \rightarrow \infty} E(z_n, r_n) > 0 \quad (32)$$

implies

$$\limsup_{n \rightarrow \infty} E_\varepsilon(z_n, r_n) > 0 \quad (33)$$

for sufficiently small ε . Furthermore, the entropy-coercive estimate (25) yields

$$D_{\text{ent}}^{(\varepsilon)} \geq c_{\text{ent}} |D(u_\varepsilon)|^2. \quad (34)$$

Integrating over $Q_{r_n}(z_n)$ and using the definitions of Θ_ε and E_ε , we obtain

$$\Theta_\varepsilon(z_n, r_n) \geq c_{\text{ent}} E_\varepsilon(z_n, r_n). \quad (35)$$

Taking the limit superior and using (31), we obtain

$$\limsup_{n \rightarrow \infty} \Theta(z_n, r_n) \geq c_{\text{ent}} \limsup_{n \rightarrow \infty} E(z_n, r_n). \quad (36)$$

Combining (26) and (36) proves (27). Therefore, every concentration sequence represented by the lifted family belongs to the admissible low-Mach realizable class and necessarily possesses a nontrivial thermodynamic trace. $\square$

A constructive realizability criterion based on admissible entropy-coercive, weakly compressible approximations is provided in Appendix B.4.

3.3. Realizability Classes and Scope of Applicability

The admissible low-Mach realizability condition introduced in Definition 1 serves as a structural representation principle for concentration sequences. Its purpose is not to postulate an abstract admissible class, but to identify the class of concentration sequences for which localized mechanical concentration admits a thermodynamic representation through an entropy-coercive, weakly compressible approximation. Let u be an incompressible velocity field satisfying

$$\nabla \cdot u = 0 \quad (37)$$

and let (ρ_n, u_n, T_n) be a weakly compressible approximation family. Assume that

$$u_n \rightarrow u \text{ strongly in } L^2_{\text{loc}}, \quad (38)$$

and

$$M_{\text{loc},n}(z, r) \rightarrow 0. \quad (39)$$

Suppose further that the thermodynamic bounds

$$0 < c_\rho \leq \rho_n \leq C_\rho \quad (40)$$

and

$$0 < c_T \leq T_n \leq C_T \quad (41)$$

hold uniformly in n , together with the constitutive admissibility conditions

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0, \quad \kappa > 0. \quad (42)$$

Assume moreover that the entropy-production density satisfies the scale-uniform coercive estimate

$$D_{\text{ent},n} \geq c_{\text{ent}}(|D(u_n)|^2 + |\nabla \log T_n|^2 + |\nabla \log \rho_n|^2), \quad (43)$$

where the constant $c_{\text{ent}} > 0$ is independent of both n and the concentration scale. Under these assumptions,

$$r^{-1} \int_{Q_r(z)} |D(u_n)|^2 \, dxdt \leq c_{\text{ent}}^{-1} r^{-1} \int_{Q_r(z)} D_{\text{ent},n} \, dxdt. \quad (44)$$

Introducing the thermodynamic trace functional

$$\Theta_n(z, r) = r^{-1} \int_{Q_r(z)} D_{\text{ent},n} \, dxdt, \quad (45)$$

we obtain

$$r^{-1} \int_{Q_r(z)} |D(u_n)|^2 \, dxdt \leq c_{\text{ent}}^{-1} \Theta_n(z, r). \quad (46)$$

Consequently, whenever

$$\limsup_{n \rightarrow \infty} M(z_n, r_n) > 0, \quad (47)$$

the represented concentration sequence necessarily satisfies

$$\limsup_{n \rightarrow \infty} \Theta_n(z_n, r_n) > 0. \quad (48)$$

Thus, admissible low-Mach realizability is precisely the structural condition ensuring that localized mechanical concentration possesses a nontrivial thermodynamic trace.

Proposition 1 (Admissible Approximation Criterion). *Let u be an incompressible velocity field and let (ρ_n, u_n, T_n) be a weakly compressible approximation family satisfying (38)–(43). Then every concentration sequence represented by this family belongs to the admissible low-Mach realizable class of Definition 1. Moreover,*

$$\limsup_{n \rightarrow \infty} M(z_n, r_n) > 0 \quad (49)$$

implies

$$\limsup_{n \rightarrow \infty} \Theta_n(z_n, r_n) > 0. \quad (50)$$

Proof. Conditions (38)–(43) coincide with the admissibility assumptions required in Lemma 1. Therefore, the approximation family constitutes an admissible low-Mach family.

Applying Lemma 1 to the represented concentration sequence, we obtain

$$\limsup_{n \rightarrow \infty} M(z_n, r_n) > 0 \Rightarrow \limsup_{n \rightarrow \infty} \Theta_n(z_n, r_n) > 0. \quad (51)$$

The implication (51) is precisely statement (50). Therefore, the represented concentration sequence is admissible low-Mach realizability. $\square$

Remark 3 (Non-Emptiness of the Realizable Class). *Proposition 1 shows that admissible low-Mach realizability is not introduced as a purely formal assumption. Indeed, every entropy-coercive, weakly compressible approximation satisfying (38)–(43) automatically generates an admissible low-Mach realization. Consequently, the admissible low-Mach realizable class contains all concentration sequences represented by admissible entropy-coercive, weakly compressible approximation families. Furthermore, if $Q_r(z) \subset Q$, then the restricted approximation family remains admissible on $Q_r(z)$ with the same constants c_ρ , C_ρ , c_T , C_T , and c_{ent} . Hence, admissible low-Mach realizability is stable under localization and is compatible with the local nature of concentration analysis.*

Scope of the Assumption

The scope of the present assumption should nevertheless be stated explicitly. The present paper does not claim that every Leray–Hopf weak solution, every potential blow-up candidate, or every scale-critical concentration sequence automatically admits an admissible low-Mach realization. The inheritance mechanism itself is established by Lemma 1, Theorem 3, and Proposition 1. The remaining open problem is the universality of admissible entropy-coercive, weakly compressible approximations. The logical content of the present framework may therefore be summarized as

Mechanical Concentration + Admissible Low-Mach Realizability

$\Rightarrow$ Thermodynamic Trace $\Rightarrow$ Thermo-Acoustic Exclusion.

Equivalently,

$$\limsup_{n \rightarrow \infty} M(z_n, r_n) > 0 \Rightarrow \limsup_{n \rightarrow \infty} \Theta_n(z_n, r_n) > 0 \Rightarrow \limsup_{n \rightarrow \infty} H(z_n, r_n) > 0. \quad (52)$$

Therefore, Theorem 8 excludes precisely those persistent concentration mechanisms that remain compatible with an admissible entropy-coercive low-Mach thermodynamic representation. Whether every potentially singular incompressible concentration sequence possesses such a representation remains an open problem.

Proposition 2 (Non-Emptiness of the Admissible Low-Mach Family Class). *The admissible low-Mach family class introduced in Definition 1 is nonempty. Let the spatial domain be the three-dimensional torus, and let $Q = \mathbb{T}^3 \times (0, T)$. Fix constants*

$$\rho_0 > 0, T_0 > 0, \quad (53)$$

and let $U_0 \in \mathbb{R}^3$ be a constant velocity vector. For each $n \in \mathbb{N}$, define

$$\rho_n(x, t) \equiv \rho_0, u_n(x, t) \equiv U_0, T_n(x, t) \equiv T_0. \quad (54)$$

Then

$$\nabla \cdot u_n = 0, D(u_n) = 0, \nabla \log \rho_n = 0, \nabla \log T_n = 0. \quad (55)$$

Assume the constitutive coefficients satisfy

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0, \quad \kappa > 0. \quad (56)$$

Define the entropy-production density by

$$D_{\text{ent},n} \equiv 0. \quad (57)$$

Then the entropy-coercive condition is satisfied, since

$$D_{\text{ent},n} = 0 \geq c_{\text{ent}}(|D(u_n)|^2 + |\nabla \log T_n|^2 + |\nabla \log \rho_n|^2) = 0. \quad (58)$$

Let the associated acoustic speeds satisfy

$$c_{s,n} \rightarrow \infty. \quad (59)$$

Then the local Mach numbers satisfy

$$M_n = \frac{|U_0|}{c_{s,n}} \rightarrow 0. \quad (60)$$

Therefore, the sequence (ρ_n, u_n, T_n) is an admissible low-Mach family in the sense of Definition 1. Consequently, the admissible low-Mach family class is nonempty:

$$\mathcal{A}_{\text{LM}} \neq \emptyset. \quad (61)$$

Proof. The fields defined in (54) are smooth, spatially periodic, and satisfy the incompressibility relation and vanishing gradient identities in (55). The thermodynamic non-degeneracy assumptions hold with constants depending only on ρ_0 and T_0 . The constitutive admissibility assumptions are imposed in (56). Since all gradient contributions in the entropy-coercive condition vanish identically, (58) holds for any $c_{\text{ent}} > 0$. The uniform local entropy-production and local energy bound also hold, because

$$\int_Q D_{\text{ent},n} \, dxdt = 0, \quad (62)$$

and

$$\int_Q |u_n|^2 \, dxdt = |U_0|^2 |Q| < \infty. \quad (63)$$

Finally, (59) implies the low-Mach condition (60). Hence, all the conditions in Definition 1 are satisfied. Therefore, the admissible low-Mach family class is nonempty. $\square$

Remark 4. Proposition 2 proves only the mathematical non-emptiness of the admissible low-Mach family class. It does not imply that every incompressible concentration sequence admits an admissible low-Mach realization, nor does it imply that nontrivial persistent concentration sequences exist in the admissible class. The distinction is

$$\mathcal{A}_{\text{LM}} \neq \emptyset \quad (64)$$

whereas the universality problem would require

$$\mathcal{R}_{\text{all}} \subset \mathcal{R}_{\text{adm}}. \quad (65)$$

The present proposition establishes (64), but not (65). Therefore, the admissible low-Mach family class is mathematically nonempty, while the universality of admissible low-Mach realizability remains an open problem.

3.4. Compactness Structure

The admissibility assumptions imply a compactness structure sufficient for concentration analysis. From the entropy-coercive estimate (8), there exists a constant C independent of n such that

$$\int_Q |D(u_n)|^2 dx dt \leq C. \quad (66)$$

Together with the local energy bound, this yields

$$u_n \rightharpoonup u \text{ weakly in } L^2_{\text{loc}}(0, T; H^1_{\text{loc}}). \quad (67)$$

By the Aubin–Lions–Simon compactness theorem [31,34], after extraction of a subsequence,

$$u_n \rightarrow u \text{ strongly in } L^2_{\text{loc}}. \quad (68)$$

The strong convergence property (68) is the key compactness mechanism underlying the inheritance principle developed below. The compactness argument and Aubin–Lions framework are summarized in Appendix B.1.

3.5. Mechanical Concentration Functional

To quantify localized mechanical concentration, we introduce a scale-normalized concentration functional. For a space–time point $z = (x, t)$ and a parabolic cylinder $Q_r(z_0)$, define the mechanical concentration functional by

$$E(z, r) = r^{-1} \int_{Q_r(z)} |D(u)|^2 dx dt. \quad (69)$$

The quantity $E(z, r)$ measures localized mechanical activity associated with the strain-rate field. A persistent concentration sequence (z_0, r_n) with $r_n \rightarrow 0$ is characterized by the condition

$$\limsup_{n \rightarrow \infty} E(z_0, r_n) > 0. \quad (70)$$

Thus, $E(z, r)$ provides a localized indicator of a mechanically sustained concentration. The persistence of concentration under strong convergence is discussed in Appendix B.2.

3.6. Thermodynamic Trace Functional

The entropy-production density introduced in Section 2 generates a corresponding thermodynamic concentration measure. Define the thermodynamic trace functional by

$$\Theta(z, r) = r^{-1} \int_{Q_r(z)} D_{\text{ent}} dx dt. \quad (71)$$

The quantity $\Theta(z, r)$ measures the thermo-acoustic footprint produced by concentration. Because entropy production records irreversible thermodynamic activity, nonvanishing values of $\Theta(z, r)$ indicate that concentration leaves a detectable thermodynamic trace. The purpose of the inheritance principle is to connect the mechanical concentration functional $E(z, r)$ with the thermodynamic trace functional $\Theta(z, r)$. The corresponding thermodynamic-trace estimate is recorded in Appendix B.3.

Physical Interpretation

From a physical viewpoint, the thermodynamic trace functional $\Theta(z, r)$ measures the irreversible thermodynamic response generated by a localized mechanical concentration.

When strong localized strain develops, entropy production, temperature gradients, and density gradients become concentrated through the entropy-coercive mechanism. The thermodynamic trace records this accumulated thermo-acoustic activity at the scale r .

Thus, $\Theta(z, r)$ should not be interpreted merely as an auxiliary mathematical functional. Rather, it represents the thermodynamic footprint left by mechanically

concentrated flow structures. In the present framework, localized mechanical concentration and thermo-acoustic activity are therefore viewed as two complementary representations of the same underlying concentration mechanism.

The low-Mach inheritance principle established below shows that persistent mechanical concentration necessarily generates a nontrivial thermodynamic trace. Consequently, $\Theta(z, r)$ serves as the key bridge connecting incompressible concentration dynamics with the thermo-acoustic admissibility framework.

3.7. Low-Mach Inheritance Principle

We now establish the fundamental bridge between mechanical concentration and thermo-acoustic activity.

Theorem 3 (Low-Mach Inheritance Principle). *Let (ρ_n, u_n, T_n) be an admissible low-Mach family. Suppose there exists a concentration sequence (z_0, r_n) such that*

$$\limsup_{n \rightarrow \infty} E(z_0, r_n) > 0. \quad (72)$$

Then

$$\limsup_{n \rightarrow \infty} \Theta(z_0, r_n) > 0. \quad (73)$$

Proof. Assume, on the contrary, that

$$\limsup_{n \rightarrow \infty} \Theta(z_0, r_n) = 0. \quad (74)$$

Then the entropy-production density becomes asymptotically negligible on the concentration sequence. By the entropy-coercive estimate (8),

$$D_{\text{ent}} \geq c_{\text{ent}} \|D(u_n)\|^2, \quad (75)$$

for some positive constant c_{ent} . Combining (74) and (75) yields

$$\limsup_{n \rightarrow \infty} E(z_0, r_n) = 0, \quad (76)$$

which contradicts (72). Therefore, (73) must hold. $\square$

A complementary inheritance proof is recorded in Appendix B.5. Theorem 3 states that a persistent mechanical concentration cannot disappear in the low-Mach limit. Instead, it necessarily generates nontrivial thermo-acoustic activity.

3.8. Inherited Concentration Control

The inheritance principle immediately yields a control statement for the thermo-acoustic concentration functional introduced in Section 2.

Corollary 1 (Inherited Concentration Control). *Let (ρ_n, u_n, T_n) be an admissible low-Mach family. If*

$$\limsup_{n \rightarrow \infty} E(z_0, r_n) > 0, \quad (77)$$

then

$$\limsup_{n \rightarrow \infty} H(z_0, r_n) > 0. \quad (78)$$

Proof. By the definition of the thermo-acoustic concentration functional,

$$\Theta(z, r) \leq H(z, r). \quad (79)$$

Theorem 3 implies

$$\limsup_{n \rightarrow \infty} \theta(z_0, r_n) > 0. \quad (80)$$

Combining (79) and (80) immediately yields (78). $\square$

Corollary 1 provides the mechanism through which a mechanically sustained concentration generates a thermo-acoustic concentration. This result will play a central role in the exclusion of Branch I in Section 7. The concentration-transfer mechanism is summarized in Appendix B.6.

4. Structural Reduction of Scale-Critical Transfer

The purpose of this section is to summarize the structural reduction theory established in [22]. For completeness, the principal reduction lemmas and transfer-core characterization are summarized in Appendix D. The central result of that theory is that, among all the nonlinear transfer mechanisms appearing in the three-dimensional incompressible Navier–Stokes equations, only a highly restricted subset of coherent High–High interactions can remain dynamically relevant near a potential concentration event.

This reduction plays a fundamental role in the present paper because it identifies the only remaining transfer mechanism that may support a persistent concentration.

4.1. Dyadic Transfer Decomposition

Let $u = \sum_j u_j$ denote the Littlewood–Paley decomposition of the velocity field, where u_j is localized near the dyadic frequency $\lambda_j = 2^j$ [35,36]. Let T_j denote the nonlinear transfer activity associated with shell j . By Bony’s paraproduct decomposition [35,36], the transfer activity admits the decomposition

$$T_j = T_j^{LL} + T_j^{LH} + T_j^{HH}. \quad (81)$$

Here, T_j^{LL} represents Low–Low interactions, T_j^{LH} represents Low–High interactions, and T_j^{HH} represents High–High interactions. The decomposition (81) separates perturbative interactions from potentially non-perturbative transfer mechanisms.

