

Article

Metric-Induced Rotations with Gielis-Based Geometry

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Abstract

In this work, we present a generalized geometric framework that extends quaternion-based rotation and translation operators within a generalized inner product and vector product setting defined on Gielis-type superquadrics. By incorporating the multiplicative shape factor $\rho(\phi)$, we formulate rotation matrices and quaternion mappings adapted to the elastic and non-Euclidean behavior of biological growth surfaces. The proposed framework extends classical Euclidean constructions to the generalized metric setting, enabling smooth, direction-dependent deformations and providing a unified description of curvature-induced growth, differential thickening, and torsional motions observed in plants. The resulting formulation offers a mathematically consistent and biologically interpretable tool with potential applications in computational botany, growth-based animation, and the design of biologically inspired structures.

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1. Introduction

1.1. On Growth and Form

Mathematics is the language of nature, its forms and processes. Natural systems, both the living (biology) and the non-living (physics), show a remarkable diversity of forms. Living organisms include archaeobacteria, bacteria, plants, fungi, and animals, and non-living structures and phenomena include snow crystals, music, seismic wave fronts, and galaxies. Universal Natural Shapes encompasses natural forms and phenomena, living and non-living, from the smallest to the largest [1,2]. From a geometric point of view, Universal Natural Shapes is a unified way to model and understand natural forms, their development, evolution, and the underlying processes. Organisms and phenomena are indeed not static, but forever in flux, whereby growth (and decay) in our universe are fundamental processes that shape both the organic and inorganic worlds, at all levels. Growth can be thought of in different ways, such as an increase in size, in complexity, or in connectivity. We focus here on an increase in size, but the methods outlined can be adapted to other forms as well.

The phenomena of growth in nature can be categorized into primary modes, internal expansion and accretion, which govern both living and non-living organisms.

Internal Expansion (Intussusception) operates from the “inside out,” incorporating new material directly into the internal matrix of an object which pushes its boundaries outwards. Biologically, this is observed in growth at the tips of plant roots and shoots,



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where rapid cell division in the root and shoot apical meristems pushes the plant body outward. The same process of cell division leads to the development of embryos. Examples of physical and chemical processes include the expansion of the universe via space–time, the swelling of hydrogels that absorb water, or the development of artificial osmotic growths [3], created from mineral salts; they absorb nutrients from their fluid medium, assimilate it, and grow internally.

Accretion (apposition) adds new material layer by layer to the external surface of an existing, typically rigid, core. On physical and cosmic scales, this blueprint governs processes as diverse as the formation of crystals, snowflakes, stalactites, electroplating, and the accretion of dust and gas to form planets in proto-planetary disks. In living organisms, this process is called bio-mineralization. Corals excel at this, building massive reefs that provide homes for many marine species. In mollusks and fossil ammonoids, calcium carbonate is added through the action of the mantle, an organ that secretes a liquid mixture of proteins and minerals at the growing edge to increase the length of the shell and across the inner surface to increase the thickness.

These two opposites represent internal processes (physiological processes for living organisms) and communication with the immediate surroundings (or the external environment at large), respectively. In open systems, these opposites occur in a hybrid form, since all systems are characterized by transport and exchange. In diatoms, glass cages of extraordinary beauty, a highly controlled internal mechanism creates rigid, glass-like nanostructures from silicon dioxide dissolved in water. Because their silica shells cannot stretch, diatoms use proteins and long-chain polyamines to internalize precise structures inside a specialized compartment called the Silica Deposition Vesicle before cell division [4]. Similarly, the process of photosynthesis in plants can be considered as accretion (capturing carbon dioxide from the air), turning it into sugars and energy, and driving growth and development through internal expansion.

A further step involves the acquisition and fusion of the same (such as fusion of gametes or black holes coalescing together) or of different types. Indeed, whether internal growth or apposition occurs when crossing organismal boundaries may also depend on the point of view. Gall formation and tumor growth illustrate complementarity across organismal boundaries. A gall parasite's physiological secretions (e.g., wasps) reprogram the host's developmental machinery, triggering rapid intussusceptive growth within the plant tissue. Under this action, the plant switches to secondary wall deposition, forming thick, rigid, lignified outer layers or capsules. From the perspective of the gall, this is an apposition or accretion [5] onto the external surface of the larval chamber.

From these primary modes and their combinations, complexity can be increased by folding or by increasing branching. The latter can be internal branching (the respiratory system in the lungs and circulatory blood systems in animals, or neuronal connections in the brain [6]), or external, such as in dendritic snowflakes [7]). For complex but structured branching in trees and shrubs, L-systems, named after Aristid Lindenmayer, are commonly used [8,9] with recent extension on curved surfaces [10]).

Ultimately, all of these growth blueprints can be analyzed through the lens of mathematics and physics. In the physical sciences, a nearly universal principle is that the equilibrium configuration of a system can be found by minimizing its total energy among all admissible configurations. Whether through internal expansion or accretion, nature seeks to find optimal solutions in surface tension, metabolic cost, and material waste. Interestingly, both inanimate matter and living systems use similar physical mechanisms to build complex forms. By studying these mechanisms, we gain a comprehensive foundation for understanding how nature constructs its endless array of shapes while maintaining structural stability [11].

1.2. A Geometrical Perspective: Universal Natural Shapes

From a geometrical perspective, it amounts to constructing an abstract, purely geometrical theory of morphogenesis, independent of the substrate of forms and the nature of the forces that create them [11]. Current mathematical models rely on a small set of idealized shapes, such as circles, spheres, and logarithmic spirals [12] for the study of natural shapes and phenomena. Indeed, when mapping these limited forms, geometry reduces the most elementary movements in nature to two basic forms: the circle and the spiral. This is determined by the relationship between the position vector of a growth trajectory X and its tension or acceleration vector X'' . If the organism grows parallel to the tension vector $X \parallel X''$, the basic figure is a circle, representing a dynamic of “going with the flow”. If the growth is orthogonal to the tension $X \perp X''$, it forms a logarithmic spiral, representing growth that “opposes the flow” or resists the tension [13]. As noted by D’Arcy Thompson, the spiral represents a constant ratio of expansion in length and width [14]. The organism grows in size but does not change its shape, a phenomenon that is visible in biological structures like shells, fossil ammonoids, horns, plant phyllotaxy and in spiral galaxies.

Most mathematical methods are all based on isotropic geometry, with no preferred directions, and while these models capture certain global features, they often fail to describe the direction-dependent characteristics that dominate biological growth and natural morphogenesis, as observed in the eggs of birds, reptiles and insects, in snowflakes and ice [7], in the shells of mollusks and fossil ammonoids, and in plant seeds, leaves and flowers [15], in square archaeobacteria and diatoms [16], in mud cracks, rocks and tectonic plates [17], in the shape of galaxies [18] and even space–time [2]. But also seismic wave propagation [19] or the spread of wildfires [20] are direction-dependent, as are many other phenomena in nature.

Furthermore, it should be noted that the application of mathematical methods to biology has primarily focused on the most common shapes or archetypes, often neglecting the many evolutionary and developmental variations and experiments, many of which have been successful. *Nautilus* shells are often presented as the quintessential example of logarithmic spirals, but nature has experimented throughout evolution with a variety of forms. Within the group of starfish, some genera are almost spherical (*Culcita* or cushion starfish) or nearly pentagonal (*Tosia* or cookie starfish). Ammonoids also form an interesting group for another reason. In contrast to the shells of snails and other mollusks, the internal chambers of many ammonoids are separated by very complex septa, which can be observed on the shell as suture lines (Figure 1). In ammonoids, one can find various examples of planispiral polygonal coiling, such as triangular coiling in *Solichymenia* and quadrangular coiling [21,22], or examples of non-planispiral coiling, as in the heteromorph shells of *Nipponites* [23,24]; its shells grow as relatively straight tubular segments, making U-turns, resulting in a three-dimensional tangle of U-shapes. From a geometrical point of view, the shapes of snowflakes and shells are no different from the shapes, and the structures of DNA and RNA, or the folding of proteins [25–27].

Universal Natural Shapes provide a deeper layer to the geometry of natural shapes, with the best possible models adapted to the shapes, their development, or evolution.

1.3. Generalized Conic Sections

A growing body of work has demonstrated that many natural shapes can be effectively described using generalized geometric frameworks based on Lamé curves or superellipses. In particular, the Gielis superformula provides a unifying parametric description capable of a continuous transformation between circular, polygonal, and star-like forms [28,29], with many examples in botany [15,30,31]. This formulation, which inspired the term Universal Natural Shapes [1,2], has been successfully applied to a wide range of biological

structures, including plant leaves [32], starfish [33], diatom frustules [4], and avian eggs [34], revealing that seemingly complex morphologies can emerge from relatively simple generative rules. All of these shapes optimize their forms using low-dimensional geometric strategies, often in only two dimensions. In many cases, superellipses, as simplified versions of the superformula, model tree rings [35,36], leaves and leaf stomata, bamboo culms and apical meristems in shoots and roots in a wide diversity of plants [37]. Moreover, from a geometric perspective, a wide range of natural shapes and phenomena at all scales can be efficiently modeled not only by combining the two opposing geometric tendencies—circle and spiral—using these Gielis transformations [13], but also by accounting for the anisotropic and direction-dependent growth and development. Other examples are the spiral growth of trees [35], winding vines, and the backbone of biomolecules [38].



Figure 1. (Left): Morphology of a planispiral ammonite shell of *Eopachydiscus marcianus* [39]. It shows a 3D model of a half cut shell (A), a single septum (B), and a suture line (C). (Right): Heteromorph ammonite *Nipponites*.

Gielis curves and surfaces are essentially a generalization of conic sections, and a key notion is that of flexibility and elasticity [40,41]. To extend this descriptive geometry to growth, development, and evolution, a kinematic and dynamic approach is needed, taking into account anisotropic expansion, torsion, curvature-driven deformation, and local reorientation over time. Many biological structures exhibit growth that is intrinsically coupled with geometry. Local curvature, tissue stiffness, and directional growth rates influence not only shape, but also the orientation and evolution of the organism. As a result, growth and rotation in biological systems cannot be adequately described by uniform Euclidean operators; instead, rotational behavior must depend explicitly on anisotropic geometric properties. To address this gap, a mathematical framework is required in which rotation is modulated by the same parameters that govern shape.

1.4. The Objective of This Study

The objective of this study is twofold.

First, we establish a unified geometric framework for modeling natural forms using Gielis-based parametric representations and transformations. Previous papers defined inner and vector products, generalizing previous work [42,43], and a complete differential geometric apparatus for curves and surfaces was developed in [41]. Since Gielis-forms are represented in polar coordinates, a representation in complex numbers is straightforward.

Second, we introduce an anisotropic rotational model in which quaternion-based transformations are intrinsically linked to the underlying geometry. Quaternions provide a natural and numerically stable representation of three-dimensional rotations and have been widely used in physics [44], biology [45,46], robotics, and computer graphics. In this work,

classical quaternion-based rotations are generalized by coupling them with anisotropic metric structures derived from superelliptic and Gielis-based geometries. Rather than introducing a new algebraic structure, the proposed formulation extends the classical Euclidean framework to the generalized metric considered in this paper. This coupling leads to a geometry-aware rotational framework in which rotational behavior adapts to local shape parameters and their evolution. The principal contribution of this work is not the introduction of new algebraic objects, but the extension of classical quaternionic and geometric constructions to the generalized metric framework associated with Gielis-type geometries. The resulting framework enables a unified treatment of anisotropic rotations, geometric transformations, and biologically motivated growth processes. Together, these components provide a mathematically consistent extension of classical geometric transformations to the proposed anisotropic metric framework for modeling growth, deformation, and orientation in biological systems, such as pentagonal starfish, heteromorph ammonoids with polygonal coiling, root and shoot meristems in plants, and biomolecules.

Quaternion-Based Rotations and Translations: Quaternions provide a compact, robust, and numerically stable representation of three-dimensional rotations. Unlike Euler-angle-based formulations, quaternion representations avoid gimbal lock and enable smooth interpolation through spherical linear interpolation. When combined with translation operators, quaternions form the mathematical foundation of rigid-body transformations widely used in robotics, biomechanics, computer graphics, and animation.

In the context of plant growth, quaternion-based rotations are particularly well suited for modeling torsional twisting of stems (e.g., climbing plants), gradual reorientation of leaves, curvature-induced rotations along growth trajectories, and helical motions observed in vines or inflorescences. The representation of orientation changes through unit quaternions provides a natural and mathematically consistent framework for describing smooth, continuous, and growth-induced rotational evolution in biological structures.

Coupling Quaternions with a Superelliptic Metric: Anisotropic rotation operators that vary across a surface can be obtained by defining quaternion-based rotations with respect to a generalized superelliptic metric. This coupling enables the incorporation of direction-dependent scaling and curvature-sensitive behavior directly into the rotational structure. As a result, the formulation naturally supports elastic response modeling, curvature-controlled deformation, position-dependent rotation associated with directional growth, and the integrated treatment of radial expansion, bending, and torsion.

Biological tissues do not undergo uniform rigid rotations during growth. Instead, deformation fields emerge that locally stretch, compress, or rotate depending on mechanical constraints, differential growth rates, and biological regulation. The proposed generalized formulation captures this behavior by extending quaternion-based transformations to the anisotropic metric framework developed in this paper, making it possible to simulate complex morphogenetic processes in a mathematically consistent and geometrically faithful manner.

2. Basic Concepts and Notions

2.1. Lamé Curves and Superellipses

In 1818, Gabriel Lamé (1795–1870) introduced a family of curves that generalizes the classical ellipse equation [47]. These are now known as *Lamé curves* or *superellipses*. The general equation is

$$\left| \frac{x}{a} \right|^n + \left| \frac{y}{b} \right|^n = 1$$

where $a > 0$, $b > 0$, and $n \neq 0$ is a real number.

Lamé showed that the four classical conic sections (circle, ellipse, parabola, hyperbola) arise as special cases of this single equation for particular values of the exponent n :

1. *Ellipse* ($n = 2$):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

2. *Circle* (special case of the ellipse when $a = b$ and $n = 2$):

$$x^2 + y^2 = a^2$$

3. *Parabola* ($n = \frac{1}{2}$):

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \quad (\text{assuming } a = b \text{ for simplicity, gives a parabolic arc}).$$

4. *Rectangular hyperbola* ($n = -1$): The curve consists of four branches of rectangular hyperbolas. In the first quadrant, the equation becomes

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{a} \quad (\text{again assuming } a = b).$$

This single parametric form unifies the classical conic sections as particular instances of a far richer family of curves.

