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A Fractional Calculus Approach to Two-Variable Function Modeling of GDP Growth Rates: Evidence from G8 Countries and Türkiye

Şeyma Beşir ¹, Nisa Özge Önal Tuğrul ^{2,*}, Ertuğrul Karaçuha ² and Vasil Tabatadze ²¹ Defense Industries Research and Development Institute, The Scientific and Technological Research Council of Türkiye, TÜBİTAK-SAGE, Ankara 06484, Türkiye² Informatics Institute, Istanbul Technical University, Istanbul 34467, Türkiye

* Correspondence: onal16@itu.edu.tr; Tel.: +90-2122857194

Abstract

This study proposes a two-variable fractional calculus-based modeling framework for analyzing gross domestic product (GDP) growth rates using key macroeconomic indicators of G8 countries and Türkiye over the period of 1998–2022. The economic variables considered include exports, imports, inflation, foreign direct investment, and unemployment rates, with data obtained from the World Bank. Caputo-type fractional derivatives combined with the least squares method are employed to construct two-variable economic models, where GDP growth rate is treated as the dependent variable. Multiple combinations of economic factors are systematically examined to evaluate their modeling performance. The accuracy of the proposed models is assessed using the Mean Absolute Percentage Error (MAPE). The results indicate that the export–inflation combination yields the lowest modeling error for Japan, while the inflation–unemployment combination produces the highest error for Russia. Overall, the findings demonstrate that fractional calculus-based bivariate models provide an effective and flexible framework for capturing nonlinear and memory-dependent dynamics in economic growth analysis.

Keywords: fractional calculus; MAPE; modeling; least squares method**MSC:** 15A09; 26A33; 44A10; 91-10

Academic Editors: Assia Guezane-Lakoud and Rabah Khaldi

Received: 1 May 2026

Revised: 25 June 2026

Accepted: 30 June 2026

Published: 6 August 2026

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1. Introduction

Fractional calculus is a branch of mathematics, considered as an extension of classical calculus, that studies the properties of differential and integral operators characterized by real or complex orders [1,2]. For this reason, for the past three centuries, the subject has been mainly of interest to mathematicians, but in recent years, there has been a marked increase in research activity into the application of fractional calculus in many scientific fields [3], including physics [4], engineering [5,6], biology [7], and economics [8,9]. In addition, fractional calculus has been used in system control applications [10,11], image processing [12], engineering solutions [6,13], and medical research, including cancer research [14–16]. This is due to the fact that differential and integral equations of fractional order can mathematically model natural and social phenomena better than those restricted to integer order [1,2,17,18].

Mathematical modeling in economics generally uses methods such as derivative and integral calculus, differential and difference equations. These mathematical methods

have enabled the description of various economic processes and phenomena by creating economic models in a mathematical form. However, economic models based on differential equations of integer order cannot describe memory and non-local processes. Therefore, important aspects of economic processes and phenomena are not taken into account [19–21].

Fractional analysis methods are powerful tools for describing non-local [21] processes and systems with memory [20]. Therefore, fractional analysis methods are applied and developed in the fields of economics and finance [18,19,22]. Economic growth refers to the increase in the growth rate of gross domestic product. There are many factors that determine economic growth. These factors affect economic growth both individually and through mutual interaction. Export, import, foreign direct investment and inflation variables are the economic factors that have statistically significant effects for the evaluation of economic growth [23,24].

In recent years, fractional calculus has also found numerous applications in the field of economics [8,9,25–28]. Modeling China's GDP growth rates between 2012 and 2016 using the Caputo-type fractional analysis method [27] and modeling Portugal's GDP growth rates between 1960 and 2012 as a function of a nine-variable vector [8] are examples of economic modeling studies that utilize fractional calculus. In addition, the study of modeling the telecommunications investment data of Germany, France, Italy, Spain, Türkiye, and OECD countries for the years 1997–2018 [29], modeling the fixed and mobile broadband subscriber data for France, Italy, Türkiye, Germany, Spain, and the United Kingdom between 2010 and 2019 [25], modeling data on the percentage of telecommunications revenue in gross domestic product for Germany, France, Italy, Türkiye, the United Kingdom, and OECD members between 2000 and 2018 [30], modeling the annual total subscriber numbers of mobile network operators in Türkiye (Turkcell, Türk Telekom, and Vodafone) for the years 2004–2018 [31], a modeling study conducted using gross domestic product growth data from G20 countries between 1970 and 2018 [32], and a simulation study was conducted using R software to analyze China's gross domestic product (GDP) growth between 2012 and 2016 [27] are other examples of economic modeling studies that use fractional calculus.