4.2. Perturbative Transfer Channels

The structural analysis of [22] shows that the Low–Low and Low–High channels are perturbative in the scale-critical regime. More precisely, there exists a constant C independent of j such that

$$\|T_j^{LL}\| \leq C \lambda_j^{-1}, \quad (82)$$

and

$$\|T_j^{LH}\| \leq C \lambda_j^{-1}. \quad (83)$$

Consequently,

$$T_j^{LL} \rightarrow 0, \quad T_j^{LH} \rightarrow 0, \quad j \rightarrow \infty. \quad (84)$$

Therefore, neither the Low–Low channel nor the Low–High channel can support a scale-critical concentration. All potentially dangerous transfer mechanisms must therefore be contained in the High–High channel [22].

4.3. Reduction to High–High Transfer

The perturbative elimination of the Low–Low and Low–High channels yields the following reduction theorem.

Theorem 4 (Reduction to High–High Transfer). *Suppose that a persistent concentration sequence exists. Then, there exists a sequence of shell indices $j_n \rightarrow \infty$ such that*

$$\limsup_{n \rightarrow \infty} |T_n^{HH}| > 0. \quad (85)$$

Proof. If

$$\limsup_{j \rightarrow \infty} |T_j^{HH}| = 0, \quad (86)$$

then (84) implies

$$\limsup_{j \rightarrow \infty} |T_j| = 0. \quad (87)$$

In this case, no nonlinear transfer mechanism remains available to sustain concentration at arbitrarily small scales. This contradicts the persistence hypothesis. Therefore, (87) follows. $\square$

The detailed reduction argument is summarized in Appendix D.2–D.7. Theorem 4 shows that any admissible persistent concentration sequence must necessarily be supported by High–High transfers.

4.4. Coherent Transfer Class

Although High–High interactions remain after Theorem 4, not all such interactions contribute equally to transfer amplification. The analysis of [22] identifies a coherent transfer class consisting of triadic interactions whose relative phase drift remains sufficiently small over a finite time interval. Let G_j denote the coherent transfer activity associated with shell j . The key result of [22] is that a persistent High–High transfer necessarily implies concentration within this coherent class.

Furthermore, a coherent transfer is governed by two structural mechanisms:

1. Residence-time compression.
2. Cumulative transfer decay.

Residence-time compression restricts the duration over which coherent amplification can persist. Cumulative transfer decay shows that the total transfer accumulated over coherent intervals decreases with an increasing shell number.

Together these mechanisms strongly constrain the possibility of sustained transfer amplification. The coherent-transfer classification used below is summarized in Appendix E.1.

4.5. Structural Consequences

Combining Theorem 4 with the coherent-transfer analysis of [22], one obtains the following implication:

$$\begin{aligned} \text{Persistent Concentration} &\Rightarrow \text{Persistent High–High Transfer} \\ &\Rightarrow \text{Coherent Transfer Class.} \end{aligned} \quad (88)$$

Equation (88) is an important consequence of the structural reduction theory. It shows that any persistent concentration mechanism must necessarily pass through a highly restricted coherent-transfer regime. Although the structural closure theorem established later does not require coherent-transfer exclusion, the coherent-transfer framework provides an additional structural interpretation of persistent modulation and is summarized in Appendix E for completeness.

The answer will be provided through the localized structural alternative developed in Sections 5 and 6, and completed by the exclusion arguments of Sections 7 and 8.

5. Localized Amplitude-Modulation Framework

The structural reduction theorem established in Section 4 shows that any persistent concentration sequence must be supported by a non-negligible High–High transfer. The purpose of this section is to introduce a localized representation of such a transfer and to separate its amplitude contribution from its phase organization. Unlike a purely formal factorization, the decomposition developed below assigns explicit quantitative meanings to the amplitude and modulation components and provides the shell-uniform bounds required for the structural alternative established in Section 6.

5.1. Localized Packet Decomposition

Let $u = \sum_j u_j$ be the Littlewood–Paley decomposition of the velocity field. For each shell index j , the shell component is further decomposed into a family of localized packets

$$u_j = \sum_{k \in K_j} u_{j,k}. \quad (89)$$

Here, K_j denotes the packet index set associated with shell j . Each packet is spatially localized inside a neighborhood whose characteristic diameter satisfies

$$r_j = \lambda_j^{-1}. \quad (90)$$

Let z_0 be a concentration center. Define

$$K_j(z_0) = \{k \in K_j : \text{supp}(u_{j,k}) \cap Q_{r_j}(z_0) \neq \emptyset\}. \quad (91)$$

The localized High–High transfer is obtained by restricting the transfer activity to packets belonging to $K_j(z_0)$. Let $T_{j,k}^{HH}(z_0)$ denote the contribution of packet $u_{j,k}$ to the localized High–High transfer. The total localized High–High transfer is

$$T_j^{HH,loc}(z_0) = \sum_{k \in K_j(z_0)} T_{j,k}^{HH}(z_0). \quad (92)$$

Estimate (90) follows from the standard helical representation of the Navier–Stokes quadratic nonlinearity. For same-scale triads, the interaction coefficient is determined by geometric products of the corresponding helical basis vectors. In particular, the coefficient vanishes in the collinear limit and is controlled by the relative orientation of the interacting wavevectors. Consequently, the interaction strength is bounded by a constant multiple of the sine of the relevant interaction angle, yielding estimate (90). Explicit formulas for the helical interaction coefficients, together with the derivation of the angular dependence underlying (90), may be found in Waleffe [37] and Lessinnes et al. [38].

5.2. Local Phase Measure and Modulation

The coherent-transfer analysis of [22] shows that transfer activity depends not only on packet amplitudes but also on the relative phase organization among packets. Let $\Phi_{j,k}$ denote the phase associated with packet $u_{j,k}$. The localized phase measure centered at z_0 is defined by

$$\mu_j^{(z_0)} = \sum_{k \in K_j(z_0)} \delta_{\Phi_{j,k}}, \quad (93)$$

where $\delta_{\Phi_{j,k}}$ is the Dirac measure concentrated at phase $\Phi_{j,k}$. The measure records the local phase distribution contributing to transfers near z_0 . A strong dispersion of phases produces cancellation, whereas a concentration of the phase measure indicates coherent alignment and enhanced transfer efficiency. To quantify this effect, we introduce the following quantities.

Definition 2 (Localized Amplitude). *The localized amplitude is defined by*

$$A_j(z_0) = \sum_{k \in K_j(z_0)} |T_{j,k}^{HH}(z_0)|. \quad (94)$$

The quantity $A_j(z_0)$ measures the total transfer strength carried by the localized packet family.

Definition 3 (Localized Modulation Factor). The localized modulation factor is defined by

$$M_j(z_0) = \frac{|T_j^{HH,loc}(z_0)|}{A_j(z_0)}, A_j(z_0) > 0, \quad (95)$$

and

$$M_j(z_0) = 0, A_j(z_0) = 0. \quad (96)$$

The quantity $M_j(z_0)$ measures the efficiency with which the localized packet amplitudes combine constructively. Values close to zero correspond to strong phase cancellation. Values close to one correspond to nearly coherent phase alignment.

Lemma 2 (Bounded Modulation). For every shell index j and concentration center z_0 ,

$$0 \leq M_j(z_0) \leq 1. \quad (97)$$

Proof. By the triangle inequality,

$$|T_j^{HH,loc}(z_0)| = \left| \sum_{k \in K_j(z_0)} T_{j,k}^{HH}(z_0) \right| \leq \sum_{k \in K_j(z_0)} |T_{j,k}^{HH}(z_0)| = A_j(z_0). \quad (98)$$

Dividing by $A_j(z_0)$ yields

$$0 \leq M_j(z_0) \leq 1. \quad (99)$$

□

5.3. Localized Amplitude-Modulation Decomposition

The definitions above yield a quantitative decomposition of localized High–High transfer into transfer strength and phase organization.

Theorem 5 (Localized Amplitude-Modulation Decomposition). For every shell index j and concentration center z_0 ,

$$|T_j^{HH,loc}(z_0)| = A_j(z_0) M_j(z_0). \quad (100)$$

Proof. If $A_j(z_0) = 0$, then $T_j^{HH,loc}(z_0) = 0$, and the identity is trivial. Assume therefore that $A_j(z_0) > 0$. Using Definition 3,

$$M_j(z_0) = \frac{|T_j^{HH,loc}(z_0)|}{A_j(z_0)}. \quad (101)$$

Multiplying by $A_j(z_0)$ gives

$$|T_j^{HH,loc}(z_0)| = A_j(z_0) M_j(z_0). \quad (102)$$

This proves the result. □

5.4. Necessary Condition and Structural Bound

The decomposition (100) separates two fundamentally distinct mechanisms governing localized High–High transfers. The amplitude quantity $A_j(z_0)$ measures the total localized transfer strength carried by the packet family.

The modulation quantity $M_j(z_0)$ measures the degree of constructive phase organization and satisfies the shell-uniform bound $0 \leq M_j(z_0) \leq 1$. Consequently, a persistent, localized transfer can arise only through:

1. Persistent growth of the transfer amplitude A_j ;
2. Persistent non-dispersive modulation represented by non-vanishing M_j .

This quantitative separation forms the basis of the localized structural alternative established in Section 6.

6. Localized Structural Alternative

The amplitude-modulation decomposition established in Section 5 provides a quantitative representation of localized High–High transfer activity in terms of transfer strength and phase organization. The purpose of this section is to show that every admissible persistent concentration sequence necessarily generates at least one of two distinct structural mechanisms. This result reduces the concentration problem to the analysis of two explicitly identifiable branches.

6.1. Persistent Concentration Hypothesis

Assume that there exists a persistent concentration sequence

$$(z_n, r_n), r_n \rightarrow 0. \quad (103)$$

By definition,

$$\limsup_{n \rightarrow \infty} E(z_n, r_n) > 0, \quad (104)$$

where $E(z, r)$ denotes the mechanical concentration functional introduced in Section 3. According to Theorem 4, a persistent concentration implies the existence of a sequence of shell indices $j_n \rightarrow \infty$ such that

$$\limsup_{n \rightarrow \infty} |T_{j_n}^{HH,loc}(z_n)| > 0. \quad (105)$$

Applying the amplitude-modulation decomposition of Theorem 5,

$$|T_{j_n}^{HH,loc}(z_n)| = A_{j_n}(z_n)M_{j_n}(z_n). \quad (106)$$

Therefore,

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n)M_{j_n}(z_n) > 0. \quad (107)$$

The persistence hypothesis thus forces the localized transfer amplitude and the modulation factor to remain jointly non-negligible.

6.2. Amplification Alternatives

The bounded modulation property established in Lemma 2 yields

$$0 \leq M_{j_n}(z_n) \leq 1. \quad (108)$$

The following elementary observation is crucial.

Lemma 3 (Persistence Alternative). *Assume that*

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n)M_{j_n}(z_n) > 0. \quad (109)$$

Then

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) > 0. \quad (110)$$

Proof. Suppose instead that

$$A_{j_n}(z_n) \rightarrow 0. \quad (111)$$

Since $0 \leq M_{j_n}(z_n) \leq 1$, we obtain

$$A_{j_n}(z_n)M_{j_n}(z_n) \rightarrow 0, \quad (112)$$

contradicting (109). Therefore, (110) holds. $\square$

The significance of Lemma 3 is that the boundedness of the modulation factor eliminates possible scale-compensation scenarios. In particular, configurations of the form

$$A_{j_n}(z_n) \rightarrow 0, M_{j_n}(z_n) \rightarrow \infty, \quad (113)$$

cannot occur because

$$M_{j_n}(z_n) \leq 1. \quad (114)$$

Consequently, persistent transfer amplification necessarily requires a non-vanishing localized transfer amplitude.

6.3. Structural Branches

Persistent transfer amplification may arise through two distinct mechanisms.

Branch I (Amplitude Concentration)

The localized transfer amplitude remains non-vanishing:

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) > 0. \quad (115)$$

This branch corresponds to the concentration of transfer strength.

Branch II (Non-Dispersive Modulation)

The modulation factor remains bounded away from complete phase cancellation:

$$\limsup_{n \rightarrow \infty} M_{j_n}(z_n) > 0. \quad (116)$$

Because $0 \leq M_j \leq 1$, condition (116) characterizes persistent constructive phase organization. In Section 8, it will be shown that non-dispersive modulation implies membership in the coherent-transfer class identified in [22].

6.4. Necessity of the Amplitude Concentration Branch

The preceding observations yield the principal result of this section.

Theorem 6 (Localized Structural Necessity of Amplitude Concentration). *Assume that a persistent concentration sequence exists. Then the amplitude-concentration branch necessarily occurs. More precisely,*

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) > 0. \quad (117)$$

Proof. By Theorem 4,

$$\limsup_{n \rightarrow \infty} |T_{j_n}^{HH,loc}(z_n)| > 0. \quad (118)$$

Using the localized amplitude-modulation decomposition of Theorem 5,

$$|T_{j_n}^{HH,loc}(z_n)| = A_{j_n}(z_n) M_{j_n}(z_n). \quad (119)$$

By Lemma 2, the localized modulation factor satisfies

$$0 \leq M_{j_n}(z_n) \leq 1. \quad (120)$$

Therefore,

$$|T_{j_n}^{HH,loc}(z_n)| \leq A_{j_n}(z_n). \quad (121)$$

Combining (118) and (121), we obtain

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) > 0. \quad (122)$$

Thus, a persistent concentration necessarily generates an amplitude concentration. $\square$

Immediately after Theorem 6, replace the explanatory paragraph with the following: Theorem 6 shows that Branch I is not merely one possible alternative, but a necessary

structural consequence of a persistent concentration. The boundedness of the modulation factor prevents phase organization from compensating for a vanishing localized amplitude. Consequently, any dynamically sustained persistent concentration sequence must generate a non-vanishing localized transfer amplitude. The non-dispersive modulation condition

$$\limsup_{n \rightarrow \infty} M_{j_n}(z_n) > 0 \quad (123)$$

may still occur as an additional coherent-transfer subcase. However, it is not needed for the proof of structural closure. Section 8 discusses this additional subcase separately under the coherent-transfer framework.

6.5. Reduction of the Problem

By Lemma 3 together with the bounded-modulation property $0 \leq M_{j_n}(z_n) \leq 1$, established in Lemma 2, a persistent concentration necessarily generates a non-vanishing localized transfer amplitude. Consequently, Branch I is a necessary structural consequence of a persistent concentration.

Theorem 6 shows that a persistent concentration necessarily generates a non-vanishing localized transfer amplitude. Indeed, it does so by the localized amplitude-modulation decomposition, $|T_{j_n}^{HH,loc}(z_n)| = A_{j_n}(z_n)M_{j_n}(z_n)$, and by the bounded modulation property, $0 \leq M_{j_n}(z_n) \leq 1$. Therefore,

$$\text{Persistent Concentration} \Rightarrow \text{Branch I.} \quad (124)$$

Thus, the amplitude-concentration branch is a necessary structural consequence of a persistent concentration. Branch II is not forced by a persistent concentration alone. Rather, it represents an additional coherent-transfer subcase that may occur when non-dispersive modulation persists. More precisely,

$$\limsup_{n \rightarrow \infty} M_{j_n}(z_n) > 0 \Rightarrow \text{Branch II.} \quad (125)$$

Consequently,

$$\text{Persistent Concentration} \Rightarrow \text{Branch I, and, if } \limsup_{n \rightarrow \infty} M_{j_n}(z_n) > 0, \text{ also Branch II.} \quad (126)$$

Section 7 excludes the necessary amplitude-concentration branch through the low-Mach inheritance principle and thermo-acoustic exclusion. Section 8 excludes the additional coherent-transfer subcase associated with non-dispersive modulation through coherent-transfer admissibility, residence-time compression, and cumulative transfer decay. Since Branch I is necessary for a persistent concentration, its exclusion already rules out a dynamically sustained persistent concentration. The exclusion of Branch II further shows that coherent non-dispersive phase organization cannot provide an alternative mechanism for sustaining concentration. This yields the structural closure theorem proved in Section 9.

7. Exclusion of Branch I

Branch I corresponds to a persistent, localized transfer-amplitude concentration. The objective of this section is to show that such a concentration necessarily generates thermo-acoustic activity and is therefore incompatible with the thermo-acoustic exclusion principle established in Section 3. The argument proceeds through four steps:

$$A_j \Rightarrow C \Rightarrow E \Rightarrow \Theta \Rightarrow H. \quad (127)$$

The first implication is provided by a localized Littlewood–Paley bridge. The second follows from localized interpolation estimates. The third is the low-Mach inheritance

principle. The fourth follows directly from the definition of the thermo-acoustic concentration functional.

7.1. Localized LP Bridge

The first step is to connect localized transfer amplitudes with localized concentration functionals. Let $C(z, r)$ denote the localized concentration functional introduced in Section 3. The following estimate provides the required bridge.

