

Article

Rigidity and Classification of Ricci Solitons on Lorentzian Hypersurfaces with Conircular Vector Fields on Minkowski Space

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Abstract

This paper examines the classification of Lorentzian hypersurfaces in Minkowski space, specifically focusing on those that are characterized as good or bad and that admit Ricci solitons with a concircular vector field. First, we demonstrate that Ricci solitons of this nature do not exist in Lorentzian spaces that have non-zero sectional curvature. Additionally, we show that the shape operator can be expressed in specific canonical forms.

Keywords: Lorentzian hypersurfaces; generalized umbilical hypersurfaces; good and bad Lorentzian hypersurfaces; Ricci solitons; conformal vector fields

MSC: 53C50; 53C24; 53B30

1. Introduction

Ricci solitons are essential in geometric analysis, as they are self-similar solutions to the Ricci flow and natural extensions of Einstein metrics. Introduced by Hamilton in the context of the Ricci flow [1], Ricci solitons provide key geometric models for studying singularity formation. They also reveal deep connections with semi-Riemannian geometry.

In recent years, the study of Ricci soliton structures on submanifolds, particularly on hypersurfaces, has gained significant traction, especially when these are embedded in ambient manifolds with distinct symmetries. A powerful strategy for achieving classification results is to utilize the tangential component of a key ambient vector field as the soliton potential vector field. This approach frequently encompasses the tangential part of a closed conformal vector field or the position vector in flat ambient spaces. By implementing this strategy, we can derive effective, necessary, and sufficient conditions that greatly enhance our classification efforts.

In [2], a classification of Ricci solitons on Euclidean hypersurfaces is provided, which arises from the position vector field of those hypersurfaces. In [3], it has been clearly demonstrated that a complete gradient Ricci soliton, characterized by a non-parallel closed conformal vector field and constant scalar curvature, is isometric to one of three distinct geometries: Euclidean space, a Euclidean sphere, or a negatively curved Einstein warped product of the real line combined with a complete non-positively curved Einstein manifold.

In [4], the authors explored Ricci solitons on Riemannian hypersurfaces that are immersed in both Riemannian and Lorentzian manifolds with constant sectional curvature. They demonstrated that, under certain conditions, these hypersurfaces must be totally



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umbilical. This conclusion implies that they possess both constant mean curvature and constant sectional curvature.

Recently, there has been significant progress in the study of Ricci solitons within the context of Lorentzian and pseudo-Riemannian geometry, as evidenced in works such as [5–10]. These studies reflect a growing interest in the relationship between Ricci solitons, hypersurface geometry, and conformal vector fields, and they motivate further investigation into new geometric contexts and classification results.

For a semi-Riemannian manifold (M, g) , a quadruple $(M, g, \mathcal{Z}, \kappa)$ is said to define a Ricci soliton if there exists a vector field $\mathcal{Z}$ and a constant κ satisfying

$$R + \frac{1}{2}\mathcal{L}_{\mathcal{Z}}g = \kappa g, \quad (1)$$

where R is the Ricci tensor of (M, g) , and $\mathcal{L}_{\mathcal{Z}}g$ is the Lie derivative of the metric g along $\mathcal{Z}$. $\mathcal{Z}$ is known as the potential vector field, while κ is called the soliton constant.

In this work, we study the Ricci solitons on Lorentzian hypersurfaces in Lorentzian ambient manifolds that admit a closed conformal vector field. The main results of the paper can be summarized as follows:

- (1) We derive intrinsic relations linking the Ricci tensor of the Lorentzian hypersurfaces, the shape operator, and the closed conformal vector field.
- (2) We prove that Ricci solitons of this type in Lorentzian space forms are uniquely found in Minkowski space $\mathbb{R}_1^{n+1}$.
- (3) We obtain classification results in Minkowski space $\mathbb{R}_1^{n+1}$, extending several recent contributions to the theory of Ricci solitons on Lorentzian hypersurfaces.

The paper is structured as follows. In Section 2, we explore essential definitions and key formulas concerning hypersurfaces in real space forms. Then, in Section 3, we present several examples of Lorentzian hypersurfaces in Minkowski space $\mathbb{R}_1^{n+1}$, providing compelling model cases that are used in our classification results. In Section 4, we establish the intrinsic equation relating the Ricci tensor, the shape operator, and the closed conformal vector field, and we show that Ricci solitons of this type in Lorentzian space forms exist only in Minkowski space $\mathbb{R}_1^{n+1}$. In this setting, we show that the shape operator must belong to one of seven canonical forms described in Theorem 3. Finally, Section 5 contains the results of the rigidity theorems and a complete classification of Ricci solitons on bad and good hypersurfaces in $\mathbb{R}_1^{n+1}$ induced by concircular vector fields.

2. Preliminaries

Throughout this article, we assume that $n \geq 3$. Let (M_1, g) be a Lorentzian hypersurface immersed in an $(n + 1)$ -dimensional Lorentzian manifold $(\overline{M}_1, \overline{g})$, and let $\mathcal{N}$ denote a local spacelike unit normal vector field along M_1 . We denote by A the shape operator corresponding to the normal field $\mathcal{N}$. The operator A is self-adjoint with respect to the metric g , namely,

$$g(A(X), Y) = g(X, A(Y)),$$

for all $X, Y \in \mathfrak{X}(M_1)$. If $\overline{\nabla}$ and ∇ stand for the Levi-Civita connections of $\overline{M}_1$ and M_1 , respectively, then the Weingarten formula is given by

$$A(X) = -\overline{\nabla}_X \mathcal{N}, \quad (2)$$

for any $X \in \mathfrak{X}(M_1)$.

The Gauss formula is expressed as

$$\bar{\nabla}_X Y = \nabla_X Y + g(A(X), Y)\mathcal{N}, \quad (3)$$

for all $X, Y \in \mathfrak{X}(M_1)$.

The Codazzi equation reads

$$(\nabla_X A)Y = (\nabla_Y A)X, \quad (4)$$

for all $X, Y \in \mathfrak{X}(M_1)$. Here, the covariant derivative of A is defined by

$$(\nabla_X A)Y = \nabla_X(A(Y)) - A(\nabla_X Y),$$

for all $X, Y \in \mathfrak{X}(M_1)$.

