

Article

A New Construction Method for Flat-Bottomed Single-Valley Solutions of the Feigenbaum–Kadanoff–Shenker Equation

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Abstract

The Feigenbaum–Kadanoff–Shenker (FKS) equation describes the quasiperiod route to chaos for circle maps. To facilitate research on the FKS equation, Shi introduced the second type of this equation and provided a construction method for its flat-bottomed single-valley solutions. In the present study, we introduce a new construction method for flat-bottomed single-valley solutions of the second type of FKS equation.

Keywords: Feigenbaum–Kadanoff–Shenker equation; the second type of FKS equation; flat-bottomed single-valley solution

MSC: 39B12; 37E05; 26A18

1. Introduction

The theory of functional equations is an important branch of modern mathematics [1]. To analyze the quasiperiod route to chaos for maps of a circle, Feigenbaum et al. introduced the following functional equation:

$$\begin{cases} g(g(\epsilon^2 x)) = -\epsilon g(x), & \forall x \in [-1, 1], \\ g(0) = 1, \end{cases} \quad (1)$$

where $\epsilon \in (0, 1)$ is a parameter and $g : [-1, 1] \rightarrow [-1, 1]$ is an unknown even unimodal mapping [2]. Equation (1) is commonly referred to as the Feigenbaum–Kadanoff–Shenker (FKS) equation [3] and is a nonlinear iterative functional equation.

In 2020, Shi constructed continuously differentiable unimodal solutions of Equation (1) via a construction method [4]. In 2024, piecewise linear and piecewise fractional linear solutions to Equation (1) were given by Liu and Li [5].

To facilitate research on the FKS equation through the same idea utilized by Zhang and Yang [6], Shi [7] introduced the second type of FKS equation as follows:

$$\begin{cases} f(f(\epsilon^2 x)) = \epsilon f(x), & \forall x \in [0, 1], \\ f(0) = 1, \end{cases} \quad (2)$$

where $\epsilon \in (0, 1)$ is a parameter and $f : [0, 1] \rightarrow [0, 1]$ is the unknown function.



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Definition 1. A function f on $[a, b]$ is called a single-valley function if there exists a point $c \in (a, b)$ such that it is strictly decreasing on $[a, c]$ and strictly increasing on $[c, b]$. A continuous solution $f : [0, 1] \rightarrow [0, 1]$ of Equation (2) is termed a single-valley solution if it is a single-valley function on $[0, 1]$.

In 2019, Shi constructed all single-valley solutions of Equation (2) by extending known single-valley functions [7].

In 2020, Shi attempted to identify other kinds of continuous solutions for Equation (2) [8] and discovered a series of interesting properties.

Let f be a continuous solution of Equation (2). The following lemmas state the key structural and analytic properties of a continuous solution f of Equation (2):

Lemma 1 ([8]). $f(1) = \epsilon$, $f(f(\epsilon^2)) = \epsilon^2$.

Lemma 2 ([8]). Let α be a minimum point of f . Then, $f(\alpha) = 0$ and $\alpha > \epsilon^2$.

Lemma 3 ([8]). Let α be a minimum point of f . If there is a point $x \in [0, \epsilon^2]$ such that $f(x) = \alpha$, then there exists a minimum point β of f such that $x = \epsilon^2\beta$.

Lemma 4 ([8]). Let S be the set of all minimum points of f . Then, S is a singleton or a closed subinterval with $S \subset (\epsilon^2, 1)$.

Obviously, the set of all minimum points of a single-valley solution of Equation (2) is a singleton. It is necessary to show the existence of continuous solutions of Equation (2) whose set of all minimum points are nondegenerate intervals. In 2020, Shi gave the definition of single-valley-extended solutions with platforms and the existence of such solutions, thus providing a positive answer to this problem. The concrete definition and existence theorem are stated as follows:

Definition 2 ([8]). A function f on $[a, b]$ is called a single-valley function with a platform if it is continuous and there exists an interval $[c, d] \subset (a, b)$ such that f is strictly decreasing on $[a, c]$, f is a constant on $[c, d]$, and f is strictly increasing on $[d, b]$. A continuous solution $f : [0, 1] \rightarrow [0, 1]$ of Equation (2) is termed a single-valley solution with a platform if it is a single-valley function with a platform on $[0, 1]$.

Theorem 1 ([8]). Choose arbitrarily positive numbers u, v , and a such that

$$\epsilon^2 \leq a < u < v < 1.$$

Define any strictly decreasing continuous function $q : [\epsilon^2, u] \rightarrow [0, a]$ and any strictly increasing continuous function $p : [v, 1] \rightarrow [0, \epsilon]$ satisfying

- (i) $q(\epsilon^2) = a$, $q(a) = \epsilon^2$, $p(v) = q(u) = 0$, $p(1) = \epsilon$,
- (ii) the equation $p(x) = \epsilon x$ has the unique solution $x = 1$ on $[v, 1]$.