In this study, the economic factors that can be used to assess the overall economic situation of the G8 countries [33,34] and Türkiye between 1998 and 2022 are Exports (annual growth %), Imports (annual growth%), GDP Growth Rate (annual %), Net Foreign Investment inflows (% of GDP), Inflation (annual %), and Unemployment Rate data were used. Data taken from the World Bank [35]. A mathematical modeling study was carried out with a two-variable function using fractional calculus and least squares method. The Caputo-type fractional derivative is used in the mathematical modeling. The parameters of the two-variable function are given as various combinations of economic factors and GDP Growth Rate (annual %) is modeled. In the modeling, the relevant economic factors of the countries as function parameters are combined as follows: import–export, import–foreign direct investment, import–unemployment rate, import–inflation, export–inflation, export–foreign direct investment, export–unemployment rate, inflation–foreign direct investment, inflation–unemployment rate and unemployment rate–foreign direct investment. MATLAB (version R2025b) was used for the implementation of the mathematical models. Modeling performance is evaluated by the Mean Absolute Percentage Error (MAPE). The number of intercepts in the modeling is taken as $M = 7$ [36].

The next section will introduce the recommended mathematical method. It will cover the definitions of fractional derivatives and integral transforms used, the Laplace transform and its properties used in solving fractional differential equations, and the least squares method used as an optimization method. The third section contains the application results and graphs obtained. The fourth section contains a discussion of the results.

2. Materials and Methods

Several definitions of fractional derivatives exist in the literature, including the Riemann–Liouville, Grünwald–Letnikov, and Caputo formulations [37]. In this study, the Caputo fractional derivative was selected because of its compatibility with Laplace and inverse Laplace transform techniques, which considerably simplify the analytical solution procedure. Furthermore, the Caputo definition allows the use of classical initial conditions and is, therefore, widely adopted in applied modeling studies. For these reasons, the proposed model was developed using the Caputo fractional derivative. In this study, modeling was performed using a two-variable function by utilizing the Caputo fractional derivative definition. This section will describe the modeling steps of the developed model step by step. Let the series expansion of the two-variable, continuous, and bounded function be expressed as follows.

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{nm} x^{n\alpha} y^{m\gamma} \quad (1)$$

The series representation employed in this study was selected to express the dependent variable as a function of two independent variables while providing a flexible framework capable of capturing nonlinear relationships between them. Classical polynomial-based models can represent nonlinear interactions to a certain extent; however, extending these models through fractional calculus offers greater flexibility and improved modeling capability. Taking the first derivative of Equation (1) with respect to the variables yields Equation (2).

$$\frac{\partial^2 f(x, y)}{\partial x \partial y} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} n\alpha x^{n\alpha-1} m\gamma y^{m\gamma-1} \quad (2)$$

Equation (2) is generalized, such that the classical first-order partial derivatives are recovered when $\alpha = \gamma = 1$. By introducing the fractional orders α and γ , the proposed formulation provides additional flexibility to capture nonlinear interactions between the variables. Furthermore, assuming that the fractional derivative of the function $f(x, y)$ is equal to the right-hand side of Equation (2), the derivation process is continued to obtain the proposed two-variable fractional model.

$$\frac{\partial^{\alpha+\gamma} f(x, y)}{\partial x^{\alpha} \partial y^{\gamma}} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} n\alpha x^{n\alpha-1} m\gamma y^{m\gamma-1} \quad (3)$$