The mean curvature of the Lorentzian hypersurface (M_1, g) is defined by

$$H = \frac{1}{n} \operatorname{tr}(A). \quad (5)$$

The Ricci tensors of the hypersurface M_1 and the ambient manifold $\bar{M}_1$ are connected through the Gauss equation (see [8,11]), which can be written as

$$R(X, Y) = \bar{R}(X, Y) - \bar{g}(\bar{\mathcal{R}}(\mathcal{N}, X)Y, \mathcal{N}) + g(A(X), nHY - A(Y)), \quad (6)$$

for all $X, Y \in \mathfrak{X}(M_1)$, where $\bar{\mathcal{R}}$ is the curvature tensor of $(\bar{M}_1, \bar{g})$.

Furthermore, taking the trace of Equation (6), we obtain the relation between the scalar curvatures S and $\bar{S}$ of (M_1, g) and $(\bar{M}_1, \bar{g})$, respectively:

$$S = \bar{S} - 2\bar{R}(\mathcal{N}, \mathcal{N}) + n^2 H^2 - |A|^2, \quad (7)$$

where $|A|^2 = \operatorname{tr}(A^2)$.

For each point $x \in M_1$, the relative nullity distribution is defined by

$$T_0(x) = \ker(A_x) \subset T_x M_1,$$

the dimension of $T_0(x)$ is called the index of relative nullity at x , while $\nu(x)$, the type number at x , is the rank of A_x .

It is well known that the shape operator A of a Lorentzian hypersurface (M_1, g) in a Lorentzian manifold $(\bar{M}_1, \bar{g})$ is not necessarily diagonalizable. In such a case, A can be classified according to a classical result (see, for example, [7,12]).

Lemma 1. *Let A be the shape operator of an n -dimensional Lorentzian hypersurface (M_1, g) of a Lorentzian manifold $(\bar{M}_1, \bar{g})$. A can be taken in one of the following four matrix forms:*

1. $A = \operatorname{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$;
2. $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \oplus \operatorname{diag}(\alpha_3, \alpha_4, \dots, \alpha_n)$, with $b \neq 0$;
3. $A = \begin{bmatrix} a & 0 \\ \epsilon & a \end{bmatrix} \oplus \operatorname{diag}(\alpha_3, \alpha_4, \dots, \alpha_n)$, where $\epsilon = \pm 1$;
4. $A = \begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha & 0 \\ 0 & 1 & \alpha \end{bmatrix} \oplus \operatorname{diag}(\alpha_4, \alpha_5, \dots, \alpha_n)$.

In cases (1) and (2), the operator A is represented in an orthonormal basis $\{e_1, \dots, e_n\}$ of (M_1, g) where $g(e_1, e_1) = -1$ and $g(e_i, e_i) = 1$ for $2 \leq i \leq n$. All other inner products in this basis are zero.

In cases (3) and (4), the operator A is represented in a pseudo-orthonormal basis $\{e_1, \dots, e_n\}$ where $g(e_1, e_2) = 1$ and $g(e_i, e_i) = 1$ for $3 \leq i \leq n$. Again, all other inner products are zero.

3. Canonical Examples of Lorentzian Hypersurfaces

The purpose of this section is to introduce the model examples that will be referenced throughout the remainder of the paper. Specifically, the classification theorems in Section 5 demonstrate that every manifold meeting the stated assumptions is locally isometric to one of the manifolds constructed in Examples 1–3. Therefore, these examples are presented before discussing the classification results. Let $\mathbb{R}_1^{n+1}$ be the $(n+1)$ -dimensional Minkowski space, that is, the vector space $\mathbb{R}^{n+1}$ equipped with a Lorentzian inner product $\langle \cdot, \cdot \rangle$ of index one defined by

$$\langle x, y \rangle = -x^0 y^0 + \sum_{j=1}^n x^j y^j,$$

for any vectors $x = (x^0, x^1, \dots, x^n)$ and $y = (y^0, y^1, \dots, y^n)$ in $\mathbb{R}^{n+1}$.

Given a positive constant c , the de Sitter space of curvature c is represented as a hypersurface.

$$\mathbb{S}_1^n(c) = \{x \in \mathbb{R}_1^{n+1} \mid \langle x, x \rangle = c^{-1}\}.$$

This space is a connected Lorentzian manifold with constant sectional curvature equal to c , and it is simply connected for $n \geq 3$. In Section 5, the notation $\mathbb{S}^n(c)$ will be used to denote the standard round sphere of radius c^{-1} in the Euclidean space $\mathbb{R}^{n+1}$.

Example 1. Let $\gamma(s)$ be a planar curve in $\mathbb{R}^3$ parametrized by arc length, whose curvature $\kappa(s)$ is strictly positive. Consider the surface obtained by translating γ along its binormal direction, namely

$$\Phi(s, t) = \gamma(s) + tB,$$

where B is the constant binormal vector field of γ (the torsion being identically zero). This construction yields an isometric embedding of the Euclidean plane into $\mathbb{R}^3$.

With respect to the coordinate vector fields $\{\partial_s, \partial_t\}$, the shape operator of Φ takes the form

$$A = \begin{pmatrix} -\kappa(s) & 0 \\ 0 & 0 \end{pmatrix}.$$

Extending Φ trivially by an identity map on $\mathbb{R}_1^{n-2}$, we obtain the product immersion

$$\Phi \times \text{Id}_{n-2} : \mathbb{R}^2 \times \mathbb{R}_1^{n-2} \longrightarrow \mathbb{R}^3 \times \mathbb{R}_1^{n-2},$$

which defines an isometric immersion of $\mathbb{R}_1^n$ into $\mathbb{R}_1^{n+1}$. The corresponding shape operator is diagonalizable, having a single nonzero principal curvature $-\kappa(s)$ and a zero eigenvalue of multiplicity $n-1$.

A similar argument applies when γ is a timelike planar curve in $\mathbb{R}_1^3$. In that case, one obtains a Lorentzian cylinder via the immersion

$$\Phi \times \text{Id}_{n-2} : \mathbb{R}_1^2 \times \mathbb{R}^{n-2} \longrightarrow \mathbb{R}_1^3 \times \mathbb{R}^{n-2}.$$

Such hypersurfaces admit non-degenerate relative nullity distributions. Moreover, by Theorem 8.7 of [13], every isometric immersion of $\mathbb{R}_1^n$ into $\mathbb{R}_1^{n+1}$ with non-degenerate relative nullity is locally congruent to one of these constructions.