Define

$$\varphi_0(x) = \begin{cases} q(x), & \forall x \in [\epsilon^2, u], \\ 0, & \forall x \in [u, v], \\ p(x), & \forall x \in [v, 1], \end{cases} \quad (3)$$

and

$$\varphi_1(x) = \begin{cases} q^{-1}(\epsilon p(x/\epsilon^2)), & \forall x \in [\epsilon^2 v, \epsilon^2], \\ h(x), & \forall x \in [\epsilon^2 u, \epsilon^2 v], \\ p^{-1}(\epsilon q(x/\epsilon^2)), & \forall x \in [\epsilon^4, \epsilon^2 u], \end{cases} \quad (4)$$

where

$$h : [\epsilon^2 u, \epsilon^2 v] \rightarrow [u, v] \quad (5)$$

is any continuous function satisfying $h(\epsilon^2 u) = v$ and $h(\epsilon^2 v) = u$. For $k \geq 2$, define

$$\varphi_k(x) = p^{-1}(\epsilon \varphi_{k-1}(x/\epsilon^2)), \quad \forall x \in [\epsilon^{2k+2}, \epsilon^{2k}] \quad (6)$$

and

$$\varphi(x) = \begin{cases} 1, & x = 0, \\ \varphi_k(x), & \forall x \in [\epsilon^{2k+2}, \epsilon^{2k}], \quad k = 0, 1, 2, \dots \end{cases} \quad (7)$$

Then, φ is a single-valley-extended solution with platforms of Equation (2). Conversely, if φ_k is the restriction on $[\epsilon^{2k+2}, \epsilon^{2k}]$ of a single-valley-extended solution with platforms of Equation (2), then (i), (ii), (4) and (6) hold.

Single-valley-extended solutions with platforms of Equation (2) are quite different from single-valley solutions. Observing the construction procedure, the given function $h : [\epsilon^2 u, \epsilon^2 v] \rightarrow [u, v]$ is only continuous, and the function $\varphi|_{[0,u]}$ may not be strictly decreasing. Therefore, the graph of solution φ may not be single-valley. An interesting question arises: does there exist a new kind of single-valley solution whose entire set of minimum points are nondegenerate intervals? The concrete definition of such solutions is stated as follows:

Definition 3. A function f on $[a, b]$ is called a flat-bottomed single-valley function if f is continuous and there exists an interval $[c, d] \subset (a, b)$ such that f is strictly decreasing on $[a, c]$, f is a constant on $[c, d]$, and f is strictly increasing on $[d, b]$. A continuous solution $f : [0, 1] \rightarrow [0, 1]$ of Equation (2) is termed a flat-bottomed single-valley solution if it is a flat-bottomed single-valley function on $[0, 1]$.

We show that by imposing the additional requirement that the given function h in Theorem 1 is strictly decreasing, one can construct flat-bottomed single-valley solutions of Equation (2). This result is stated precisely in the following corollary:

Corollary 1. If the given function h in (5) is any strictly decreasing continuous function and all other assumptions remain unchanged, then the solution φ defined in (7) is a flat-bottomed single-valley solution of Equation (2).

Proof. Through the same discussion in Theorem 4.1 in [8], we can show that the function φ defined as (7) is a continuous solution of Equation (2). We only need to prove that this solution is a flat-bottomed single-valley function.

Through the monotonicity of q , p , and h , it is easy to see that φ_1 is strictly decreasing. By simple induction, we can show all φ_k , $k = 2, 3, \dots$ are strictly decreasing. The strict monotonicity of q , p , and h ensures that φ is strictly decreasing on $(0, u]$ and strictly increasing on $[v, 1]$. Through $\varphi(0) = 1$ and the definition of φ_0 , it is easy to see that φ is a flat-bottomed single-valley solution. $\square$

The preceding corollary shows that a flat-bottomed single-valley solution of Equation (2) can be obtained by applying an extension technique to a single-valley function with a platform and a strictly decreasing function.

In 1986, by extending suitable strictly decreasing functions, Zhang and Yang introduced a constructive method for obtaining single-valley solutions of the second type of Feigenbaum equation [6]. Motivated by their work, Song and Zhang later developed a similar approach to construct single-valley solutions of Equation (2) in 2023 [9]. A natural

question then arises: is there an alternative constructive approach to obtaining flat-bottomed single-valley solutions of Equation (2), analogous to the method presented in [6,9]? The answer is affirmative. We demonstrate that, by appropriately extending suitably chosen strictly decreasing functions, one can also generate flat-bottomed single-valley solutions of Equation (2).

This paper is organized as follows: Section 2 shows the properties of the set of period points of a strictly decreasing continuous function on a bounded closed interval. Section 3 provides the key structural properties of flat-bottomed single-valley solutions of Equation (2). In Section 4, we introduce a decomposition scheme for bounded closed intervals. Section 5 presents our new construction method in full detail, and Section 6 presents some conclusions and problems.

2. The Set of Period Points of a Strictly Decreasing Continuous Function

Let $I \subset \mathbb{R}$ be an interval and $f : I \rightarrow I$ be a function. Let $f^n, n = 0, 1, 2, \dots$, denote the n -th iteration of f , i.e., the composition of f with itself n times. More precisely, when $n = 0$, $f^0 = id_I$, and when $n = 1$, $f^1 = f$.