Remark 1. In [37], superellipses and superparabolas both model apical meristems very well. Indeed, there is a very close relationship between these curves. Supercircles can also be considered as the combination of the two extremal values of the expansion $(x + y)^n$, namely x^n and y^n . This is modular arithmetic and only works for n prime. It works, however, for all real numbers in tropical geometry where ordinary addition and multiplication are replaced by operations that turn the real line into a semiring. The tropical operations $\oplus$ (addition) and $\odot$ (multiplication) are:

$$x \oplus y := \max(x, y)$$

$$x \odot y := x + y$$

The tropical power is then defined as: $x^n := \underbrace{x \odot x \odot \cdots \odot x}_{n \text{ times}} = n \cdot x$. A superparabola of the type $y^n = x^m$ can be written as $ny = mx$ and the tropical binomial as $(x \oplus y)^n = n \odot (x \oplus y) = n \cdot \max(x, y) = \max(nx, ny)$. Hence, the identity $(x \oplus y)^n = x^n \oplus y^n$ holds exactly for every real number $n > 0$.

2.2. Gielis-Curves, Surfaces and Transformations

In polar coordinates, superellipses can be defined as:

$$r = \left\{ \left| \frac{\cos \theta}{a} \right|^n + \left| \frac{\sin \theta}{b} \right|^n \right\}^{\frac{-1}{n}}$$

They have an inherent restriction in symmetry, but can be generalized for any symmetry [28]:

$$r(\phi) = \left[\left| \frac{\cos \frac{m\phi}{4}}{a} \right|^{n_2} + \left| \frac{\sin \frac{m\phi}{4}}{b} \right|^{n_3} \right]^{\frac{-1}{n_1}} \quad (1)$$

This equation was called Superformula (as extension of supercircles), where $\phi \in [-\pi, \pi]$, m, a, b are real parameters and n_1, n_2, n_3 are real constants, with $a, b, n_1 \neq 0$. For $m = 4$ and $n_1 = n_2 = n_3 = n$ supercircles and superellipses result.

The independence of powers reflects the generalization of Euzet on Lamé curves for different exponents on x and y [30]. The parameter m controls the rotational symmetry, while the exponents n_1, n_2, n_3 independently adjust the curvature and concavity in different directions. The symmetry parameter m can also be non-integer, leading to self-intersecting shapes. Furthermore, when using other relation operators like $<$, or $>$, one can define the disk enclosed by the curve and the outside of the shape.

In general, the shapes generated by Equation (1) are called supershapes [1,28,30]. “Gielis formula,” and “Gielis curves” are other names used in the literature for “superformula” and “supershapes”. This allows the formula to generate a wide range of natural and abstract shapes and structures, while still a generalization of Lamé curves and the classic conic sections. Because it can simulate a large range of natural shapes with a single, compact parametric expression, Gielis’ Superformula is regarded as one of the most versatile geometrical tools for the natural sciences.

Equation (2) can also act as a transformation on other planar functions $f(\phi)$. This leads to Superformula Transformations or Gielis Transformations:

$$\rho(\phi) = f(\phi) \left[\left| \frac{\cos \frac{m\phi}{4}}{a} \right|^{n_2} + \left| \frac{\sin \frac{m\phi}{4}}{b} \right|^{n_3} \right]^{\frac{-1}{n_1}} \quad (2)$$

The radial function $r(\phi)$ can operate on any planar function $f(\phi)$ and this function can take various forms depending on the phenomenon to be modeled. The parametric flexibility of the Gielis superformula allows not only the modeling of new curves and surfaces but also the construction of a new geometric algebra specific to these structures. The “shape-specific vector product” and differential toolbox can naturally model and calculate local rotations along the curve, shape-dependent variations of the surface normal, and orientation differences in asymmetric structures. This feature provides a significant advantage, particularly in the modeling of complex organic structures.

Quaternions are classically based on the standard vector product in Euclidean space. However, by defining products dependent on the superformula, the quaternion structure can be extended with these new products. These extended quaternions can represent local rotations along the curves of the superformula, non-uniform rotations resulting from parameter-dependent deformations, and shape-oriented transformation symmetries in organic or fractal-like structures without the need for another transformation matrix. This approach allows the construction of a new transformation algebra, which can be called “Gielis Space” and constructed as follows:

2.3. Gielis-Induced Inner and Vector Products

In three-dimensional space, we presented the symmetric and positive definite bilinear form [43], which is formulated as follows: for any two vectors $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$

$$\langle \cdot, \cdot \rangle_{\mathbb{S}_{\mathcal{E}}} : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}; \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{S}_{\mathcal{E}}} = \frac{1}{\rho(\phi)^2} \eta_1 u_1 v_1 + \frac{1}{\rho(\phi)^2} \eta_2 u_2 v_2 + \frac{1}{\rho(\phi)^2} \eta_3 u_3 v_3.$$

where $\rho(\phi) = f(\phi) \left[\left| \frac{\cos \frac{m\phi}{4}}{a} \right|^{n_2} + \left| \frac{\sin \frac{m\phi}{4}}{b} \right|^{n_3} \right]^{\frac{-1}{n_1}}$, $f(\phi) \neq 0$ and $\eta_i \in \mathbb{R}^+$, $i = \{1, 2, 3\}$. Here, we define the vectors $\mathbf{u}$ and $\mathbf{v}$ as superelliptic vectors. This multiplication is called a superelliptic inner product. The space $\mathbb{R}^3$ endowed with the superelliptic inner product is called the superelliptic 3-space and is represented by $\mathbb{R}_{\mathbb{S}_{\mathcal{E}}}^3 = (\mathbb{R}^3, \langle \cdot, \cdot \rangle_{\mathbb{S}_{\mathcal{E}}})$. The superelliptic inner product $\langle \cdot, \cdot \rangle_{\mathbb{S}_{\mathcal{E}}}$ can also be written as $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{S}_{\mathcal{E}}} = \mathbf{u}^t G \mathbf{v}$, where the associated matrix is

$$G = \begin{bmatrix} \frac{1}{\rho(\phi)^2} \eta_1 & 0 & 0 \\ 0 & \frac{1}{\rho(\phi)^2} \eta_2 & 0 \\ 0 & 0 & \frac{1}{\rho(\phi)^2} \eta_3 \end{bmatrix}.$$

The length of a superelliptical vector $\mathbf{u} \in \mathbb{R}_{\mathbb{S}_\varepsilon}^3$ associated with the superelliptic inner product is given by

$$\|\mathbf{u}\|_{\mathbb{S}_\varepsilon} = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{S}_\varepsilon}}.$$

In $\mathbb{R}_{\mathbb{S}_\varepsilon}^3$, any two superelliptical vectors $\mathbf{u}$ and $\mathbf{v}$ are called superelliptic orthogonal if and only if $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{S}_\varepsilon} = 0$. Additionally, if the superelliptic norm of a superelliptic vector is equal to 1, then this vector is called a superelliptic orthonormal vector. Let us denote the standard basis of Euclidean 3-space as $\{e_1, e_2, e_3\}$. Then, the base vectors of $\mathbb{R}_{\mathbb{S}_\varepsilon}^3$ can be denoted as $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ and satisfy

$$\tilde{e}_1 = \frac{\rho(\phi)}{\sqrt{\eta_1}} e_1 = \left(\frac{\rho(\phi)}{\sqrt{\eta_1}}, 0, 0 \right), \tilde{e}_2 = \frac{\rho(\phi)}{\sqrt{\eta_2}} e_2 = \left(0, \frac{\rho(\phi)}{\sqrt{\eta_2}}, 0 \right), \tilde{e}_3 = \frac{\rho(\phi)}{\sqrt{\eta_3}} e_3 = \left(0, 0, \frac{\rho(\phi)}{\sqrt{\eta_3}} \right)$$

and any vector $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}_{\mathbb{S}_\varepsilon}^3$ can be written in the following form:

$$\begin{aligned} \mathbf{u} &= u_1 \tilde{e}_1 + u_2 \tilde{e}_2 + u_3 \tilde{e}_3 \\ &= u_1 \frac{\rho(\phi)}{\sqrt{\eta_1}} e_1 + u_2 \frac{\rho(\phi)}{\sqrt{\eta_2}} e_2 + u_3 \frac{\rho(\phi)}{\sqrt{\eta_3}} e_3. \end{aligned}$$

Let $\mathbf{u}$ and $\mathbf{v}$ be two non-zero superelliptical vectors; the cosine function of the angle between them

$$\cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{S}_\varepsilon}}{\|\mathbf{u}\|_{\mathbb{S}_\varepsilon} \|\mathbf{v}\|_{\mathbb{S}_\varepsilon}},$$

where θ is in agreement with the angular parametric equations of a superellipse or a superellipsoid. The superelliptic cross product of two vectors $\{\mathbf{u}, \mathbf{v}\} \in \mathbb{R}_{\mathbb{S}_\varepsilon}^3$ is given by

$$\mathbf{u} \times_{\mathbb{S}_\varepsilon} \mathbf{v} = \Lambda^* \begin{vmatrix} \frac{\rho(\phi)^2 e_1}{\eta_1} & \frac{\rho(\phi)^2 e_2}{\eta_2} & \frac{\rho(\phi)^2 e_3}{\eta_3} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \quad (3)$$

where $\Lambda^* = \frac{1}{\rho(\phi)^3} \sqrt{\eta_1 \eta_2 \eta_3}$, $\eta_1, \eta_2, \eta_3 \in \mathbb{R}^+$. The new inner product–vector product–quaternion triad can model not only static shapes but also the transformation of shapes over time. Thus, growth orientations in starfish arms, rotational symmetry breaking in shell spirals, and combinations of rotation and extension in biological limbs can be expressed within a single algebraic framework. Representing rotational motions with quaternions, in particular, enables singularity-free and stable calculations of motions on complex surfaces.

Remark 2. The proposed rotation operator is an isometry with respect to the superelliptic metric, preserving the distances and vector norms defined by that metric. Therefore, the anisotropy considered in this work is not a property of the rotation operator itself, but is induced by the underlying metric structure. In the intrinsic geometry of the proposed metric space, the set of all points at a fixed distance from a given point is regarded as a circle. However, when this intrinsically defined circle is viewed through the Euclidean embedding, it is generally represented as a Gielis curve or another non-Euclidean shape. Hence, the apparent anisotropy originates from the Euclidean

representation of the metric geometry, while the rotation remains metric-preserving within the proposed geometric framework.

2.4. Generalized Superelliptic Rotation Matrices and Rodrigues and Cayley Formulas

Generalized superelliptic rotation matrices can be constructed using the new inner product structure based on Gielis's superformula. This new metric enables a shape-sensitive redefinition of the vector product, which underlies both the length-angle concepts and the rotation operators. Thus, the generated rotation matrices acquire a shape-adaptive structure that directly incorporates the parametric deformations and symmetry breakings of the superformula.

The main contribution of this chapter is the re-expression of two important tools of classical rotation theory, the Rodrigues transformation and the Cayley transformation, within the framework of this new inner product and vector product. This generalization allows the rotation axis and angle to be defined along with the metric distortions and orientation changes specific to the superformula, producing more flexible and realistic rotation models.

In the rest of the section, the derivation of superelliptic rotation matrices is first presented, then the forms that Rodrigues and Cayley formulas take under this new metric are explained in detail. A superelliptic rotation motion in two-dimensional space can be described as follows: If we take the matrix

$$A := \begin{bmatrix} 0 & -\frac{\sqrt{\eta_2}}{\sqrt{\eta_1}} \\ \frac{\sqrt{\eta_1}}{\sqrt{\eta_2}} & 0 \end{bmatrix} \quad (4)$$

and using the following equation

$$R := I + \sin \theta A + (\cos \theta - 1)A^2, \quad (5)$$

we obtain

$$R := \begin{bmatrix} \cos \theta & -\frac{\sqrt{\eta_2}}{\sqrt{\eta_1}} \sin \theta \\ \sin \theta \frac{\sqrt{\eta_1}}{\sqrt{\eta_2}} & \cos \theta \end{bmatrix}. \quad (6)$$

The matrix R gives a superelliptic rotation on the superellipse $\frac{1}{\rho(\phi)^2} \eta_1 x_1^2 + \frac{1}{\rho(\phi)^2} \eta_2 x_2^2 = 1$. If we want to generate the superellipse, we can rotate the vector $x = \tilde{e}_1 = \frac{\rho(\phi)}{\sqrt{\eta_1}} e_1 \in \mathbb{R}_{\mathbb{S}_\mathcal{E}}^2$ through the matrix R with the angle $\theta = \phi$. Namely, the equations of the superellipses are obtained as follows:

$$\alpha(\phi) = \begin{bmatrix} \cos \theta & -\frac{\sqrt{\eta_2}}{\sqrt{\eta_1}} \sin \theta \\ \sin \theta \frac{\sqrt{\eta_1}}{\sqrt{\eta_2}} & \cos \theta \end{bmatrix} \begin{bmatrix} \frac{\rho(\phi)}{\sqrt{\eta_1}} \\ 0 \end{bmatrix} = \left[\frac{\rho(\phi) \cos \theta}{\sqrt{\eta_1}}, \frac{\rho(\phi) \sin \theta}{\sqrt{\eta_2}} \right].$$

The images of the superelliptic rotation and the related superellipses are visualized in Figure 2 and 3.

The examples presented above are only special cases obtained for certain parameter choices; however, the flexible parametric structure of generalized Gielis curves allows for the modeling of a much larger transformation space. This structure allows for both highly symmetric and significantly symmetry-broken rotational motions, depending on the metric and symmetry properties specified by the superformula. Local or global variation of parameters allows the rotation axis to vary from position to position along the curve, the rotation angle to be defined in interaction with geometric deformations, and rotation operators to

acquire a shape-adaptive character. Thus, generalized Gielis curves offer a rich kinematic modeling framework that encompasses not only standard symmetric structures but also irregular, asymmetric, or fractal-like transformation behaviors frequently encountered in natural forms. This feature provides a significant advantage in both theoretical differential geometry studies and shape-based rotation analysis of biological growth, morphological transformation, and mechanical systems. Figures 2 and 3 illustrate the symmetric and asymmetric generalized superelliptic rotational motions, respectively.