Here, α and γ are fractional exponents in the interval $[0, 1]$. That is, the conditions $(0 < \alpha < 1 \text{ and } 0 < \gamma < 1)$ are satisfied. For a two-variable function, $f(x, y)$, the Caputo fractional derivative is defined as in Equation (4) below [38]:

$$\begin{aligned} {}^c_y D_{0^+}^{\gamma} {}^c_x D_{0^+}^{\alpha} f(x, y) &= \frac{1}{\Gamma(n-\alpha)\Gamma(m-\gamma)} \int_0^y \int_0^x (y-\tau)^{m-\gamma-1} (x-\xi)^{n-\alpha-1} \\ &\times \frac{\partial^{m+n} f(\xi, \tau)}{\partial \tau^m \partial \xi^n} d\xi d\tau \end{aligned} \quad (4)$$

Here, $n-1 < \alpha \leq n$, $m-1 < \gamma \leq m$ and $n, m \in \mathbb{N}$.

Caputo's fractional derivative has Laplace transform properties [39,40]. Using these properties, the Laplace transform of the proposed fractional model is obtained as in Equation (5) below [40].

$$\bar{F}(p, q) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \frac{\Gamma(n\alpha+1)}{p^{n\alpha+\alpha}} \frac{\Gamma(m\gamma+1)}{q^{m\gamma+\gamma}} + \frac{1}{q} \bar{F}(p, 0) + \frac{1}{p} \bar{F}(0, q) - \frac{1}{pq} f(0, 0) \quad (5)$$

From here, to obtain the $f(x, y)$ function, the inverse Laplace transform of both sides of Equation (5) is taken.

$$\mathcal{L}^{-1}\{\bar{F}(p, q)\} = \mathcal{L}^{-1}\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{nm}\frac{\Gamma(n\alpha+1)}{p^{n\alpha+\alpha}}\frac{\Gamma(m\gamma+1)}{q^{m\gamma+\gamma}} + \frac{1}{q}\bar{F}(p, 0) + \frac{1}{p}\bar{F}(0, q) - \frac{1}{pq}f(0, 0)\right\} \quad (6)$$

In Equation (7), the $f(x, y)$ function has been obtained using Caputo's fractional derivative's inverse Laplace transform properties [39,40].

$$f(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{nm}x^{n\alpha+\alpha-1}y^{m\gamma+\gamma-1}\frac{\Gamma(n\alpha+1)\Gamma(m\gamma+1)}{\Gamma(n\alpha+\alpha)+\Gamma(m\gamma+\gamma)} + f(x, 0) + f(0, y) - f(0, 0) \quad (7)$$

In order to obtain finite discrete data, a truncation number, which will be denoted by M , is needed. Furthermore, A_0 represents the number of the constant coefficient. Therefore, we express the function $f(x, y)$ as in Equation (8) below.

$$f(x, y) \cong \sum_{n=1}^M\sum_{m=1}^Ma_{nm}C_{nm}(x, y) + A_0 \quad (8)$$

$$C_{nm}(x, y) = \frac{\Gamma(n\alpha+1)+\Gamma(m\gamma+1)}{\Gamma(n\alpha+\alpha)+\Gamma(m\gamma+\gamma)}x^{n\alpha+\alpha-1}y^{m\gamma+\gamma-1} \quad (9)$$

After obtaining the analytical form of the proposed two-variable fractional model, the next step is to determine the unknown coefficients (a_{nm} , A_0) and fractional orders (α and γ) of unknown value obtained with Equation (9), which will allow modeling with the lowest error. For this purpose, the Least Squares Method will be used to find the unknown coefficients and fractional orders that will give the lowest error value.

Let P_i denote the observed (actual) value corresponding to the i_{th} observation in the dataset, while $f(x_i, y_i)$ represents the value estimated by the proposed model.