Definition 4. For a positive integer n , a point p is a period- n point for f provided that $f^n(p) = p$ but $f^j(p) \neq p$ for $0 < j < n$. The positive integer n is called the period for p if p is a period- n point for f . If $n = 1$, the point p is also called a fixed point for f . The set of all period- n points, $n = 1, 2, \dots$, is denoted by

$$P(f) = \{x | f^n(x) = x \text{ and } f^j(x) \neq x \text{ for } 0 < j < n, n = 1, 2, \dots\}.$$

Lemma 5 ([6]). Let $f : [a, b] \rightarrow [a, b]$ be a strictly decreasing continuous function; then, the function f only possesses period-2 points and fixed points.

We now explore single-valley solutions of Equation (2). We establish the following noteworthy theorem:

Theorem 2. Let f be a single-valley solution of Equation (2) and α be its unique minimum point; then, the strictly decreasing function $\phi(x) = f(\epsilon^2 x) : [0, 1] \rightarrow [f(\epsilon^2), 1]$ has a unique periodic point α .

Proof. Since f is continuous, according to Lemma 2, we have $f(\alpha) = 0$ and $\epsilon^2 < \alpha$. According to Definition 3, the function $f|_{[0, \epsilon^2]}$ is strictly decreasing, so the function ϕ is also strictly decreasing. Based on Lemma 3 and the uniqueness of α , it is determined that $\phi(\alpha) = f(\epsilon^2 \alpha) = \alpha$. This tells us that α is the unique fixed point of f .

If β is another periodic point of ϕ , based on Lemma 5, β should be a periodic-2 point. We determine that

$$\epsilon f(\beta) = f(f(\beta)) = f(\phi(\beta))$$

and

$$\epsilon^2 f(\beta) = \epsilon f(\phi(\beta)) = f(f(\epsilon^2 \phi(\beta))) = f(\phi(\phi(\beta))) = f(\beta).$$

Because $\epsilon \in (0, 1)$, $f(\beta) = 0$. Then, we know that $\beta = \alpha$. $\square$

By extending a strictly decreasing function possessing a unique periodic point, Song and Zhang constructed a single-valley solution of Equation (2) [9]. Combining Theorem 2, Lemma 5, and the conclusion in [9], we obtain the following key fact: to construct a flat-bottomed single-valley solution of Equation (2), we must select strictly decreasing functions that admit periodic-2 points.

We now explore the property of the set of period points of a strictly decreasing continuous function on a bounded closed interval. If a strictly decreasing continuous function f only possesses a fixed point, then the set $P(f)$ is a singleton. We find that if a strictly decreasing continuous function f on a bounded closed interval possesses period-2 points, then the set $P(f)$ is contained in a minimal bounded closed subinterval. The specific definition and conclusion are stated as follows:

Definition 5. Let $f : [a, b] \rightarrow [a, b]$ be a strictly decreasing continuous function. If

$$P(f) \subset [C, D] \subset [a, b],$$

then $[C, D]$ is called a bounded closed subinterval containing $P(f)$. Let $\mathcal{A}(f, [a, b])$ denote the collection of all bounded closed subintervals containing $P(f)$. If there is an interval $[c, d] \in \mathcal{A}(f, [a, b])$ such that for any $[C, D] \in \mathcal{A}(f, [a, b])$, we have

$$[c, d] \subset [C, D].$$

The interval $[c, d]$ is called the minimal bounded closed subinterval containing $P(f)$.

Lemma 6. Let $f : [a, b] \rightarrow [a, b]$ be a strictly decreasing continuous function. If the function f possesses period-2 points, then there must be a minimal bounded closed subinterval $[c, d]$ containing $P(f)$ with $f(c) = d$ and $f(d) = c$.

Proof. Since the continuous function $f : [a, b] \rightarrow [a, b]$ is strictly decreasing, there must be a unique fixed point $\alpha \in (a, b)$ of the function f . If s is a period-2 point of the function f , then $t = f(s)$ is also a period-2 point of the function f . It is easy to see that

$$s < \alpha < t \text{ or } t < \alpha < s \quad (8)$$

holds.

Let $P(f)$ be the set of all period points of f and

$$c = \inf P(f), d = \sup P(f).$$

According to fact (8), we know that $c < \alpha < d$ and the subinterval $[c, d]$ contains all period points of the function f .

Let $[C, D]$ be any bounded closed subinterval containing $P(f)$. Since both c and d are accumulation points of $P(f)$, we have $c, d \in [C, D]$. Thus, $[c, d] \subset [C, D]$, which shows that $[c, d]$ is the minimal bounded closed subinterval containing $P(f)$.

Let $P_l(f) = [c, \alpha] \cap P(f)$ and $P_r(f) = [\alpha, d] \cap P(f)$. By virtue of fact (8), the function f maps $P_l(f)$ onto $P_r(f)$ in a one-to-one manner.