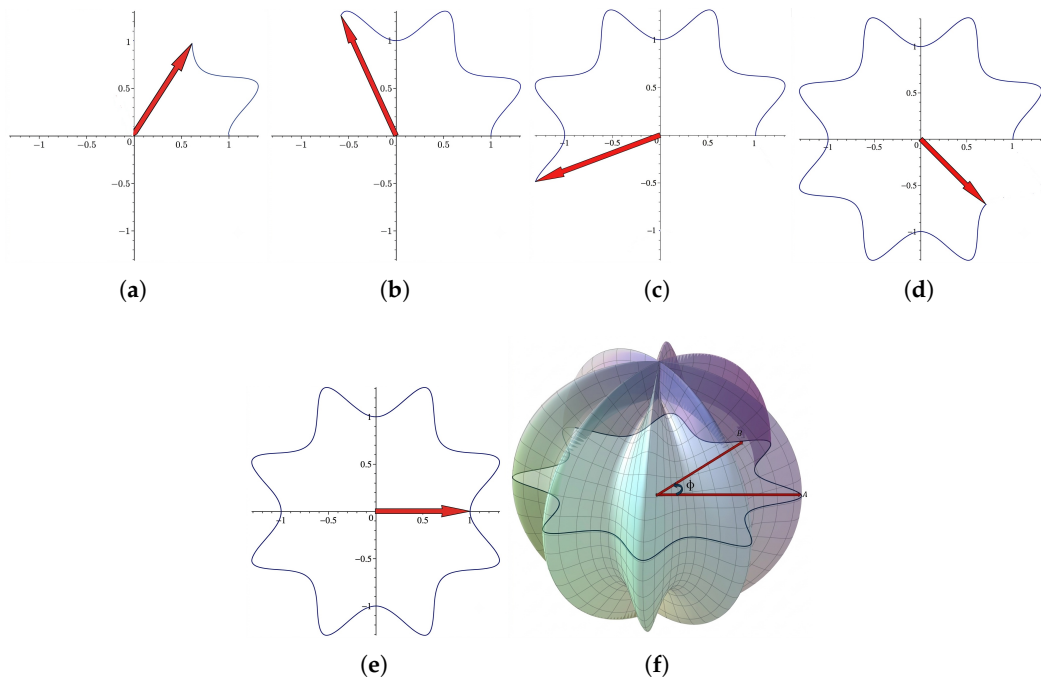


Figure 2. Superelliptic rotation model for the values of $m = 8$, $n_1 = 4$, $n_2 = n_3 = 6$, $a = b = \eta_i = 1$, $i = \{1, 2, 3\}$, and $f(\phi) = 1$. (a) Initial rotation position ($\phi = 1$); (b) rotation at $\phi = 2$; (c) rotation at $\phi = 3.5$; (d) rotation at $\phi = 5.5$; (e) final rotation position ($\phi = 6.3$); (f) superelliptic rotation model of the rotation radius (red) vector about the angle ϕ on the corresponding superellipsoid. The labels A and B indicate the corresponding characteristic points.

In a three-dimensional space, we can give the rotation matrix by using the following theorems obtained by Rodrigues and Cayley's formulas.

Theorem 1. Let A be a $\mathbb{S}_\mathcal{E}$ -skew-symmetric matrix such that $\langle Au, v \rangle_{\mathbb{S}_\mathcal{E}} = -\langle u, Av \rangle_{\mathbb{S}_\mathcal{E}}$ and $w = (w_1, w_2, w_3) \in \mathbb{R}_{\mathbb{S}_\mathcal{E}}^n$ be a unit vector; then the matrix exponential is written as follows:

$$R_\theta^A = e^{A\theta} = I + (\sin \theta)A + (1 - \cos \theta)A^2.$$

This achieves superelliptic rotation about the angle ϕ around the axis $w = (w_1, w_2, w_3)$ on the superellipsoid $\frac{1}{\rho(\phi)^2}\eta_1x^2 + \frac{1}{\rho(\phi)^2}\eta_2y^2 + \frac{1}{\rho(\phi)^2}\eta_3z^2 = 1$ and the superelliptic matrix form can be given as follows:

$$R_\phi^A = \begin{bmatrix} \frac{1}{\rho(\phi)^2}\eta_1w_1^2 + \left(1 - \frac{1}{\rho(\phi)^2}\eta_1w_1^2\right)\cos\theta & -\frac{\Lambda\rho(\phi)^2w_3\sin\theta}{\eta_1} - \frac{1}{\rho(\phi)^2}\eta_2w_1w_2(\cos\theta - 1) & -\frac{\Lambda\rho(\phi)^2w_2\sin\theta}{\eta_1} - \frac{1}{\rho(\phi)^2}\eta_3w_1w_3(\cos\theta - 1) \\ -\frac{\Lambda\rho(\phi)^2w_3\sin\theta}{\eta_2} - \frac{1}{\rho(\phi)^2}\eta_1w_1w_2(\cos\theta - 1) & \frac{1}{\rho(\phi)^2}\eta_2w_2^2 + \left(1 - \frac{1}{\rho(\phi)^2}\eta_2w_2^2\right)\cos\theta & -\frac{\Lambda\rho(\phi)^2w_1\sin\theta}{\eta_2} - \frac{1}{\rho(\phi)^2}\eta_3w_2w_3(\cos\theta - 1) \\ -\frac{\Lambda\rho(\phi)^2w_2\sin\theta}{\eta_3} - \frac{1}{\rho(\phi)^2}\eta_1w_1w_3(\cos\theta - 1) & \frac{\Lambda\rho(\phi)^2w_1\sin\theta}{\eta_3} - \frac{1}{\rho(\phi)^2}\eta_2w_2w_3(\cos\theta - 1) & \frac{1}{\rho(\phi)^2}\eta_3w_3^2 + \left(1 - \frac{1}{\rho(\phi)^2}\eta_3w_3^2\right)\cos\theta \end{bmatrix}.$$

Proof. Let w be a unit vector in $(\mathbb{R}_{\mathbb{S}_\mathcal{E}}^n)$. It follows that the associated matrix A satisfies the identity $(A^3 = -A)$. Consequently, the matrix exponential defining the rotation can be expressed as

$$\begin{aligned} R_\theta^A &= e^{\theta A} = I + \theta A + \frac{\theta^2 A^2}{2!} - \frac{\theta^3 A}{3!} - \frac{\theta^4 A^2}{4!} + \frac{\theta^5 A}{5!} + \frac{\theta^6 A^2}{6!} \\ &= I + A\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) + A^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \dots\right) \\ &= I + A\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) + A^2\left(1 - \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right)\right) \\ &= I + (\sin \theta)A + (1 - \cos \theta)A^2. \end{aligned}$$

Here, the reduction $(A^3 = -A)$ allows the exponential series to separate naturally into its odd and even components, which correspond to the Taylor series expansions of the sine and cosine functions, respectively. This immediately yields the compact representation above. Finally, the explicit form of the superelliptic rotation matrix is obtained by employing the identity

$$\frac{1}{\rho(\phi)^2} \eta_2 w_2^2 + \frac{1}{\rho(\phi)^2} \eta_3 w_3^2 + \frac{1}{\rho(\phi)^2} \eta_1 w_1^2 = 1,$$

followed by expanding the resulting expression and simplifying the corresponding matrix entries. $\square$

Theorem 2. Let A be a $\mathbb{S}_\mathcal{E}$ -skew-symmetric matrix such that $\langle Au, v \rangle_{\mathbb{S}_\mathcal{E}} = -\langle u, Av \rangle_{\mathbb{S}_\mathcal{E}}$ and $w = (w_1, w_2, w_3) \in \mathbb{R}_{\mathbb{S}_\mathcal{E}}^n$ be a vector; then

$$R^{A,u} = (I + A)(I - A)^{-1}$$

performs a superelliptic rotation about the angle ϕ around the axis $w = (w_1, w_2, w_3)$ on the superellipsoid $\frac{1}{\rho(\phi)^2} \eta_1 x^2 + \frac{1}{\rho(\phi)^2} \eta_2 y^2 + \frac{1}{\rho(\phi)^2} \eta_3 z^2 = 1$. Additionally, the matrix $R^{A,w}$ can be expressed as follows:

$$R^{A,w} = \frac{1}{1 + \|w\|_{\mathbb{S}_\mathcal{E}}^2} \begin{bmatrix} \frac{1}{\rho(\phi)^2} \eta_1 w_1^2 - \frac{1}{\rho(\phi)^2} \eta_2 w_2^2 - \frac{1}{\rho(\phi)^2} \eta_3 w_3^2 + 1 & 2 \frac{1}{\rho(\phi)^2} \eta_2 w_1 w_2 - \frac{2\Lambda\rho(\phi)^2 w_3}{\eta_1} & \frac{2\Lambda\rho(\phi)^2 w_2}{\eta_1} + 2 \frac{1}{\rho(\phi)^2} \eta_3 w_1 w_3 \\ \frac{2\Lambda\rho(\phi)^2 w_3}{\eta_2} + 2 \frac{1}{\rho(\phi)^2} \eta_1 w_1 w_2 & \frac{1}{\rho(\phi)^2} \eta_2 w_2^2 - \frac{1}{\rho(\phi)^2} \eta_1 w_1^2 - \frac{1}{\rho(\phi)^2} \eta_3 w_3^2 + 1 & 2 \frac{1}{\rho(\phi)^2} \eta_3 w_2 w_3 - \frac{2\Lambda\rho(\phi)^2 w_1}{\eta_2} \\ 2 \frac{1}{\rho(\phi)^2} \eta_1 w_1 w_3 - \frac{2\Lambda\rho(\phi)^2 w_2}{\eta_3} & \frac{2\Lambda\rho(\phi)^2 w_1}{\eta_3} + 2 w_2 w_3 \frac{1}{\rho(\phi)^2} \eta_2 & \frac{1}{\rho(\phi)^2} \eta_3 w_3^2 - \frac{1}{\rho(\phi)^2} \eta_1 w_1^2 - \frac{1}{\rho(\phi)^2} \eta_2 w_2^2 + 1 \end{bmatrix}$$

where the rotation angle can be computed through the equation $\tan \phi = \frac{2\|w\|_{\mathbb{S}_\mathcal{E}}}{1 - \|w\|_{\mathbb{S}_\mathcal{E}}^2}$.

Proof. By leveraging the fundamental properties inherent to $\mathbb{S}_\mathcal{E}$ -skew-symmetric matrices, we establish that $A^t G = -GA$. Consequently, the following relations immediately follow:

$$(I + A)^t G = G(I - A) \text{ and } (I - A)^t G = -G(I + A).$$

The validity of this formulation is directly confirmed by evaluating the subsequent expression:

$$(R^{A,u})^t G (R^{A,u}) = ((I + A)(I - A)^{-1})^t G (I + A)(I - A)^{-1} = G.$$

Furthermore, the condition $\det R^{A,w} = 1$ is explicitly deduced from the constituent determinants $\det(I + A) = 1 + \|w\|_{\mathbb{S}_\mathcal{E}}^2$ and $\det(I - A)^{-1} = \frac{1}{1 + \|w\|_{\mathbb{S}_\mathcal{E}}^2}$. This ultimately demonstrates that the operator $\det R^{A,w}$ constitutes a well-defined superelliptic rotation matrix. $\square$

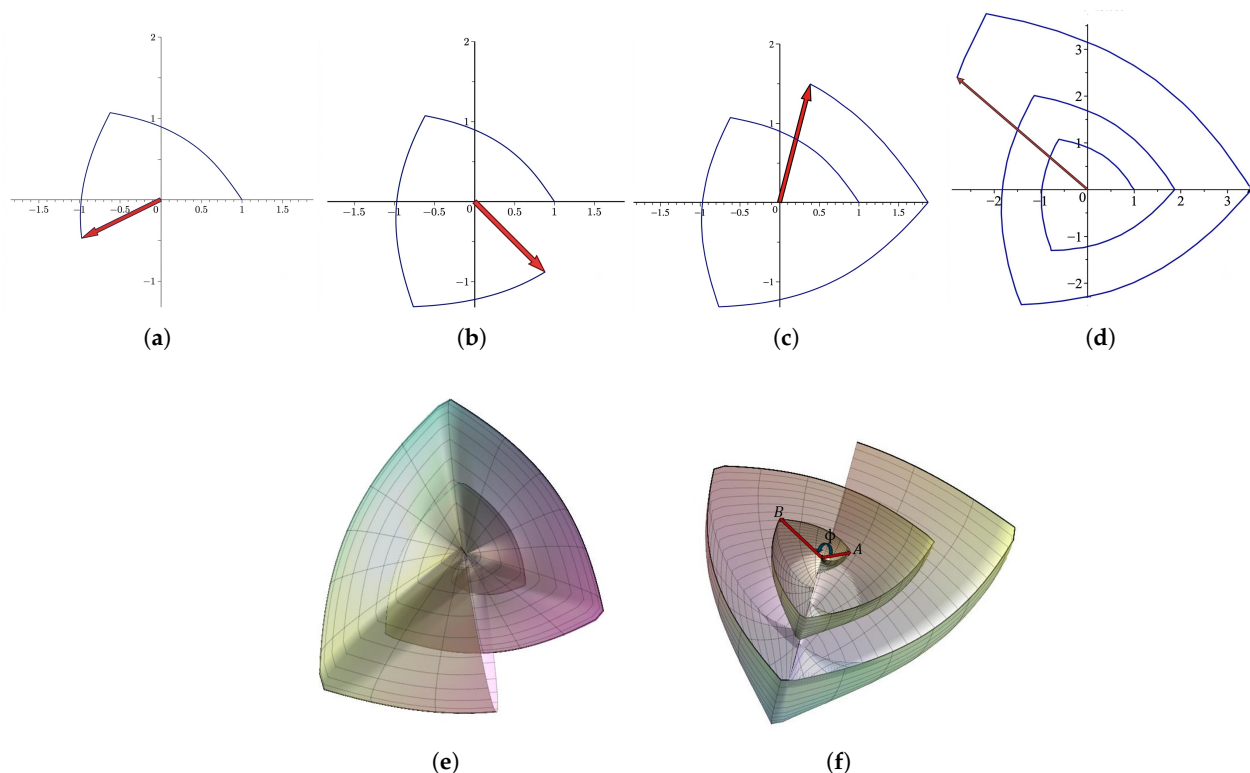


Figure 3. Superelliptic rotation model for the values of $m = 3$, $a = b = n_i = \eta_i = 1$, $i = \{1, 2, 3\}$, and $f(\phi) = \exp(0.10\phi)$. (a) Initial rotation position ($\phi = 3.6$); (b) rotation at $\phi = 7.05$; (c) rotation at $\phi = 15$; (d) rotation at $\phi = 5.5$; (e) complete superellipsoid; (f) hemispherical superellipsoid with the front half removed to provide a clearer view of the rotation angle ϕ . The red vector represents the rotation radius, and the labels A and B indicate the corresponding characteristic points.