Lemma 4 (Localized LP Bridge). *There exist positive constants $C_B, C_R, \beta > 0$, independent of the shell index j , such that*

$$A_j(z_0) \leq C_B C(z_0, C_R r_j) + R_j(z_0), \quad (128)$$

where

$$|R_j(z_0)| \leq C_B \lambda_j^{-\beta} C(z_0, C_R r_j). \quad (129)$$

Consequently,

$$R_j(z_0) \rightarrow 0, j \rightarrow \infty. \quad (130)$$

Proof. The complete proof is given in Appendix C. More precisely, Appendix C first establishes the finite-overlap estimate (Lemma A2), the Bernstein estimates (Lemma A3), the localized Korn–Bernstein estimate (Lemma A4), the localized strain-control estimate (Lemma A5), the boundary commutator estimate (Lemma A6), and the shell-uniform bounds (Proposition A3). These results are proved independently of Lemma 4 and are subsequently combined to derive the localized Littlewood–Paley bridge. In particular, estimate (128) follows from the localized LP bridge estimate established in Appendix C, while the remainder bound (129) follows from the boundary localization error estimate. No step of the Appendix C argument depends on Lemma 4 itself. $\square$

Appendix C therefore provides a self-contained derivation of the bridge estimate from packet localization, finite-overlap bounds, Bernstein inequalities, Korn-type control, localization estimates, and shell-uniform constants.

As a consequence,

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) > 0 \quad (131)$$

implies

$$\limsup_{n \rightarrow \infty} C(z_n, C_R r_{j_n}) > 0. \quad (132)$$

This implication is proved in Corollary A2 of Appendix C. Thus, a persistent transfer-amplitude concentration necessarily generates a concentration of the localized concentration functional. Through the interpolation estimate of Section 7.2, this yields a persistent mechanical concentration.

7.2. Mechanical Concentration

The concentration functional is connected to mechanical concentration through localized interpolation estimates. By the localized Caffarelli–Kohn–Nirenberg interpolation estimate,

$$C(z, r) \leq C_I E(z, r), \quad (133)$$

where C_I is independent of the scale. Combining (132) and (133) yields

$$\limsup_{n \rightarrow \infty} E(z_n, r_n) > 0. \quad (134)$$

Therefore, Branch I necessarily generates a persistent mechanical concentration.

7.3. Low-Mach Inheritance

The next step uses the admissible low-Mach inheritance framework introduced in Section 3.

Lemma 5 (Application of the Low-Mach Inheritance Principle). *Let (z_n, r_n) be a low-Mach realizable admissible concentration sequence. Assume*

$$\limsup_{n \rightarrow \infty} E(z_n, r_n) > 0. \quad (135)$$

Then

$$\limsup_{n \rightarrow \infty} \Theta(z_n, r_n) > 0. \quad (136)$$

Proof. The result is an immediate consequence of Theorem 3, but we record it here in the form required for the Branch I exclusion argument. Suppose instead that

$$\limsup_{n \rightarrow \infty} \Theta(z_n, r_n) = 0. \quad (137)$$

By the entropy-coercive admissibility condition,

$$D_{\text{ent}} \geq c_{\text{ent}} |D(u)|^2, \quad (138)$$

where $c_{\text{ent}} > 0$ is independent of the shell scale and independent of the low-Mach parameter. Integrating over $Q_{r_n}(z_n)$ and using the definition of $\Theta(z, r)$, one obtains

$$E(z_n, r_n) \leq c_{\text{ent}}^{-1} \Theta(z_n, r_n). \quad (139)$$

Taking the limit superior yields

$$\limsup_{n \rightarrow \infty} E(z_n, r_n) = 0, \quad (140)$$

contradicting (135). Hence, (136) follows. $\square$

7.4. Thermo-Acoustic Activation

The thermodynamic trace functional is a component of the thermo-acoustic concentration functional. By definition,

$$H(z, r) = \Theta(z, r) + F(z, r), \quad (141)$$

where $F(z, r)$ denotes the localized free-energy contribution. Since $F(z, r) \geq 0$, we obtain

$$H(z, r) \geq \Theta(z, r). \quad (142)$$

Combining (136) and (142) yields

$$\limsup_{n \rightarrow \infty} H(z_n, r_n) > 0. \quad (143)$$

Thus, Branch I necessarily activates thermo-acoustic concentration. The entire mechanism may therefore be summarized by

$$A_j \Rightarrow C \Rightarrow E \Rightarrow \Theta \Rightarrow H. \quad (144)$$

7.5. Contradiction with Thermo-Acoustic Exclusion

The thermo-acoustic exclusion principle (Theorem 2) states that admissible concentration sequences satisfy

$$H(z_n, r_n) \rightarrow 0. \quad (145)$$

A complete derivation of the thermo-acoustic ε -regularity and exclusion framework is recorded in Appendix A. However, Branch I implies

$$\limsup_{n \rightarrow \infty} H(z_n, r_n) > 0. \quad (146)$$

Equations (145) and (146) are incompatible. Therefore, Branch I cannot occur.

7.6. Exclusion Theorem

Theorem 7 (Amplitude-Branch Exclusion). *Let (z_n, r_n) be a low-Mach realizable admissible concentration sequence. Then the amplitude-concentration branch,*

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) > 0, \quad (147)$$

is impossible.

Proof. Let r_n denote the concentration scale associated with shell j_n , satisfying $r_n = \lambda_{j_n}^{-1}$. Assume, for contradiction, that

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) > 0. \quad (148)$$

By Lemma 4, an amplitude concentration implies a localized concentration:

$$\limsup_{n \rightarrow \infty} C(z_n, r_n) > 0. \quad (149)$$

By the localized interpolation estimate (133),

$$\limsup_{n \rightarrow \infty} E(z_n, r_n) > 0. \quad (150)$$

Since the concentration sequence is low-Mach realizable, Lemma 5 yields

$$\limsup_{n \rightarrow \infty} \Theta(z_n, r_n) > 0. \quad (151)$$

Using (142), namely the inclusion of the thermodynamic trace in the thermo-acoustic concentration functional, we obtain

$$\limsup_{n \rightarrow \infty} H(z_n, r_n) > 0. \quad (152)$$

This contradicts the thermo-acoustic exclusion principle (145), which rules out a non-vanishing thermo-acoustic concentration for admissible persistent concentration sequences. Therefore,

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) = 0. \quad (153)$$

Thus, the amplitude-concentration branch is excluded. $\square$

Immediately after the proof, add Theorem 7, which is independent of the averaged phase-drift non-degeneracy assumption (A1). Its proof uses only the localized LP bridge, the mechanical concentration estimate, the low-Mach inheritance mechanism, and the thermo-acoustic exclusion principle. Consequently, the exclusion of the necessary amplitude branch is a purely thermo-acoustic consequence within the admissible low-Mach realizable class.

8. Additional Coherent-Transfer Results

Theorem 6 showed that every admissible persistent concentration sequence necessarily generates Branch I. Section 7 excluded Branch I through thermo-acoustic activation and thermo-acoustic exclusion. The purpose of the present section is to record an additional coherent-transfer result associated with persistent non-dispersive modulation. The argument proceeds by showing that persistent non-dispersive modulation necessarily generates coherent-transfer activity. The coherent-transfer reduction theory of [22] then implies simultaneous residence-time compression and cumulative transfer decay, leading to a contradiction.

8.1. Non-Dispersive Modulation

Branch II is characterized by

$$\limsup_{n \rightarrow \infty} M_{j_n}(z_n) > 0. \quad (154)$$

By Lemma 2,

$$0 \leq M_j(z) \leq 1. \quad (155)$$

Therefore, (154) means that the localized transfer activity does not become asymptotically dominated by phase cancellation. Since

$$M_j(z) = \frac{|T_j^{HH,loc}(z)|}{A_j(z)}, \quad (156)$$

The persistent positivity of M_j implies that a non-vanishing fraction of the localized transfer amplitude survives cancellation. Consequently, Branch II represents a non-dispersive transfer regime in which coherent phase organization persists at arbitrarily small scales.

8.2. Coherent-Transfer Admissibility

The following lemma provides the crucial link between modulation persistence and coherent-transfer theory.

Lemma 6 (Non-Dispersive Modulation Implies Coherent-Transfer Admissibility). *Assume*

$$\limsup_{n \rightarrow \infty} M_{j_n}(z_n) > 0. \quad (157)$$

Then there exists a subsequence, again denoted by j_n , together with a coherent packet family such that the associated transfer belongs to the coherent-transfer class of [22].

Proof. Condition (157) implies the existence of a constant $m_0 > 0$ and a subsequence satisfying

$$M_{j_n}(z_n) \geq m_0. \quad (158)$$

Using Definition 3,

$$|T_{j_n}^{HH,loc}(z_n)| \geq m_0 A_{j_n}(z_n). \quad (159)$$

Thus, a fixed fraction of the localized transfer amplitude survives phase cancellation. If the phase distribution were asymptotically dispersive, cancellation would force $M_{j_n}(z_n) \rightarrow 0$, contradicting (158). Therefore, a non-negligible fraction of transfer activity must be carried by a coherent packet family. By the coherent-transfer classification theorem of [22], this implies membership in the coherent-transfer class. $\square$

The modulation–coherence link is summarized in Appendix E.1. As a consequence,

$$\text{Branch II} \Rightarrow \text{Coherent-Transfer Class}. \quad (160)$$

8.3. Coherent Transfer Activity

Let G_j denote the coherent-transfer activity associated with shell j . By Lemma 6 and the coherent-transfer classification theorem of [22],

$$\limsup_{j \rightarrow \infty} G_j > 0. \quad (161)$$

Thus, Branch II necessarily produces persistent coherent-transfer activity. The problem therefore is reduced to the analysis of a coherent transfer.

8.4. Residence-Time Compression

The underlying phase-drift mechanism is summarized in Appendix E.2. The first structural restriction on coherent transfers is residence-time compression.

Proposition 3 (Residence-Time Compression). *There exist positive constants $C_\tau, \alpha > 0$, such that every coherent-transfer interval satisfies*

$$\tau_j \leq C_\tau 2^{-\alpha j}. \quad (162)$$

Proof. See Theorem 1 of [22]. The estimate follows from the coherent phase drift lower bound

$$|\Omega_\tau| \geq c_\Omega \lambda_j^\alpha, \quad (163)$$

which implies that coherent phase alignment cannot persist longer than $O(\lambda_j^{-\alpha})$. $\square$

See Appendix E.3 for the corresponding residence-time compression estimate. Proposition 3 is a restatement of Proposition A6 in Appendix E, included for completeness as part of the coherent-transfer framework inherited from [22]. Consequently,

$$\tau_j \rightarrow 0, j \rightarrow \infty. \quad (164)$$

Therefore, coherent amplification intervals become asymptotically negligible.

8.5. Cumulative Transfer Decay

The second structural restriction is cumulative transfer decay.

Proposition 4 (Cumulative Transfer Decay). *There exists a constant $C_G > 0$ such that*

$$G_j \leq C_G 2^{-\alpha j}. \quad (165)$$

Proof. See the cumulative coherent-transfer decay theorem established in [22]. $\square$

Proposition 4 is the main-text formulation of Proposition A7 in Appendix E, included for completeness as part of the coherent-transfer framework inherited from [22].

The underlying coherent-transfer exclusion theorem inherited from [22] is summarized in Appendix E.5. Since $\alpha > 0$, it follows that

$$G_j \rightarrow 0, j \rightarrow \infty. \quad (166)$$

Therefore, coherent-transfer activity necessarily decays at sufficiently small scales.

8.6. Contradiction

Branch II implies

$$\limsup_{j \rightarrow \infty} G_j > 0. \quad (167)$$

However, Proposition 4 yields

$$G_j \rightarrow 0. \quad (168)$$

Equations (167) and (168) are incompatible. Therefore, under Assumption A1, the coherent non-dispersive modulation subcase cannot occur.

9. Structural Closure

Section 7 excluded the necessary amplitude-concentration branch through thermo-acoustic activation and thermo-acoustic exclusion. Section 8 records an additional coherent-transfer exclusion result for persistent non-dispersive modulation through coherent-

transfer admissibility, residence-time compression, and cumulative transfer decay. This supplementary result is independent of the proof of Theorem 8 and is included primarily for structural comparison with the thermo-acoustic closure mechanism. The purpose of the present section is to combine Theorems 6 and 7 and establish the structural closure theorem.

9.1. Elimination of Branch I

Branch I is characterized by

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) > 0. \quad (169)$$

Theorem 7 showed that a persistent transfer-amplitude concentration necessarily generates thermo-acoustic activity. More precisely,

$$A_j \Rightarrow C \Rightarrow E \Rightarrow \Theta \Rightarrow H. \quad (170)$$

Consequently,

$$\limsup_{n \rightarrow \infty} H(z_n, r_n) > 0. \quad (171)$$

However, the thermo-acoustic exclusion principle implies

$$H(z_n, r_n) \rightarrow 0. \quad (172)$$

Equations (171) and (172) are incompatible. Therefore, Branch I cannot occur.

9.2. Structural Closure Theorem

The combination of the structural necessity principle and the amplitude-branch exclusion yield the principal result of this paper.

Theorem 8 (Conditional Structural Closure Theorem within the Admissible Low-Mach Realizable Class). *Let (z_n, r_n) be a low-Mach realizable admissible concentration sequence belonging to the admissible low-Mach inheritance class introduced in Section 3. Then, a dynamically sustained persistent concentration cannot occur. Equivalently,*

$$\text{Persistent Concentration} \Rightarrow \perp. \quad (173)$$

Proof. Assume that a persistent concentration sequence exists. By Theorem 6, a persistent concentration necessarily generates an amplitude concentration. Hence,

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) > 0. \quad (174)$$

However, Theorem 7 shows that the amplitude-concentration branch is impossible for admissible low-Mach realizable concentration sequences. Therefore,

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) = 0. \quad (175)$$

Relations (174) and (175) are contradictory. Consequently,

$$\text{Persistent Concentration} \Rightarrow \perp. \quad (176)$$

Therefore, no admissible low-Mach realizable persistent concentration sequence can exist. $\square$

Theorem 8 should be interpreted as a conditional structural closure theorem within the admissible low-Mach realizable class introduced in Section 3.

Its proof relies on two ingredients:

$$\text{Persistent Concentration} \Rightarrow \text{Amplitude Concentration}, \quad (177)$$

established in Theorem 6, and

$$\text{Amplitude Concentration} \Rightarrow \perp, \quad (178)$$

established in Theorem 7 through the concentration-representation framework, the low-Mach inheritance mechanism, and the thermo-acoustic exclusion principle. Consequently, the structural closure mechanism of the present paper is entirely thermo-acoustic in nature and does not rely on the averaged phase-drift non-degeneracy condition (A1).

The coherent-transfer exclusion result of Section 8 remains valid as an additional conditional result under Assumption A1, but it is not required for the proof of the structural closure theorem.

9.3. Structural Interpretation

Theorem 8 establishes a structural closure principle for admissible concentration sequences. The argument does not proceed through a global norm estimate, a continuation criterion, or an a priori smoothness assumption. Instead, the proof identifies the structural mechanisms capable of sustaining a scale-critical concentration and excludes each mechanism individually.

The logical structure is

$$\text{Persistent Concentration} \Rightarrow \text{High} - \text{High Transfer}, \quad (179)$$

and by Theorem 4,

$$\text{Persistent Concentration} \Rightarrow \text{Amplitude Concentration}. \quad (180)$$

By Theorem 6,

$$\text{Amplitude Concentration} \Rightarrow \perp, \quad (181)$$

and by Theorem 7

$$\text{Persistent Concentration} \Rightarrow \perp. \quad (182)$$

Therefore,

$$\text{Persistent Concentration} \Rightarrow \perp. \quad (183)$$

The closure mechanism combines four structural ingredients:

1. Thermo-acoustic admissibility and entropy-coercive exclusion.
2. Low-Mach inheritance of mechanical concentration.
3. Structural reduction of a scale-critical transfer.
4. Localized amplitude-modulation decomposition and a structural alternative.

Together these ingredients provide a unified framework linking thermodynamic constraints, concentration representation, and nonlinear transfer dynamics.

9.4. Scope of the Result

Theorem 8 is a conditional result. The theorem applies to concentration sequences that are realizable within the admissible low-Mach inheritance class introduced in Section 3. The result does not assert unconditional global regularity for arbitrary, weak solutions of the three-dimensional incompressible Navier–Stokes equations.

Rather, it establishes that within the admissible low-Mach realizable class considered in this work, no dynamically sustained persistent concentration sequence can occur. More precisely, Theorem 8 combines the structural necessity principle established in Theorem 6 with the thermo-acoustic exclusion mechanism established in Theorem 7. Consequently, a persistent concentration is incompatible with the admissible low-Mach realizable framework developed in the present paper.

The conditional nature of the theorem arises entirely from the admissible low-Mach realizability assumption. The universality of this realizability class remains an open problem and represents the principal limitation of the present structural closure framework.

10. Discussion

10.1. Relation to Structural Transfer Reduction

The present work builds directly upon the structural transfer-reduction theory developed in [22]. The principal conclusion of that theory is that most nonlinear transfer pathways become asymptotically perturbative in the scale-critical regime. In particular,

$$T_j^{LL} \rightarrow 0, \quad (184)$$

and

$$T_j^{LH} \rightarrow 0, \quad (185)$$

as $j \rightarrow \infty$. Consequently,

$$T_j \sim T_j^{HH}, \quad (186)$$

and any dynamically relevant concentration mechanisms must ultimately be supported by High–High transfers. The present work begins precisely at this point. Rather than revisiting the reduction argument itself, we analyze the internal structure of the surviving High–High transfer. Using the localized packet framework introduced in Section 5, the transfer activity is decomposed into amplitude and modulation components,

$$|T_j^{HH,loc}| = A_j M_j, \quad (187)$$

with

$$0 \leq M_j \leq 1. \quad (188)$$

This decomposition transforms the concentration problem into a structural alternative involving transfer-amplitude concentration and non-dispersive modulation. Accordingly, the present work may be viewed as a continuation of the reduction program initiated in [22]. Whereas [22] identifies the only remaining non-perturbative transfer mechanism, the present work analyzes its internal structure and establishes its incompatibility with a persistent concentration within the admissible class considered here.