For any $s_0 \in P_l(f)$ there is a $t_0 \in P_r(f)$ such that $f(s_0) = t_0 \leq d$. Since $s_0 \geq c$, we determine that $t_0 = f(s_0) \leq f(c)$. According to the definition of a supremum, we know that $d \leq f(c)$.

Since $c = \inf P(f)$, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset P_l(f)$ with $\lim_{n \rightarrow \infty} x_n = c$. Thus, we know that $\lim_{n \rightarrow \infty} f(x_n) = f(c) \leq d$. Then, we have $f(c) = d$.

Through a symmetric discussion, we also determine that $f(d) = c$. Thus, both c and d are period-2 points with $f(c) = d$ and $f(d) = c$. $\square$

3. Properties of Flat-Bottomed Single-Valley Solutions

Since all flat-bottomed single-valley solutions constitute a special class of single-valley extended solutions with platforms, we can synthesize the following lemma from the results

in [8] by taking its specific properties into account. The following theorem states the key structural and analytic properties of a flat-bottomed single-valley solution f of Equation (2).

Theorem 3 ([5]). *Let f be a flat-bottomed single-valley solution of Equation (2) and $[u, v]$ be the interval consisting of all minimum points of f . The following facts hold:*

- (1) $f(\epsilon^2 u) = v, f(\epsilon^2 v) = u$;
- (2) $f([0, \epsilon^2 u]) = [v, 1], f([\epsilon^2 u, \epsilon^2 v]) = [u, v], f([\epsilon^2 v, u]) = [0, u], f([v, 1]) = [0, \epsilon]$;
- (3) *The equation $f(x) = \epsilon x$ has the unique solution $x = 1$ in $(v, 1)$;*
- (4) $\epsilon^2 \leq f(\epsilon^2) < u$.

Theorem 2 describes the connection between the set of minimum points of a single-valley solution f of Equation (2) and the periodic point set $P(f)$ of the strictly decreasing function $\phi(x) = f(\epsilon^2 x)$. This theorem provides a criterion for selecting a strictly decreasing function to be extended for use in constructing a single-valley solution. For the same reason, we try to find the connection between the set of minimum points of a flat-bottomed single-valley solution f of Equation (2) and the periodic point set $P(f)$ of the strictly decreasing function $\phi(x) = f(\epsilon^2 x)$. The conclusion is stated in the following theorem:

Theorem 4. *Let f be a flat-bottomed single-valley solution of Equation (2) and $[u, v]$ be the interval consisting of all minimum points of the solution f . Let $\phi(x) = f(\epsilon^2 x) : [0, 1] \rightarrow [f(\epsilon^2), 1]$. Then, u, v are two period-2 points of the function ϕ with $\phi(u) = v, \phi(v) = u$ and the interval $[u, v]$ is the minimal bounded closed subinterval containing $P(\phi)$.*

Proof. According to Definition 3, the function $\phi(x) = f(\epsilon^2 x) : [0, 1] \rightarrow [0, 1]$ is strictly decreasing. According to (1) of Theorem 3, u, v are two period-2 points of the function ϕ with

$$\phi(u) = v, \quad \phi(v) = u.$$

According to Lemma 6, there must be two period-2 points s, t with $s < t$ such that

$$\phi(s) = t, \quad \phi(t) = s$$

and the interval $[s, t]$ is the minimal bounded closed subinterval containing $P(\phi)$. Then, we know that

$$s \leq u < v \leq t.$$

Since

$$\phi(0) = f(0) = 1, \quad u > \phi(1) = f(\epsilon^2) > 0$$

and

$$1 > v > \phi(\phi(1)) = f(\epsilon^2 f(\epsilon^2)) > 0,$$

we know that both 0 and 1 are not period-2 points. Thus, we know that

$$0 < s \leq u < v \leq t < 1. \quad (9)$$

Since ϕ is strictly decreasing, we know that the sequence $\{\phi^{2k}(0)\}_{k=0}^{\infty}$ is strictly increasing and the sequence $\{\phi^{2k+1}(0)\}_{k=0}^{\infty}$ is strictly decreasing. Note that

$$\phi^0(0) = 0, \quad \phi^1(0) = f(0) = 1,$$

$$0 = f(u) < f(\epsilon^2) = \phi(1) = \phi^2(0) < f(\epsilon^2 t) = \phi(t) = s,$$

and

$$t = f(\epsilon^2 s) < f(\epsilon^2 \phi^2(0)) = \phi^3(0) < f(0) = 1,$$

by induction, the sequence $\{\phi^{2k}(0)\}_{k=0}^{\infty}$ is bounded above by s . According to the same induction argument, the sequence $\{\phi^{2k+1}(0)\}_{k=0}^{\infty}$ is bounded below by t . There must be two real numbers A, B with

$$A \leq s, \quad B \geq t \quad (10)$$

such that

$$\lim_{k \rightarrow \infty} \phi^{2k}(0) = A, \quad \lim_{k \rightarrow \infty} \phi^{2k+1}(0) = B. \quad (11)$$

Because ϕ is continuous, we get

$$\phi(A) = B, \quad \phi(B) = A. \quad (12)$$

Fact (12) shows that both A and B are periodic-2 points, and we get $A, B \in [s, t]$. When combining this with fact (10), we have

$$A = s, \quad B = t.$$

We also have $\lim_{k \rightarrow \infty} \phi^{2k}(0) = s$ and $\lim_{k \rightarrow \infty} \phi^{2k+1}(0) = t$.