The fractional orders α and γ were determined through a grid-search procedure. Both parameters were varied between 0 and 1 with a step size of 0.01. For each α and γ combination, the model coefficients were estimated using the least-squares method and the corresponding Mean Absolute Percentage Error (MAPE) was calculated. The parameter combination yielding the minimum MAPE value was selected as the optimal model configuration. Using the optimal fractional orders and the unknown coefficients are determined by the least-squares method.

$$\begin{aligned} P_i &= [P_1 \ P_2 \ \dots \ P_K] \\ x_i &= [x_1 \ x_2 \ \dots \ x_K] \\ y_i &= [y_1 \ y_2 \ \dots \ y_K]. \end{aligned} \quad (10)$$

K is the size of the data set used and represents the number of years in the data set used. Due to the lack of inflation data for Russia for the year 2022, the value of K is taken as 24 for Russia. For the other countries, $K = 25$.

x_i and y_i represent countries' import, export, unemployment rate, inflation, and foreign direct investment data, which vary by year.

P_i shows the GDP growth (annual %) data of the countries by year. i represents the years in the data set used. $i = 1, 2, 3, \dots, K$. For example, for the year 1998, it corresponds to $i = 1$, for the year 1999, it corresponds to $i = 2$, and for 2022, it corresponds to $i = K$.

$$P = [P_{1998}, P_{1999}, \dots, P_{2022}]$$

The error between P_i and $f(x_i, y_i)$ is defined as ϵ_i in Equation (11). Our aim is to minimize the total error at each data point.

$$\epsilon_i^2 = [P_i - f(x_i, y_i)]^2 \quad (11)$$

The sum of the squares of the errors is given in Equation (12). To minimize the total error in (12), the least squares method given in Equation (13) is applied [41].

$$\epsilon_T^2 = \sum_{i=1}^K \epsilon_i^2 = \sum_{i=1}^K [P_i - f(x_i, y_i)]^2 = \sum_{i=1}^K [P_i - \{ \sum_{n=1}^M \sum_{m=1}^M a_{nm} C_{nm}(x_i, y_i) + A_0 \}]^2 \quad (12)$$

$$\frac{\partial \epsilon_T^2}{\partial A_0} = 0, \quad \frac{\partial \epsilon_T^2}{\partial a_{11}} = 0, \quad \frac{\partial \epsilon_T^2}{\partial a_{12}} = 0, \quad \frac{\partial \epsilon_T^2}{\partial a_{21}} = 0, \quad \dots \quad \frac{\partial \epsilon_T^2}{\partial a_{rp}} = 0 \quad (13)$$

According to Equations (12) and (13), ϵ_T^2 is expressed as in Equation (14).

$$\epsilon_T^2 = \sum_{i=1}^K \left[\left(P_i - \{ (a_{11} C_{11}(x_i, y_i) + a_{12} C_{12}(x_i, y_i) + a_{21} C_{21}(x_i, y_i) + \dots + a_{MM} C_{MM}(x_i, y_i)) \} + A_0 \right) \right]^2 \quad (14)$$

Here, by sequentially substituting $(1 \leq r \leq M)$ and $(1 \leq p \leq M)$, $(M \times M)$ equations are obtained. An example equation for $p = M$ and $r = M$ is given below.

$$\begin{aligned} \sum_{i=1}^K P_i C_{MM}(x_i, y_i) = & \left(a_{11} \sum_{i=1}^K C_{11}(x_i, y_i) C_{MM}(x_i, y_i) \right. \\ & + a_{12} \sum_{i=1}^K C_{12}(x_i, y_i) C_{MM}(x_i, y_i) \\ & + \dots + a_{MM} \sum_{i=1}^K C_{MM}(x_i, y_i) C_{MM}(x_i, y_i) \Big) \\ & + A_0 \sum_{i=1}^K C_{MM}(x_i, y_i) \end{aligned} \quad (15)$$

The equations obtained to find the coefficients a_{rp} and A_0 , using the least square method approach, are written in matrix form. This yields the linear equation system in Equation (16). The [B] matrix contains the unknown coefficients a_{rp} and A_0 .

$$[A][B] = [C] \quad (16)$$

The [A], [B], and [C] matrices are shown in Equation (17), Equation (18), and Equation (19), respectively.