Example 2. Let $\gamma(s)$ be a lightlike curve in $\mathbb{R}_1^3$, which we may assume, without loss of generality, is defined on the whole real line $\mathbb{R}$, and set $T(s) = \gamma'(s)$. Since $T'(s) = \gamma''(s)$ is spacelike, one defines the curvature by

$$\kappa(s) = \|T'(s)\|.$$

The associated principal normal vector $N(s)$ is determined through

$$T'(s) = \kappa(s)N(s),$$

and satisfies

$$\langle T, T \rangle = 0, \quad \langle T, N \rangle = 0, \quad \langle N, N \rangle = 1.$$

The orthogonal complement of $N'(s)$ is a Lorentzian 2-plane containing $T(s)$ and another null vector field $B(s)$ uniquely characterized by $\langle B, T \rangle = -1$. The torsion function is then defined by

$$\tau(s) = \langle B'(s), N(s) \rangle.$$

With respect to the frame $\{T, N, B\}$, the Frenet equations for null curves are expressed as

$$\begin{aligned} T'(s) &= \kappa(s)N(s), \\ N'(s) &= \kappa(s)B(s) + \tau(s)T(s), \\ B'(s) &= \tau(s)N(s). \end{aligned}$$

When the torsion vanishes identically, the curve γ is uniquely determined (up to isometry) by its initial data and is referred to as a generalized cubic.

For such a curve, the mapping

$$\rho(s, u) = \gamma(s) + u B(s)$$

defines an isometric immersion of the Lorentzian plane into $\mathbb{R}_1^3$, called a B-scroll. Its shape operator, with respect to a suitable basis, is represented by

$$A = \begin{pmatrix} 0 & 0 \\ -\kappa(s) & 0 \end{pmatrix}, \quad \kappa(s) > 0.$$

Higher-dimensional examples are obtained by taking the orthogonal product

$$\text{Id}_{n-2} \times \rho : \mathbb{R}^{n-2} \times \mathbb{R}_1^2 \longrightarrow \mathbb{R}^{n-2} \times \mathbb{R}_1^3.$$

These hypersurfaces exhibit degenerate relative nullity. According to Theorems 9.7 and 9.8 in [13], every isometric immersion of $\mathbb{R}_1^2$ into $\mathbb{R}_1^3$ with degenerate relative nullity is locally a B-scroll, and all higher-dimensional cases arise via such a product construction.

Example 3. Generalized umbilical hypersurfaces in Minkowski space were first introduced by Magid [14]. Let $\tau \neq 0$ be a given constant, and let $\gamma(s)$ denote a lightlike curve in $\mathbb{R}_1^{n+1}$, which we may assume, without loss of generality, is defined on the whole real line $\mathbb{R}$. Along γ , consider a pseudo-orthonormal frame

$$\{T, B, N, E_1, \dots, E_{n-2}\},$$

satisfying

$$\langle T, B \rangle = -1, \quad \langle N, N \rangle = 1, \quad \langle E_i, E_i \rangle = 1,$$

with all remaining inner products equal to zero.

This frame evolves according to the differential system

$$\begin{aligned}T &= \gamma'(s), \\ \nabla_s T &= \kappa(s)N, \\ \nabla_s B &= \tau N, \\ \nabla_s N &= \kappa(s)B + \tau T, \\ \nabla_s E_i &= 0,\end{aligned}$$

where ∇_s denotes covariant differentiation along γ . Such a frame is known as a generalized Cartan frame of type 1.

Assume $|\tau| \neq 1$ and define a map $f: \mathbb{R}^n \rightarrow \mathbb{R}_1^{n+1}$ by

$$f(s, u, y) = \gamma(s) + uB(s) - \sigma(y)N(s) - \sum_{i=1}^{n-2} y^i E_i(s),$$

where $y = (y^1, \dots, y^{n-2})$, and

$$\sigma(y) = \frac{1}{\tau} + \sqrt{\frac{1}{\tau^2} + \sum_{i=1}^{n-2} (y^i)^2}.$$

A direct computation yields

$$\begin{aligned}f_*(\partial_s) &= (1 - \tau\sigma(y))T - \kappa(s)\sigma(y)B + u\tau N, \\ f_*(\partial_u) &= B, \\ f_*(\partial_{y^i}) &= E_i + \frac{\tau y^i}{\tau\sigma(y) - 1}N.\end{aligned}$$

The image of f is referred to as a generalized umbilical hypersurface of $\mathbb{R}_1^{n+1}$. Its shape operator decomposes as

$$A = \begin{pmatrix} \tau & 0 \\ \kappa(s) & \tau \end{pmatrix} \oplus \tau \text{Id}_{n-2}.$$

4. Ricci Solitons on Lorentzian Hypersurfaces in Lorentzian Space Forms

Let (M_1, g) represent an orientable Lorentzian hypersurface in a Lorentzian space form $(\overline{M}_1(\bar{c}), \bar{g})$ of dimension $(n+1)$. We will consider $\tilde{Z}$ as a closed conformal vector field on $\overline{M}_1(\bar{c})$ that satisfies the equation:

$$\overline{\nabla}_X \tilde{Z} = \rho X,$$

for $X \in \mathfrak{X}(\overline{M}_1(\bar{c}))$. Here, ρ is a smooth conformal function. In particular, when $\rho = 0$, the vector field $\tilde{Z}$ is said to be parallel. If ρ is a nonzero constant, then $\tilde{Z}$ is called concircular, and ρ is referred to as the concircular factor. Moreover, when $\rho = 1$, the vector field $\tilde{Z}$ is said to be concurrent.

Let $\mathcal{Z}$ be the restriction of $\tilde{Z}$ to M_1 , and let $\mathcal{N}$ be a unit spacelike normal vector field on M_1 , we define $\theta = \bar{g}(\mathcal{Z}, \mathcal{N})$, leading us to express $\mathcal{Z}$ as

$$\mathcal{Z} = \mathcal{Z}^T + \theta \mathcal{N},$$

here $\mathcal{Z}^T$ is the tangential component of $\mathcal{Z}$.

Using the Gauss and Weingarten formulas, we derive:

$$\nabla_X \mathcal{Z}^T = \rho X + \theta A(X), \quad (8)$$

and

$$A(\mathcal{Z}^T) = -\nabla\theta. \quad (9)$$

From (8), we obtain the divergence:

$$\operatorname{div} \mathcal{Z}^T = n(\rho + \theta H). \quad (10)$$

Building on Formula (1), we can now derive a crucial result that plays a pivotal role in our analysis.