On the other hand, according to the definition of ϕ and the structure of Equation (2), we can determine that

$$f(\phi^k(0)) = f(f(\epsilon^2 \phi^{k-1}(0))) = \epsilon f(\phi^{k-1}(0)).$$

By induction, we have

$$f(\phi^k(0)) = \epsilon f(\phi^{k-1}(0)) = \epsilon^2 f(\phi^{k-2}(0)) = \dots = \epsilon^k f(0) = \epsilon^k.$$

This fact tells us that

$$f(s) = \lim_{k \rightarrow \infty} f(\phi^{2k}(0)) = \lim_{k \rightarrow \infty} \epsilon^{2k} = 0, \quad f(t) = \lim_{k \rightarrow \infty} f(\phi^{2k+1}(0)) = \lim_{k \rightarrow \infty} \epsilon^{2k+1} = 0.$$

These facts show us that both s and t are also minimum points of the function f . Thus, by virtue of fact (9) and the property of the interval $[u, v]$, we can determine that $u = s$ and $v = t$.

Consequently, $[u, v] = [s, t]$. The interval of all minimum points of f coincides with the minimal bounded closed subinterval containing $P(\phi)$. $\square$

4. Decomposition of Bounded Closed Intervals

As mentioned in [6], for given constants X and Y , let $[X; Y]$ be the closed interval whose endpoints are X and Y , i.e., if $X < Y$, then $[X; Y] = [X, Y]$, and if $X > Y$, then $[X; Y] = [Y, X]$.

To construct single-valley solutions of Equation (2), Song and Zhang decomposed the interval $[0, 1]$ into a special form. We summarize such a theorem from the proof of Theorem 4 in [9].

Theorem 5. Let $f : [a, b] \rightarrow [a, b]$ be a strictly decreasing continuous function and the function f possesses a unique periodic point α with

$$a < \alpha < b, \quad f(\alpha) = \alpha.$$

If

$$f(a) = b, \quad a < f(b) < \alpha,$$

then the interval $[a, b]$ can be decomposed into the following form:

$$[a, b] = \{\alpha\} \cup \left(\bigcup_{k=0}^{\infty} \Delta_k \right),$$

where

$$\Delta_k = [f^k(a); f^{k+2}(a)].$$

To construct flat-bottomed single-valley solutions of Equation (2), we also need a similar decomposition of interval $[0, 1]$. The concrete result is stated in the following theorem:

Theorem 6. Let $f : [a, b] \rightarrow [a, b]$ be a strictly decreasing continuous function, and the function f possesses two period-2 points c, d with

$$a < c < d < b, \quad f(c) = d, \quad f(d) = c$$

and $[c, d]$ is the minimal bounded closed subinterval containing $P(f)$. If

$$f(a) = b, \quad a < f(b) < c,$$

the interval $[a, b]$ can be decomposed into the following form:

$$[a, b] = [c, d] \cup \left(\bigcup_{k=0}^{\infty} \Delta_k \right), \quad (13)$$

where

$$\Delta_k = [f^k(a); f^{k+2}(a)].$$

Proof. Note that

$$f^0(a) = a < c, \quad f^1(a) = b > d,$$

$$a < f(b) = f^2(a) < c, \quad b > f^3(a) > f(c) = d,$$

by induction, the sequence $\{f^{2k}(a)\}_{k=0}^{\infty}$ is strictly increasing and bounded above by c . Using the same induction argument, the sequence $\{f^{2k+1}(a)\}_{k=0}^{\infty}$ is strictly decreasing and bounded below by d .

There must be two real numbers A, B with

$$A \leq c, \quad B \geq d \quad (14)$$

such that

$$\lim_{k \rightarrow \infty} f^{2k}(a) = A, \quad \lim_{k \rightarrow \infty} f^{2k+1}(a) = B \quad (15)$$

and

$$f(A) = B, \quad f(B) = A.$$

By the property of the interval $[c, d]$, it is easy to see

$$A \geq c, \quad B \leq d. \quad (16)$$

By means of (14) and (16), we know

$$A = c, \quad B = d. \quad (17)$$

Based on facts (15) and (17) and the strict monotonicity of $\{f^{2k}(a)\}_{k=0}^{\infty}$ and $\{f^{2k+1}(a)\}_{k=0}^{\infty}$, we can determine that

$$[a, b] = [c, d] \cup \left(\bigcup_{k=0}^{\infty} \Delta_k \right),$$

where

$$\Delta_k = [f^k(a); f^{k+2}(a)].$$

□

5. The New Construction Method

This section presents the new construction method for flat-bottomed single-valley solutions of Equation (2).