$$A = \begin{bmatrix} K & \Sigma C_{11} & \Sigma C_{12} & \dots & \Sigma C_{1M} & \Sigma C_{21} & \dots & \Sigma C_{2M} & \dots & \Sigma C_{MM} \\ \Sigma C_{11} & \Sigma C_{11}C_{11} & \Sigma C_{12}C_{11} & \dots & \Sigma C_{1M}C_{11} & \Sigma C_{21}C_{11} & \dots & \Sigma C_{2M}C_{11} & \dots & \Sigma C_{MM}C_{11} \\ \Sigma C_{12} & \Sigma C_{11}C_{12} & \Sigma C_{12}C_{12} & \dots & \Sigma C_{1M}C_{12} & \Sigma C_{21}C_{12} & \dots & \Sigma C_{2M}C_{12} & \dots & \Sigma C_{MM}C_{12} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Sigma C_{1M} & \Sigma C_{11}C_{1M} & \Sigma C_{12}C_{1M} & \dots & \Sigma C_{1M}C_{1M} & \Sigma C_{21}C_{1M} & \dots & \Sigma C_{2M}C_{1M} & \dots & \Sigma C_{MM}C_{1M} \\ \Sigma C_{21} & \Sigma C_{11}C_{21} & \Sigma C_{12}C_{21} & \dots & \Sigma C_{1M}C_{21} & \Sigma C_{21}C_{21} & \dots & \Sigma C_{2M}C_{21} & \dots & \Sigma C_{MM}C_{21} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Sigma C_{2M} & \Sigma C_{11}C_{2M} & \Sigma C_{12}C_{2M} & \dots & \Sigma C_{1M}C_{2M} & \Sigma C_{21}C_{2M} & \dots & \Sigma C_{2M}C_{2M} & \dots & \Sigma C_{MM}C_{2M} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Sigma C_{MM} & \Sigma C_{11}C_{MM} & \Sigma C_{12}C_{MM} & \dots & \Sigma C_{1M}C_{MM} & \Sigma C_{21}C_{MM} & \dots & \Sigma C_{2M}C_{MM} & \dots & \Sigma C_{MM}C_{MM} \end{bmatrix} \quad (17)$$

$$B = \begin{bmatrix} A_0 & a_{11} & a_{12} & \dots & a_{1M} & a_{21} & \dots & a_{2M} & \dots & a_{MM} \end{bmatrix}^T \quad (18)$$

$$C = \begin{bmatrix} \sum_{i=1}^K P_i & \sum_{i=1}^K P_i C_{11} & \sum_{i=1}^K P_i C_{12} & \dots & \sum_{i=1}^K P_i C_{1M} & \sum_{i=1}^K P_i C_{21} & \dots & \sum_{i=1}^K P_i C_{2M} & \dots & \sum_{i=1}^K P_i C_{MM} \end{bmatrix}^T \quad (19)$$

T denotes the transpose of the matrix. After the linear equations are solved, the coefficients a_{rp} and A_0 are obtained. Once $f(x, y)$ is obtained, the discrete data is modeled as a continuous curve with minimum error.

3. Results

Modeling Results

In this study, the GDP growth data of the G8 countries and Türkiye for the years 1998–2022 (data for Russia for the years 1998–2021 were used) were modeled with a value of $M = 7$. Based on the results of the performed optimization studies, it was concluded that the number M should be set to 7. The countries' relevant economic factors were combined as x and y function parameters as follows: imports–exports, imports–foreign direct investment, imports–unemployment rate, imports–inflation, exports–inflation, exports–foreign direct investment, exports–unemployment rate, inflation–foreign direct investment, inflation–unemployment rate, and unemployment rate–foreign direct investment. The economic factors used in the modeling are presented in Table 1. Modeling performance was evaluated using the Mean Absolute Percentage Error (MAPE). Equation (40) provides the MAPE formula [9,26]. In the formula, $v(i)$ represents the actual data, $\hat{v}(i)$ represents the modeled data, and k represents the data size.

$$MAPE = \frac{1}{k} \sum_{i=1}^k \left| \frac{v(i) - \hat{v}(i)}{v(i)} \right| \times 100 \quad (20)$$

In the fractional-calculus literature, fractional orders are often associated with memory and hereditary effects because fractional derivatives are defined through integral operators that depend on the past behavior of a function. Consequently, the current state of a system may be influenced not only by its present condition but also by its historical evolution.