2.5. From Polar Coordinates to Superelliptic Complex Numbers

To better understand the emergence of superelliptic quaternions and their geometric applications, it is essential to first examine the superelliptic complex number system together with its geometric interpretation. The set of the superelliptic complex numbers is given by

$$\mathbb{C} = \{z = a + i\mathbf{b} : \mathbf{b} = \frac{\rho(\phi)}{\sqrt{\eta_1}}b, a, b \in \mathbb{R}, i^2 = -\frac{\eta_1}{\rho(\phi)^2}\} \quad (7)$$

The angle that the complex number $\mathbf{z}$ makes with the positive real axis is called the *argument of $\mathbf{z}$* , denoted by $\arg(\mathbf{z})$. Based on the relations

$$a = \cos \theta \quad \text{and} \quad b = \sin \theta,$$

we obtain

$$\theta = \arg(a + i\mathbf{b}) = \arctan\left(\frac{b}{a}\right).$$

Furthermore,

$$1 = \sqrt{a^2 + \frac{\eta_1}{\rho^2(\phi)}b^2}.$$

On the other hand, the complex number $\mathbf{z} = a + i\mathbf{b}$ can be expressed as

$$\mathbf{z} = a + i\mathbf{b} = \cos \theta + i \sin \theta, i^2 = -\frac{\eta_1}{\rho^2(\phi)}.$$

This representation is known as the *polar (or trigonometric) form* of the superelliptic complex number $\mathbf{z}$, and it is often written in a more concise way

$$\mathbf{z} = \text{cis } \theta.$$

For a complex number $\mathbf{z} = a_1 + i\mathbf{b}_1$, the number $\bar{\mathbf{z}} = a_1 - i\mathbf{b}_1$ is called the complex conjugate of $\mathbf{z}$. The set of complex numbers forms a two-dimensional vector space over the field of real numbers. On this vector space, the inner product is defined for $\mathbf{z}, \mathbf{w} \in \mathbb{C}$ as

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\mathbb{S}_{\mathcal{E}}} : \mathbb{C}_{\mathbb{S}_{\mathcal{E}}} \times \mathbb{C}_{\mathbb{S}_{\mathcal{E}}} &\rightarrow \mathbb{R} \\ (\mathbf{z}, \mathbf{w}) &\mapsto \langle \mathbf{z}, \mathbf{w} \rangle_{\mathbb{S}_{\mathcal{E}}} = \frac{\mathbf{z}\bar{\mathbf{w}} + \mathbf{w}\bar{\mathbf{z}}}{2} = \text{Re}(\bar{\mathbf{z}}\mathbf{w}) \end{aligned}$$

Accordingly, for $\mathbf{z} = a_1 + i\mathbf{b}_1$ and $\mathbf{w} = a_2 + i\mathbf{b}_2$, we obtain

$$\langle \mathbf{z}, \mathbf{w} \rangle_{\mathbb{S}_{\mathcal{E}}} = a_1 a_2 + \frac{\eta_1}{\rho^2(\phi)} \mathbf{b}_1 \mathbf{b}_2.$$

This expression is exactly the superelliptic inner product in the superelliptic plane.

Therefore, complex numbers provide a natural algebraic framework that encodes the vector structure of Euclidean geometry, allowing geometric transformations and motions in the plane to be studied effectively within the framework of complex analysis. With respect to this inner product, the norm of a complex number $\mathbf{z} = a + i\mathbf{b}$ is defined as

$$|\mathbf{z}|_{\mathbb{S}_{\mathcal{E}}} = \sqrt{\langle \mathbf{z}, \mathbf{z} \rangle_{\mathbb{S}_{\mathcal{E}}}} = \sqrt{\text{Re}(\bar{\mathbf{z}}\mathbf{z})} = \sqrt{\bar{\mathbf{z}}\mathbf{z}} = \sqrt{a^2 + \frac{\eta_1}{\rho^2(\phi)} \mathbf{b}^2}.$$

The quantity $|\mathbf{z}|_{\mathbb{S}_{\mathcal{E}}}$ is called the *norm* or *modulus* of the complex number $\mathbf{z}$. It is clear that this value corresponds geometrically to the length of the position vector of $\mathbf{z}$ in the complex plane, and it is fully consistent with geometric intuition. Moreover, it satisfies the norm axioms. Hence, we can write

$$\mathbf{z}\bar{\mathbf{z}} = |\mathbf{z}|_{\mathbb{S}_{\mathcal{E}}}^2.$$

Furthermore, the Cauchy–Schwarz inequality can be expressed in this setting as

$$|\langle \mathbf{z}, \mathbf{w} \rangle|_{\mathbb{S}_{\mathcal{E}}} = |\text{Re}(\bar{\mathbf{z}}\mathbf{w})|_{\mathbb{S}_{\mathcal{E}}} \leq |\mathbf{z}| |\mathbf{w}|_{\mathbb{S}_{\mathcal{E}}}.$$

For any complex number $\mathbf{z} = a + i\mathbf{b}$, the number $\mathbf{z}^{-1}$ satisfying

$$\mathbf{z}\mathbf{z}^{-1} = \mathbf{z}^{-1}\mathbf{z} = 1$$

is called the *multiplicative superelliptic inverse* of $\mathbf{z}$.

Multiplying these identities by $\bar{\mathbf{z}}$ and using the relation $\mathbf{z}\bar{\mathbf{z}} = |\mathbf{z}|^2$, we obtain

$$(\bar{\mathbf{z}}\mathbf{z})\mathbf{z}^{-1} = \mathbf{z}^{-1}(\bar{\mathbf{z}}\mathbf{z}) = \bar{\mathbf{z}}.$$

From this, it follows that

$$\mathbf{z}^{-1} = \frac{\bar{\mathbf{z}}}{|\mathbf{z}|_{\mathbb{S}_{\mathcal{E}}}^2}.$$

Accordingly, for any complex number $\mathbf{z} = a + i\mathbf{b}$, its inverse can be explicitly written as

$$\mathbf{z}^{-1} = \frac{a - i\mathbf{b}}{a^2 + \frac{\eta_1}{\rho(\phi)^2} \mathbf{b}^2}.$$

The angle between two complex numbers is understood as the angle between their position vectors in the complex plane. According to the relation

$$\langle \mathbf{z}, \mathbf{w} \rangle_{\mathbb{S}_{\mathcal{E}}} = |\mathbf{z}|_{\mathbb{S}_{\mathcal{E}}} |\mathbf{w}|_{\mathbb{S}_{\mathcal{E}}} \cos \theta = \operatorname{Re}(\bar{\mathbf{z}}\mathbf{w}),$$

the angle θ between the complex numbers $\mathbf{z} = a_1 + i\mathbf{b}_1$ and $\mathbf{w} = a_2 + i\mathbf{b}_2$ is given by

$$\cos \theta = \frac{\langle \mathbf{z}, \mathbf{w} \rangle_{\mathbb{S}_{\mathcal{E}}}}{|\mathbf{z}|_{\mathbb{S}_{\mathcal{E}}} |\mathbf{w}|_{\mathbb{S}_{\mathcal{E}}}} = \frac{\operatorname{Re}(\bar{\mathbf{z}}\mathbf{w})}{|\mathbf{z}| |\mathbf{w}|_{\mathbb{S}_{\mathcal{E}}}} = \frac{a_1 a_2 + \frac{\eta_1}{\rho(\phi)^2} \mathbf{b}_1 \mathbf{b}_2}{\sqrt{a_1^2 + \frac{\eta_1}{\rho(\phi)^2} \mathbf{b}_1^2} \sqrt{a_2^2 + \frac{\eta_1}{\rho(\phi)^2} \mathbf{b}_2^2}}.$$

The vector product of two complex numbers $\mathbf{z}$ and $\mathbf{w}$ is defined as

$$\mathbf{z} \times_{\mathbb{S}_{\mathcal{E}}} \mathbf{w} = (|\mathbf{z}|_{\mathbb{S}_{\mathcal{E}}} |\mathbf{w}|_{\mathbb{S}_{\mathcal{E}}} \sin \theta) i = \operatorname{Im}(\bar{\mathbf{z}}\mathbf{w}) i = \frac{i}{2}(\bar{\mathbf{z}}\mathbf{w} - \mathbf{z}\bar{\mathbf{w}}).$$

Consequently,

$$|\mathbf{z} \times_{\mathbb{S}_{\mathcal{E}}} \mathbf{w}| = |\mathbf{z}|_{\mathbb{S}_{\mathcal{E}}} |\mathbf{w}|_{\mathbb{S}_{\mathcal{E}}} \sin \theta.$$

For $\mathbf{z} = a_1 + i\mathbf{b}_1$ and $\mathbf{w} = a_2 + i\mathbf{b}_2$, we obtain

$$\mathbf{z} \times_{\mathbb{S}_{\mathcal{E}}} \mathbf{w} = (a_1 \mathbf{b}_2 - a_2 \mathbf{b}_1) i \quad \text{and} \quad |\mathbf{z} \times_{\mathbb{S}_{\mathcal{E}}} \mathbf{w}| = |a_1 \mathbf{b}_2 - a_2 \mathbf{b}_1|.$$

Moreover,

$$\sin \theta = \frac{|\mathbf{z} \times \mathbf{w}|}{|\mathbf{z}| |\mathbf{w}|} = \frac{|a_1 \mathbf{b}_2 - a_2 \mathbf{b}_1|}{\sqrt{a_1^2 + \frac{\eta_1}{\rho(\phi)^2} \mathbf{b}_1^2} \sqrt{a_2^2 + \frac{\eta_1}{\rho(\phi)^2} \mathbf{b}_2^2}}.$$

From this formulation, it follows directly that the identity $\cos^2 \theta + \sin^2 \theta = 1$ is satisfied, confirming the consistency of the geometric interpretation. The algebraic product of any two complex numbers $\mathbf{z}$ and $\mathbf{w}$ can be expressed as

$$\mathbf{z}\mathbf{w} = \langle \mathbf{z}, \mathbf{w} \rangle_{\mathbb{S}_{\mathcal{E}}} + \mathbf{z} \times_{\mathbb{S}_{\mathcal{E}}} \mathbf{w} = |\mathbf{z}|_{\mathbb{S}_{\mathcal{E}}} |\mathbf{w}|_{\mathbb{S}_{\mathcal{E}}} \cos \theta + i(|\mathbf{z}|_{\mathbb{S}_{\mathcal{E}}} |\mathbf{w}|_{\mathbb{S}_{\mathcal{E}}} \sin \theta).$$

If $\mathbf{z}$ and $\mathbf{w}$ are unit complex numbers, then their product is also a unit complex number, and we obtain

$$\mathbf{z}\mathbf{w} = \cos \theta + i \sin \theta.$$

For a superelliptic complex number

$$\mathbf{z} = a + \mathbf{b}i, \quad a \in \mathbb{R}, \text{ and } \mathbf{b} \in \mathbb{R}_{\mathbb{S}_{\mathcal{E}}}$$

there exists an equivalent 2×2 real matrix representation given by

$$\mathbf{z} \longleftrightarrow \begin{bmatrix} a & -\mathbf{b} \frac{\sqrt{\eta_2}}{\rho(\phi)} \\ \frac{\mathbf{b}}{\sqrt{\eta_2}} \frac{\eta_1}{\rho(\phi)} & a \end{bmatrix}.$$

This representation preserves both addition and multiplication operations, establishing an isomorphism between the field of complex numbers and a subset of real matrices.

In particular, the imaginary unit i is represented as

$$i \longleftrightarrow \begin{bmatrix} 0 & -\frac{\sqrt{\eta_2}}{\rho(\phi)} \\ \frac{1}{\sqrt{\eta_2}} \frac{\eta_1}{\rho(\phi)} & 0 \end{bmatrix},$$

which satisfies

$$\begin{bmatrix} 0 & -\frac{\sqrt{\eta_2}}{\rho(\phi)} \\ \frac{1}{\sqrt{\eta_2}} \frac{\eta_1}{\rho(\phi)} & 0 \end{bmatrix}^2 = \begin{bmatrix} -\frac{\eta_1}{\rho^2(\phi)} & 0 \\ 0 & -\frac{\eta_1}{\rho^2(\phi)} \end{bmatrix} = -\frac{\eta_1}{\rho^2(\phi)} I_2,$$

consistent with the identity $i^2 = -\frac{\eta_1}{\rho(\phi)^2}$.

Furthermore, the rotation matrix associated with a counterclockwise rotation by an angle θ is defined by

$$R(\theta) = \begin{bmatrix} \cos \theta & -\frac{\sqrt{\eta_2}}{\sqrt{\eta_1}} \sin \theta \\ \sin \theta \frac{\sqrt{\eta_1}}{\sqrt{\eta_2}} & \cos \theta \end{bmatrix}$$

This matrix form is directly related to Euler's formula,

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

through the correspondence

$$e^{i\theta} \longleftrightarrow \begin{bmatrix} \cos \theta & -\frac{\sqrt{\eta_2}}{\sqrt{\eta_1}} \sin \theta \\ \sin \theta \frac{\sqrt{\eta_1}}{\sqrt{\eta_2}} & \cos \theta \end{bmatrix}$$

Hence, multiplication by a unit complex number geometrically corresponds to a planar generalized superelliptic rotation. The composition property of rotations is expressed naturally as

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2),$$

demonstrating that matrix multiplication corresponds to the addition of rotation angles. The matrix R gives a superelliptic rotation on the generalized superellipse $\frac{1}{\rho(\phi)^2} \eta_1 x_1^2 + \frac{1}{\rho(\phi)^2} \eta_2 x_2^2 = 1$, as in Section 2.4. For some values of the super formula, the images of the generalized superelliptic rotation models are visualized in Figure 4.

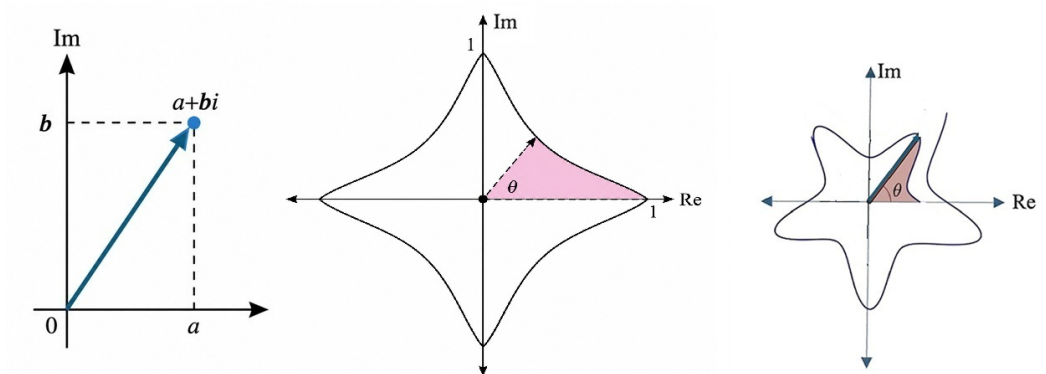


Figure 4. Geometric representation of a complex number $z = a + ib$ on the Gielis superellipses ($f(\phi) = 1, m = 4, n_1 = 0.5, n_2 = n_3 = a = b = 1$ and $f(\phi) = \exp(0.10 * \phi), m = 5, n_1 = 2, n_2 = n_3 = 6, a = b = \eta_i = 1, i = \{1, 2, 3\}$, respectively). The shaded region denotes the sectorial area defined by the polar angle θ within the super-boundary.