Lemma 2. *Let $(\overline{M}_1, \overline{g})$ be an $(n+1)$ -dimensional Lorentzian manifold with a closed conformal vector field $\tilde{\mathcal{Z}}$. If a Lorentzian hypersurface (M_1, g) in $(\overline{M}_1, \overline{g})$ admits a Ricci soliton $(M_1, g, \mathcal{Z}^T, \kappa)$, then the Ricci tensor R of (M_1, g) satisfies the following equation:*

$$R(X, Y) = (\kappa - \rho)g(X, Y) - \theta g(AX, Y), \quad (11)$$

for all $X, Y \in \mathfrak{X}(M_1)$.

In contrast to the Einstein case (cf. [11]), Lorentzian hypersurfaces supporting a Ricci soliton in space forms cannot possess complex eigenvalues.

Lemma 3. *Let $(\overline{M}_1(\bar{c}), \bar{g})$ be an $(n+1)$ -dimensional Lorentzian space form with a closed conformal vector field $\tilde{\mathcal{Z}}$, and (M_1, g) a Lorentzian hypersurface in $(\overline{M}_1(\bar{c}), \bar{g})$. If $(M_1, g, \mathcal{Z}^T, \kappa)$ is a Ricci soliton, then the shape operator A cannot admit a complex eigenvalue and cannot take the fourth form in Lemma 1.*

Proof. Assume, for contradiction, that A has a complex eigenvalue at point $x \in M_1$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_x M_1$ such that A takes the second form of Lemma 1. By Lemma 2, we have

$$\begin{aligned} R(e_1, e_1) &= (\kappa - \rho)g(e_1, e_1) - \theta g(Ae_1, e_1) \\ &= -\kappa + \rho - \theta g(ae_1 - be_2, e_1) \\ &= -\kappa + \rho + a\theta, \end{aligned}$$

and

$$\begin{aligned} R(e_2, e_2) &= (\kappa - \rho)g(e_2, e_2) - \theta g(Ae_2, e_2) \\ &= \kappa - \rho - \theta g(be_1 + ae_2, e_2) \\ &= \kappa - \rho - a\theta, \end{aligned}$$

and

$$\begin{aligned} R(e_1, e_2) &= (\kappa - \rho)g(e_1, e_2) - \theta g(Ae_1, e_2) \\ &= -\theta g(ae_1 - be_2, e_2) \\ &= b\theta, \end{aligned}$$

and for $i > 2$, we have

$$R(e_i, e_i) = \kappa - \rho - \theta\alpha_i.$$

On the other hand, by utilizing Formula (6), we obtain the following

$$\begin{aligned}
R(e_1, e_1) &= \bar{R}(e_1, e_1) - \bar{g}(\bar{\mathcal{R}}(\mathcal{N}, e_1)e_1, \mathcal{N}) + g(A(e_1), nHY - A(e_1)) \\
&= -(n+1)\bar{c} - anH + a^2 - b^2 \\
&= -(n+1)\bar{c} - a(2a + \sum_{k=3}^n \alpha_k) + a^2 - b^2 \\
&= -(n+1)\bar{c} - a^2 - b^2 - a \sum_{k=3}^n \alpha_k,
\end{aligned}$$

$$\begin{aligned}
R(e_2, e_2) &= \bar{R}(e_2, e_2) - \bar{g}(\bar{\mathcal{R}}(\mathcal{N}, e_2)e_2, \mathcal{N}) + g(A(e_2), nHY - A(e_2)) \\
&= (n+1)\bar{c} + anH - a^2 + b^2 \\
&= (n+1)\bar{c} + a(2a + \sum_{k=3}^n \alpha_k) - a^2 + b^2 \\
&= (n+1)\bar{c} + a^2 + b^2 + a \sum_{k=3}^n \alpha_k,
\end{aligned}$$

$$\begin{aligned}
R(e_1, e_2) &= \bar{R}(e_1, e_2) - \bar{g}(\bar{\mathcal{R}}(\mathcal{N}, e_1)e_2, \mathcal{N}) + g(A(e_1), nHe_2 - A(e_2)) \\
&= -bnH + 2ab \\
&= -b(2a + \sum_{k=3}^n \alpha_k) + 2ab \\
&= -b \sum_{k=3}^n \alpha_k,
\end{aligned}$$

and for $i > 2$, we have

$$\begin{aligned}
R(e_i, e_i) &= \bar{R}(e_i, e_i) - \bar{g}(\bar{\mathcal{R}}(\mathcal{N}, e_i)e_i, \mathcal{N}) + g(A(e_i), nHe_i - A(e_i)) \\
&= (n+1)\bar{c} + \alpha_i nH - \alpha_i^2 \\
&= (n+1)\bar{c} + \alpha_i(2a + \sum_{k=3}^n \alpha_k) - \alpha_i^2 \\
&= (n+1)\bar{c} + 2a\alpha_i + \alpha_i \sum_{k \neq i} \alpha_k.
\end{aligned}$$

By examining the previous equations, we can establish the following equations

$$\kappa - \rho - a\theta = (n+1)\bar{c} + a^2 + b^2 + a \sum_{k=3}^n \alpha_k, \quad (12)$$

$$\theta = - \sum_{k=3}^n \alpha_k, \quad (13)$$

and for $i > 2$

$$\kappa - \rho - \alpha_i\theta = (n+1)\bar{c} + 2a\alpha_i + \alpha_i \sum_{k \neq i} \alpha_k. \quad (14)$$

Substituting (13) into (12), we have

$$\kappa - \rho = (n+1)\bar{c} + a^2 + b^2, \quad (15)$$

and by substituting (13) and (15) into (14), we deduce that

$$(a - \alpha_i)^2 + b^2 = 0,$$

which is a contradiction, as $b \neq 0$. Thus, A cannot have complex eigenvalues.

Suppose $\{e_1, \dots, e_n\}$ is a pseudo-orthonormal basis of M_1 such that A takes the third form of Lemma 1. By Lemma 2, we have

$$R(e_1, e_1) = R(e_2, e_2) = 0.$$

On the other hand, utilizing Formula (6) yields the following.

$$R(e_2, e_2) = -1,$$

which is an abuse. Hence, A cannot take the third form of Lemma 1. $\square$

We now state a result determining the structure of the shape operator for Lorentzian hypersurfaces supporting a Ricci soliton in Lorentzian space forms with a closed conformal vector field, in analogy with the main theorem of [11] established for the Einstein case.