10.2. Relation to Thermo-Acoustic Admissibility

The second essential ingredient is the thermo-acoustic admissibility framework developed in [23]. That framework introduced the thermo-acoustic concentration functional $H(z, r)$, together with the concentration-exclusion mechanism

$$H(z_n, r_n) \rightarrow 0 \Rightarrow \text{concentration exclusion}. \quad (189)$$

However, the theory of [23] was formulated for admissible compressible Navier–Stokes–Fourier dynamics. The present work does not directly employ compressible evolution equations. Instead, thermo-acoustic admissibility enters through the concentration-representation framework developed in Section 3.

The key observation is that a mechanically sustained concentration admits a thermodynamic representation whenever an admissible entropy-coercive low-Mach realization exists. Lemma 1 and Proposition A1 establish that mechanically concentrated states generate a nontrivial thermodynamic trace. More precisely,

$$E(z, r) \Rightarrow \Theta(z, r), \quad (190)$$

and therefore

$$E(z, r) \Rightarrow H(z, r). \quad (191)$$

Consequently, thermo-acoustic exclusion becomes applicable to the amplitude-concentration branch of the incompressible transfer problem. From this viewpoint, the role of thermo-acoustic admissibility is not merely to provide an exclusion criterion. Rather, it supplies the concentration-control functional at the final stage of the representation hierarchy. The present work therefore provides a structural connection between incompressible transfer dynamics and thermo-acoustic concentration control.

Thermodynamic Trace versus Mechanical Concentration

A conceptual distinction should be emphasized. The mechanical concentration functional $E(z, r)$ measures the intensity of localized strain activity, whereas the thermodynamic trace functional $\Theta(z, r)$ measures the corresponding irreversible thermodynamic response. The concentration-representation framework developed in this paper shows that these quantities are different representations of the same concentration event viewed from mechanical and thermodynamic perspectives, respectively.

10.3. Relation to Low-Mach Realizability and Inheritance

Classical low-Mach theory is primarily concerned with the convergence of compressible flows toward incompressible limits. The objective is typically to establish the convergence of velocity, density, and pressure variables as the Mach number tends to zero. The present work uses the low-Mach regime in a different way. The central issue is not convergence itself but the representation and preservation of concentration information.

More precisely, the question is whether localized mechanical concentration remains detectable after passage through an admissible low-Mach realization. Lemma 1 and Proposition A1 answer this question positively. See Appendix B.4 and Appendix B.5 for the constructive realizability and inheritance mechanisms. In particular,

$$\limsup_{n \rightarrow \infty} E(z_n, r_n) > 0 \Rightarrow \limsup_{n \rightarrow \infty} \Theta(z_n, r_n) > 0. \quad (192)$$

Thus, concentration information is preserved rather than destroyed. The role of low-Mach realizability is therefore not to establish concentration, but to provide a representation mechanism through which mechanical concentration becomes observable at the thermodynamic level. In this sense, low-Mach inheritance should be viewed as a realization mechanism for the concentration-representation principle established in Section 3.

Without this representation step, the transfer-reduction theory of [22] and the thermo-acoustic exclusion theory of [23] would remain disconnected. The present framework connects them through the chain

$$\begin{aligned} \text{Mechanical Concentration} &\Rightarrow \text{Thermodynamic Trace} \\ &\Rightarrow \text{Thermo-Acoustic Control.} \end{aligned} \quad (193)$$

Consequently, admissible low-Mach realizability provides the structural bridge linking incompressible transfer dynamics and thermo-acoustic concentration exclusion.

10.4. Relation to Classical Regularity Criteria

The present framework differs fundamentally from the classical regularity theories of Beale–Kato–Majda, Prodi–Serrin, and Caffarelli–Kohn–Nirenberg. The distinction is not merely technical but concerns the quantity through which concentration is controlled. The Beale–Kato–Majda criterion is formulated through vorticity amplification:

$$\int_0^T \|\omega(t)\|_{L^\infty} dt < \infty. \quad (194)$$

The Prodi–Serrin criterion is formulated through velocity integrability:

$$u \in L^p(0, T; L^q(\Omega)), \frac{2}{p} + \frac{3}{q} \leq 1, q > 3. \quad (195)$$

The Caffarelli–Kohn–Nirenberg theory is formulated through localized scale-invariant quantities:

$$r^{-2} \iint_{Q_r(z)} |u|^3 \, dx \, dt + r^{-1} \iint_{Q_r(z)} |\nabla u|^2 \, dx \, dt. \quad (196)$$

All three approaches begin with analytical control quantities and deduce regularity from their boundedness or smallness. By contrast, the present theory is organized around a concentration-transfer pathway. The localized Littlewood–Paley bridge established in Section 7 yields

$$\limsup_{j \rightarrow \infty} A_j > 0 \Rightarrow \limsup_{n \rightarrow \infty} E(z_n, r_n) > 0. \quad (197)$$

The low-Mach inheritance principle then gives

$$\limsup_{n \rightarrow \infty} E(z_n, r_n) > 0 \Rightarrow \limsup_{n \rightarrow \infty} \Theta(z_n, r_n) > 0. \quad (198)$$

Since

$$H(z, r) = F(z, r) + \Theta(z, r), \quad (199)$$

one obtains

$$\limsup_{j \rightarrow \infty} A_j > 0 \Rightarrow \limsup_{n \rightarrow \infty} H(z_n, r_n) > 0. \quad (200)$$

Consequently, a persistent concentration necessarily activates a thermo-acoustic concentration. The resulting control hierarchy may therefore be summarized as

$$A_j \Rightarrow E(z, r) \Rightarrow \Theta(z, r) \Rightarrow H(z, r). \quad (201)$$

This hierarchy has no direct analogue in the classical regularity theories. The logical structure of the present framework is therefore fundamentally different. Classical regularity criteria have the form

$$\text{Analytical Control Quantity} \Rightarrow \text{Regularity}. \quad (202)$$

The present theory instead establishes

$$\begin{aligned} &\text{Persistent Concentration} \Rightarrow \text{Transfer-Core Activation} \\ &\Rightarrow \text{Thermo-Acoustic or Coherent Activity} \Rightarrow \text{Structural Contradiction}. \end{aligned} \quad (203)$$

In particular, the proof of Theorem 8 does not proceed through a continuation criterion, a global norm estimate, or a priori smoothness assumptions. Rather, it identifies the only the remaining mechanisms capable of sustaining a scale-critical concentration and excludes each mechanism individually. From this viewpoint, Theorem 8 should be interpreted not as a new analytical regularity criterion, but as a structural concentration-exclusion theorem within the admissible low-Mach realizable class.

An interesting open question is whether the thermo-acoustic concentration functional $H(z, r)$ admits a quantitative relationship with the control quantities appearing in the Beale–Kato–Majda, Prodi–Serrin, or Caffarelli–Kohn–Nirenberg theories. Establishing such a connection could clarify whether the present structural framework and the classical analytical criteria possess a deeper common concentration-control structure.

More broadly, concentration-control and stabilization mechanisms have also been investigated in several nonlinear PDE settings beyond incompressible fluid dynamics, including Keller–Segel–Navier–Stokes type models [39] and chemotaxis–haptotaxis systems [40]. Although the governing equations differ substantially from the present framework, these studies illustrate the growing importance of concentration-control mechanisms in nonlinear evolution equations.

Recent studies have also examined stabilization, asymptotic behavior, and reduced-model dynamics in other fluid-mechanical settings, including Navier–Stokes–Voigt systems in thin domains [41] and generalized non-Newtonian lubrication and Reynolds-type models [42]. Although these problems differ substantially from the concentration-representation framework developed here, they further illustrate the broad relevance of concentration-control, asymptotic stabilization, and reduced-scale description mechanisms across nonlinear fluid dynamics.

10.5. Scope, Conditionality, and Limitations

The present result remains subject to several important limitations.

Admissible Class Restriction

The analysis applies only to the admissible thermodynamic class introduced in Section 2. The admissibility assumptions include entropy coercivity, constitutive non-degeneracy, and uniform thermodynamic control. Consequently, Theorem 8 is not a statement concerning arbitrary, weak solutions of the three-dimensional incompressible Navier–Stokes equations.

Low-Mach Realizability Requirement

Theorem 8 applies only to concentration sequences that admit an admissible low-Mach realization. However, Definition 1, Proposition 1, and Proposition A1 show that low-Mach realizability is not introduced as a purely abstract assumption. Rather, realizability follows whenever an admissible entropy-coercive, weakly compressible approximation satisfying the approximation criteria of Section 3 and Appendix B exists. More precisely,

$$\begin{aligned} \text{Entropy-Coercive Approximation} &\Rightarrow \text{Low-Mach Realizability} \\ &\quad \text{Thermodynamic Trace.} \end{aligned} \tag{204}$$

Therefore, the present theory reduces the realizability problem to verifiable approximation properties rather than to an independent admissibility postulate.

Inheritance Mechanism versus Universality

A fundamental distinction should be made between the inheritance mechanism and the universality problem. The inheritance mechanism is established in the present paper through Definition 1, Lemma 1, Proposition 1, and Proposition A1. In particular, once an admissible entropy-coercive, low-Mach approximation is available, mechanical concentration necessarily generates a nontrivial thermodynamic trace. Accordingly, the concentration-representation mechanism itself is not an open issue within the admissible realizable class. The remaining unresolved question is the universality of admissible approximation constructions. The present paper does not establish that every potentially singular incompressible concentration sequence admits an admissible entropy-coercive, low-Mach realization.

The admissible class considered here is expected to include concentration scenarios compatible with a weakly compressible thermodynamic origin and uniform entropy-coercive control. Concentration mechanisms involving a loss of entropy coercivity, degeneration of thermodynamic bounds, exact acoustic locking, or a breakdown of uniform local energy control may lie outside the present framework.

Additional Coherent-Transfer Result

Section 8 contains an additional coherent-transfer exclusion result inherited from the coherent-transfer framework of [22]. The present work establishes the structural connection between persistent non-dispersive modulation and a coherent transfer, while the coherent-transfer exclusion mechanism itself relies on the residence-time compression and

cumulative transfer decay theory developed in [22]. This coherent-transfer result is supplementary and is not required for the proof of Theorem 8.

Conditional Nature of Theorem 8

Theorem 8 remains a conditional structural closure theorem. More precisely, Theorem 8 $\neq$ General Navier–Stokes Regularity, and

$$\begin{aligned} &\text{Persistent Concentration} + \text{Admissible Low-Mach Realizability} \\ &\Rightarrow \text{Contradiction.} \end{aligned} \quad (205)$$

within the admissible class. Accordingly, Theorem 8 excludes a dynamically sustained persistent concentration only within the admissible low-Mach realizable class described above. Whether every potentially singular incompressible concentration sequence belongs to this class remains an open problem. Resolving this universality question would substantially enlarge the scope of the present structural closure framework and clarify its relation to general incompressible concentration mechanisms. The principal unresolved issue is therefore the universality of admissible entropy-coercive, low-Mach realizability. Establishing such universality would provide a direct bridge between the present structural closure framework and general incompressible concentration scenarios.

10.6. Toward a Concentration Representation Theory

Beyond the specific exclusion result, the present work suggests a broader structural interpretation of concentration phenomena. The central conceptual outcome is the emergence of a concentration-representation hierarchy. The analysis establishes the sequence

$$A_j \Rightarrow E(z, r) \Rightarrow \Theta(z, r) \Rightarrow H(z, r). \quad (206)$$

Each quantity represents the same underlying concentration mechanism at a different structural level. The transfer amplitude A_j represents a concentration in Fourier-transfer space. The localized mechanical functional $E(z, r)$ represents the corresponding concentration in physical space. The thermodynamic trace $\Theta(z, r)$ represents the same concentration through admissible thermo-acoustic activity.

Finally, the concentration-control functional $H(z, r)$ converts this information into a form suitable for concentration exclusion. The key structural step is the representation principle established through Lemma 1 and Proposition A1,

$$E(z, r) \Rightarrow \Theta(z, r), \quad (207)$$

which converts a localized mechanical concentration into a thermodynamic concentration. From this viewpoint, the present framework may be interpreted not merely as a concentration-exclusion theory, but as a concentration-representation theory. A natural question is whether an entirely incompressible representation hierarchy exists.

More specifically, one may ask whether there exists a purely incompressible concentration-control functional $I(z, r)$ such that

$$E(z, r) \Rightarrow I(z, r), \quad (208)$$

and

$$I(z_n, r_n) \rightarrow 0 \Rightarrow \text{concentration exclusion.} \quad (209)$$

If such a mechanism can be identified, the low-Mach framework would no longer be necessary. The resulting theory would establish structural closure entirely within incompressible Navier–Stokes dynamics.

The broader significance of the present framework is that concentration is not treated as a single object but as a phenomenon admitting multiple representations. Transfer

activity, localized mechanical concentration, thermodynamic traces, and concentration-control functionals describe the same underlying structure at different levels of observation.

The structural closure theorem may therefore be interpreted as a representation theorem for concentration mechanisms. The exclusion result arises because every admissible representation ultimately enters a regime in which concentration becomes incompatible with the governing structural constraints. The overall mechanism may be summarized schematically as

$$\begin{aligned} \text{Transfer} \Rightarrow \text{Mechanical Concentration} \Rightarrow \text{Thermodynamic Trace} \\ \Rightarrow \text{Concentration Exclusion.} \end{aligned} \quad (210)$$

Whether an entirely incompressible representation hierarchy exists remains an open problem. Nevertheless, the present work demonstrates that transfer reduction, concentration representation, and concentration exclusion can be assembled into a unified structural framework. This suggests that the persistent concentration problem may be approachable through a structural analysis of concentration mechanisms rather than solely through global norm estimates or continuation criteria.

11. Conclusions

This paper developed a conditional structural framework for the exclusion of a dynamically sustained persistent concentration in three-dimensional incompressible flow. The central contribution is not a new analytical regularity criterion, but a structural mechanism linking transfer dynamics, localized concentration, and thermo-acoustic admissibility within a single concentration-representation framework.

A key outcome of the analysis is the identification of a structural pathway through which a persistent concentration, if present, must generate a localized transfer-amplitude concentration and subsequently a nontrivial thermo-acoustic trace. Within the admissible low-Mach realizable class, this pathway is incompatible with the thermo-acoustic exclusion mechanism, leading to the conditional structural closure theorem established in Section 9.

The significance of the present framework lies in its representation viewpoint. Rather than controlling concentration directly through global norms or continuation criteria, the theory tracks concentration across multiple structural levels, connecting transfer activity, localized mechanical concentration, thermodynamic traces, and concentration-control functionals. This suggests a possible alternative perspective on concentration phenomena based on structural representations and exclusion mechanisms.

The principal limitation of the theory is its dependence on admissible low-Mach realizability. The inheritance and concentration-representation mechanisms are established once such realizations exist, but the present work does not prove that every potentially singular incompressible concentration sequence admits an admissible entropy-coercive low-Mach realization. Consequently, the structural closure theorem should be interpreted as a conditional result within the realizable class considered here rather than as an unconditional regularity theorem for the full three-dimensional incompressible Navier–Stokes equations.

Several directions remain open. The most important is the universality of admissible low-Mach realizability. A second question is whether the thermo-acoustic representation framework can be replaced by a purely incompressible concentration-control mechanism. More broadly, it would be valuable to determine whether the concentration-representation viewpoint can be connected to classical regularity theories and established

concentration criteria. Progress on any of these problems would substantially clarify the scope and significance of the present structural approach.

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Nomenclature

Roman Symbols

A_j	Localized transfer amplitude in shell j .
c	Generic positive constant.
c_{ent}	Entropy-coercivity constant.
$D(u)$	Symmetric strain-rate tensor.
D_{ent}	Entropy-production density.
$E(z, r)$	Mechanical concentration functional.
F	Relative free-energy density.
G_j	Coherent transfer activity in shell j .
$H(z, r)$	Thermo-acoustic concentration functional.
I_j	Packet index set associated with shell j .
k	Fourier wavenumber.
$M_j(z)$	Localized modulation factor.
M_{loc}	Local Mach number.
p	Pressure field.
$Q_r(z)$	Parabolic cylinder centered at z with radius r .
r	Spatial scale.
r_n	Concentration scale.
T_j	Shellwise transfer activity.
T_j^{HH}	High–High transfer activity.
u	Velocity field.
u_j	Littlewood–Paley shell component.
x	Spatial coordinate.
z	Space–time point.
z_n	Concentration point.

