

Article

An Explicit Eventual p^4 -Divisibility Theorem for an Apéry-Type Numerator Family

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Abstract

For a nonnegative integer m , let $u_m(n)$ denote the reduced numerator of $\sum_{k=1}^n \frac{\binom{n}{k}^2 \binom{n+k}{k}^2}{k^{2m+1}}$. OEIS A357513 originally posed the finite-exception conjecture that $u_m(p-1) \equiv 0 \pmod{p^4}$ for all but finitely many primes p . We prove the explicit uniform sufficient threshold $p > 2m + 6$. Thus, every exceptional prime satisfies $p \leq 2m + 6$, while $\{0, 1, \dots, 2m + 6\}$ serves only as a finite witness in the ambient set $\mathbb{N}$. The proof reduces the Apéry-type summand to two inverse-power sums in $\mathbb{Z}/p^4\mathbb{Z}$, applies finite-field and pairing cancellations, and transfers the result to the reduced numerator through a common denominator prime to p . An immutable supplementary Lean 4 package was used to verify a theorem whose type directly exposes the threshold $2m + 6 < p$.

Keywords: Apéry-type sums; supercongruences; harmonic sums; reduced numerators; p -adic congruences; Lean formalization

MSC: 11A07; 11B65; 11B83; 68V20

1. Introduction

The Apéry numbers and related sequences exhibit many striking supercongruences. OEIS A357513 concerns the reduced numerators of

$$a(n) = \text{num} \left(\sum_{k=1}^n \frac{\binom{n}{k}^2 \binom{n+k}{k}^2}{k^3} \right),$$

with $a(0) = 0$. The same entry originally posed an m -parameter finite-exception conjecture.

$$u_m(n) = \text{num} \left(\sum_{k=1}^n \frac{\binom{n}{k}^2 \binom{n+k}{k}^2}{k^{2m+1}} \right),$$

This conjecture predicts that $u_m(p-1) \equiv 0 \pmod{p^4}$ outside a finite set of primes depending on m . This historical formulation was qualitative and did not supply a uniform threshold.

We prove that the congruence holds for every prime $p > 2m + 6$. Consequently, every exceptional prime belongs to

$$\{p : p \text{ is prime and } p \leq 2m + 6\}.$$



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The interval $E_m = \{0, 1, \dots, 2m + 6\}$ is therefore a convenient finite witness in $\mathbb{N}$ for the existential source statement. Its nonprime elements are not exceptional primes, and the bound need not be sharp for a fixed m .

The theorem proved here gives an explicit uniform sufficient threshold for this exact reduced-numerator family. A stronger result that was posted while this manuscript was under review now classifies all exceptional primes; the Section 2 gives the precise comparison. The present contribution is therefore the direct threshold proof for this family. The argument does not introduce a general method for Apéry-type supercongruences, and the accompanying Lean formalization provides reproducible validation rather than a separate source of mathematical novelty.

The proof is a direct calculation in $\mathbb{Z}/p^4\mathbb{Z}$. At $n = p - 1$, the summand reduces to

$$p^2 k^{-(2m+3)} - 2p^3 k^{-(2m+4)} \pmod{p^4}.$$

A finite-field power-sum identity eliminates the even-exponent sum modulo p . Pairing k with $p - k$ eliminates the odd-exponent sum modulo p^2 . A common denominator prime to p then transfers the residue calculation to the reduced numerator.

2. Source and Related Work

OEIS A357513 records the base sequence and the special $m = 1$ congruence, and it originally posed the general finite-exception conjecture [1]. The current entry now states that the general conjecture has been proved and discusses the exact criterion below. The pinned FormalConjectures file formalizes the same $k = 1$ family [2].

Several broader theories explain why congruences arise near the classical Apéry kernel. Osburn and Sahu study integer-valued Apéry-like sums at indices related by powers of a prime [3]. Straub develops multivariate supercongruences for coefficients of rational functions [4]. These results concern unweighted integer sums or coefficient families. They do not introduce the reciprocal factor $k^{-(2m+1)}$ followed by reduction to a rational numerator.

The proof is also adjacent to binomial and harmonic-sum methods. McCarthy gives binomial-harmonic identities associated with supercongruences [5]. Rosen and Sun–Zhao study Wolstenholme-type and higher harmonic-sum congruences [6,7]. Sun and Ye apply Beukers-type methods to products of three binomial coefficients [8]. These works motivate the finite-field and pairing tools used below, but none of their stated results gives the present m -parameter reduced-numerator threshold.