Theorem 1. Let $(\overline{M}_1(\bar{c}), \bar{g})$ be an $(n+1)$ -dimensional Lorentzian space form that admits a closed conformal vector field $\bar{Z}$ with a conformal function ρ . Let (M_1, g) be a Lorentzian hypersurface in $(\overline{M}_1(\bar{c}), \bar{g})$. If (M_1, g, Z^T, κ) is a Ricci soliton, then the shape operator A is either

- i. $A = \alpha I$, with $\alpha = \pm\sqrt{\rho - \kappa + (n+1)\bar{c}}$, and $\theta = -(n-2)\alpha$. In particular, $\kappa \geq \rho + (n+1)\bar{c}$,
- ii. $A = \alpha I_p \oplus \beta I_{n-p}$, with $\alpha \neq \beta$, $\alpha, \beta = \frac{nH + \theta \pm \sqrt{\kappa - \rho - (n+1)\bar{c}}}{2}$, and $\theta = -(p-1)\alpha - (n-p-1)\beta$.
- iii. $A = \begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix} \oplus a I_{n-2}$, with $\kappa - \rho - (n+1)\bar{c} = a^2$, and $\theta = -(n-2)a$.

Proof. According to Lemma 3, the shape operator A does not have any complex eigenvalues. Furthermore, the fourth canonical form listed in Lemma 1 cannot occur

Case 1. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis on M_1 such that A is diagonal, that is, A takes the first form of Lemma 1. By Lemma 2, we have, for all i ,

$$R(e_i, e_i) = \kappa - \rho - \alpha_i \theta. \quad (16)$$

On the other hand, utilizing Formula (6) yields the following

$$\begin{aligned} R(e_i, e_i) &= (n+1)\bar{c} + \alpha_i nH - \alpha_i^2 \\ &= (n+1)\bar{c} + \alpha_i \sum_{k \neq i} \alpha_k. \end{aligned}$$

It follows that for every i , we have

$$\alpha_i^2 - (\theta + nH)\alpha_i + \kappa - \rho - (n+1)\bar{c} = 0. \quad (17)$$

By examining (17), we can confidently establish the following equation

$$(\alpha_i - \alpha_j)(\theta + \sum_{k \neq i, j} \alpha_k) = 0, \quad (18)$$

for $i \neq j$. If $\alpha_i \neq \alpha_j$, and $\alpha_i \neq \alpha_k$ for $i \neq k$, then by (18), we deduce $\alpha_j = \alpha_k$. It follows that there are at most two distinct eigenvalues, say α and β . If $\alpha \neq \beta$, we deduce from (17), which represents a second-degree polynomial having two distinct roots, that

$$\alpha, \beta = \frac{nH + \theta \pm \sqrt{\kappa - \rho - (n+1)\bar{c}}}{2}.$$

In particular, $\alpha + \beta = \theta + nH$, which implies that $\theta = -(p-1)\alpha - (n-p-1)\beta$, where p is the multiplicity of α . If $\alpha = \beta$, we deduce that $\theta + nH = \pm 2\sqrt{\rho - \kappa + (n+1)\bar{c}}$, and consequently, $\alpha = \pm\sqrt{\rho - \kappa + (n+1)\bar{c}}$.

Case 2. Let $\{e_1, \dots, e_n\}$ be a pseudo-orthonormal basis of M_1 such that A is represented by a matrix taking the third form of Lemma 1. By Lemma 2, we have

$$R(e_1, e_1) = -\theta,$$

$$R(e_1, e_2) = \kappa - \rho - a\theta,$$

and for $i > 2$, we have

$$R(e_i, e_i) = \kappa - \rho - \alpha_i \theta.$$

On the other hand, utilizing formula (6) yields the following

$$R(e_1, e_1) = nH - 2a$$

$$= \sum_{k=3}^n \alpha_k,$$

$$R(e_1, e_2) = (n+1)\bar{c} + anH - a^2$$

$$= (n+1)\bar{c} + a^2 + a \sum_{k=3}^n \alpha_k,$$

and for $i > 2$, we have

$$\begin{aligned} R(e_i, e_i) &= (n+1)\bar{c} + \alpha_i nH - \alpha_i^2 \\ &= (n+1)\bar{c} + 2a\alpha_i + \alpha_i \sum_{k \neq i} \alpha_k. \end{aligned}$$

By examining the previous equations, we can confidently establish the following equations

$$\theta = - \sum_{k=3}^n \alpha_k, \quad (19)$$

$$\kappa - \rho - a\theta = (n+1)\bar{c} + a^2 + a \sum_{k=3}^n \alpha_k, \quad (20)$$

and for $i > 2$

$$\kappa - \rho - \alpha_i \theta = (n+1)\bar{c} + 2a\alpha_i + \alpha_i \sum_{k \neq i} \alpha_k. \quad (21)$$

Substituting (19) into (20), we have

$$\kappa - \rho = (n+1)\bar{c} + a^2, \quad (22)$$

and substituting (19) and (22) into (21), it implies that $a = \alpha_i$ for each $i \geq 3$. Hence, the shape operator A has the form

$$A = \begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix} \oplus aI_{n-2}.$$

□

Corollary 1. Let $(\bar{M}_1(\bar{c}), \bar{g})$ be a Lorentzian space form that admits a closed conformal vector field $\bar{Z}$. Let (M_1, g) be a Lorentzian hypersurface in $(\bar{M}_1(\bar{c}), \bar{g})$. If $(M_1, g, \bar{Z}^T, \kappa)$ is a Ricci soliton, then the shape operator has almost two distinct real eigenvalues.

In terms of scalar curvatures, Theorem 1 can be stated as follows:

Corollary 2. Let $(\overline{M}_1(\bar{c}), \bar{g})$ be a Lorentzian space form that admits a closed conformal vector field $\bar{Z}$. Let (M_1, g) be a Lorentzian hypersurface in $(\overline{M}_1(\bar{c}), \bar{g})$. If $(M_1, g, \mathcal{Z}^T, \kappa)$ is a Ricci soliton, then the shape operator is either

- i $A = \alpha I$, with $S - \bar{S} = n(n-1)\alpha^2$,
- ii $A = \alpha I_p \oplus \beta I_{n-p}$, $\alpha \neq \beta$, with

$$S - \bar{S} = p(p-1)\alpha^2 + (n-p)(n-p-1)\beta^2 + 2p(n-p)\alpha\beta,$$
- iii $A = \begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix} \oplus a I_{n-2}$, with $S - \bar{S} = n(n-1)a^2$.