Greek Symbols

α	Coherent-transfer decay exponent.
ε_*	Thermo-acoustic ε -regularity threshold.
κ	Thermal conductivity.
λ	Second viscosity coefficient.
λ_j	Dyadic frequency associated with shell j .
μ	Dynamic viscosity coefficient.
ρ	Density.
ρ_n	Density field of the n -th low-Mach family.
σ	Viscous stress tensor.
τ	Triadic interaction index.
Φ_τ	Relative triadic phase.
Φ_*	Coherence threshold.
Ω_τ	Phase-drift rate.

Appendix A. Thermo-Acoustic ε -Regularity Framework

This appendix develops the thermo-acoustic ε -regularity mechanism used in Sections 2, 3, and 7. The purpose is to replace the use of informal “standard arguments” by an explicit chain of estimates:

Entropy coercivity $\Rightarrow$ localized free-energy control
 $\Rightarrow$ Caccioppoli estimate $\Rightarrow$ compactness
 $\Rightarrow$ Campanato decay $\Rightarrow$ ε -regularity $\Rightarrow$ concentration exclusion.

Appendix A.1. Uniform Admissible Structure

Let ρ denote the density, u the velocity field, T the absolute temperature, and D_{ent} the entropy-production density. The admissible thermo-acoustic class is assumed to satisfy

$$0 < c_\rho \leq \rho \leq C_\rho, \quad 0 < c_T \leq T \leq C_T. \quad (\text{A1})$$

The constitutive coefficients satisfy

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0, \quad \kappa > 0. \quad (\text{A2})$$

The entropy-production density satisfies the uniform coercive lower bound

$$D_{\text{ent}} \geq c_{\text{ent}}(|D(u)|^2 + |\nabla \log T|^2 + |\nabla \log \rho|^2). \quad (\text{A3})$$

Here

$$D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \quad (\text{A4})$$

is the symmetric strain-rate tensor. The constant c_{ent} is assumed to be independent of the concentration scale and of the low-Mach parameter. This uniformity is essential: without it, the ε -regularity threshold below could degenerate along a concentration sequence. The local entropy balance is written as

$$\partial_t(\rho s) + \nabla \cdot (\rho s u) = D_{\text{ent}}. \quad (\text{A5})$$

Thus, entropy production controls the mechanical strain, the temperature-gradient contribution, and the density-gradient contribution simultaneously.

Appendix A.2. Thermo-Acoustic Concentration Functional

Let

$$z_0 = (x_0, t_0), \quad Q_r(z_0) = B_r(x_0) \times (t_0 - r^2, t_0]. \quad (\text{A6})$$

Let F_{rel} denote the relative free-energy density. Define

$$H(z_0, r) = \frac{1}{r^3} \int_{Q_r(z_0)} F_{\text{rel}} \, dx \, dt + \frac{1}{r} \int_{Q_r(z_0)} D_{\text{ent}} \, dx \, dt. \quad (\text{A7})$$

The thermodynamic trace functional is

$$\Theta(z_0, r) = \frac{1}{r} \int_{Q_r(z_0)} D_{\text{ent}} \, dx \, dt. \quad (\text{A8})$$

By definition,

$$\Theta(z_0, r) \leq H(z_0, r). \quad (\text{A9})$$

Combining (A3) and (A8), we obtain

$$\Theta(z_0, r) \geq c_{\text{ent}} \frac{1}{r} \int_{Q_r(z_0)} (|D(u)|^2 + |\nabla \log T|^2 + |\nabla \log \rho|^2) \, dx \, dt. \quad (\text{A10})$$

Hence

$$H(z_0, r) \geq c_0 \frac{1}{r} \int_{Q_r(z_0)} (|D(u)|^2 + |\nabla \log T|^2 + |\nabla \log \rho|^2) \, dx \, dt. \quad (\text{A11})$$

Define

$$\Phi(z_0, r) = \frac{1}{r} \int_{Q_r(z_0)} (|D(u)|^2 + |\nabla \log T|^2 + |\nabla \log \rho|^2) \, dx \, dt. \quad (\text{A12})$$

Then (A11) becomes

$$H(z_0, r) \geq c_0 \Phi(z_0, r). \quad (\text{A13})$$

This is the first key estimate: small thermo-acoustic activity forces small entropy-coercive gradients.

Appendix A.3. Localized Free-Energy Inequality

Let ϕ be a smooth cutoff satisfying

$$0 \leq \phi \leq 1, \phi \equiv 1 \text{ on } Q_r(z_0), \text{ supp } \phi \subset Q_{2r}(z_0), \quad (\text{A14})$$

and

$$|\nabla \phi| \leq \frac{C}{r}, |\partial_t \phi| \leq \frac{C}{r^2}. \quad (\text{A15})$$

By multiplying the localized relative free-energy balance by ϕ^2 and integrating over $Q_{2r}(z_0)$, we obtain

$$\begin{aligned} & \sup_{t_0 - r^2 < t < t_0} \int_{B_r(x_0)} F_{\text{rel}} \phi^2 dx + \int_{Q_r(z_0)} D_{\text{ent}} \phi^2 dx dt \\ & \leq C \int_{Q_{2r}(z_0)} F_{\text{rel}} (|\nabla \phi|^2 + |\partial_t \phi|) dx dt. \end{aligned} \quad (\text{A16})$$

Using (A15),

$$|\nabla \phi|^2 + |\partial_t \phi| \leq \frac{C}{r^2}. \quad (\text{A17})$$

Substituting this into (A16) gives

$$\sup_{t_0 - r^2 < t < t_0} \int_{B_r(x_0)} F_{\text{rel}} dx + \int_{Q_r(z_0)} D_{\text{ent}} dx dt \leq CH(z_0, 2r). \quad (\text{A18})$$

Thus $H(z_0, 2r)$ controls both the free-energy content and entropy production on the smaller cylinder $Q_r(z_0)$.

Appendix A.4. Constructive Realization Criterion

The admissible low-Mach realizability condition introduced in Definition 1 is intended as a representation criterion rather than an abstract admissibility postulate.

The following proposition provides a sufficient condition under which admissible low-Mach realizability follows from verifiable approximation properties.

Proposition A1 (Constructive Realization Criterion). *Let u be an incompressible velocity field and let (ρ_n, u_n, T_n) be a weakly compressible approximation family satisfying*

$$u_n \rightarrow u \text{ strongly in } L^2_{loc}, \quad (\text{A19})$$

$$M_{loc,n}(z, r) \rightarrow 0, \quad (\text{A20})$$

$$0 < c_\rho \leq \rho_n \leq C_\rho, \quad (\text{A21})$$

$$0 < c_T \leq T_n \leq C_T, \quad (\text{A22})$$

and

$$D_{ent,n} \geq c_{ent} (|D(u_n)|^2 + |\nabla \log T_n|^2 + |\nabla \log \rho_n|^2), \quad (\text{A23})$$

uniformly on admissible local cylinders. Then every concentration sequence represented by this approximation family belongs to the admissible low-Mach realizable class. Moreover,

$$\limsup_{n \rightarrow \infty} M(z_n, r_n) > 0 \quad (\text{A24})$$

implies

$$\limsup_{n \rightarrow \infty} \Theta_n(z_n, r_n) > 0. \quad (\text{A25})$$

Proof. By (A23),

$$r^{-1} \int_{Q_r(z)} |D(u_n)|^2 \, dx dt \leq c_{\text{ent}}^{-1} r^{-1} \int_{Q_r(z)} D_{\text{ent},n} \, dx dt. \quad (\text{A26})$$

Using the definition of the thermodynamic trace functional,

$$\Theta_n(z, r) = r^{-1} \int_{Q_r(z)} D_{\text{ent},n} \, dx dt, \quad (\text{A27})$$

we obtain

$$r^{-1} \int_{Q_r(z)} |D(u_n)|^2 \, dx dt \leq c_{\text{ent}}^{-1} \Theta_n(z, r). \quad (\text{A28})$$

Together with the strong convergence (A19), the concentration-representation estimate of Lemma 1 yields

$$M(z_n, r_n) \leq C \Theta_n(z_n, r_n) + o(1). \quad (\text{A29})$$

Taking the limit superior proves (A25). Therefore, the approximation family generates an admissible low-Mach realization of the concentration sequence. $\square$

Remark A1. Proposition A1 shows that admissible low-Mach realizability is not introduced as an independent abstract assumption. Rather, realizability follows whenever an entropy-coercive weakly compressible approximation satisfying (A19)–(A23) exists. Consequently, admissible realizability is reduced to verifiable approximation properties.

Appendix A.5. Caccioppoli Estimate

From (A3),

$$\begin{aligned} \int_{Q_r(z_0)} (|D(u)|^2 + |\nabla \log T|^2 + |\nabla \log \rho|^2) \, dx dt \\ \leq c_{\text{ent}}^{-1} \int_{Q_r(z_0)} D_{\text{ent}} \, dx dt. \end{aligned} \quad (\text{A30})$$

Multiplying (A30) by r^{-1} and using the entropy-production component of (A18), together with the definition of $H(z_0, 2r)$, we obtain

$$\Phi(z_0, r) \leq CH(z_0, 2r). \quad (\text{A31})$$

This is the thermo-acoustic Caccioppoli estimate. It shows that if

$$H(z_0, 2r) < \varepsilon, \quad (\text{A32})$$

then

$$\Phi(z_0, r) < C\varepsilon. \quad (\text{A33})$$

Thus, the small thermo-acoustic concentration on $Q_{2r}(z_0)$ forces small mechanical, thermal, and density-gradient concentrations on $Q_r(z_0)$.

Appendix A.6. Blow-Up Normalization

Assume that ε -regularity fails. Then there exists a sequence

$$(z_n, r_n), r_n \rightarrow 0, \quad (\text{A34})$$

such that

$$H(z_n, 2r_n) \rightarrow 0, \quad (\text{A35})$$

but

$$(\rho, u, T) \notin C^\alpha(Q_{r_n}(z_n)). \quad (\text{A36})$$

Introduce the normalized variables

$$x = x_n + r_n y, t = t_n + r_n^2 s. \quad (\text{A37})$$

Define

$$\tilde{\rho}_n(y, s) = \rho(x_n + r_n y, t_n + r_n^2 s), \quad (\text{A38})$$

$$\tilde{u}_n(y, s) = u(x_n + r_n y, t_n + r_n^2 s), \quad (\text{A39})$$

$$\tilde{T}_n(y, s) = T(x_n + r_n y, t_n + r_n^2 s). \quad (\text{A40})$$

The cylinders $Q_{r_n}(z_n)$ are transformed into the fixed unit cylinder

$$Q_1 = B_1 \times (-1, 0]. \quad (\text{A41})$$

Because H is a scale invariant,

$$H(z_n, 2r_n) = H_n(0, 2), \quad (\text{A42})$$

and therefore

$$H_n(0, 2) \rightarrow 0. \quad (\text{A43})$$

Using (A31),

$$\Phi_n(0, 1) \rightarrow 0. \quad (\text{A44})$$

Consequently,

$$\int_{Q_1} |D(\tilde{u}_n)|^2 \rightarrow 0, \quad (\text{A45})$$

$$\int_{Q_1} |\nabla \log \tilde{T}_n|^2 \rightarrow 0, \quad (\text{A46})$$

$$\int_{Q_1} |\nabla \log \tilde{\rho}_n|^2 \rightarrow 0. \quad (\text{A47})$$

Appendix A.7. Compactness Lemma

The admissibility assumptions imply

$$0 < c_\rho \leq \tilde{\rho}_n \leq C_\rho, \quad (\text{A48})$$

$$0 < c_T \leq \tilde{T}_n \leq C_T. \quad (\text{A49})$$

Furthermore,

$$\tilde{u}_n \text{ is bounded in } L_t^\infty L_x^2 \cap L_t^2 H_x^1. \quad (\text{A50})$$

The rescaled equations imply

$$\partial_s \tilde{u}_n \text{ is bounded in } L^2(-1, 0; H^{-1}(B_1)). \quad (\text{A51})$$

Hence, by the Aubin–Lions lemma,

$$\tilde{u}_n \rightarrow u_\infty \text{ strongly in } L^2(Q_1). \quad (\text{A52})$$

Similarly,

$$\tilde{T}_n \rightarrow T_\infty, \quad (\text{A53})$$

$$\tilde{\rho}_n \rightarrow \rho_\infty. \quad (\text{A54})$$

after the extraction of a subsequence.

Appendix A.8. Limiting Profile

Passing to the limit in (A45)–(A47),

$$D(u_\infty) = 0, \quad (\text{A55})$$

$$\nabla \log T_\infty = 0, \quad (\text{A56})$$

$$\nabla \log \rho_\infty = 0. \quad (\text{A57})$$

Therefore,

$$u_\infty = a + b \times x, \quad (\text{A58})$$

for constants a, b . Moreover,

$$T_\infty = \text{constant}, \quad (\text{A59})$$

$$\rho_\infty = \text{constant}. \quad (\text{A60})$$

The limiting profile is therefore smooth.

Appendix A.9. Campanato Decay Lemma

Lemma A1 (Campanato Decay). *There exist constants*

$$\theta \in (0,1), \beta > 0, \quad (\text{A61})$$

such that

$$H(z, \theta r) \leq \frac{1}{2} \theta^\beta H(z, r) \quad (\text{A62})$$

whenever

$$H(z, r) < \varepsilon_*. \quad (\text{A63})$$

Proof. Assume that (A63) fails. Then there exists a violating sequence. After normalization and compactness, the limiting profile satisfies (A55)–(A60). Since the limit is smooth,

$$H_\infty(0, \theta) = 0. \quad (\text{A64})$$

Strong convergence implies

$$H_n(0, \theta) \rightarrow 0. \quad (\text{A65})$$

This contradicts the assumed failure of (A62). Therefore, the decay estimate holds. $\square$

Appendix A.10. Iterative Decay

Applying (A62) repeatedly,

$$H(z, \theta^k r) \leq 2^{-k} \theta^{k\beta} H(z, r). \quad (\text{A66})$$

Hence,

$$H(z, \rho) \leq C \left(\frac{\rho}{r} \right)^\beta H(z, r) \quad (\text{A67})$$

for every

$$0 < \rho < r. \quad (\text{A68})$$

This is the Campanato decay estimate.

Appendix A.11. Thermo-Acoustic ε -Regularity Theorem

Theorem A1. *There exist*

$$\varepsilon_* > 0, \alpha \in (0,1), \quad (\text{A69})$$

such that

$$H(z_0, 2r) < \varepsilon_* \quad (\text{A70})$$

implies

$$(\rho, u, T) \in C^\alpha(Q_r(z_0)). \quad (\text{A71})$$

Proof. By (A67),

$$H(z_0, \rho) \leq C\rho^\beta. \quad (\text{A72})$$

Campanato's characterization theorem yields

$$u \in C^\alpha, \quad (\text{A73})$$

$$T \in C^\alpha, \quad (\text{A74})$$

$$\rho \in C^\alpha. \quad (\text{A75})$$

Hence, $(\rho, u, T) \in C^\alpha(Q_r(z_0))$. $\square$

Appendix A.12. Thermo-Acoustic Exclusion Principle

Theorem A2 (Thermo-Acoustic Exclusion Consequence). *Let*

$$(z_n, r_n), r_n \rightarrow 0, \quad (\text{A76})$$

be an admissible concentration sequence. Then

$$H(z_n, r_n) \rightarrow 0. \quad (\text{A77})$$

Proof. Assume instead

$$\limsup_{n \rightarrow \infty} H(z_n, r_n) > 0. \quad (\text{A78})$$

Then there exists $\delta > 0$ such that

$$H(z_n, r_n) \geq \delta \quad (\text{A79})$$

along a subsequence. Applying the blow-up procedure of Sections A.5-A.8 produces a limiting profile satisfying

$$H_\infty = 0. \quad (\text{A80})$$

Strong convergence implies

$$H(z_n, r_n) \rightarrow 0, \quad (\text{A81})$$

contradicting (A79). Therefore,

$$H(z_n, r_n) \rightarrow 0. \quad (\text{A82})$$

$\square$

Appendix A.13. Consequence for Branch I

Combining Theorem A2 with the low-Mach inheritance principle,

$$E(z_n, r_n) \rightarrow 0 \Rightarrow \Theta(z_n, r_n) \rightarrow 0 \Rightarrow H(z_n, r_n) \rightarrow 0. \quad (\text{A83})$$

However, Theorem A2 gives

$$H(z_n, r_n) \rightarrow 0. \quad (\text{A84})$$

This contradiction yields

$$\text{Branch I} \Rightarrow \perp. \quad (\text{A85})$$

Therefore, a persistent concentration cannot be sustained through the amplitude branch.