3. Quaternion-Based Asymmetric Rotation Induced by the Gielis Superformula

This formulation extends the classical rotational symmetry of the Gielis superformula to a quaternion-based anisotropic rotational framework, enabling the modeling of shells and biological structures exhibiting broken or direction-dependent rotational symmetry.

A superelliptic quaternion can be defined by $s = s_0 + s_1i + s_2j + s_3k$, where the basis elements $\{1, i, j, k\}$ satisfy

$$ij = \frac{\Lambda}{\eta_3}k = -ji, \quad jk = \frac{\Lambda}{\eta_1}i = -kj, \quad ki = \frac{\Lambda}{\eta_2}j = -ik$$

where $\eta_1, \eta_2, \eta_3 \in \mathbb{R}^+$ and $\Lambda = \frac{1}{\rho(\phi)}\sqrt{\eta_1\eta_2\eta_3}$. The set of superelliptic quaternions is an associative, non-commutative division ring with the basis elements $\{1, i, j, k\}$ and is denoted by

$$\mathbb{H}_{\mathbb{S}_E}(\eta_1, \eta_2, \eta_3) = \{s : s = s_0 + s_1i + s_2j + s_3k, s_i \in \mathbb{R}, i^2 = -\frac{1}{\rho(\phi)^2}\eta_1, j^2 = -\frac{1}{\rho(\phi)^2}\eta_2, k^2 = -\frac{1}{\rho(\phi)^2}\eta_3, i = 0, 1, 2, 3\}.$$

The multiplication rule table for superelliptic quaternions is given as follows:

In the Table 1, if $\rho(\phi) = 1$, the elliptic quaternion algebra is obtained, and if $\rho(\phi) = \eta_i = 1, i \in \{1, 2, 3\}$, the real quaternion algebra is obtained. A superelliptic quaternion can also be expressed in the form $s = s_s + s_v$, where $s_s = s_0$ is the scalar part and $s_v = s_1i + s_2j + s_3k$ is the vector part of s . If $s_0 = 0$, then s is called a pure superelliptic quaternion.

Table 1. Superelliptic quaternion multiplication table.

	1	i	j	k
1	1	i	j	k
i	i	$-\frac{\eta_1}{\rho(\phi)^2}$	$\frac{\Lambda}{\eta_3}k$	$-\frac{\Lambda}{\eta_2}j$
j	j	$-\frac{\Lambda}{\eta_3}k$	$-\frac{\eta_2}{\rho(\phi)^2}$	$\frac{\Lambda}{\eta_1}i$
k	k	$\frac{\Lambda}{\eta_2}j$	$-\frac{\Lambda}{\eta_1}i$	$-\frac{\eta_3}{\rho(\phi)^2}$

The definition of the superelliptic quaternion product of two superelliptic quaternions $s = s_0 + s_1i + s_2j + s_3k$ and $\check{s} = \check{s}_0 + \check{s}_1i + \check{s}_2j + \check{s}_3k$ is as follows:

$$s \otimes \check{s} = s_0\check{s}_0 - \langle s_v, \check{s}_v \rangle_{\mathbb{S}_E} + s_0\check{s}_v + \check{s}_0s_v + s_v \times_{\mathbb{S}_E} \check{s}_v.$$

where $\langle s_v, \check{s}_v \rangle_{\mathbb{S}_E}$ and $s_v \times_{\mathbb{S}_E} \check{s}_v$ are the superelliptic scalar product and the superelliptic vector product. If s and $\check{s}$ are pure, then we obtain

$$s \otimes \check{s} = -\langle s_v, \check{s}_v \rangle_{\mathbb{S}_E} + s_v \times_{\mathbb{S}_E} \check{s}_v.$$

The quaternion product is also given in matrix form as follows:

$$s \otimes \check{s} = \begin{bmatrix} s_0 & -\frac{1}{\rho(\phi)^2}\eta_1s_1 & -\frac{1}{\rho(\phi)^2}\eta_2s_2 & -\frac{1}{\rho(\phi)^2}\eta_3s_3 \\ s_1 & s_0 & \frac{-s_3\Lambda}{\eta_1} & \frac{s_2\Lambda}{\eta_2} \\ s_2 & \frac{s_3\Lambda}{\eta_2} & s_0 & -\frac{s_1\Lambda}{\eta_3} \\ s_3 & -\frac{s_2\Lambda}{\eta_3} & \frac{s_1\Lambda}{\eta_1} & s_0 \end{bmatrix} \begin{bmatrix} \check{s}_0 \\ \check{s}_1 \\ \check{s}_2 \\ \check{s}_3 \end{bmatrix}.$$

The conjugate of the superelliptic quaternion $s = s_0 + s_1i + s_2j + s_3k$ is denoted by $\bar{s} = s_0 - s_1i - s_2j - s_3k = s_s - s_v$. Then the norm of s is defined as

$$N_s = \sqrt{s \otimes \bar{s}} = \sqrt{\bar{s} \otimes s} = \sqrt{s_0^2 + \frac{1}{\rho(\phi)^2}\eta_1s_1^2 + \frac{1}{\rho(\phi)^2}\eta_2s_2^2 + \frac{1}{\rho(\phi)^2}\eta_3s_3^2}.$$

The inverse of the superelliptic quaternion s is in the form of $s^{-1} = \frac{\bar{s}}{N_s^2}$, where $N_s^2 \neq 0$. Additionally, every superelliptic quaternion has a polar form written as

$$s = N_s(\cos \theta + \xi_0 \sin \theta),$$

$$\text{where } \cos \theta = \frac{s_0}{N_s}, \sin \theta = \frac{\sqrt{\frac{1}{\rho(\phi)^2} \eta_1 s_1^2 + \frac{1}{\rho(\phi)^2} \eta_2 s_2^2 + \frac{1}{\rho(\phi)^2} \eta_3 s_3^2}}{N_s}, \text{ and } \xi_0 = \frac{(s_1, s_2, s_3)}{\sqrt{\frac{1}{\rho(\phi)^2} \eta_1 s_1^2 + \frac{1}{\rho(\phi)^2} \eta_2 s_2^2 + \frac{1}{\rho(\phi)^2} \eta_3 s_3^2}}.$$

A unit superelliptic quaternion $s = \cos \theta + \xi_0 \sin \theta$ accompanies a superelliptic rotation about the angle 2θ around the axis ξ_0 on the superellipsoid $\frac{1}{\rho(\phi)^2} \eta_1 x^2 + \frac{1}{\rho(\phi)^2} \eta_2 y^2 + \frac{1}{\rho(\phi)^2} \eta_3 z^2 = 1$. Then, we can give the following main theorem.

Theorem 3. Let $s = s_0 + s_1 i + s_2 j + s_3 k = \cos \theta + \xi_0 \sin \theta$ be a unit superelliptic quaternion; then the linear transformation $R_s(\mathbf{u}) = s \otimes \mathbf{u} \otimes s^{-1}$ rotates the any vector $\mathbf{u} \in \mathbb{R}_{\mathbb{S}_\varepsilon}^3$ about the angle 2θ around the axis ξ_0 on the superellipsoid $\frac{1}{\rho(\phi)^2} \eta_1 x^2 + \frac{1}{\rho(\phi)^2} \eta_2 y^2 + \frac{1}{\rho(\phi)^2} \eta_3 z^2 = 1$. Then, superelliptic rotation can be represented by the following matrix form:

$$R_s = \begin{bmatrix} s_0^2 + \frac{1}{\rho(\phi)^2} \eta_1 s_1^2 - \frac{1}{\rho(\phi)^2} \eta_2 s_2^2 - \frac{1}{\rho(\phi)^2} \eta_3 s_3^2 & 2 \frac{1}{\rho(\phi)^2} \eta_2 s_1 s_2 - 2 \frac{\Lambda^*}{\rho(\phi)^2} s_0 s_3 & 2 \frac{1}{\rho(\phi)^2} \eta_3 s_1 s_3 + 2 \frac{\Lambda^*}{\rho(\phi)^2} s_0 s_2 \\ 2 \frac{1}{\rho(\phi)^2} \eta_1 s_1 s_2 + 2 \frac{\Lambda^*}{\rho(\phi)^2} s_0 s_3 & s_0^2 - \frac{1}{\rho(\phi)^2} \eta_1 s_1^2 + \frac{1}{\rho(\phi)^2} \eta_2 s_2^2 - \frac{1}{\rho(\phi)^2} \eta_3 s_3^2 & 2 \frac{1}{\rho(\phi)^2} \eta_3 s_2 s_3 - 2 \frac{\Lambda^*}{\rho(\phi)^2} s_0 s_1 \\ 2 \frac{1}{\rho(\phi)^2} \eta_1 s_1 s_3 - 2 \frac{\Lambda^*}{\rho(\phi)^2} s_0 s_2 & 2 \frac{1}{\rho(\phi)^2} \eta_2 s_2 s_3 + 2 \frac{\Lambda^*}{\rho(\phi)^2} s_0 s_1 & s_0^2 - \frac{1}{\rho(\phi)^2} \eta_1 s_1^2 - \frac{1}{\rho(\phi)^2} \eta_2 s_2^2 + \frac{1}{\rho(\phi)^2} \eta_3 s_3^2 \end{bmatrix}$$

Proof. Let us take the linear transformation R_s and an orthonormal base vectors $\{\xi_0, \zeta_0, \mu_0\}$ in $\mathbb{R}_{\mathbb{S}_\varepsilon}^n$ such that they satisfy

$$\xi_0 \times_{\mathbb{S}_\varepsilon} \zeta_0 = \mu_0, \mu_0 \times_{\mathbb{S}_\varepsilon} \xi_0 = \zeta_0, \zeta_0 \times_{\mathbb{S}_\varepsilon} \mu_0 = \xi_0$$

Let us denote the $\overrightarrow{OP}$ vector corresponding to the point $P \in \mathbb{R}_{\mathbb{S}_\varepsilon}^n$ as x . In this case, it is written in the form of

$$x = \cos \alpha \xi_0 + \sin \alpha \zeta_0.$$

Then, first, let us see how the transformation R_s changes the vectors ξ_0 and ζ_0 . Since the vector part v_s of s is parallel to the vector ξ_0 , we get

$$s \otimes \xi_0 = \xi_0 \otimes s.$$

Thus, we calculate

$$R_s(\xi_0) = s \otimes \xi_0 \otimes s^{-1} = \xi_0 \otimes s \otimes s^{-1} = \xi_0.$$

This shows that the transformation R_s does not change the vector ξ_0 . Namely, ξ_0 is the axis of rotation. The rotation motion occurs in a plane perpendicular to the vector ξ_0 . On the other hand, we compute

$$\begin{aligned} R_s(\zeta_0) &= s \otimes \zeta_0 \otimes s^{-1} \\ &= \zeta_0 \cos^2 \theta - \cos \theta \sin \theta (\zeta_0 \otimes \xi_0) + \cos \theta \sin \theta (\xi_0 \otimes \zeta_0) \\ &\quad - (\xi_0 \otimes \zeta_0) \otimes \xi_0 \sin^2 \theta. \end{aligned}$$

If $\zeta_0 \perp_{\langle, \rangle_{\mathbb{S}_\varepsilon}} \xi_0$ is also taken into account in this equality, we reach

$$\zeta_0 \otimes \xi_0 = -\langle \zeta_0, \xi_0 \rangle_{\mathbb{S}_\varepsilon} + \zeta_0 \times_{\mathbb{S}_\varepsilon} \xi_0 = -\xi_0 \times_{\mathbb{S}_\varepsilon} \zeta_0 = -\xi_0 \otimes \zeta_0 = -\mu_0.$$

Thus, we obtain

$$\begin{aligned} R_s(\zeta_0) &= \zeta_0 \cos^2 \theta + (\zeta_0 \otimes \zeta_0) \otimes \zeta_0 \sin^2 \theta + 2\mu_0 \cos \theta \sin \theta \\ &= \zeta_0 \cos 2\theta + \mu_0 \sin 2\theta. \end{aligned}$$

Next, for $x = \zeta_0 \cos \alpha + \zeta_0 \sin \alpha$, we compute the following equation:

$$\begin{aligned} R_s(x) &= s \otimes (\zeta_0 \cos \alpha + \zeta_0 r(\alpha) \sin \alpha) \otimes s^{-1} \\ &= \zeta_0 \cos \alpha + (\zeta_0 \cos 2\theta + \mu_0 \sin 2\theta) \sin \alpha. \end{aligned}$$

This means that the vector x rotates around the vector ζ_0 about the angle 2θ . Now, let us find the rotation matrix corresponding to the linear transformation R_s . For this, from the equality of $R_s(x) = s \otimes x \otimes s^{-1}$, with $s = s_0 + s_1 i + s_2 j + s_3 k$, we compute

$$\begin{aligned} R_s(i) &= (s_0^2 + \frac{1}{\rho(\phi)^2} \eta_1 s_1^2 - s_2^2 \frac{1}{\rho(\phi)^2} \eta_2 - s_3^2 \frac{1}{\rho(\phi)^2} \eta_3) i + 2(\frac{1}{\rho(\phi)^2} \eta_2 s_1 s_2 - \frac{s_0 s_3 \Lambda^*}{\frac{1}{\rho(\phi)^2} \eta_1}) j \\ &\quad + 2(s_1 s_3 \frac{1}{\rho(\phi)^2} \eta_3 + \frac{s_0 s_2 \Lambda^*}{\frac{1}{\rho(\phi)^2} \eta_1}) k \\ R_s(j) &= 2(\frac{1}{\rho(\phi)^2} \eta_1 s_1 s_2 + \frac{s_0 s_3 \Lambda^*}{\frac{1}{\rho(\phi)^2} \eta_2}) i + (s_0^2 - \frac{1}{\rho(\phi)^2} \eta_1 s_1^2 + \frac{1}{\rho(\phi)^2} \eta_2 s_2^2 - \frac{1}{\rho(\phi)^2} \eta_3 s_3^2) j \\ &\quad + 2(s_2 s_3 \frac{1}{\rho(\phi)^2} \eta_3 - \frac{s_0 s_1 \Lambda^*}{\frac{1}{\rho(\phi)^2} \eta_2}) k \\ R_s(k) &= 2(\frac{1}{\rho(\phi)^2} \eta_1 s_1 s_3 - \frac{s_0 s_2 \Lambda^*}{\frac{1}{\rho(\phi)^2} \eta_3}) i + 2(\frac{1}{\rho(\phi)^2} \eta_2 s_2 s_3 + \frac{s_0 s_1 \Lambda^*}{\frac{1}{\rho(\phi)^2} \eta_3}) j \\ &\quad + (s_0^2 - \frac{1}{\rho(\phi)^2} \eta_1 s_1^2 - \frac{1}{\rho(\phi)^2} \eta_2 s_2^2 + \frac{1}{\rho(\phi)^2} \eta_3 s_3^2) k. \end{aligned}$$