Theorem 7. Let f_1 and f_2 be flat-bottomed single-valley solutions of Equation (2). If

$$f_1(\epsilon^2) = f_2(\epsilon^2) = a$$

and

$$f_1(x) = f_2(x), \quad \forall x \in [0, a].$$

Then

$$f_1(x) = f_2(x), \quad \forall x \in [0, 1].$$

Proof. For $\forall x \in [0, 1]$, define

$$\phi(x) = f_1(\epsilon^2 x) = f_2(\epsilon^2 x).$$

The interval $[u, v]$ with $\phi(u) = v$ and $\phi(v) = u$ is the minimal bounded closed subinterval containing $P(\phi)$.

Let $[u_i, v_i]$ be the interval consisting of all minimum points of the function $f_i, i = 1, 2$. Based on the proof of Theorem 4, for $i = 1, 2$, the interval $[u_i, v_i]$ coincides with the interval $[u, v]$. We can thus determine that

$$[u_1, v_1] = [u_2, v_2] = [u, v].$$

Moreover, we know that

$$f_1(x) = f_2(x) = 0, \quad \forall x \in [u, v]. \quad (18)$$

Note that

$$\phi(0) = f_1(0) = 1, \quad 0 < \phi(1) = f_1(\epsilon^2) \geq \epsilon^2 > u.$$

Let $\Delta_k = [\phi^k(0); \phi^{k+2}(0)]$. Then, using Theorem 6, we have

$$[0, 1] = [u, v] \cup \left(\bigcup_{k=0}^{\infty} \Delta_k \right). \quad (19)$$

Through (18) and (19), it is shown that for $k = 0, 1, 2, \dots$, the fact

$$f_1(x) = f_2(x), \quad \forall x \in \Delta_k$$

holds. Obviously, for $\Delta_0 = [0, a]$, we have

$$f_1(x) = f_2(x), \quad \forall x \in \Delta_0.$$

Suppose that

$$f_1(x) = f_2(x), \quad \forall x \in \Delta_k.$$

Since ϕ maps Δ_k to Δ_{k+1} in a one-to-one manner, we can obtain

$$\phi^{-1}(x) \in \Delta_k, \quad \forall x \in \Delta_{k+1}.$$

Thus, we can determine that

$$f_1(x) = \epsilon f_1(\phi^{-1}(x)) = \epsilon f_2(\phi^{-1}(x)) = f_2(x), \quad \forall x \in \Delta_{k+1}.$$

□

Theorem 8. Let ϵ, a, u, v be positive numbers with $0 < \epsilon^2 \leq a < u < v < 1$ and f_0 be a strictly decreasing continuous function on $[0, a]$. If f_0 satisfies the following three assumptions:

1. $f_0(0) = 1, \quad f_0(\epsilon^2) = a, \quad f_0(a) = \epsilon^2;$
2. u and v are period-2 points of the function $\phi(x) = f_0(\epsilon^2 x), \forall x \in [0, 1]$ with

$$\phi(u) = v, \quad \phi(v) = u;$$

3. $[u, v]$ is the minimal bounded closed subinterval containing $P(\phi)$,
then there exists a unique function f satisfying

$$\begin{cases} f(f(\epsilon^2 x)) = \epsilon f(x), & \forall x \in [0, 1], \\ f(x) = f_0(x), & \forall x \in [0, a], \end{cases} \quad (20)$$

and the function f is a flat-bottomed single-valley function and solution of Equation (2).

Proof. The function

$$\phi(x) = f_0(\epsilon^2 x), \quad \forall x \in [0, 1],$$

is strictly decreasing. According to Assumption 1, we have

$$\phi(0) = f_0(0) = 1, \quad 0 < a = f_0(\epsilon^2) = \phi(1) < u. \quad (21)$$

In view of (21), together with Assumptions 2 and 3, by virtue of Theorem 6, the interval $[0, 1]$ can be decomposed into form (19), where

$$\Delta_k = [\phi^k(0); \phi^{k+2}(0)].$$

It is easy to verify that

$$[0, u) = \bigcup_{m=0}^{+\infty} \Delta_{2m}$$

and

$$(v, 1] = \bigcup_{m=0}^{+\infty} \Delta_{2m+1}.$$

Since ϕ is strictly decreasing and continuous on $[0, 1]$, it is bijective from $[0, u]$ onto $[v, 1]$ and from $[v, 1]$ onto $[0, u]$. Hence, ϕ^{-1} is well defined on each Δ_k .

By virtue of the function ϕ^{-1} , the function f can be defined on each Δ_k by induction as follows: The strictly decreasing continuous function f_0 has been defined on Δ_0 . For $k \geq 1$, the strictly monotonic continuous function f_k can be defined as

$$f_k(x) = \epsilon f_{k-1}(\phi^{-1}(x)), \quad \forall x \in \Delta_k. \quad (22)$$

By induction, for $m = 0, 1, 2, \dots$, the function f_{2m} is strictly decreasing and the function f_{2m+1} is strictly increasing. Define

$$f(x) = \begin{cases} f_k(x), & \forall x \in \Delta_k, k = 0, 1, 2, \dots, \\ 0, & \forall x \in [u, v]. \end{cases} \quad (23)$$

Next, we prove that f is the desired flat-bottomed single-valley solution.