Appendix B. Low-Mach Inheritance Proofs

Appendix B.1. Compactness Framework

Let (ρ_n, u_n, T_n) be an admissible low-Mach family satisfying the assumptions of Section 3. The associated Mach number satisfies

$$M_n \rightarrow 0. \quad (\text{A86})$$

The entropy-coercive estimate yields

$$\int_Q (|D(u_n)|^2 + |\nabla \log T_n|^2 + |\nabla \log \rho_n|^2) dx dt \leq C. \quad (\text{A87})$$

Furthermore,

$$\sup_n \|u_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C. \quad (\text{A88})$$

Hence,

$$u_n \text{ is bounded in } L^2(0,T;H^1(\Omega)). \quad (\text{A89})$$

From the momentum equation and admissibility assumptions,

$$\partial_t u_n \text{ is bounded in } L^2(0,T;H^{-1}(\Omega)). \quad (\text{A90})$$

Applying the Aubin–Lions–Simon theorem gives

$$u_n \rightarrow u \text{ strongly in } L^2_{\text{loc}}(Q). \quad (\text{A91})$$

Moreover,

$$\rho_n \rightarrow \rho, T_n \rightarrow T. \quad (\text{A92})$$

The strong convergence (A91) is the compactness mechanism used below.

Appendix B.2. Strong Convergence and Concentration Persistence

Let

$$Q_r(z) = B_r(x) \times (t - r^2, t]. \quad (\text{A93})$$

For every fixed cylinder,

$$D(u_n) \rightharpoonup D(u) \text{ weakly in } L^2(Q_r(z)). \quad (\text{A94})$$

By weak lower semi continuity,

$$\int_{Q_r(z)} |D(u)|^2 dx dt \leq \liminf_{n \rightarrow \infty} \int_{Q_r(z)} |D(u_n)|^2 dx dt. \quad (\text{A95})$$

Consequently, a concentration cannot disappear through strong convergence. Indeed, if

$$E(z_n, r_n) = \frac{1}{r_n} \int_{Q_{r_n}(z_n)} |D(u_n)|^2 dx dt \quad (\text{A96})$$

satisfies

$$\limsup_{n \rightarrow \infty} E(z_n, r_n) > 0, \quad (\text{A97})$$

then the associated strain-energy concentration remains detectable in the limit.

Appendix B.3. Thermodynamic Trace

Define

$$E(z, r) = \frac{1}{r} \int_{Q_r(z)} |D(u)|^2 \, dx \, dt. \quad (\text{A98})$$

The thermodynamic trace functional is

$$\Theta(z, r) = \frac{1}{r} \int_{Q_r(z)} D_{\text{ent}} \, dx \, dt. \quad (\text{A99})$$

Entropy coercivity gives

$$D_{\text{ent}} \geq c_{\text{ent}} |D(u)|^2. \quad (\text{A100})$$

Integrating over $Q_r(z)$,

$$\Theta(z, r) \geq c_{\text{ent}} E(z, r). \quad (\text{A101})$$

Equation (A101) is the fundamental bridge between mechanical concentration and thermodynamic concentration.

Appendix B.4. Constructive Realizability Criterion

Proposition A2 (Constructive Realizability Criterion). *Let*

$$u \in L^\infty(0, T; H^s(\Omega)), s > \frac{5}{2}, \quad (\text{A102})$$

and let (z_n, r_n) be a concentration sequence associated with u . Assume that there exists a weakly compressible approximation $(\rho_\varepsilon, u_\varepsilon, T_\varepsilon)$ satisfying

$$u_\varepsilon \rightarrow u \text{ strongly in } L^2_{\text{loc}}(Q), \quad (\text{A103})$$

$$D(u_\varepsilon) \rightarrow D(u) \text{ strongly in } L^2_{\text{loc}}(Q), \quad (\text{A104})$$

$$\rho_\varepsilon \rightarrow \rho_0 > 0, \, T_\varepsilon \rightarrow T_0 > 0, \quad (\text{A105})$$

and

$$M_{\text{loc}}^{(\varepsilon)}(z, r) \rightarrow 0 \quad (\text{A106})$$

uniformly on admissible cylinders. Suppose further that

$$0 < c_\rho \leq \rho_\varepsilon \leq C_\rho, \, 0 < c_T \leq T_\varepsilon \leq C_T, \quad (\text{A107})$$

$$\mu > 0, \, 2\mu + 3\lambda \geq 0, \, \kappa > 0, \quad (\text{A108})$$

and

$$D_{\text{ent}}^{(\varepsilon)} \geq c_{\text{ent}} (|D(u_\varepsilon)|^2 + |\nabla \log T_\varepsilon|^2 + |\nabla \log \rho_\varepsilon|^2), \quad (\text{A109})$$

with a constant $c_{\text{ent}} > 0$ independent of ε . Then the concentration sequence (z_n, r_n) admits an admissible low-Mach realization. Moreover,

$$\limsup_{n \rightarrow \infty} E(z_n, r_n) > 0 \quad (\text{A110})$$

implies

$$\limsup_{n \rightarrow \infty} \Theta(z_n, r_n) > 0. \quad (\text{A111})$$

Proof. Conditions (A103)–(A109) coincide with the hypotheses of Lemma 1. Therefore, the approximation family $(\rho_\varepsilon, u_\varepsilon, T_\varepsilon)$ forms an admissible low-Mach family representing the

concentration sequence. Lemma 1 then yields admissible low-Mach realizability. Furthermore, Lemma 1 gives

$$\limsup_{n \rightarrow \infty} \Theta(z_n, r_n) \geq c_{\text{ent}} \limsup_{n \rightarrow \infty} E(z_n, r_n). \quad (\text{A112})$$

Hence, (A110) implies (A111). $\square$

Remark A2 (Realizability versus Inheritance). *Proposition A2 clarifies the logical role of low-Mach realizability. The present theory does not assume realizability as a purely abstract property. Rather, realizability follows whenever an admissible entropy-coercive weakly compressible approximation exists. In particular,*

$$\begin{aligned} \text{Entropy-Coercive Approximation} &\Rightarrow \text{Low-Mach Realizability} \\ &\Rightarrow \text{Thermodynamic Trace.} \end{aligned} \quad (\text{A113})$$

Thus, the remaining open question is not the inheritance mechanism itself, but the existence and universality of admissible entropy-coercive approximations.

Appendix B.5. Low-Mach Inheritance Principle

Theorem A3 (Low-Mach Inheritance Principle). *Let (ρ_n, u_n, T_n) be an admissible low-Mach family. Suppose*

$$\limsup_{n \rightarrow \infty} E(z_n, r_n) > 0. \quad (\text{A114})$$

Then

$$\limsup_{n \rightarrow \infty} \Theta(z_n, r_n) > 0. \quad (\text{A115})$$

Proof. Assume instead

$$\Theta(z_n, r_n) \rightarrow 0. \quad (\text{A116})$$

Using (A113),

$$E(z_n, r_n) \leq c_{\text{ent}}^{-1} \Theta(z_n, r_n). \quad (\text{A117})$$

Passing to the limit,

$$E(z_n, r_n) \rightarrow 0. \quad (\text{A118})$$

This contradicts (A114). Hence, $\limsup_{n \rightarrow \infty} \Theta(z_n, r_n) > 0$. $\square$

Appendix B.6. Concentration Transfer

The inheritance principle immediately yields thermo-acoustic activation.

Corollary A1 (Concentration Transfer). *If*

$$\limsup_{n \rightarrow \infty} E(z_n, r_n) > 0, \quad (\text{A119})$$

then

$$\limsup_{n \rightarrow \infty} H(z_n, r_n) > 0. \quad (\text{A120})$$

Proof. By definition,

$$H(z, r) = \frac{1}{r^3} \int_{Q_r(z)} F_{\text{rel}} \, dx \, dt + \Theta(z, r). \quad (\text{A121})$$

Since

$$F_{\text{rel}} \geq 0, \quad (\text{A122})$$

we have

$$H(z, r) \geq \Theta(z, r). \quad (\text{A123})$$

Using Theorem A3,

$$\limsup_{n \rightarrow \infty} \Theta(z_n, r_n) > 0. \quad (\text{A124})$$

Hence,

$$\limsup_{n \rightarrow \infty} H(z_n, r_n) > 0. \quad (\text{A125})$$

□

Therefore,

$$E(z_n, r_n) > 0 \Rightarrow \Theta(z_n, r_n) > 0 \Rightarrow H(z_n, r_n) > 0. \quad (\text{A126})$$

Equation (A126) is the concentration-transfer mechanism used in Section 7 to convert localized amplitude concentration into thermo-acoustic activation.

Appendix C. Localized Littlewood–Paley Estimates

This appendix establishes the localized Littlewood–Paley estimates used in Section 7. The purpose is to derive a shell-uniform bridge between localized High–High transfer amplitudes and the localized concentration functional. Particular attention is devoted to finite-overlap estimates, Bernstein inequalities, localization errors, and shell-uniform constants.

Appendix C.1. Packet Localization

Let

$$u = \sum_j u_j \quad (\text{A127})$$

be the Littlewood–Paley decomposition.

For each shell j ,

$$u_j = \sum_{\alpha \in I_j} u_{j,\alpha}. \quad (\text{A128})$$

Each packet satisfies

$$\text{diam}(\text{supp } u_{j,\alpha}) \leq C_P \lambda_j^{-1}, \quad (\text{A129})$$

where

$$\lambda_j = 2^j. \quad (\text{A130})$$

Let

$$Q_r(z_0) = B_r(x_0) \times (t_0, -r^2 t_0) \quad (\text{A131})$$

be a parabolic cylinder. Define the localized packet index set

$$I_j(z_0, r) = \{\alpha \in I_j : \text{supp}(u_{j,\alpha}) \cap Q_r(z_0) \neq \emptyset\}. \quad (\text{A132})$$

The localized shell field is

$$u_j^{\text{loc}} = \sum_{\alpha \in I_j(z_0, r)} u_{j,\alpha}. \quad (\text{A133})$$

Appendix C.2. Finite Overlap Property

The packet family may be chosen so that the overlap number is uniformly bounded.

Lemma A2 (Finite Overlap). *There exists a constant $N_P > 0$, independent of j , such that*

$$\sum_{\alpha} 1_{\text{supp}(u_{j,\alpha})} \leq N_P. \quad (\text{A134})$$

Proof. The packets are constructed from a partition of unity in physical space with the characteristic scale λ_j^{-1} . After dyadic rescaling,

$$x \mapsto \lambda_j x, \quad (\text{A135})$$

the packet geometry becomes independent of j . Since only finitely many neighboring partition elements intersect any given point, the overlap number remains uniformly bounded.

Hence,

$$N_P = \sup_x \#\{\alpha: x \in \text{supp}(u_{j,\alpha})\} < \infty. \quad (\text{A136})$$

□

Consequently,

$$\|u_j^{\text{loc}}\|_{L^2(Q_r)}^2 \leq N_P \sum_{\alpha \in I_j(\mathbb{Z}_0 r)} \|u_{j,\alpha}\|_{L^2(Q_r)}^2. \quad (\text{A137})$$

Appendix C.3. Bernstein Chain

The next step is to estimate the derivatives of localized packets.

Lemma A3 (Bernstein Estimates). *For every packet $u_{j,\alpha}$,*

$$\|\nabla u_{j,\alpha}\|_{L^2} \leq C_B \lambda_j \|u_{j,\alpha}\|_{L^2}, \quad (\text{A138})$$

and

$$\|u_{j,\alpha}\|_{L^\infty} \leq C_B \lambda_j^{3/2} \|u_{j,\alpha}\|_{L^2}. \quad (\text{A139})$$

Consequently,

$$\|\nabla u_{j,\alpha}\|_{L^\infty} \leq C_B \lambda_j^{5/2} \|u_{j,\alpha}\|_{L^2}. \quad (\text{A140})$$

Proof. The Fourier support of each packet satisfies

$$\text{supp } \hat{u}_{j,\alpha} \subset \{c_1 \lambda_j \leq |\xi| \leq c_2 \lambda_j\}. \quad (\text{A141})$$

Standard Bernstein inequalities imply

$$\|D^m f\|_{L^q} \leq C \lambda_j^{m+3(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}. \quad (\text{A142})$$

Choosing $(p, q, m) = (2, 2, 1)$ gives (A138).

Choosing $(p, q, m) = (2, \infty, 0)$ gives (A139).

Applying (A138) to $\nabla u_{j,\alpha}$ yields (A140). □

Appendix C.4. Localized Korn–Bernstein Estimate

The next estimate provides the Korn-type control required for the localized LP bridge.

Lemma A4 (Localized Korn–Bernstein Estimate). *Let $B_R^* \supset B_R$ be a fixed smooth enlargement. Then*

$$\|u_j^{\text{loc}}\|_{L^2(B_R)} \leq C_K \lambda_j^{-1} (\|D(u_j)\|_{L^2(B_R^*)} + \|u_j\|_{L^2(B_R^*)}). \quad (\text{A143})$$

Proof. Since u_j is spectrally localized at frequency λ_j ,

$$\|u_j^{\text{loc}}\|_{L^2(B_R)} \leq C \lambda_j^{-1} \|\nabla u_j\|_{L^2(B_R^*)}. \quad (\text{A144})$$

Because B_R^* is a bounded Lipschitz domain, the local Korn inequality yields

$$\|\nabla u_j\|_{L^2(B_R^*)} \leq C(\|D(u_j)\|_{L^2(B_R^*)} + \|u_j\|_{L^2(B_R^*)}). \quad (\text{A145})$$

See, for example, Acosta-Durán [38] and Galdi [43] for Korn inequalities on bounded Lipschitz domains. Combining (A144) and (A145) yields (A143). $\square$

Appendix C.5. Localized Strain Control

Define

$$C(z_0, r) = r^{-1} \iint_{Q_r(z_0)} |D(u)|^2 \, dx \, dt. \quad (\text{A146})$$

Lemma A5 (Localized Strain Control). *There exists a constant $C > 0$ such that*

$$\|u_j^{\text{loc}}\|_{L^2}^2 \leq C\lambda_j^{-2} \iint_{Q_r^*} |D(u)|^2 + C\lambda_j^{-2} \|u_j\|_{L^2(Q_r^*)}^2, \quad (\text{A147})$$

where

$$r = C_R \lambda_j^{-1}. \quad (\text{A148})$$

Proof. Squaring (A143) gives

$$\|u_j^{\text{loc}}\|_{L^2}^2 \leq C\lambda_j^{-2} (\|D(u_j)\|_{L^2(Q_r^*)}^2 + \|u_j\|_{L^2(Q_r^*)}^2). \quad (\text{A149})$$

Choose cutoffs $\chi_1, \chi_2 \in C_c^\infty(Q_r^*)$, satisfying

$$\chi_1 \equiv 1 \text{ on a neighborhood of } \text{supp } \chi_2. \quad (\text{A150})$$

The standard off-diagonal decay property of the Littlewood–Paley projections implies

$$\|(1 - \chi_1)P_j(\chi_2 f)\|_{L^2} \leq C_N \lambda_j^{-N} \|f\|_{L^2}, \quad N \geq 1. \quad (\text{A151})$$

By decomposing

$$f = \chi_1 f + (1 - \chi_1)f, \quad (\text{A152})$$

one obtains the almost-locality estimate

$$\|\chi_2 P_j f\|_{L^2} \leq C \|f\|_{L^2(Q_r^*)} + C_N \lambda_j^{-N} \|f\|_{L^2}. \quad (\text{A153})$$

Applying (A153) to $f = D(u)$,

$$\|D(u_j)\|_{L^2(Q_r)} \leq C \|D(u)\|_{L^2(Q_r^*)} + C_N \lambda_j^{-N} \|D(u)\|_{L^2}. \quad (\text{A154})$$

Hence,

$$\|D(u_j)\|_{L^2(Q_r)}^2 \leq C \iint_{Q_r^*} |D(u)|^2 + C_N \lambda_j^{-N} \|D(u)\|_{L^2}^2. \quad (\text{A155})$$

Substituting into (A149),

$$\|u_j^{\text{loc}}\|_{L^2}^2 \leq C\lambda_j^{-2} \iint_{Q_r^*} |D(u)|^2 + C\lambda_j^{-2} \|u_j\|_{L^2(Q_r^*)}^2 + C_N \lambda_j^{-N-2}. \quad (\text{A156})$$

The final term is rapidly decaying and may be absorbed into the localization error.

Therefore, (A147) follows. $\square$

Appendix C.6. Boundary Packet Error

The localized transfer amplitude only involves packets fully contained in the localization cylinder. Boundary-intersecting packets generate an error term.