The matrix form R_s is a superelliptic orthogonal rotation matrix; that is, $\det R_s = 1$ and $R_s^t G R_s = G$, where

$$G(\phi) = \frac{1}{\rho(\phi)^2} \text{diag}(\eta_1, \eta_2, \eta_3),$$

Thus, we can say that for a unit superelliptic quaternion $s = s_0 + s_1 i + s_2 j + s_3 k = \cos \theta + \zeta_0 \sin \theta$ the transformation $R_s : \mathbb{R}_{\mathbb{S}_\varepsilon}^3 \rightarrow \mathbb{R}_{\mathbb{S}_\varepsilon}^3; x \rightarrow R_s(x) = s \otimes x \otimes s^{-1}$ rotates the vector x superelliptically in the plane perpendicular to the vector ζ_0 by an angle 2θ . Here, the endpoint of the rotating vector draws a superellipse via the Gielis function that is changed the length of the vector during the rotation. $\square$

From the results summarized in Table 2, the following classes of motion can be identified:

1. *Isotropic motion*: Standard ambient rotations $R(\theta)$ or unit quaternions ensure numerical robustness and rotational symmetry, with the tip tracing circular or great-circle trajectories.
2. *Anisotropic motion*: Elliptical rotations introduced via axis-dependent scaling of rotation matrices or quaternions generate anisotropic motion. The resulting trajectories are elliptical and aligned with the principal axes.
3. *Lobed geometries*: Replacing the constant radius with a shape function $r(\phi)$ yields multi-lobed tip trajectories. The parameters m, n_i, a , and b control the number, sharpness, and symmetry of the lobes.

4. *Growth and asymmetry*: Multiplying the radius by a growth function $f(\phi)$ to define $\rho(\phi)$ introduces asymmetry. When ϕ evolves concurrently with an increasing $f(\phi)$, the trajectories can model spiral-form rotation.

In this context, while real quaternions describe circular rotational motion, elliptic quaternions can express both circular and elliptical rotations. In general, superelliptic quaternions can produce rotational motion expressed by many different geometric shapes, including circular and elliptical rotations.

Table 2. Hierarchy of quaternion-based rotational models and their associated anisotropic geometries.

Quaternion Model	Normalized Rotation Axis	Associated Geometry and Interpretation
Classical unit quaternion	(s_1, s_2, s_3)	Unit sphere $x^2 + y^2 + z^2 = 1$, representing isotropic rotational symmetry in Euclidean space.
Elliptic unit quaternion	$\left(\frac{s_1}{\sqrt{\eta_1}}, \frac{s_2}{\sqrt{\eta_2}}, \frac{s_3}{\sqrt{\eta_3}}\right)$	Ellipsoidal geometry $\eta_1 x^2 + \eta_2 y^2 + \eta_3 z^2 = 1$, describing anisotropic rotational scaling along principal axes.
Superelliptic quaternion	$\left(\frac{r(\phi)}{\sqrt{\eta_1}} s_1, \frac{r(\phi)}{\sqrt{\eta_2}} s_2, \frac{r(\phi)}{\sqrt{\eta_3}} s_3\right)$	Gielis-modulated ellipsoidal manifold $\frac{\eta_1}{r(\phi)^2} x^2 + \frac{\eta_2}{r(\phi)^2} y^2 + \frac{\eta_3}{r(\phi)^2} z^2 = 1$, allowing superelliptic morphologies, rounded polygonal structures, and star-like rotational geometries.
Generalized superelliptic quaternion	$\left(\frac{\rho(\phi)}{\sqrt{\eta_1}} s_1, \frac{\rho(\phi)}{\sqrt{\eta_2}} s_2, \frac{\rho(\phi)}{\sqrt{\eta_3}} s_3\right)$	Generalized Gielis geometry $\frac{\eta_1}{\rho(\phi)^2} x^2 + \frac{\eta_2}{\rho(\phi)^2} y^2 + \frac{\eta_3}{\rho(\phi)^2} z^2 = 1$, capturing asymmetric, adaptive, and spiral-form rotation motion.

Unlike classical quaternion rotations in Euclidean space, this transformation is intrinsically anisotropic (but see Remark 2) due to the Gielis-induced inner and vector products. As a consequence, the rotational motion may be asymmetric, allowing different angular responses along distinct geometric directions.

The final generalized model is well suited for morphogenetic and biomimetic simulations where the tip of a generating vector both rotates and records anisotropic accretion. Complex numbers are defined as

$$z = a + \mathbf{b}i, \quad a, b \in \mathbb{R}, \quad \mathbf{b} = \frac{\rho(\phi)}{\sqrt{\eta_1}} b, \quad i^2 = -\frac{1}{\rho(\phi)^2} \eta_1. \quad (8)$$

Superelliptic quaternions extend this structure by introducing three imaginary units:

$$i^2 = -\frac{1}{\rho(\phi)^2} \eta_1, \quad j^2 = -\frac{1}{\rho(\phi)^2} \eta_2, \quad k^2 = -\frac{1}{\rho(\phi)^2} \eta_3, \quad (9)$$

with non-commutative multiplication rules:

$$ij = \frac{\Lambda}{\eta_3} k = -ji, \quad jk = \frac{\Lambda}{\eta_1} i = -kj, \quad ki = \frac{\Lambda}{\eta_2} j = -ik \quad (10)$$

The key observation is that the complex plane is embedded in the quaternion algebra as follows:

$$\mathbb{C}_{\mathbb{S}_{\mathcal{E}}} \subset \mathbb{H}_{\mathbb{S}_{\mathcal{E}}}, \quad \text{via } i \equiv i, \quad j = k = 0. \quad (11)$$

Since

$$a + \mathbf{b}i + \mathbf{c}j + \mathbf{d}k = \underbrace{a + \mathbf{b}i}_{z \in \mathbb{C}_{\mathbb{S}_\varepsilon}} + \underbrace{(\mathbf{c} + \mathbf{d}i)}_{w \in \mathbb{C}_{\mathbb{S}_\varepsilon}} \frac{\rho(\phi)}{\sqrt{\eta_2}} \mathbf{j} = z + w\mathbf{j},$$

the set $\mathcal{B}_{\mathbb{S}_\varepsilon} = \{1, \mathbf{j}\}$ is a $\mathbb{C}_{\mathbb{S}_\varepsilon}$ -basis for $\mathbb{H}_{\mathbb{S}_\varepsilon}$, so that the dimension of $\mathbb{H}_{\mathbb{S}_\varepsilon}$ over $\mathbb{C}_{\mathbb{S}_\varepsilon}$ is 2.

4. Geometry-Driven Modeling of Starfish and Shell Morphologies

Natural systems exhibit a remarkable diversity of forms that arise from a limited set of generative principles. While classical geometric models often rely on isotropic and idealized shapes, many biological structures display pronounced anisotropy, curvature-driven growth, and direction-dependent expansion. The Gielis superformula and its simplified variants provide a unifying parametric description capable of representing both smooth and polygonal morphologies using a small number of shape parameters [28,30,48]. These formulations have been successfully applied to a wide range of biological and physical systems, demonstrating that complex forms can emerge from simple generative rules. However, while the Gielis descriptive framework effectively captures static geometry, biological growth is inherently dynamic. Processes such as torsion, curvature-induced reorientation, and anisotropic expansion cannot be modeled using classical Euclidean rotations alone. This limitation motivates the introduction of quaternion-based rotations coupled to geometry-dependent, anisotropic metric structures.

These characteristics are particularly evident in marine organisms such as starfish and other echinoderms, and in the shells of mollusks and fossil ammonoids, where geometry, mechanics, and growth are tightly coupled. Echinoderms provide a clear example of how a common organizational plan can give rise to a wide range of morphologies. This may be the case in starfish, for example, as despite sharing a similar pentaradial structure, different species exhibit substantial variation in symmetry, curvature, arm thickness, and overall proportions. Nature has experimented throughout evolution with almost spherical and almost pentagonal starfish (Figure 5).

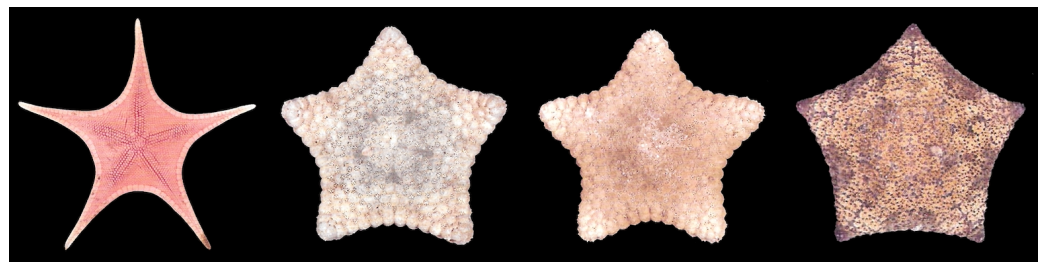


Figure 5. *Stellaster*, a typical starfish (left), and cookie starfish (*Tosia*) [33].

A uniform description of shape and growth is also key to understanding evolutionary transitions from ancestral to modern body organization. The pentaradial Bauplan of echinoderms is derived from an ancestral bilateral axis, so a uniform model has to be able to account for such transitions. Interestingly, recent research in adult sea stars (*Patiria miniata*) [49] and brittle stars [50] revealed the genes involved in coding for the “head” run along the midline of each arm and the central disk, appearing more towards the outer edges of the arms. In this sense, echinoderms are essentially “all head and no body”.

Shell morphogenesis of mollusks and ammonoids represents another classic, yet challenging case of biological shape evolution. Although logarithmic and equiangular spiral models successfully describe global growth trends, they fail to capture the intrinsic anisotropy observed in many gastropods and cephalopods, including fossil ammonoids. Anisotropic expansion, and non-circular cross sections—commonly observed in modern

shells of mollusks and snails, and polygonal coiling in fossil ammonoids—require a more general geometric framework (see, Figure 6).

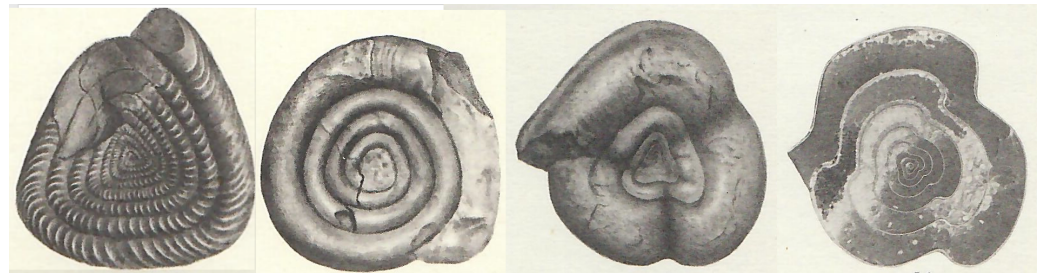


Figure 6. Triangular coiling in, from left to right: *Soliclymenia paradoxa*, *Kamptoclymenia trigona*, *Parawocklumeria patens* and *Parawocklumeria paradoxa* [51].

In the following applications, we employ the asymmetric rotation model introduced in the previous section to simulate the growth and shape evolution of starfish and shell-like structures.

This figure provides a kinematic interpretation of starfish-like geometry. The construction does not aim to model biological growth itself, but rather to illustrate the geometric motion associated with tissue deposition. As successive layers are added, points undergo a geometry-driven rotation modulated by a Gielis-type curve, resulting in a star-shaped configuration. The red arrows indicate the local rotational direction during the deposition process. By coupling quaternion-based rotations to Gielis-modulated geometries, we demonstrate how anisotropic rotation naturally emerges from shape parameters and leads to biologically plausible morphologies. Figure 7 presents the step-by-step construction of a starfish model using a starfish-shaped generalized superelliptic rotational motion.

Let $s = (s_0, s_1, s_2, s_3)$ be a superelliptic unit quaternion and R_s be the rotation matrix associated with s and $\tilde{e}_i$ be the standard basis vectors in the space $\mathbb{R}_{\mathbb{S}_\varepsilon}^3$, and for $i = 1, 2, 3$ define the quaternion $\tilde{e}_i = (0, \tilde{e}_i)$. For $i = 1, 2, 3$, the quaternion product is

$$s \tilde{e}_i \bar{s} = R_s \tilde{e}_i^T = (0, R_s \tilde{e}_i) = R_s \tilde{e}_i.$$

This expression is equal to the i -th column of the matrix R_s . Consequently, the vector part of the expression $s \tilde{e}_i \bar{s}$ yields a pure quaternion corresponding to the three superelliptic orthonormal columns of the matrix R_s . Therefore, the components of the vectors to be transformed must first be expressed in a form consistent with the basis of the superelliptic space. Then, multiplication with the superelliptic rotation matrix is performed using this consistent representation. This approach ensures that the transformation is both algebraically and geometrically consistent and eliminates inconsistencies that may arise during the computational process.