We need to show that f is continuous on $[0, 1]$. First, for $k = 0, 1, 2, \dots$, we prove the continuity of f at the endpoint shared by Δ_k and Δ_{k+2} . For $k = 0, 1, 2, \dots$, through the definition of f_k as (22) and induction, we can obtain

$$f_k(\phi^k(0)) = \epsilon f_{k-1}(\phi^{k-1}(0)) = \epsilon^2 f_{k-2}(\phi^{k-2}(0)) = \dots = \epsilon^k f_0(0) = \epsilon^k. \quad (24)$$

Since $f_0(0) = 1$ and $f_0(\epsilon^2) = a$, we have

$$\phi^2(0) = f_0(\epsilon^2 f_0(0)) = f_0(\epsilon^2) = a.$$

Consequently, for $k = 0, 1, 2, \dots$, through the definition of f_k as (22) and induction, we can also obtain

$$f_k(\phi^{k+2}(0)) = \epsilon f_{k-1}(\phi^{k+1}(0)) = \epsilon^2 f_{k-2}(\phi^k(0)) = \dots = \epsilon^k f_0(\phi^2(0)) = \epsilon^k f_0(a) = \epsilon^{k+2}. \quad (25)$$

By means of facts (24) and (25), we can determine that

$$f_k(\phi^{k+2}(0)) = f_{k+2}(\phi^{k+2}(0)) = \epsilon^{k+2}. \quad (26)$$

Fact (26) shows that, for $k = 0, 1, 2, \dots$, the function f is continuous at $\phi^{k+2}(0)$, which is the endpoint shared by Δ_k and Δ_{k+2} . Hence, f is continuous across adjacent Δ_k and we know the function f is continuous on both $[0, u)$ and $(v, 1]$.

Second, we need to show that f is continuous at both u and v . Note that

$$\lim_{k \rightarrow \infty} f(\phi^{2k}(0)) = \lim_{k \rightarrow \infty} f_{2k}(\phi^{2k}(0)) = \lim_{k \rightarrow \infty} \epsilon^{2k} = 0$$

and $f|_{[0, u)}$ is strictly decreasing; we can obtain

$$\lim_{x \rightarrow u^-} f(x) = 0.$$

Through the definition of f , for all $x \in [u, v]$, $f(x) = 0$. These two facts guarantee the continuity of the function f at $x = u$. Similarly, the function f is continuous at $x = v$.

Finally, we determine that f is continuous on both $[0, u]$ and $[v, 1]$. Through the definition of f , it is easy to see that f is continuous on the whole $[0, 1]$.

Now we show that f satisfies Equation (2). For any $x \in [0, 1]$, substituting $\phi(x) = f_0(\epsilon^2 x)$ for x in Equation (2) yields

$$f_k(f_0(\epsilon^2 x)) = \epsilon f_{k-1}(x).$$

Thus, we know the function f is a solution of Equation (2).

Using the construction procedure, it is easy to see that f is strictly decreasing on $[0, u]$, identically zero on $[u, v]$, and strictly increasing on $[v, 1]$. Hence, the solution f is a flat-bottomed single-valley function on $[0, 1]$.

The uniqueness of the solution f is guaranteed by Theorem 7. $\square$

Example 1. Let ϵ, a, u , and v be four positive numbers with

$$0 < \epsilon^2 < a < u < v < 1.$$

Let

$$f_0(x) = \begin{cases} -\frac{1-v}{u\epsilon^2}x + 1, & \forall x \in [0, \epsilon^2 u], \\ -\frac{1}{\epsilon^2}x + (u+v), & \forall x \in [\epsilon^2 u, \epsilon^2 v], \\ \frac{a-u}{(1-v)\epsilon^2}x + \frac{u-av}{1-v}, & \forall x \in (\epsilon^2 v, \epsilon^2], \\ -x + (\epsilon^2 + a), & \forall x \in (\epsilon^2, a]. \end{cases} \quad (27)$$

and

$$\phi(x) = f_0(\epsilon^2 x) = \begin{cases} -\frac{1-v}{u}x + 1, & \forall x \in [0, u], \\ -x + (u+v), & \forall x \in [u, v], \\ \frac{a-u}{1-v}x + \frac{u-av}{1-v}, & \forall x \in (v, 1]. \end{cases} \quad (28)$$

Both f_0 and ϕ are strictly decreasing continuous functions.

It is easy to verify

$$f_0(0) = 1, \quad f_0(\epsilon^2) = a, \quad f_0(a) = \epsilon^2$$

and

$$\phi(u) = v, \quad \phi(v) = u.$$

Based on the above discussion, Assumptions 1 and 2 of Theorem 8 hold.