While this manuscript was under review, Kutal posted a mathematical proof and Lean formalization of the exact classification [9]:

$$p^4 \mid u_m(p-1) \iff p-1 \nmid 2m+4 \text{ or } p \mid 2m+7.$$

For the nonnegative parameters considered here, the exceptional primes are therefore exactly those satisfying $p-1 \mid 2m+4$ and $p \nmid 2m+7$; in particular, each is at most $2m+5$. This later result is stronger than Theorem 1 because it classifies every exceptional prime, whereas the theorem here supplies a uniform sufficient threshold. The claim retained in this paper is the direct threshold proof and its reproducible formal validation; we make no claim to the subsequent exact classification.

3. Statement

For a rational number q , write $\text{num}(q)$ for the absolute value of its numerator in lowest terms. For $m, n \geq 0$, define

$$u_m(n) = \text{num} \left(\sum_{k=1}^n \frac{\binom{n}{k}^2 \binom{n+k}{k}^2}{k^{2m+1}} \right).$$

Theorem 1. Let $m \geq 0$, and let p be a prime. If $p > 2m + 6$, then

$$u_m(p-1) \equiv 0 \pmod{p^4}.$$

Consequently,

$$\{p : p \text{ is an exceptional prime for } m\} \subseteq \{p : p \text{ is prime and } p \leq 2m + 6\}.$$

Equivalently, $E_m = \{0, 1, \dots, 2m + 6\} \subset \mathbb{N}$ is a finite witness for the existential statement originally recorded in OEIS A357513.

The final passage to the reduced numerator uses the following elementary lemma.

Lemma 1. Let p be prime, and let $q = N/D \in \mathbb{Q}$, where $N, D \in \mathbb{Z}$, $D \neq 0$, and $\gcd(D, p) = 1$. If $N \equiv 0 \pmod{p^4}$, then

$$\text{num}(q) \equiv 0 \pmod{p^4}.$$

Proof. Let $g = \gcd(|N|, |D|)$. The reduced numerator is $|N|/g$. Since $g \mid D$ and $\gcd(D, p) = 1$, the factor g is a unit modulo p^4 . Thus, $p^4 \mid N$ implies $p^4 \mid |N|/g$. $\square$

4. Proof of the Main Theorem

The proof uses two standard congruence inputs: the nontrivial power-sum vanishing over $\mathbb{F}_p^\times$, and the first-order binomial expansion $(1-x)^{-e} \equiv 1+ex \pmod{x^2}$ applied with $x = p/k$. The remaining work is the bookkeeping needed to reduce the Apéry-type numerator sum to these two elementary facts.

Lemma 2 (Summand expansion). Let $m \geq 0$, let p be prime, and let $1 \leq k \leq p-1$. In $\mathbb{Z}/p^4\mathbb{Z}$,

$$\binom{p-1}{k}^2 \binom{p-1+k}{k}^2 k^{-(2m+1)} \equiv p^2 k^{-(2m+3)} - 2p^3 k^{-(2m+4)} \pmod{p^4}.$$

Proof. Set $R = \mathbb{Z}/p^4\mathbb{Z}$. Since $1 \leq k \leq p-1$, every $1, \dots, k$ is a unit in R . In R , the binomial coefficients admit the product expansions

$$\binom{p-1}{k} = (-1)^k \prod_{j=1}^k \left(1 - \frac{p}{j}\right), \quad \binom{p-1+k}{k} = \frac{p}{k} \prod_{j=1}^{k-1} \left(1 + \frac{p}{j}\right).$$

After squaring and separating the factor with $j = k$, we obtain

$$\begin{aligned} \binom{p-1}{k}^2 \binom{p-1+k}{k}^2 &= p^2 k^{-2} \left(1 - \frac{p}{k}\right)^2 \prod_{j=1}^{k-1} \left(1 - \frac{p}{j}\right)^2 \left(1 + \frac{p}{j}\right)^2 \\ &= p^2 k^{-2} \left(1 - \frac{p}{k}\right)^2 \prod_{j=1}^{k-1} \left(1 - \frac{p^2}{j^2}\right)^2. \end{aligned}$$

Set

$$P_k = \prod_{j=1}^{k-1} \left(1 - \frac{p^2}{j^2}\right)^2.$$

Each factor satisfies $\left(1 - \frac{p^2}{j^2}\right)^2 \equiv 1 \pmod{p^2}$. A finite product of elements congruent to 1 modulo p^2 is again congruent to 1 modulo p^2 , so

$$P_k \equiv 1 \pmod{p^2}.$$

Hence, $P_k = 1 + p^2 C_k$ for some $C_k \in R$. Multiplying by the leading factor $p^2 k^{-2}$, the $p^2 C_k$ error contributes a multiple of p^4 , and we get

$$p^2 k^{-2} \left(1 - \frac{p}{k}\right)^2 P_k \equiv p^2 k^{-2} \left(1 - \frac{p}{k}\right)^2 \pmod{p^4}.$$