Firstly, we generate superellipsoids using a unit superelliptic quaternion and its corresponding matrix form. Let us consider the unit superelliptic quaternion $s = \cos \phi + \tilde{e}_3 \sin \phi$. Using this unit superelliptic quaternion, we rotate the curve $\alpha(\psi) := [\sin \psi, 0, \cos \psi]$. Writing this curve in form as $\alpha(\psi) := \sin \psi \tilde{e}_1 + \cos \psi \tilde{e}_3$, the corresponding starfish surface is obtained as follows:

$$\text{Starfish}(\theta, \psi) := s \otimes \alpha(\psi) \otimes s^{-1}$$

or it can be represented as the following matrix form

$$R_s \alpha(\phi) = \begin{bmatrix} \cos \phi & -\frac{\sqrt{\eta_2}}{\sqrt{\eta_1}} \sin \phi & 0 \\ \frac{\sqrt{\eta_1}}{\sqrt{\eta_2}} \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin \psi \tilde{e}_1 \\ 0 \\ \cos \psi \tilde{e}_3 \end{bmatrix}$$

$$= [\sin \psi \cos \phi \frac{\rho(\phi)}{\sqrt{\eta_1}}, \sin \psi \sin \phi \frac{\rho(\phi)}{\sqrt{\eta_2}}, \cos \psi \frac{\rho(\phi)}{\sqrt{\eta_3}}].$$

Then, we have obtained the following starfish surfaces by superelliptically rotating the superellipse curve around the $[0, 0, \tilde{e}_3]$ axis, in Figure 8. For actual parameters fitted to real starfish, we refer to [33].

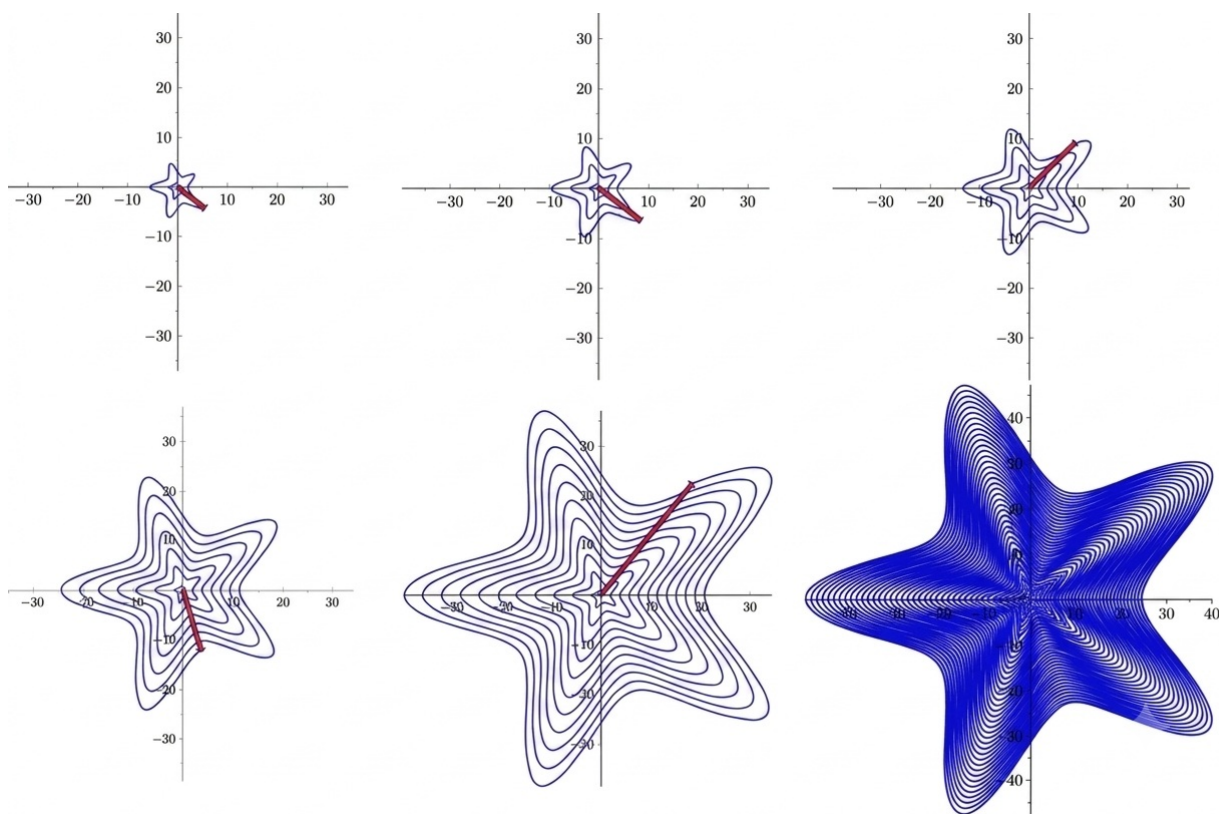


Figure 7. Kinematic visualization of a starfish-like shape generated by successive deposition under a Gielis-modulated rotational motion. Red arrows indicate local rotation directions for the values $m = 5, n_1 = 2, n_2 = 6, n_3 = 6, f(\phi) = 0.3\phi, \phi > 0$.

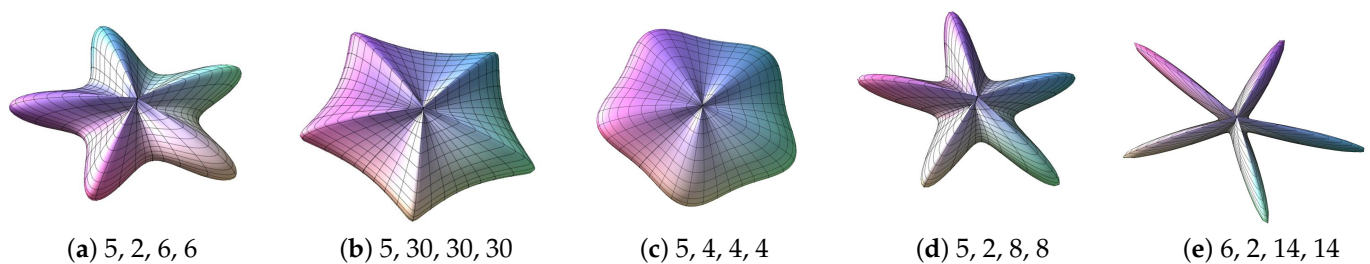


Figure 8. Starfish models for different parameter sets (m, n_1, n_2, n_3) , as indicated below each subfigure, with $(f(\phi) = \eta_i = 1, i \in \{1, 2, 3\})$.

Shell surfaces can also be constructed using a generalized superelliptic rotational motion derived from the generalized Gielis function. In this framework, a superellipse is

subjected to a generalized rotational transformation. During this process, $r(\psi)$ defines the cross-sectional curve of the shell, while $\rho(\phi)$ governs the superelliptic trajectory traced by this cross-section under the generalized rotational motion.

By systematically varying the parameters of these two generalized superformulas, it is possible to generate millions of distinct shell surfaces. This approach provides a compelling mathematical framework for explaining the remarkable diversity observed in natural shell morphologies. In particular, polygonal coiling is quite common in ammonoids [21,51], but is absent in all models of ammonoid shells [52]. Triangular or quadrangular coiling can easily be modeled (Figures 9 and 10).

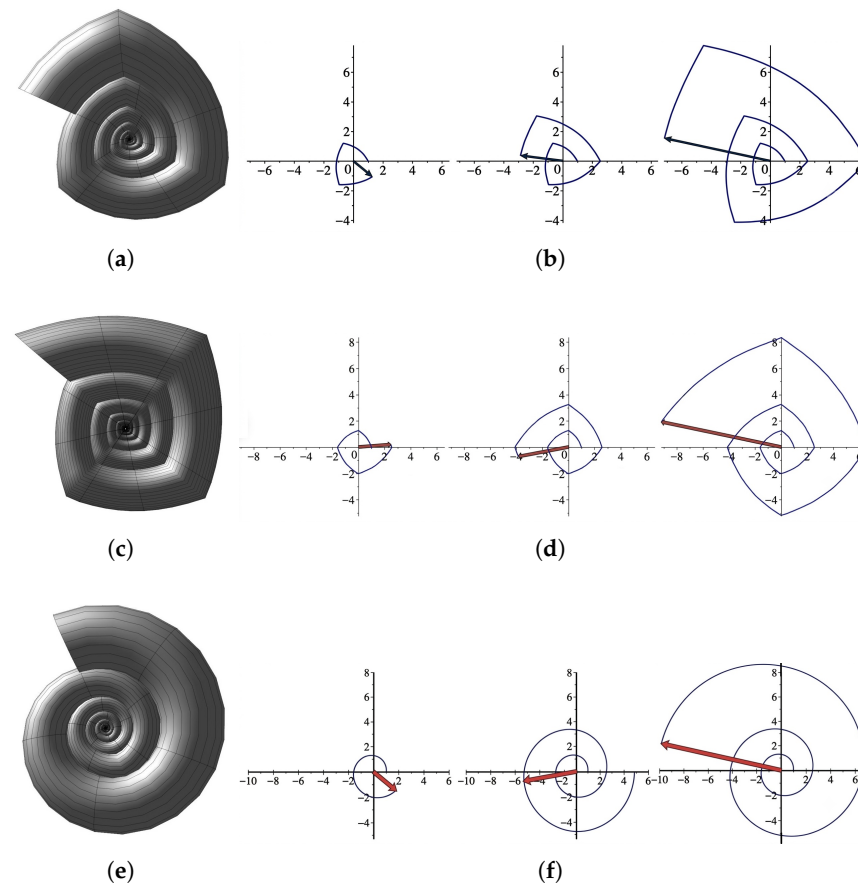


Figure 9. Superelliptic polygonal ammonoid shell surfaces with non-planar coiling generated using the generalized Gielis formulas. (a) Triangular polygonal ammonoid shell surface. (b) Generalized superelliptic rotational motion during the formation of the triangular polygonal ammonoid shell surface. (c) Square polygonal ammonoid shell surface. (d) Generalized superelliptic rotational motion during the formation of the square polygonal ammonoid shell surface. (e) Circular polygonal ammonoid shell surface. (f) Generalized superelliptic rotational motion during the formation of the circular polygonal ammonoid shell surface. The corresponding parameter values are listed in Table 3.

Shell surfaces can be generated using a unit generalized superelliptic quaternion and its corresponding matrix form. Consider the unit superelliptic quaternion $s = \cos \phi + \tilde{e}3 \sin \phi$. Using this unit superelliptic quaternion, we rotate the curve $\alpha(\psi) = [R_1 - r(\psi) \cos \psi, 0, R_2 + r(\psi) \sin \psi]$. Writing this curve in the form as $\alpha(\psi) := (R_1 - r(\psi) \cos \psi)\tilde{e}1 + (R_2 + r(\psi) \sin \psi)\tilde{e}2$, the corresponding seashell surface is obtained as follows:

$$\text{Shell}(\psi, \phi) := s \otimes \alpha(\phi) \otimes s^{-1}$$

or it can be represented as the following matrix form

$$R_s \alpha(\phi) = \begin{bmatrix} \cos \phi & -\frac{\sqrt{\eta_2}}{\sqrt{\eta_1}} \sin \phi & 0 \\ \frac{\sqrt{\eta_1}}{\sqrt{\eta_2}} \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (R_1 - r(\psi) \cos \psi) \tilde{e}_1 \\ 0 \\ (R_2 + r(\psi) \sin \psi) \tilde{e}_3 \end{bmatrix}$$

$$= [(R_1 - r(\psi) \cos \psi) \cos \phi \frac{\rho(\phi)}{\sqrt{\eta_1}}, (R_1 - r(\psi) \cos \psi) \sin \phi \frac{\rho(\phi)}{\sqrt{\eta_2}}, (R_2 + r(\psi) \sin \psi) \frac{\rho(\phi)}{\sqrt{\eta_3}}],$$

where $r(\psi) = g(\psi)[|\frac{\cos \frac{p\phi}{4}}{a_1}|^{m_2} + |\frac{\sin \frac{p\phi}{4}}{b_1}|^{m_3}]^{\frac{-1}{m_1}}$ and $\rho(\phi) = f(\phi)[|\frac{\cos \frac{m\phi}{4}}{a}|^{n_2} + |\frac{\sin \frac{m\phi}{4}}{b}|^{n_3}]^{\frac{-1}{n_1}}$. If we choose the following parameter sets with $\eta_i = 1$, $i \in \{1, 2, 3\}$ we generate the following surfaces.

The mathematical modeling of seashell morphogenesis involves two distinct rotational motions (see Figure 11). The first is a superelliptic rotation that occurs during the generation of the cross-sectional curve (see Figure 11b). This motion traces the boundary of the generating aperture, transitioning from an asymmetric initial state to a fully formed, symmetric closed superelliptic profile. The second is a generalized superelliptic rotation (see Figure 11c), which governs the overall trajectory and transformation of this cross-sectional curve to generate the complete three-dimensional shell surface. This motion traces a logarithmic spiraling pathway in the global coordinate system, dictating the spatial evolution, winding rate, and overall logarithmic helico-spiral morphology of the complete three-dimensional seashell structure.

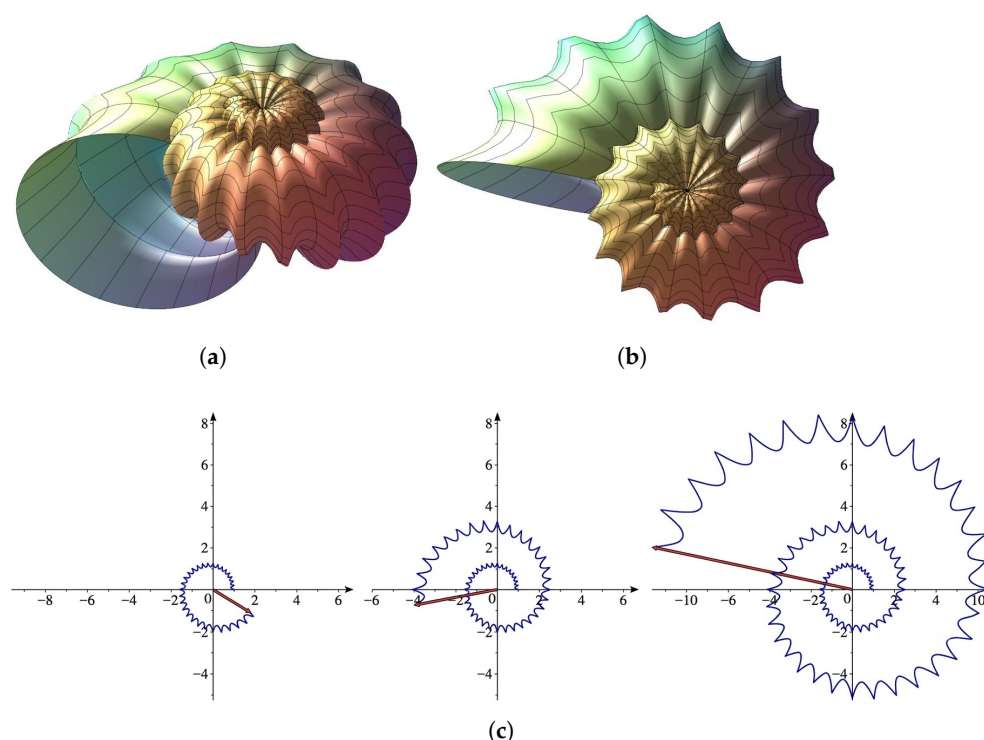


Figure 10. Superelliptic seashell surfaces for some values of the generalized superformulas. (a) Front view of the surface. (b) Top view of the surface. (c) Generalized superelliptic rotation motion during the formation of the surface. The corresponding parameter values are listed in Table 3.