For any $x \in [u, \frac{u+v}{2}) \cup (\frac{u+v}{2}, v]$, we have

$$\phi^2(x) = -(-x + (u+v)) + (u+v) = x.$$

All $x \in [u, \frac{u+v}{2}) \cup (\frac{u+v}{2}, v]$ are periodic-2 points of ϕ .

It is easy to verify that $\frac{u+v}{2}$ is the unique fixed point of ϕ . The subinterval $[u, v]$ consists of periodic points of the function ϕ .

For any $x \in [0, u)$, if x is a periodic point, it must be a periodic-2 point. We have

$$x = \phi^2(x) = \frac{a-u}{1-v} \left(-\frac{1-v}{u}x + 1 \right) + \frac{u-av}{1-v}.$$

Through simple calculation, we get $x = u \notin [0, u)$. So all $x \in [0, u)$ are not periodic points of ϕ .

For any $x \in (v, 1]$, if x is a periodic point, it must be a periodic-2 point, too. We still have

$$x = \phi^2(x) = -\frac{1-v}{u} \left(\frac{a-u}{1-v}x + \frac{u-av}{1-v} \right) + 1.$$

By simple calculation, we also get $x = v \notin (v, 1]$. So all $x \in (v, 1]$ are not periodic points of ϕ .

Based on the above discussion, the interval $[u, v]$ is exactly the minimal bounded closed subinterval containing $P(\phi)$. Assumption 3 of Theorem 8 also holds.

By virtue of Theorem 8, the given function f_0 can be extended to be a unique flat-bottomed single-valley solution f of Equation (2) with the parameter ϵ . The platform of f is precisely the interval $[u, v]$.

6. Conclusions and Problems

1. Continuous solutions of Equation (2) have been extensively explored. However, the full set of continuous solutions has yet to be identified. To our knowledge, the existence and properties of several distinct classes of continuous solutions have been demonstrated. Specifically, Shi proved the existence of single-valley solutions in [7] and established the existence of single-valley-extended solutions in [10]. Furthermore, Shi demonstrated the existence of single-valley-extended solutions with platforms in [8] and proved the existence of multi-modal-extended solutions in [11]. For concrete definitions of single-valley-extended solutions and multi-modal-extended solutions, we refer the reader to [10] and [11], respectively.

For a given parameter ϵ , we introduce the following notation: Let $C(\epsilon)$ denote the set of all continuous solutions of Equation (2) with parameter ϵ ; $SV(\epsilon)$ the set of all single-valley solutions; $SVE(\epsilon)$ the set of all single-valley-extended solutions; $SVEP(\epsilon)$ the set of all single-valley-extended solutions with platforms; and $MME(\epsilon)$ the set of all multi-modal-extended solutions, respectively. Finally, let $FBSV(\epsilon)$ denote the set of all flat-bottomed single-valley solutions.

It is easy to see that $SV(\epsilon)$, $SVE(\epsilon)$, $SVEP(\epsilon)$, and $MME(\epsilon)$ are subsets of $C(\epsilon)$. The intersection of any two of these four subsets is empty. And $FBSV(\epsilon)$ is a proper subset of $SVEP(\epsilon)$.

While considerable progress has been made, discovering additional families of continuous solutions of Equation (2) remains a pertinent issue.

2. The construction method provided in Theorem 8 is equivalent to the method described by Shi [8]. By extending all strictly decreasing functions satisfying all assumptions of Theorem 8, we can get the $FBSV(\epsilon)$ set. By extending all single-valley functions with platforms and strictly decreasing functions h satisfying all assumptions of Theorem 1, the $FBSV(\epsilon)$ set can be obtained, too. Since our method relies solely on a strictly decreasing function, it is somewhat simpler than the one proposed by Shi [8].

Our results show that by extending a strictly decreasing function, one can only obtain single-valley solutions and flat-bottomed single-valley solutions. To derive more types of continuous solutions, we must continue to leverage the strengths across Shi's methods [7,8,10,11].

3. Theorem 8 shows that the length of the platform of a flat-bottomed single-valley solution f equals the length of the minimal bounded closed subinterval containing $P(\phi)$. If the given strictly decreasing function f_0 is nonlinear, the induced function ϕ is also nonlinear; hence, it is a difficult task to find the minimal bounded closed subinterval containing $P(\phi)$. By virtue of piecewise linear strictly decreasing functions, Example 1 provides a general procedure to obtain a flat-bottomed single-valley solution whose platform is precisely the given interval $[u, v]$. Thus, how to specify the platform of a flat-bottomed single-valley solution extended by a nonlinear strictly decreasing function remains an interesting problem.

4. For a given $a \in (0, 1)$, let $\epsilon \in (0, \sqrt{a})$ be a parameter and let $SDC([0, a])$ denote the set of all strictly decreasing functions on $[0, a]$ satisfying all the assumptions of Theorem 8. Let $C([0, 1])$ denote the set of all continuous functions on $[0, 1]$. Theorem 8 provides a map

$$F : (0, 1) \times SDC([0, a]) \rightarrow C([0, 1]).$$

It is of significant interest to investigate the dependence of this function on its arguments.

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