Now,

$$\left(1 - \frac{p}{k}\right)^2 = 1 - \frac{2p}{k} + \frac{p^2}{k^2},$$

so

$$p^2 k^{-2} \left(1 - \frac{p}{k}\right)^2 = p^2 k^{-2} - 2p^3 k^{-3} + p^4 k^{-4} \equiv p^2 k^{-2} - 2p^3 k^{-3} \pmod{p^4}.$$

Multiplying by $k^{-(2m+1)}$ yields the stated congruence. $\square$

Lemma 3 (Inverse-power cancellation). *Let $m \geq 0$, let p be prime, and suppose $p > 2m + 6$. In $\mathbb{Z}/p^4\mathbb{Z}$, set*

$$H_a = \sum_{k=1}^{p-1} k^{-a}.$$

Then,

$$H_{2m+4} \equiv 0 \pmod{p} \quad \text{and} \quad H_{2m+3} \equiv 0 \pmod{p^2}.$$

Proof. Let $e = 2m + 3$. The hypothesis $p > 2m + 6$ is used here to obtain

$$0 < e + 1 = 2m + 4 < p - 1.$$

Reduction modulo p identifies H_{e+1} with

$$\sum_{x \in \mathbb{F}_p^\times} x^{-(e+1)}.$$

Because inversion permutes $\mathbb{F}_p^\times$, this equals $\sum_{x \in \mathbb{F}_p^\times} x^{e+1}$. The multiplicative group $\mathbb{F}_p^\times$ is cyclic of order $p - 1$, so this power sum is 0 whenever $0 < e + 1 < p - 1$. Hence,

$$H_{e+1} \equiv 0 \pmod{p}.$$

For the odd-exponent sum, the map $k \mapsto p - k$ permutes $\{1, \dots, p - 1\}$. Therefore,

$$2H_e = \sum_{k=1}^{p-1} (k^{-e} + (p - k)^{-e}).$$

Now, e is odd, so in $\mathbb{Z}/p^2\mathbb{Z}$, we have

$$\begin{aligned}(p-k)^{-e} &= (-k)^{-e} \left(1 - \frac{p}{k}\right)^{-e} \\ &\equiv -k^{-e} \left(1 + \frac{ep}{k}\right) \pmod{p^2},\end{aligned}$$

because terms of order $(p/k)^2$ vanish modulo p^2 . Therefore,

$$k^{-e} + (p-k)^{-e} \equiv -epk^{-(e+1)} \pmod{p^2},$$

and summing gives

$$2H_e \equiv -epH_{e+1} \pmod{p^2}.$$

Since $H_{e+1} \equiv 0 \pmod{p}$, the right-hand side is 0 modulo p^2 . Because $p > 2m+6 \geq 6$, the prime p is odd and 2 is a unit modulo p^2 . Thus, $H_e \equiv 0 \pmod{p^2}$. Substituting $e = 2m+3$ proves both claims. $\square$

Proof of Theorem 1. Fix $m \geq 0$ and a prime $p > 2m+6$. Set

$$R = \mathbb{Z}/p^4\mathbb{Z}.$$

For $1 \leq k \leq p-1$, the residue class of k is a unit in R . Thus, every occurrence of k^{-a} below is well-defined in R .

Define the hypergeometric residue sum

$$S_m(p) = \sum_{k=1}^{p-1} \binom{p-1}{k}^2 \binom{p-1+k}{k}^2 k^{-(2m+1)}.$$

By Lemma 2,

$$S_m(p) = p^2 H_{2m+3} - 2p^3 H_{2m+4} \quad \text{in } R,$$

where $H_a = \sum_{k=1}^{p-1} k^{-a}$. By Lemma 3, $H_{2m+3} \equiv 0 \pmod{p^2}$ and $H_{2m+4} \equiv 0 \pmod{p}$. Hence, both terms on the right vanish in R , and therefore,

$$S_m(p) = 0 \quad \text{in } R.$$

It remains to pass from the residue sum to the reduced numerator. Set

$$r = 2m+1, \quad A_k = \binom{p-1}{k}^2 \binom{p-1+k}{k}^2, \quad D = \prod_{k=1}^{p-1} k^r.$$