Let $\chi_{z_0, r}$ be a smooth cutoff satisfying

$$\chi_{z_0, r} = 1 \text{ on } Q_r(z_0), \quad (\text{A157})$$

and

$$|\nabla \chi_{z_0, r}| \leq C r^{-1}. \quad (\text{A158})$$

Define

$$u_j^{\text{int}} = \chi_{z_0, r} u_j, u_j^{\text{bd}} = (1 - \chi_{z_0, r}) u_j. \quad (\text{A159})$$

Lemma A6 (Boundary Commutator Estimate). *There exists a constant $C > 0$, independent of j , such that*

$$\| [\Delta_j, \chi_{z_0, r}] u \|_{L^2} \leq C \lambda_j^{-1} r^{-1} \| u \|_{L^2}. \quad (\text{A160})$$

Proof. Let

$$\Delta_j f(x) = \int K_j(x - y) f(y) dy, \quad (\text{A161})$$

where

$$K_j(x) = \lambda_j^3 K(\lambda_j x). \quad (\text{A162})$$

The commutator may be written as

$$[\Delta_j, \chi] u(x) = \int (\chi(x) - \chi(y)) K_j(x - y) u(y) dy. \quad (\text{A163})$$

Using the mean-value estimate

$$| \chi(x) - \chi(y) | \leq \| \nabla \chi \|_{L^\infty} | x - y |, \quad (\text{A164})$$

together with

$$\int | x | | K_j(x) | dx = \lambda_j^{-1} \int | y | | K(y) | dy, \quad (\text{A165})$$

Young's inequality yields (A160). $\square$

Consequently,

$$| R_j(z_0) | \leq C \lambda_j^{-1} C(z_0, C_R \lambda_j^{-1}) \quad (\text{A166})$$

Hence,

$$R_j(z_0) \rightarrow 0, j \rightarrow \infty. \quad (\text{A167})$$

Appendix C.7. Localized LP Bridge

We now establish the principal estimate connecting localized High–High transfer amplitudes with the localized concentration functional.

Recall

$$C(z_0, r) = r^{-1} \iint_{Q_r(z_0)} | D(u) |^2 dx dt. \quad (\text{A168})$$

Let

$$A_j(z_0) = \sum_{(\alpha, \beta, \gamma) \in T_j(z_0)} | \int (u_{j, \alpha} \cdot \nabla) u_{j, \beta} \cdot u_{j, \gamma} dx |, \quad (\text{A169})$$

where $T_j(z_0)$ denotes the family of packet triples intersecting $Q_{C_R \lambda_j^{-1}}(z_0)$.

Lemma A7 (Localized LP Bridge). *There exist positive constants $C_B, C_R, \beta > 0$, independent of j , such that*

$$A_j(z_0) \leq C_B C(z_0, C_R \lambda_j^{-1}) + R_j(z_0), \quad (\text{A170})$$

with

$$| R_j(z_0) | \leq C_B \lambda_j^{-\beta} C(z_0, C_R \lambda_j^{-1}). \quad (\text{A171})$$

Proof. From (A127),

$$A_j(z_0) \leq \sum_{T_j(z_0)} \int |u_{j,\alpha}| |\nabla u_{j,\beta}| |u_{j,\gamma}| dx. \quad (\text{A172})$$

Applying Hölder's inequality,

$$A_j(z_0) \leq \sum_{T_j(z_0)} \|u_{j,\alpha}\|_{L^2} \|\nabla u_{j,\beta}\|_{L^\infty} \|u_{j,\gamma}\|_{L^2}. \quad (\text{A173})$$

Using (A140),

$$\|\nabla u_{j,\beta}\|_{L^\infty} \leq C\lambda_j^{5/2} \|u_{j,\beta}\|_{L^2}. \quad (\text{A174})$$

Substituting (A174) into (A173),

$$A_j(z_0) \leq C\lambda_j^{5/2} \sum_{T_j(z_0)} \|u_{j,\alpha}\|_{L^2} \|u_{j,\beta}\|_{L^2} \|u_{j,\gamma}\|_{L^2}. \quad (\text{A175})$$

Because of the finite-overlap property (A134), only finitely many packet triples contribute.

Hence,

$$A_j(z_0) \leq C\lambda_j^{5/2} \|u_j^{\text{loc}}\|_{L^2}^3. \quad (\text{A176})$$

Using the strain-control estimate (A147),

$$\|u_j^{\text{loc}}\|_{L^2}^2 \leq C\lambda_j^{-2} \iint_{Q_r^*} |D(u)|^2 + \text{Err}_j. \quad (\text{A177})$$

Therefore,

$$\|u_j^{\text{loc}}\|_{L^2}^3 \leq C\lambda_j^{-3} \left(\iint_{Q_r^*} |D(u)|^2 \right)^{3/2} + \text{Err}_j. \quad (\text{A178})$$

Using

$$\iint_{Q_r^*} |D(u)|^2 = r C(z_0, r), \quad (\text{A179})$$

with $r = C_R \lambda_j^{-1}$,

$$\|u_j^{\text{loc}}\|_{L^2}^3 \leq C\lambda_j^{-9/2} C(z_0, r)^{3/2} + \text{Err}_j. \quad (\text{A180})$$

Substituting (A180) into (A176),

$$A_j(z_0) \leq C\lambda_j^{-2} C(z_0, r)^{3/2} + \text{Err}_j. \quad (\text{A181})$$

Since $r = C_R \lambda_j^{-1}$, the right-hand side is bounded by

$$C_B C(z_0, C_R \lambda_j^{-1}) + R_j(z_0), \quad (\text{A182})$$

where R_j satisfies (A171). This proves the result. $\square$

Appendix C.8. Shell-Uniform Constants

A crucial point is that all constants appearing above are independent of the shell number.

Proposition A3 (Shell-Uniform Bounds). *There exists a constant $C_U > 0$ such that*

$$C_B \leq C_U, C_R \leq C_U, \quad (\text{A183})$$

for every shell index j .

Proof. The constants arise from:

1. The finite overlap number N_P ;
2. Bernstein inequalities;
3. Local Korn inequalities;
4. Littlewood–Paley projection bounds;
5. Commutator kernel moments.

After dyadic rescaling,

$$x \mapsto \lambda_j x, \quad (\text{A184})$$

and all packet geometries become scale independent. The overlap number remains bounded by N_p . The Bernstein constants depend only on dimension. The local Korn constant depends only on the fixed Lipschitz geometry of the localization domains.

Finally,

$$\int |x| |K_j(x)| dx = \lambda_j^{-1} \int |y| |K(y)| dy \quad (\text{A185})$$

shows that the commutator constants are shell-uniform.

Therefore,

$$C_U = C_U(d, N_p, \phi) \quad (\text{A186})$$

is independent of j .

Hence,

$$C_B \leq C_U, C_R \leq C_U. \quad (\text{A187})$$

□

As a consequence,

$$A_j(z_0) \leq C_U C(z_0, C_U \lambda_j^{-1}) + R_j(z_0) \quad (\text{A188})$$

uniformly in j .

Appendix C.9. Consequence

The localized LP bridge immediately implies a concentration transfer from localized transfer amplitudes to the localized mechanical concentration functional.

Corollary A2. *If*

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) > 0, \quad (\text{A189})$$

then

$$\limsup_{n \rightarrow \infty} C(z_n, C_U \lambda_{j_n}^{-1}) > 0. \quad (\text{A190})$$

Proof. Assume instead that

$$\limsup_{n \rightarrow \infty} C(z_n, C_U \lambda_{j_n}^{-1}) = 0. \quad (\text{A191})$$

Then there exists a subsequence, not relabeled, such that

$$C(z_n, C_U \lambda_{j_n}^{-1}) \rightarrow 0. \quad (\text{A192})$$

Using the shell-uniform estimate (A188),

$$A_{j_n}(z_n) \leq C_U C(z_n, C_U \lambda_{j_n}^{-1}) + R_{j_n}(z_n). \quad (\text{A193})$$

From (A171),

$$|R_{j_n}(z_n)| \leq C_U \lambda_{j_n}^{-\beta} C(z_n, C_U \lambda_{j_n}^{-1}). \quad (\text{A194})$$

Since

$$\lambda_{j_n} = 2^{j_n} \rightarrow \infty, \quad (\text{A195})$$

it follows from (A192) and (A194) that

$$R_{j_n}(z_n) \rightarrow 0. \quad (\text{A196})$$

Substituting (A192) and (A196) into (A193) yields

$$A_{j_n}(z_n) \rightarrow 0. \quad (\text{A197})$$

This contradicts (A189). Therefore,

$$\limsup_{n \rightarrow \infty} C(z_n, C_U \lambda_{j_n}^{-1}) > 0. \quad (\text{A198})$$

□

Localized Concentration Transfer

The preceding corollary yields the transfer-concentration implication required in Section 7.

Since

$$r_n = \lambda_{j_n}^{-1}, \quad (\text{A199})$$

there exists a shell-uniform constant $C_* > 0$ such that

$$C(z_n, C_U \lambda_{j_n}^{-1}) \leq C_* C(z_n, r_n). \quad (\text{A200})$$

Combining (A198) with (A200),

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) > 0 \Rightarrow \limsup_{n \rightarrow \infty} C(z_n, r_n) > 0. \quad (\text{A201})$$

Mechanical Concentration Consequence

Using the interpolation estimate employed in Section 7,

$$C(z, r) \leq C_I E(z, r), \quad (\text{A202})$$

we obtain

$$\limsup_{n \rightarrow \infty} C(z_n, r_n) > 0 \Rightarrow \limsup_{n \rightarrow \infty} E(z_n, r_n) > 0. \quad (\text{A203})$$

Combining (A201) and (A203),

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) > 0 \Rightarrow \limsup_{n \rightarrow \infty} E(z_n, r_n) > 0. \quad (\text{A204})$$

Final Localized Bridge

Therefore,

$$\limsup_{n \rightarrow \infty} A_{j_n}(z_n) > 0 \Rightarrow \limsup_{n \rightarrow \infty} C(z_n, r_n) > 0 \Rightarrow \limsup_{n \rightarrow \infty} E(z_n, r_n) > 0. \quad (\text{A205})$$

Equation (A205) is the localized concentration-transfer bridge used in Section 7 to convert a persistent transfer-amplitude concentration into a mechanical concentration and subsequently into thermo-acoustic activation.

Appendix D. High–High Transfer Reduction

This appendix summarizes the High–High core reduction results established in [22] and records only those components required for the present closure theory. The results of Appendix D.2–D.7 are inherited from [22] and are included here solely for completeness of exposition.

Appendix D.1. Dyadic Transfer Structure

Let $u = \sum_j u_j$ be the Littlewood–Paley decomposition of the velocity field. The shell index j corresponds to the characteristic dyadic frequency

$$\lambda_j = 2^j. \quad (\text{A206})$$

Applying Bony's paraproduct decomposition to the nonlinear term yields

$$(u \cdot \nabla)u = T_{LL} + T_{LH} + T_{HL} + T_{HH}. \quad (\text{A207})$$

The shellwise transfer activity is defined by

$$T_j = \int \Delta_j ((u \cdot \nabla)u) \cdot u_j \, dx. \quad (\text{A208})$$

Accordingly,

$$T_j = T_j^{LL} + T_j^{LH} + T_j^{HL} + T_j^{HH}. \quad (\text{A209})$$

The High–Low and Low–High channels differ only through commutator contributions.

Appendix D.2. Spectral Separation Lemma

Lemma A8 (Spectral Separation). *Let $M \geq 4$ be fixed. Then*

$$\text{supp}((u_{\leq j-M} \cdot \nabla)u_{\leq j-M}) \subset \{|\xi| \leq C\lambda_{j-M}\}. \quad (\text{A210})$$

Moreover,

$$\text{supp} \Delta_j \subset \{c_0\lambda_j \leq |\xi| \leq C_0\lambda_j\}. \quad (\text{A211})$$

Since

$$\lambda_{j-M} = 2^{-M}\lambda_j, \quad (\text{A212})$$

the supports satisfy

$$\text{dist}(\text{supp} \Delta_j, \text{supp}((u_{\leq j-M} \cdot \nabla)u_{\leq j-M})) \geq c\lambda_j. \quad (\text{A213})$$

Proof. Because

$$\text{supp} u_{\leq j-M} \subset \{|\xi| \leq C\lambda_{j-M}\}, \quad (\text{A214})$$

the convolution property of Fourier supports implies

$$\text{supp}((u_{\leq j-M} \cdot \nabla)u_{\leq j-M}) \subset \{|\xi| \leq C\lambda_{j-M}\}. \quad (\text{A215})$$

Combining (A211) and (A186) yields (A213). $\square$

Appendix D.3 Commutator Estimate

Lemma A9 (Commutator Estimate). *There exists a constant $C > 0$ independent of j such that*

$$|T_j^{HL} - T_j^{LH}| \leq C\lambda_j^{-1}. \quad (\text{A216})$$

Proof. The Littlewood–Paley commutator satisfies

$$[\Delta_j, f] \partial_k g = \Delta_j(f \partial_k g) - f \Delta_j(\partial_k g). \quad (\text{A217})$$

Standard commutator estimates give

$$\|[\Delta_j, f] \partial_k g\|_{L^2} \leq C\lambda_j^{-1} \|\nabla f\|_{L^\infty} \|\nabla g\|_{L^2}. \quad (\text{A218})$$

Applying (A218) to the nonlinear transfer integrals yields

$$|T_j^{HL} - T_j^{LH}| \leq C\lambda_j^{-1}. \quad (\text{A219})$$

$\square$

Therefore,

$$T_j^{HL} = T_j^{LH} + R_j^{(1)} \quad (\text{A220})$$

with

$$|R_j^{(1)}| \leq C\lambda_j^{-1}. \quad (\text{A221})$$

Substituting (A220) into (A209),

$$T_j = T_j^{LL} + T_j^{LH} + T_j^{HH} + R_j, \quad (\text{A222})$$

where

$$|R_j| \leq C\lambda_j^{-1}. \quad (\text{A223})$$

Appendix D.4. Low–Low Suppression

Proposition A4 (Low–Low Suppression). *There exists a constant $C_{LL} > 0$ such that*

$$|T_j^{LL}| \leq C_{LL}\lambda_j^{-1}. \quad (\text{A224})$$

Proof. By Lemma A9, $\Delta_j((u_{\leq j-M} \cdot \nabla)u_{\leq j-M})$ acts on frequencies separated from the support of the argument by a distance comparable to λ_j . Smooth Littlewood–Paley cutoffs therefore imply rapid decay:

$$\|\Delta_j((u_{\leq j-M} \cdot \nabla)u_{\leq j-M})\|_{L^2} \leq C_N \lambda_j^{-N} \quad (\text{A225})$$

for every $N > 0$. Choosing $N = 1$,

$$\|\Delta_j((u_{\leq j-M} \cdot \nabla)u_{\leq j-M})\|_{L^2} \leq C\lambda_j^{-1}. \quad (\text{A226})$$

Consequently,

$$|T_j^{LL}| \leq C_{LL}\lambda_j^{-1}. \quad (\text{A227})$$

□

The Sobolev quantities appearing in the Low–Low estimate serve only as auxiliary regularity parameters inherited from the transfer-reduction framework of [22]. They are not derived from the structural closure theorem proved in the present paper and are not used to establish the thermo-acoustic exclusion mechanism. Consequently, the Low–Low suppression estimate is logically independent of the concentration-exclusion argument developed in Sections 7–9 and does not create a circular dependence.

Appendix D.5. Low–High Suppression

Proposition A5 (Low–High Suppression). *There exists a constant $C_{LH} > 0$ such that*

$$|T_j^{LH}| \leq C_{LH}\lambda_j^{-1}. \quad (\text{A228})$$

Proof. Bernstein’s inequality gives

$$\|u_{\leq j-M}\|_{L^\infty} \leq C\lambda_j^{-1} \|\nabla u\|_{L^\infty}. \quad (\text{A229})$$

Hence,

$$|T_j^{LH}| \leq \|u_{\leq j-M}\|_{L^\infty} \|\nabla u_j\|_{L^2} \|u_j\|_{L^2}. \quad (\text{A230})$$

Using

$$\|\nabla u_j\|_{L^2} \leq C\lambda_j \|u_j\|_{L^2}, \quad (\text{A231})$$

we obtain

$$|T_j^{LH}| \leq C\lambda_j^{-1} \|\nabla u\|_{L^\infty} \|u_j\|_{L^2}^2. \quad (\text{A232})$$

Absorbing bounded factors into the constant yields

$$|T_j^{LH}| \leq C_{LH}\lambda_j^{-1}. \quad (\text{A233})$$

□

Combining (A224) and (A228),

$$|T_j^{LL}| + |T_j^{LH}| = O(\lambda_j^{-1}). \quad (\text{A234})$$

Therefore,

$$\lim_{j \rightarrow \infty} (T_j^{LL} + T_j^{LH}) = 0. \quad (\text{A235})$$

Appendix D.6. High–High Core Reduction

Theorem A4 (High–High Core Reduction). *Suppose a persistent concentration sequence exists. Then there exists a sequence*

$$j_n \rightarrow \infty \quad (\text{A236})$$

such that

$$\limsup_{n \rightarrow \infty} |T_{j_n}^{HH}| > 0. \quad (\text{A237})$$

Proof. Assume instead that

$$T_j^{HH} \rightarrow 0. \quad (\text{A238})$$

Using (A222) together with (A238),

$$T_j = T_j^{LL} + T_j^{LH} + R_j + o(1). \quad (\text{A239})$$

Applying (A223), (A224), and (A228),

$$T_j = O(\lambda_j^{-1}). \quad (\text{A240})$$

Hence,

$$T_j \rightarrow 0. \quad (\text{A241})$$

However, a persistent concentration requires non-vanishing scale-local transfer activity. Equation (A241) therefore contradicts the concentration hypothesis. Hence, (A238) is impossible, proving (A237). □

Appendix D.7. Transfer-Core Characterization

Corollary A3. *If a persistent concentration sequence exists, then*

$$\text{Persistent Concentration} \Rightarrow \text{High–High Transfer}. \quad (\text{A242})$$

Furthermore, by coherent-transfer reduction,

$$\text{High–High Transfer} \Rightarrow \text{Coherent High–High Transfer}. \quad (\text{A243})$$

Consequently,

$$\text{Persistent Concentration} \Rightarrow \text{Coherent High–High Transfer}. \quad (\text{A244})$$

Equation (A244) is the final output of the reduction theory and forms the starting point of the localized amplitude-modulation framework developed in Sections 5 and 6.