Based on Corollary 2 and Formula (7), we have the following:

Theorem 2. Let $(\overline{M}_1(\bar{c}), \bar{g})$ be a Lorentzian space form that admits a closed conformal vector field $\bar{Z}$. Let (M_1, g) be a Lorentzian hypersurface in $(\overline{M}_1(\bar{c}), \bar{g})$. If $(M_1, g, \mathcal{Z}^T, \kappa)$ is a Ricci soliton, then necessarily $\bar{c} = 0$, that is $(\overline{M}_1(\bar{c}), \bar{g})$ is the Minkowski space $\mathbb{R}_1^{n+1}$.

Proof. It is clear that case (ii) in Corollary 2 includes case (i). So, it suffices to consider case (ii). In this case, it follows (7) that

$$n(n-1)\alpha^2 = -2n\bar{c} + n^2\alpha^2 - n\alpha^2,$$

which simplifies to $\bar{c} = 0$.

In the case the shape operator takes the form (iii), we get:

$$n(n-1)a^2 = -2n\bar{c} + n^2a^2 - na^2,$$

which also leads to the conclusion that $\bar{c} = 0$. $\square$

Based on the conclusion that $\bar{c} = 0$, we find that we are dealing with Lorentzian hypersurfaces in the $(n+1)$ -dimensional Minkowski space $\mathbb{R}_1^{n+1}$. Closed conformal vector fields of $\mathbb{R}_1^{n+1}$ are well-known (see [15]). We will focus on the case where $\mathcal{Z}$ is a concircular vector field.

Theorem 3. Let (M_1, g) be a Lorentzian hypersurface in $\mathbb{R}_1^{n+1}$. Let $\mathcal{Z}$ be a concircular vector field in $\mathbb{R}_1^{n+1}$, with concircular function ρ . If $(M_1, g, \mathcal{Z}^T, \kappa)$ is a Ricci soliton, then the shape operator A takes one of the following:

1. $A = \alpha I$, $\alpha \neq 0$. In this case, $\kappa > \rho$.
2. $A = 0$. In this case, $\kappa = \rho$.
3. A is diagonalizable and possesses two distinct nonzero eigenvalues, α and β , each of multiplicity greater than 1.
4. A is diagonalizable and possesses two distinct nonzero eigenvalues, α and β , one of them of multiplicity 1.
5. A is diagonalizable and possesses two distinct eigenvalues $\alpha \neq 0$ of multiplicity greater than 1, and $\beta = 0$. In this case, we have $\kappa = \rho$.
6. A is nondiagonalizable with a single eigenvalue $\alpha = 0$ and a minimal polynomial of t^2 . In this case, $\kappa = \rho$.
7. A is nondiagonalizable with a unique eigenvalue $\alpha \neq 0$, and its minimal polynomial is $(t - \alpha)^2$. In this case, $\kappa = \rho + \alpha^2$.

Let M_1 be a Lorentzian hypersurface in $\mathbb{R}_1^{n+1}$. According to [8,16], a point $x \in M_1$ is classified as a bad point if the matrix A at x is nonsingular and has a simple eigenvalue.

Conversely, if A at x does not meet these criteria, we categorize x as a good point. If all points in M_1 are good, we call M_1 a good hypersurface.

We classify points $x \in M_1$ based on the specific form of A_x . If A_x corresponds to forms 1, 2, ..., or 7 of Theorem 3, x is said to be of type 1, 2, ..., or 7, respectively. The collection of points belonging to type i is referred to as C_i . Also, we say that $x \in M_1$ is a point of type 1, 2, ..., or 7 according to whether A_x has the form 1, 2, ..., or 7, respectively. The set of points of type i will be denoted by C_i . We will refer to C_4 as the “bad set.”

Theorem 4. Let (M_1, g) be a Lorentzian hypersurface in $\mathbb{R}_1^{n+1}$. Let $\mathcal{Z}$ be a concircular vector field in $\mathbb{R}_1^{n+1}$. If $(M_1, g, \mathcal{Z}^T, \kappa)$ is a Ricci soliton, then the bad set is open.

Proof. Let α and β be the two eigenvalues of A , and let x be a bad point, so that $\alpha(x)$ and $\beta(x)$ are two distinct nonzero eigenvalues of multiplicity 1 and $n - 1$, respectively. Then, since A is nonsingular, there is a neighborhood U of x on which A is nonsingular. By Theorem 3, U consists of points of type 1, 3, 4, and 7. We may assume without loss of generality that

$$\alpha = \alpha_1(x) > \beta = \alpha_2(x) = \dots = \alpha_n(x).$$

By continuity, we may assume that $\alpha_1 > \alpha_2 \geq \alpha_3 \geq \dots \geq \alpha_n$ on U . Since A has at most two distinct eigenvalues, we necessarily have $\alpha_1 > \alpha_2 = \dots = \alpha_n$. Hence, $U \subset C_4$, meaning that C_4 is open. $\square$

Remark 1. Assume that x is a point of type 3, so that $\alpha(x)$ and $\beta(x)$ are two distinct nonzero eigenvalues of multiplicity $p > 1$ and $n - p > 1$, respectively. Then, since A is nonsingular, there is a neighborhood U of x on which A is nonsingular. By Theorem 3, U consists of points of type 1, 3, 4 and 7. We may assume without loss of generality that

$$\alpha = \alpha_1(x) = \alpha_2(x) = \dots = \alpha_p(x) > \beta = \alpha_{p+1}(x) = \dots = \alpha_n(x).$$

By continuity, we may assume that $\alpha_1 > \alpha_2 \geq \alpha_3 \geq \dots \geq \alpha_n$ on U . Since A has at most two distinct eigenvalues, we necessarily have $\alpha_1 = \alpha_2 = \dots = \alpha_p > \alpha_{p+1} = \dots = \alpha_n$. Hence, $U \subset C_3$, meaning that C_3 is open.

Theorem 5. Let (M_1, g) be a Lorentzian hypersurface in $\mathbb{R}_1^{n+1}$. Let $\mathcal{Z}$ be a concircular vector field in $\mathbb{R}_1^{n+1}$, with concircular function c . If $(M_1, g, \mathcal{Z}^T, \kappa)$ is a Ricci soliton with $\kappa < \rho$, then the bad set is closed.