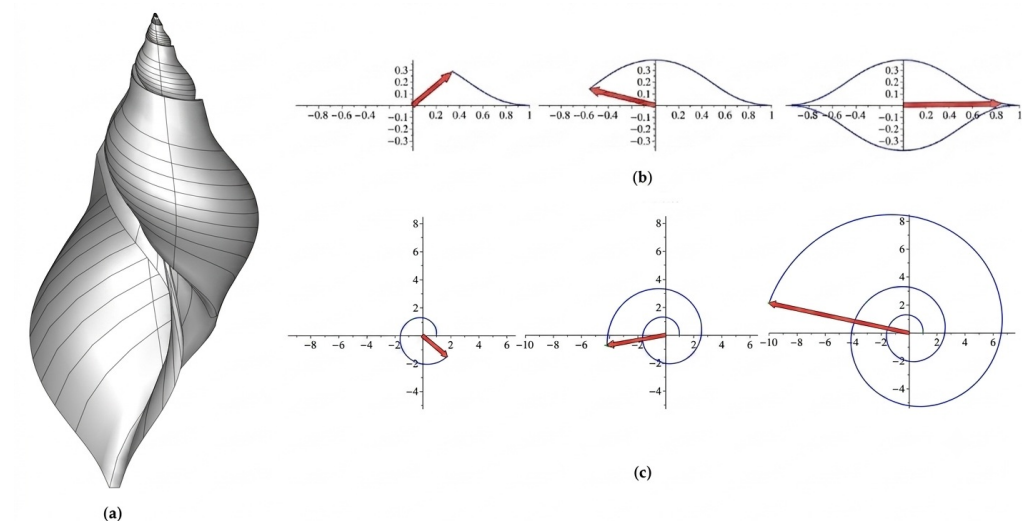


Figure 11. (a) Superelliptic seashell surfaces generated using the generalized superformulas. (b) Superelliptic rotation model of the cross-sectional curve of surface. (c) Generalized superelliptic rotational motion during the formation of the surface. The corresponding parameter values are listed in Table 3.

Table 3. Parameter values and function coefficients used to generate the shell surfaces shown in Figures 9–11.

Figure	$r(\psi)$ Parameters						$\rho(\phi)$ Parameters						f, g	
	a	b	m	n_1	n_2	n_3	a_1	b_1	p	m_1	m_2	m_3	f	g
Figure 9a,b	1	1	3	1.5	1	1	1	1	2	2	2	2	$\exp(0.10 * \phi)$	1
Figure 9c,d	1	1	4	2	1	1	1	1	2	2	2	2	$\exp(0.10 * \phi)$	1
Figure 9e,f	1	1	4	2	2	2	1	1	2	2	2	2	$\exp(0.10 * \phi)$	1
Figure 10	1	1	16	2	1	1	1	1	1	2	1	1	$\exp(0.15\phi)$	1
Figure 11	1	1	2	2	2	2	1	1	2	0.55	0.5	2.5	$\exp(0.15 * \phi)$	1

5. The Heart

Evolution has experimented with coiling in many different ways, and one of the clearest examples is the heart. A tube-heart evolved into the very complex structure of mammalian hearts. The concept of the helical heart however, removes most if not all, of the complexity. The Spanish cardiologist F. Torrent-Guasp found through elementary dissection of the heart, that it unrolls as a helical band [53–55]. Torrent-Guasp's model of the heart is a rope model, which is a Gielis curve with symmetry parameter $m = 1/2$ [56]. Figure 10 shows the unfolding of the helical heart (top) and the corresponding rope model (top). Figure 11a is a half-angle Gielis surface ($m = 1/2$) and Figure 11b shows a thickened version. This shape also determines the fiber orientation and movement of the heart [55,57], where function and form have co-evolved.

Consider the curve $\alpha(\phi) := [R + r \cos(\phi/2)) \cos \phi, (R + r \cos(\phi/2)) \sin \phi, r \sin(\phi/2)]$, then the corresponding superelliptic curve can be written in the form $\alpha(\phi) = (R + r \cos(\phi/2)) \cos \phi \tilde{e}_1 + (R + r \cos(\phi/2)) \sin \phi \tilde{e}_2 + r \sin(\phi/2) \tilde{e}_3$ and the related Frenet frame $\{T_{\mathbb{S}_\mathcal{E}}, N_{\mathbb{S}_\mathcal{E}}, B_{\mathbb{S}_\mathcal{E}}\}$ is adopted from [41], can be computed below:

$$T_{\mathbb{S}_\mathcal{E}}(\phi) = \frac{\alpha(\phi)^*}{\|\alpha(\phi)^*\|_{\mathbb{S}_\mathcal{E}}}, B_{\mathbb{S}_\mathcal{E}}(\phi) = \frac{\alpha(\phi)^* \times_{\mathbb{S}_\mathcal{E}} \alpha(\phi)^{**}}{\|\alpha(\phi)^* \times_{\mathbb{S}_\mathcal{E}} \alpha(\phi)^{**}\|_{\mathbb{S}_\mathcal{E}}}, N_{\mathbb{S}_\mathcal{E}} = B_{\mathbb{S}_\mathcal{E}} \times_{\mathbb{S}_\mathcal{E}} T_{\mathbb{S}_\mathcal{E}},$$

where $\alpha(\phi)^* = \sum_{i=1}^3 \alpha(\phi)' \frac{\rho(\phi)}{\sqrt{\eta_i}} = \sum_{i=1}^3 \alpha(\phi)' \tilde{e}_i$. Thus, using the following unit superelliptic quaternion $s := (s_0, s_1, s_2, s_3) = \cos \psi + T_{\mathbb{S}_{\mathcal{E}}}(\phi) \sin \phi$, we have

$$\begin{aligned} s_0 &= \cos \psi, \\ s_1 &= \sin \psi \alpha'_1(\phi) \tilde{e}_1, \\ s_2 &= -\sin \psi \alpha'_2(\phi) \tilde{e}_2, \\ s_3 &= 0. \end{aligned}$$

The heart surface is obtained by translating the rotated normal vector along the generating curve. More precisely, the superelliptic rotation of the normal vector is defined by

$$R_s(N_{\mathbb{S}_{\mathcal{E}}}(\phi)) = s \otimes N_{\mathbb{S}_{\mathcal{E}}}(\phi) \otimes s^{-1},$$

Accordingly, the heart surface is parametrized by

$$\text{Heart}(\psi, \phi) = \alpha(\phi) + r(\psi) R_s(N_{\mathbb{S}_{\mathcal{E}}}(\phi)),$$

where the addition is the standard vector addition in the ambient superelliptic space $\mathbb{R}_{\mathbb{S}_{\mathcal{E}}}^3$, $\rho(\phi)$ defines the shape of the profile curve, and $r(\psi) = g(\psi) [|\frac{\cos \frac{p\phi}{4}}{a_1}|^{m_2} + |\frac{\sin \frac{p\phi}{4}}{b_1}|^{m_3}]^{\frac{1}{m_1}}$ defines the shape of the cross-section curve. By varying the Gielis parameters, profile curves with different forms can be obtained, leading to heart surfaces of various shapes. The heart surface shown in Figure 12 is obtained using the following parameter values.

Remark 3. Non-integer values of m model various self-intersecting curves in the plane or the planar projection of helices and spirals as examples of space curves [26]. The half-integer value ($p = 0.5$), corresponding to the parameter m in [58], is used to define the symmetry of the heart and has also been used to model flowers. This unveils deep connections in nature between the living and non-living. A full 720° (4π) turn is needed for the quaternion with $\theta/2$ itself to return to its original value. V.I Arnold referred to this as Rodrigues' halving, and the reason for spins in physics [59].

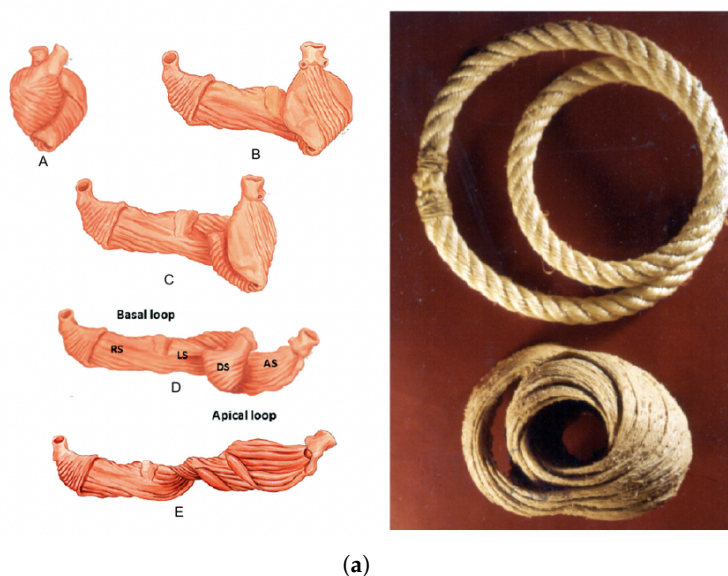
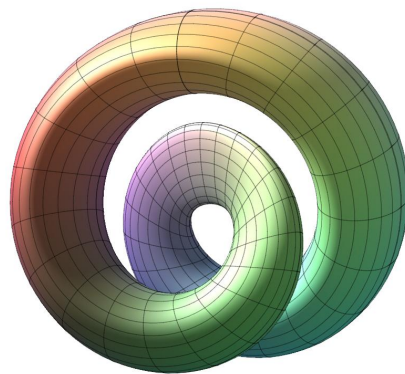


Figure 12. Cont.



(b)

Figure 12. (a) Unfolding of the helical heart and the corresponding rope model [53]. The uppercase labels (A–E) denote the successive unfolding stages of the helical heart, where A represents the initial helical heart configuration and E corresponds to the final stage in which the heart is unfolded into a tubular surface. (b) Half-angle Gielis surface corresponding to the half-angle case, obtained by setting the Gielis parameter $p = 0.5$. The parameters are $R_1 := 1$, $R_2 := 0.5$; $p := 0.5$, $m_1 := 2$, $m_2 := 0.9$, $m_3 := 8$, $g(\psi) := 1$, and $\eta_i = 1$, $i = \{1, 2, 3\}$.

6. Conclusions

6.1. Metric-Induced Rotations

This study presents a generalized geometric approach to modeling biological growth processes by combining the flexible geometric structure of Gielis surfaces with quaternion-based rotation–translation operators. The generalized inner product and vector product structures defined in the study directly integrate metric properties changing along the surface into the rotation space, producing a geometry that goes beyond classical Euclidean transformations. Thus, orientation changes, elastic bending, differential expansions, and torsional movements that occur during growth can be modeled in a metric-consistent manner.

The proposed metric-adaptive quaternion operators enable the creation of transformation fields that are both computationally stable and compatible with biological processes. These structures, in particular, enable the representation of complex morphogenetic processes in living organisms and all natural structures. In plants, morphogenetic phenomena such as stem bending, leaf expansion, and the evolution of phyllotactic patterns can be studied within a single geometric framework. It also allows for extending the original algorithmic L-systems for building complete plants. In shells, modulations can be applied to efficiently deal with complex septa and suture lines. It also allows for dealing with heteromorph structures such as the U-turns in *Nipponites* (Figure 1), and by extension with biomolecules, their basic structures and their folding. Most importantly, it turns a descriptive mathematical methodology into a kinematic one, allowing one to model evolution and development, but also movements as in the case of the heart.

This mathematical bridge between superquadrics, elasticity theory, and orientational fields contributes to a more accurate, flexible, and biologically realistic representation of growth-related shape changes in a uniform way.

In conclusion, this study presents a powerful method, both theoretically and computationally, to describe and simulate the complex forms observed in plants. The developed approach is expected to find new applications in future disciplines such as growth-based animation, botanical modeling, biomimetic design, and biological tissue engineering. Furthermore, the extension of the identified metric structure to differential geometric analysis,

shape optimization, and the study of organismal growth mechanics demonstrates the extensibility of the study.

6.2. Mathematical Significance of the Proposed Framework

Beyond its applications in the natural sciences and engineering, the proposed framework has important mathematical implications. Rather than introducing only a new distance function, the Gielis-based metric establishes a generalized geometric setting in which the classical Euclidean concepts of distance, circles, and rotations are naturally extended. Within this framework, generalized rotational transformations arise from the intrinsic metric structure and are no longer restricted to Euclidean circles. Instead, they preserve Gielis-type metric circles, including generalized supercircular, polygonal, and spiral-like geometries. Consequently, the proposed metric provides a unified description of a much broader class of rotational phenomena than those permitted by classical Euclidean geometry. Since rotational structures are ubiquitous in natural systems and quaternion-based rotations play a central role in geometry, physics, robotics, computer graphics, and engineering, this generalized rotational geometry offers a mathematically consistent foundation for modeling complex rotational behaviors across a wide range of disciplines.

An equally important consequence concerns the distinction between intrinsic and extrinsic geometry. Many Gielis curves, such as astroid-like shapes or other curves with corner-like features, appear anisotropic or geometrically singular when viewed from the Euclidean perspective. Likewise, spiral-like trajectories appear as open curves in the Euclidean plane. Within the intrinsic geometry induced by the proposed metric, however, these objects may be interpreted as metric circles, namely, loci of constant generalized distance from a center. Their apparent cusps, anisotropy, or openness are therefore not intrinsic properties of the geometry but consequences of their Euclidean embedding. From this intrinsic viewpoint, generalized rotations preserve these metric circles in the same way that Euclidean rotations preserve ordinary circles. This reinterpretation substantially broadens the classical concept of rotational symmetry by replacing Euclidean circles with generalized metric circles and provides a natural geometric framework in which geometries that appear singular or non-closed in Euclidean space acquire a regular intrinsic interpretation. In this sense, the proposed Gielis metric extends the classical notion of distance and provides a natural framework for the generalization of rotational geometry beyond the Euclidean setting.

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