Define

$$N = \sum_{k=1}^{p-1} A_k \frac{D}{k^r}.$$

Here, each quotient D/k^r is an integer because k^r is one factor of the product defining D . Moreover, D is prime to p , and

$$\sum_{k=1}^{p-1} \frac{A_k}{k^r} = \frac{N}{D}.$$

Let a bar denote reduction modulo p^4 . Since $D = k^r(D/k^r)$ and $\bar{k}^r$ is a unit in R , we have

$$\overline{D/k^r} = \bar{D} (\bar{k}^r)^{-1}.$$

Therefore,

$$\begin{aligned}\bar{N} &= \sum_{k=1}^{p-1} \bar{A}_k \overline{D/k^r} \\ &= \bar{D} \sum_{k=1}^{p-1} \bar{A}_k (\bar{k}^r)^{-1} \\ &= \bar{D} S_m(p).\end{aligned}$$

The residue sum was already shown to be $S_m(p) = 0$ in R , so $N \equiv 0 \pmod{p^4}$. Since D is prime to p , the numerator transfer lemma applies to the rational sum N/D . Its reduced numerator, with the absolute-value convention of `num`, is therefore also 0 modulo p^4 . This reduced numerator is exactly $u_m(p-1)$, proving the congruence.

Finally, take $E_m = \{0, 1, \dots, 2m+6\}$. If p is prime and $p \notin E_m$, then $p > 2m+6$, so the congruence just proved applies. Thus, E_m is a finite witness for the source statement. $\square$

5. Formal Verification

The threshold theorem was formalized in Lean 4 with `mathlib` [10,11]. The manuscript-specific immutable reproduction package is archived on Zenodo as version 1.0.0 [12]. The archive contains a checksummed snapshot of the complete Lean source, its Lake configuration and dependency lock, the Lean toolchain, and the `FormalConjectures` source. The accompanying `ARTIFACT_MANIFEST.json` and `SHA256SUMS` identify every archived file. The same reproduction package is provided separately in the Supplementary Materials.

The formal development includes a declaration whose type directly exposes the threshold in Theorem 1:

```
lemma zmod_u_vanish_mod_p4_of_large
  (p m : Nat) (hp : p.Prime) (hlarge : 2 * m + 6 < p) :
  ((0e1sA357513.u m (p - 1) : Nat) : ZMod (p ^ 4)) = 0
```

The declaration `general_supercongruence_eventual` packages this threshold theorem with the witness `Finset.Icc 0 (2 * m + 6)`. The later `general_supercongruence_source_statement` supplies the existential interface used by `FormalConjectures`. Thus, the explicit inequality can be observed as a theorem type rather than only inside an existential proof term.

After extracting the archive into a directory without a `.lake` cache, the recorded verification uses the following:

```
lake exe cache get
target='AmraLibrary.OpenProblemBatches.TrueOpenNextRound20260606.'
target='${target}<04_general_supercongruence_zmod_cast>'
lake build "$target"
lake env lean ../AxiomAudit.lean
```

The clean build completes all 1248 jobs. The unabridged `build_reports/CLEAN_BUILD.LOG.txt` records the fresh extraction, dependency retrieval, exact commands, timestamps, terminal output, and zero exit status for each step; `build_reports/run_clean_build.sh` reproduces the workflow. The `AxiomAudit.lean` run reports only `propext`, `Classical.choice`, and `Quot.sound` for the four paper-facing declarations. The hashes, dependency revisions, theorem manifest, and audit output remain in the supplementary package.

6. Conclusions and Scope

For every nonnegative integer m , the A357513 reduced-numerator congruence holds at each prime $p > 2m+6$. The proof combines a $\text{mod-}p^4$ expansion, two inverse-power cancellations, and a common-denominator transfer. Hence, every exceptional prime is at most $2m+6$, while $\{0, 1, \dots, 2m+6\}$ is a finite witness in $\mathbb{N}$.

The theorem is a sufficient-threshold result; the subsequent work cited in the Section 2 goes further by giving the exact exceptional set. The proof here is specific to the squared

Apéry kernel, the reciprocal odd-power weight, the specialization $n = p - 1$, and the modulus p^4 . It does not provide a general classification for other kernels or weights. The supplementary Lean artifact verifies the threshold-form theorem and its existential FormalConjectures wrapper.

Supplementary Materials: The following supporting information can be downloaded at <https://www.mdpi.com/article/10.3390/math14152832/s1>. The checksummed Lean source archive, dependency lock, unabridged clean-build log, reusable build driver, axiom audit, manifest, and reproduction instructions are archived as version 1.0.0 on Zenodo [12].

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