Proof. Assume $\{x_i\} \subset C_4$ converges to $x \in M_1$. Then, A at x_i is diagonalizable and its eigenvalues $\alpha(x_i)$ and $\beta(x_i)$ are distinct and nonzero, and moreover $\kappa - \rho = \alpha(x_i)\beta(x_i)$. By continuity of the eigenvalues, $\kappa - \rho = \alpha(x)\beta(x)$. Since $\kappa < \rho$, it follows that $\alpha(x) \neq \beta(x)$. Thus, A is diagonalizable at x and has two distinct nonzero eigenvalues, which means that $x \in C_3 \cup C_4$. However, since C_3 is open, it follows that $x \in C_4$. $\square$

Remark 2. Assume $\{x_i\} \subset C_3$ converges to $x \in M_1$. Then, A at x_i is diagonalizable and its eigenvalues $\alpha(x_i)$ and $\beta(x_i)$ are distinct and nonzero, and moreover $\kappa - \rho = \alpha(x_i)\beta(x_i)$. By continuity of the eigenvalues, $\kappa - \rho = \alpha(x)\beta(x)$. Since $\kappa < \rho$, it follows that $\alpha(x) \neq \beta(x)$. Thus, A is diagonalizable at x and has two distinct nonzero eigenvalues, which means that $x \in C_3 \cup C_4$. However, since C_4 is open, it follows that $x \in C_3$.

Theorem 6. Let (M_1, g) be a connected Lorentzian hypersurface in $\mathbb{R}_1^{n+1}$. Let $\mathcal{Z}$ be a concircular vector field in $\mathbb{R}_1^{n+1}$, with concircular function ρ . If $(M_1, g, \mathcal{Z}^T, \kappa)$ is a Ricci soliton with $\kappa < c$ and M_1 has at least one good point, then M_1 consists entirely of points of C_3 .

Proof. Since $\kappa < \rho$, then according to Theorem 3 $M_1 \subseteq C_3 \cup C_4$. Since M_1 is connected, we deduce from Theorems 4 and 5 that M_1 is a good hypersurface, and $M_1 = C_3$. $\square$

5. Rigidity and Classification of Ricci Solitons on Lorentzian Hypersurfaces

In this section, we investigate rigidity and Classification results for Ricci solitons on Lorentzian hypersurfaces of Minkowski space with a concircular vector field.

Theorem 7. Let $\mathcal{Z}$ be a concircular vector field on $\mathbb{R}_1^{n+1}$, with concircular function ρ . Let (M_1, g) be a complete, connected Lorentzian hypersurface in $\mathbb{R}_1^{n+1}$ with at least one good point. If $(M_1, g, \mathcal{Z}^T, \kappa)$ is a Ricci soliton with $\kappa < \rho$, then M_1 is one of the following products:

1. $\mathbb{S}_1^p(\alpha^2) \times \mathbb{S}^{n-p}(\beta^2)$, or
2. $\mathbb{S}^p(\alpha^2) \times \mathbb{S}_1^{n-p}(\beta^2)$, where $\kappa - \rho = \alpha\beta$, and $1 < p < n - 1$.

Proof. Since (M_1, g) is connected with at least one good point, then (M_1, g) is good, and since the multiplicities of α and β are greater than one, it follows that $X(\alpha) = 0$ (resp. $X(\beta) = 0$) for all $X \in T_\alpha$ (resp. T_β), where T_α and T_β are the smooth subbundles of TM_1 given by $T_\alpha = \ker(A - \alpha I)$ and $T_\beta = \ker(A - \beta I)$ (see Lemma 3.3 in [17]) see also proposition D4 in [18]. Accordingly, we deduced from the equation $\kappa - \rho = \alpha\beta$ that α and β are constant. The result now follows from Theorem 3.3 in [17] (see also [18]). $\square$

In the study of bad Lorentzian hypersurfaces within $\mathbb{R}_1^{n+1}$, we present a significant theorem. See Theorem 5.10 in [19].

Theorem 8. Let $\mathcal{Z}$ be a concircular vector field on $\mathbb{R}_1^{n+1}$, with concircular function ρ . Let (M_1, g) be a complete and connected Lorentzian hypersurface in $\mathbb{R}_1^{n+1}$ with at least one bad point. If $(M_1, g, \mathcal{Z}^T, \kappa)$ is a Ricci soliton with $\kappa < \rho$, then M_1 is foliated by $(n - 1)$ dimensional spaces of constant curvature > 0 .

Proof. According to Theorem 4, the set M_1 consists entirely of bad points. Let α and β represent the two distinct eigenvalues of the shape operator A , with multiplicities 1 and $n - 1$, respectively. By Proposition 2.3 in [8], the distributions T_α and T_β are both differentiable and integrable. Moreover, β remains constant on each leaf M_β ; thus, M_β is a non-degenerate, totally geodesic hypersurface of M_1 . The curvature tensor on M_β aligns with the restriction of R . Given that the dimension of M_β is $n - 1$, this means that M_β has a constant curvature of β^2 . $\square$

When $\kappa > \rho$, the complexities become challenging. We will discuss several partial results from our research. We can illustrate these results using the methodology from the theorem in [16], as it is relevant to our case.

Theorem 9. Let $\mathcal{Z}$ be a concircular vector field on $\mathbb{R}_1^{n+1}$, with concircular function ρ . Let (M_1, g) be a good connected Lorentzian hypersurface in $\mathbb{R}_1^{n+1}$. If $(M_1, g, \mathcal{Z}^T, \kappa)$ is a Ricci soliton with $\kappa > \rho$, then M_1 is either one of the following Lorentzian manifolds:

1. $\mathbb{S}_1^n(\alpha^2)$.
2. $\mathbb{S}_1^p(\alpha^2) \times \mathbb{S}^{n-p}(\beta^2)$, with $\kappa - \rho = \alpha\beta$ and $1 < p < n$.
3. $\mathbb{S}^p(\alpha^2) \times \mathbb{S}_1^{n-p}(\beta^2)$, with $\kappa - \rho = \alpha\beta$ and $1 < p < n$.
4. A generalized umbilical hypersurface of type 1 of $\mathbb{R}_1^{n+1}$, Example 3.