Because every triad satisfies the geometric constraint $k + p + q = 0$, the three wavevectors form a closed interaction triangle. Consequently, the three pairwise angles are not independent. For same-scale triads, a positive lower bound on one interaction angle is equivalent, up to universal geometric constants, to a positive lower bound on the remaining pairwise angles. Therefore, a core-region condition formulated using $|\sin \theta(k, p)| \geq c_0$

can equally be expressed using $\theta(p,q)$ or $\theta(q,k)$. The transfer-core characterization is thus independent of the particular pair of wavevectors chosen to define the angular condition.

Appendix E. Coherent Transfer Results

This appendix records supplementary coherent-transfer results inherited from [22]. The results collected here are not proven independently in the present paper. Rather, they are included in a summarized form to clarify their relation to the concentration framework developed in the main text. Detailed derivations, quantitative estimates, and the underlying coherent-transfer theory are contained in [22].

These results are not used in the proof of the structural closure theorem established in Section 9. Their role is solely to provide an additional coherent-transfer interpretation of persistent non-dispersive modulation and to facilitate comparison with the thermo-acoustic closure mechanism developed in the present work.

In particular, none of the results recorded in this appendix are required for the proof of Theorem 8.

Appendix E.1. Coherent Transfer Class

Let $G_j(t)$ denote the coherent transfer activity associated with shell j . For each triad $\tau = (k, \ell, m)$, let $\Phi_\tau(t)$ denote the corresponding relative phase. Fix a coherence threshold

$$\Phi_* > 0. \quad (\text{A245})$$

The coherent triad family is defined by

$$\mathcal{C}_j(t) = \{\tau : |\Phi_\tau(t)| \leq \Phi_*\}. \quad (\text{A246})$$

The coherent transfer activity is

$$G_j(t) = \sum_{\tau \in \mathcal{C}_j(t)} T_\tau(t), \quad (\text{A247})$$

where $T_\tau(t)$ denotes the transfer contribution of triad τ . Let $M_j(z)$ be the modulation factor introduced in Section 5. By construction,

$$0 \leq M_j(z) \leq 1. \quad (\text{A248})$$

Persistent non-dispersive modulation means

$$\limsup_{j \rightarrow \infty} M_j > 0. \quad (\text{A249})$$

The coherent-transfer reduction theory implies the following.

Lemma A10 (Modulation–Coherence Link). *There exists a constant $c_M > 0$ such that*

$$M_j \geq c_M \quad (\text{A250})$$

implies

$$G_j \geq c_M A_j. \quad (\text{A251})$$

Consequently,

$$M_j \rightarrow 0 \Rightarrow G_j \rightarrow 0. \quad (\text{A252})$$

Proof. By Definition 3,

$$M_j = \frac{|T_j^{HH,loc}|}{A_j}. \quad (\text{A253})$$

Therefore, (A250) implies

$$|T_j^{HH,loc}| \geq c_M A_j \quad (\text{A254})$$

The quantity A_j represents the total localized transfer amplitude, whereas $T_j^{HH,loc}$ is obtained after all cancellation effects between packet contributions have taken place.

Hence, (A254) shows that at least a fraction c_M of the total transfer amplitude survives destructive cancellation.

By the coherent-transfer decomposition developed in [22], the surviving transfer component is precisely carried by the coherent transfer family contributing to G_j . If the coherent transfer contribution were $o(A_j)$, then the surviving transfer amplitude would also be $o(A_j)$, contradicting (A254).

Therefore, the coherent transfer activity satisfies

$$G_j \geq c_M A_j. \quad (\text{A255})$$

This proves the result. $\square$

The coherent-transfer amplitudes are assumed to possess bounded variation on every compact coherent interval. This property follows from the shell-localized transfer estimates and the regularity assumptions underlying the coherent-transfer framework of [22]. In particular, the total variation of the amplitude remains finite on each coherent interval. Consequently, the amplitude functions admit well-defined endpoint traces, and the integration-by-parts arguments used in the residence-time analysis are justified without requiring additional smoothness assumptions.

Appendix E.2. Phase-Drift Lower Bound

The key obstruction to indefinite coherent amplification is the cumulative phase drift. For each coherent triad,

$$\frac{d\Phi_\tau}{dt} = \Omega_\tau, \quad (\text{A256})$$

where Ω_τ denotes the phase-drift rate. The coherent-transfer framework assumes the following non-degeneracy condition.

Assumption A1 (Averaged Phase-Drift Non-Degeneracy). *There exists a constant $c_\Omega > 0$ and an exponent $\alpha > 0$ such that, for every coherent triad and every time interval contained in a coherent interval,*

$$\left| \int_{t_0}^t \Omega_\tau(s) ds \right| \geq c_\Omega \lambda_j^\alpha |t - t_0|. \quad (\text{A257})$$

This condition excludes exact resonant locking and prevents cancellation of the accumulated phase drift over coherent intervals. We emphasize that no assumption is made that the amplitude-active set P_ε occupies the entire observation interval. The complement $(0, T) \setminus P_\varepsilon$ may contain intervals on which the coherent-transfer amplitude vanishes identically. Such intervals contribute no coherent-transfer activity and therefore play no role in the residence-time compression argument. Throughout Appendix E, all phase-drift estimates are applied only on connected components of P_ε where the amplitude-active condition is satisfied. Consequently, the behavior on zero-amplitude intervals does not affect the cumulative phase-drift mechanism or the coherent-transfer exclusion argument.

Integrating (A27) yields

$$\Phi_\tau(t) - \Phi_\tau(t_0) = \int_{t_0}^t \Omega_\tau(s) ds. \quad (\text{A258})$$

Using (A257), we obtain

$$|\Phi_\tau(t) - \Phi_\tau(t_0)| \geq c_\Omega \lambda_j^\alpha |t - t_0|. \quad (\text{A259})$$

Thus, phase coherence becomes progressively more difficult to maintain as $j \rightarrow \infty$.

Remark A3 (Status of Assumption A1). *Assumption A1 should be interpreted as a structural non-degeneracy assumption governing cumulative phase drift. The present paper does not derive Assumption A1 directly from the three-dimensional incompressible Navier–Stokes equations. Rather, A1 is inherited from the coherent-transfer framework developed in [22], where it serves as the fundamental hypothesis underlying residence-time compression, cumulative transfer decay, and coherent-transfer exclusion.*

More precisely, A1 excludes exact long-time phase locking by requiring that cumulative phase drift remains non-degenerative on sufficiently large modulation intervals. Under this assumption, a coherent transfer cannot persist indefinitely, and cumulative transfer activity necessarily decays.

Accordingly, the results of Appendix E should be interpreted as conditional upon Assumption A1. The present work does not claim that A1 holds universally for all Navier–Stokes flows or for all possible concentration scenarios.

The role of Appendix E is to record an additional coherent-transfer exclusion mechanism inherited from [22]. Within that framework, persistent non-dispersive modulation generates a coherent transfer, while coherent-transfer exclusion follows from residence-time compression together with cumulative transfer decay under Assumption A1.

The derivation, relaxation, or replacement of Assumption A1 by conditions arising directly from Navier–Stokes dynamics remains an important open problem.

Remark A4 (Physical Interpretation of Assumption A1). *Assumption A1 should not be interpreted as a generic prohibition of resonant or near-resonant triadic interactions. The coherent-transfer framework does not require that the phase-drift rate $\Omega_\tau(t)$ remain uniformly bounded away from zero at every instant. In particular, temporary phase alignment, near-resonant interactions, and finite-time coherent structures are not excluded by A1.*

The role of A1 is more specific. It requires that the cumulative phase drift over coherent intervals remain non-degenerative. Equivalently,

$$\left| \int_{t_0}^t \Omega_\tau(s) ds \right| \geq c_\Omega \lambda_j^\alpha |t - t_0|. \quad (\text{A260})$$

Thus, A1 excludes the possibility that coherent triads remain locked with an asymptotically vanishing cumulative phase drift over arbitrarily many dyadic scales. More precisely, A1 allows localized or transient near-resonant behavior, but excludes persistent exact phase locking capable of maintaining coherent transfer amplification indefinitely as

$$j \rightarrow \infty. \quad (\text{A261})$$

The coherent-transfer exclusion theory developed in [22] relies only on this cumulative phase-drift property. Residence-time compression and cumulative transfer decay are derived from the growth of the integrated phase drift

$$\int_{t_0}^t \Omega_\tau(s) ds, \quad (\text{A262})$$

rather than from a pointwise lower bound on the phase-drift rate itself. Consequently, coherent-transfer exclusion is based on cumulative phase evolution rather than on instantaneous non-resonance. In particular, finite-time near-resonant episodes are compatible with the framework, provided that the cumulative phase drift remains non-degenerative. Accordingly, A1 should be viewed as a structural non-degeneracy condition on cumulative coherent phase evolution rather than as a blanket exclusion of resonant dynamics. The logical role of A1 in the present theory may be summarized as

$$\text{Branch II} \Rightarrow \text{Coherent Transfer} \Rightarrow \text{Residence-Time Compression}$$

$$\Rightarrow \text{Transfer Decay} \Rightarrow \perp. \quad (\text{A263})$$

The second implication in (A263) relies on Assumption A1 through the coherent-transfer theory of [22]. Whether a condition of the form (A260) can be derived, relaxed, or replaced directly from incompressible Navier–Stokes dynamics remains an important open problem.

Appendix E.3. Residence-Time Compression

Define the coherent interval

$$I_j = \{t: |G_j(t)| \geq \eta\}, \quad (\text{A264})$$

where $\eta > 0$ is fixed. Its residence time is

$$\tau_j = |I_j|. \quad (\text{A265})$$

For the residence-time argument, I_j is understood to denote a maximal connected component of the low-drift (coherent) set under consideration. By maximality, every endpoint of I_j either coincides with the boundary of the observation interval or satisfies the threshold condition defining the coherent set. Consequently, the integration-by-parts argument used below generates boundary contributions only at such endpoints. These boundary terms remain uniformly controlled by the coherence threshold and therefore do not affect the residence-time compression estimate.

Proposition A6 (Residence-Time Compression). *There exist constants $C_R > 0, \alpha > 0$, such that*

$$\tau_j \leq C_R \lambda_j^{-\alpha}. \quad (\text{A266})$$

Proof. Coherence can persist only while

$$|\Phi_\tau| \leq \Phi_*. \quad (\text{A267})$$

Using (A259),

$$c_\Omega \lambda_j^\alpha |t - t_0| \leq 2\Phi_*. \quad (\text{A268})$$

Hence,

$$|t - t_0| \leq \frac{2\Phi_*}{c_\Omega} \lambda_j^{-\alpha}. \quad (\text{A269})$$

Taking the maximal coherent interval gives

$$\tau_j \leq C_R \lambda_j^{-\alpha}. \quad (\text{A270})$$

The coercivity estimate underlying Proposition A6 is used only on amplitude-active coherent intervals. If the amplitude-active condition fails, the associated coherent-transfer activity is already negligible and cannot support persistent coherent amplification. Consequently, such intervals do not enter the residence-time argument. The compression estimate is therefore applied exclusively in the amplitude-active regime where the coercivity condition holds, and no additional case distinction is required. $\square$

Corollary A4 (Scale-Invariant Energy Bounds for Dyadic Shells).

$$\tau_j \rightarrow 0 \quad (j \rightarrow \infty). \quad (\text{A271})$$

Thus, coherent amplification cannot persist over macroscopic time intervals.

Appendix E.4. Cumulative Transfer Decay

Define the cumulative coherent transfer

$$\mathcal{G}_j = \int_{I_j} |G_j(t)| \, dt. \quad (\text{A272})$$

Proposition A7 (Cumulative Transfer Decay). *There exists a constant $C_D > 0$ such that*

$$G_j \leq C_D \lambda_j^{-\alpha/2}. \quad (\text{A273})$$

Proof. Applying Hölder's inequality,

$$G_j = \int_{I_j} |G_j(t)| \, dt \leq |I_j|^{\frac{1}{2}} \left(\int_{I_j} |G_j(t)|^2 \, dt \right)^{\frac{1}{2}}. \quad (\text{A274})$$

By Proposition A6,

$$|I_j| = \tau_j \leq C_R \lambda_j^{-\alpha}. \quad (\text{A275})$$

The coherent-transfer framework established in [22] provides a shell-uniform coherent-transfer admissibility estimate. In the notation of the present paper, this estimate may be written as

$$\int_{I_j} |G_j(t)|^2 \, dt \leq C_E, \quad (\text{A276})$$

where the constant C_E is independent of the shell index j .

Equation (A276) is not proven in the present paper; it is an inherited estimate from the coherent-transfer framework of [22] and is included here solely to record the cumulative transfer-decay mechanism used in that theory.

Combining (A274)–(A276),

$$G_j \leq (C_R C_E)^{1/2} \lambda_j^{-\alpha/2}. \quad (\text{A277})$$

By defining

$$C_D = (C_R C_E)^{1/2}, \quad (\text{A278})$$

we obtain (A244). $\square$

Corollary A5 (Inheritance of Thermodynamic Concentration Traces).

$$\mathcal{G}_j \rightarrow 0 (j \rightarrow \infty). \quad (\text{A279})$$

Hence, the total coherent transfer budget vanishes at sufficiently small scales.

Appendix E.5. Coherent Transfer Exclusion

The preceding results imply that coherent amplification cannot sustain a persistent concentration.

Theorem A5 (Coherent Transfer Exclusion). *A dynamically sustained coherent-transfer sequence cannot exist. Equivalently,*

$$\limsup_{j \rightarrow \infty} \mathcal{G}_j = 0. \quad (\text{A280})$$

Proof. Assume the contrary. Then there exist $\gamma > 0$ and a subsequence

$$j_n \rightarrow \infty \quad (\text{A281})$$

such that

$$\mathcal{G}_{j_n} \geq \gamma \forall n. \quad (\text{A282})$$

However, Corollary A5 gives

$$\mathcal{G}_{j_n} \rightarrow 0. \quad (\text{A283})$$

Equations (A282) and (A283) are incompatible. Therefore,

$$\limsup_{j \rightarrow \infty} G_j = 0. \quad (\text{A284})$$

□

Appendix E.6. Conditional Coherent-Transfer Exclusion

Suppose that persistent non-dispersive modulation occurs,

$$\limsup_{j \rightarrow \infty} M_j > 0. \quad (\text{A285})$$

By Lemma A10,

$$M_j \rightarrow 0 \Rightarrow G_j \rightarrow 0. \quad (\text{A286})$$

Therefore,

$$\limsup_{j \rightarrow \infty} G_j > 0. \quad (\text{A287})$$

However, by Theorem A5,

$$\limsup_{j \rightarrow \infty} G_j = 0. \quad (\text{A288})$$

Equations (A287) and (A288) are incompatible. Hence,

$$\limsup_{j \rightarrow \infty} M_j = 0. \quad (\text{A289})$$

Thus, persistent non-dispersive modulation cannot be sustained under Assumption A1.

The result of this appendix should be interpreted as a supplementary coherent-transfer exclusion theorem inherited from [22]. The proof essentially relies on the averaged phase-drift non-degeneracy assumption (A1) and is logically independent of the thermo-acoustic concentration-representation framework developed in the main text.

Accordingly, the coherent-transfer exclusion theory recorded in Appendix E is not used in the proof of the structural closure theorem established in Section 9. Its role is instead to record an additional coherent-transfer exclusion mechanism and to clarify its relation to the structural concentration analysis developed in the present work.

The logical structure of the appendix may be summarized as

$$\begin{aligned} \text{Persistent Non-Dispersive Modulation} &\Rightarrow \text{Coherent Transfer} \\ &\Rightarrow \text{Transfer Decay} \Rightarrow \perp. \end{aligned} \quad (\text{A290})$$

The second implication in (A290) relies on Assumption A1 through the residence-time compression and cumulative transfer-decay theory inherited from [22].

The estimates (A251) and (A276) should be interpreted as inherited components of the coherent-transfer framework developed in [22]. In particular, (A255) records the quantitative relationship between persistent modulation and coherent transfer, while (A276) records the shell-uniform coherent-transfer admissibility bound used in the cumulative transfer-decay argument. Neither estimate is used in the proof of the structural closure theorem established in Section 9. Their role is solely to document the supplementary coherent-transfer exclusion mechanism inherited from [22], whose inclusion is intended for completeness and comparison with the thermo-acoustic closure framework developed in the main text.

Whether Assumption A1 can be derived, relaxed, or replaced by conditions arising directly from incompressible Navier–Stokes dynamics remains an important open problem. However, this question concerns the supplementary coherent-transfer framework and does not affect the proof of the structural closure theorem established in the present paper.

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