Proof. Assume α and β are the only eigenvalues of A . Since M_1 is a good manifold, Theorem 3 ensures that $M_1 \subseteq C_1 \cup C_3 \cup C_7$. Suppose that there exists a point $x_0 \in M_1$ of

type 3, where the eigenvalues $\alpha(x_0)$ and $\beta(x_0)$ have multiplicities p and $n - p$, respectively, with $p > 1$ and $n - p > 1$. Set

$$W = \{x \in M_1 : \alpha(x) = \alpha(x_0), \beta(x) = \beta(x_0)\}.$$

W is closed. Furthermore, the continuity of the eigenvalues implies that there exists an open neighborhood U of x_0 such that $U \subseteq C_3$. Indeed, since A admits exactly two distinct eigenvalues, both α and β are differentiable on U and have constant multiplicities p and $n - p$, respectively (see [19], Proposition). For notational simplicity, we set

$$\alpha_1 = \cdots = \alpha_p = \alpha, \quad \alpha_{p+1} = \cdots = \alpha_n = \beta.$$

Since $p > 1$ and $n - p > 1$, Proposition D.4 in [18] guarantees that the distributions T_α and T_β are differentiable and involutive. Moreover, the functions α and β are constant along the leaves M_α and M_β , respectively. On the set W , the relation $\alpha\beta = \kappa - \rho$ holds, which implies that α and β are locally constant near x_0 . Consequently, W is also open.

Now, from Proposition 4.4 in [8], and Wu's extension [20] of the de Rham decomposition theorem to pseudo-Riemannian manifolds, we can conclude that M_1 is locally isometric to a product of a Riemannian manifold and a Lorentzian manifold, both of which have constant positive curvature. This establishes statements (2) and (3) of the theorem.

Suppose that M_1 has no points of type 3. It follows that $M_1 \subseteq C_1 \cup C_7$. Let α denote the unique eigenvalue of A . α is a constant, by Proposition D.4 in [18]. Furthermore, Theorem 3 yields

$$\alpha = \pm\sqrt{\kappa - \rho}.$$

Set

$$W_1 = \{x \in M_1 : A_x = \sqrt{|\alpha|} I_n\},$$

and

$$W_2 = \left\{x \in M_1 : A_x = \begin{bmatrix} \sqrt{|\alpha|} & 0 \\ 1 & \sqrt{|\alpha|} \end{bmatrix} \oplus \sqrt{|\alpha|} I_{n-2}\right\}.$$

This leads to the conclusion that W_1 and W_2 are disjoint closed subsets. By connectedness of M_1 , it follows that either $M_1 = W_1$ or $M_1 = W_2$. This implies statements (2) and (3). $\square$

Theorem 10. Let $\mathcal{Z}$ be a concircular vector field on $\mathbb{R}_1^{n+1}$, with concircular function c . Let (M_1, g) be a complete, connected Lorentzian hypersurface in $\mathbb{R}_1^{n+1}$. If $(M_1, g, \mathcal{Z}^T, \kappa)$ is a Ricci soliton with $\kappa = \rho$ and θ vanishes on M_1 , then (M_1, g) is locally flat and isometric to one of the following Lorentzian manifolds:

1. $\mathbb{R}_1^n$.
2. $\mathbb{R}_1^{n-2} \times f(\mathbb{R}^2)$, where $f(\mathbb{R}^2)$ is a Euclidean cylinder in the 3-dimensional Euclidean space
3. $\mathbb{R}^{n-2} \times f(\mathbb{R}_1^2)$, where $f(\mathbb{R}_1^2)$ is a Lorentz cylinder or a B-scroll in $\mathbb{R}_1^3$ (Example 2).

Proof. According to Theorem 3, all points $x \in M_1$ are of type 2, type 5, or type 6 with the multiplicity of α equal to 1. This implies that the type number $\nu(x) \leq 1$ for all $x \in M_1$. Furthermore, by Proposition 5.3 in [19], we conclude that M_1 is locally flat. $\square$

We now present a result that extends Theorem 5.5 in [19] by replacing the assumption $\nu(x) \geq 3$ with the less stringent condition $\nu(x) \geq 2$.

Theorem 11. Let $\mathcal{Z}$ be a concircular vector field on $\mathbb{R}_1^{n+1}$, with concircular function c . Let (M_1, g) be a complete, connected Lorentzian hypersurface in $\mathbb{R}_1^{n+1}$. If $(M_1, g, \mathcal{Z}^T, \kappa)$ is a Ricci soliton with $\kappa = \rho$ and θ does not vanish on M_1 , then (M_1, g) is isometric to either $\mathbb{S}_1^p(\alpha^2) \times \mathbb{R}^{n-p}$ or $\mathbb{S}^p(\alpha^2) \times \mathbb{R}_1^{n-p}$, where $1 < p < n$, and $\mathbb{S}^p(\alpha^2)$.

Proof. By Theorem 3, M_1 consists of points of type 5. Let α be the nonzero eigenvalue of A of multiplicity p , since $\theta \neq 0$, then $p > 1$. From Proposition 2.2 in [8], the eigenvalue α has a constant multiplicity. Also, Proposition 2.3 in [8] the distributions $T_\alpha = \{X|A(X) = \alpha X\}$ and $T_0 = \{X|A(X) = 0\}$ are differentiable and integrable, and since $\dim T_\alpha > 1$, then α is constant on each leaf of M_α . Also from Lemma 5.7 in [19] (also see Lemma 5.6 in [21]), α is constant on M_1 . $\square$

6. Conclusions

In this paper, we investigated the classification of Ricci solitons on Lorentzian hypersurfaces within Lorentzian space forms that admit a closed conformal vector field. We established an intrinsic relationship connecting the Ricci tensor, the shape operator, and the conformal vector field, which serves as an effective tool for analyzing the geometric structure of these hypersurfaces. Our findings demonstrated that Ricci solitons of this type can only exist in the flat case, specifically in Minkowski space. Furthermore, we showed that the shape operator can be represented in specific canonical forms, which play a critical role in the classification process. As a result, we achieved a complete classification of so-called “good” and “bad” Lorentzian hypersurfaces in Minkowski space that admit Ricci solitons generated by concircular vector fields. These results reveal strong rigidity phenomena and provide a deeper understanding of the geometric restrictions imposed by the existence of such Ricci solitons.

Several promising directions for future research naturally arise from this work. One potential extension is to explore Ricci solitons on Lorentzian hypersurfaces that admit more general conformal or affine vector fields without the closedness condition. Another interesting direction is the study of higher-codimension submanifolds in Lorentzian space forms, where richer geometric structures and new classification phenomena may arise. Additionally, it would be interesting to extend the current analysis to pseudo-Riemannian ambient spaces with different signatures or to manifolds of non-constant sectional curvature. Finally, it would be interesting to explore how the geometric results obtained in this work might provide new insights into problems in mathematical relativity, where Lorentzian geometry plays a fundamental role.

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