In the proposed model, however, the fractional orders α and γ are primarily interpreted as parameters controlling the flexibility of the functional representation rather than direct quantitative measures of economic memory. When $\alpha = \gamma = 1$, the proposed formulation reduces to its corresponding integer-order form. Values different from unity indicate that the relationships among the variables are better represented by a fractional-order structure, providing additional flexibility for capturing complex nonlinear interactions. Therefore, the fractional orders may be viewed as indicators of the extent to which the observed system departs from a purely integer-order representation and requires a more flexible fractional description.

Table 1. Economic factors used in modeling [35].

Economic Factors
GDP growth (annual %)
Imports of goods and services (annual % growth)
Exports of goods and services (annual % growth)
Foreign direct investment, net inflows (% of GDP)
Unemployment, total (% of total labor force)
Inflation, consumer prices (annual %)

Table 2 presents the modeling results for the GDP data of the G8 countries and Türkiye for $M = 7$. The table shows the fractional degrees (α , γ) and MAPE values. Figures 1 and 2 show graphs related to modeling. Figure 1 contains graphs related to the combinations that yield the highest MAPE results, while Figure 2 contains graphs related to the combinations that yield the lowest MAPE results. As a result, high modeling success was achieved for all countries throughout the study.

As presented in Table 3, if we look at the results obtained in terms of countries, for Canada, the lowest modeling error was obtained for import–export economic factors with a value of $1.1450 \times 10^{-7}\%$, while the highest modeling error is obtained for unemployment rate–foreign direct investment economic factors with a value of $4.4413 \times 10^{-2}\%$; for France, the lowest modeling error was obtained for export–inflation economic factors with a value of $3.2137 \times 10^{-6}\%$, while the highest modeling error was obtained for import–unemployment rate economic factors with a value of $4.0608 \times 10^{-2}\%$; for Russia, the lowest modeling error was obtained for the import–export economic factors with a value of $7.8435 \times 10^{-8}\%$, while the highest modeling error was obtained for the inflation–unemployment rate economic factors with a value of $3.6727 \times 10^{-1}\%$; for Germany, the lowest modeling error was obtained for the import–inflation economic factors with a value of $9.3484 \times 10^{-6}\%$, while the highest modeling error was obtained for the inflation–foreign direct investment economic factors with a value of $1.0887 \times 10^{-2}\%$; for Japan, the lowest modeling error was obtained for export–inflation economic factors with a value of $1.0252 \times 10^{-8}\%$, while the highest modeling error was obtained for import–unemployment rate economic factors with a value of $1.0003 \times 10^{-4}\%$; for Italy, the lowest modeling error was obtained for export–foreign direct investment economic factors with a value of $6.8832 \times 10^{-7}\%$, while the highest modeling error was obtained for inflation–unemployment rate economic factors with a value of $5.9027 \times 10^{-4}\%$; for the United Kingdom, the lowest modeling error was obtained for export–foreign direct investment economic factors with a value of $3.0153 \times 10^{-7}\%$, while the highest modeling error was obtained for import–inflation economic factors with a value of $1.1989 \times 10^{-3}\%$; for Türkiye, the lowest modeling error was obtained for import–export economic factors with a value of $2.4654 \times 10^{-8}\%$, while the highest modeling error was obtained for inflation–unemployment rate economic factors with a value of $5.5321 \times 10^{-2}\%$; and for the United States, the lowest modeling error was obtained for import–export economic

factors with a value of $3.2077 \times 10^{-7}\%$, while the highest modeling error was obtained for inflation–unemployment rate economic factors with a value of $6.4021 \times 10^{-4}\%$ [36].

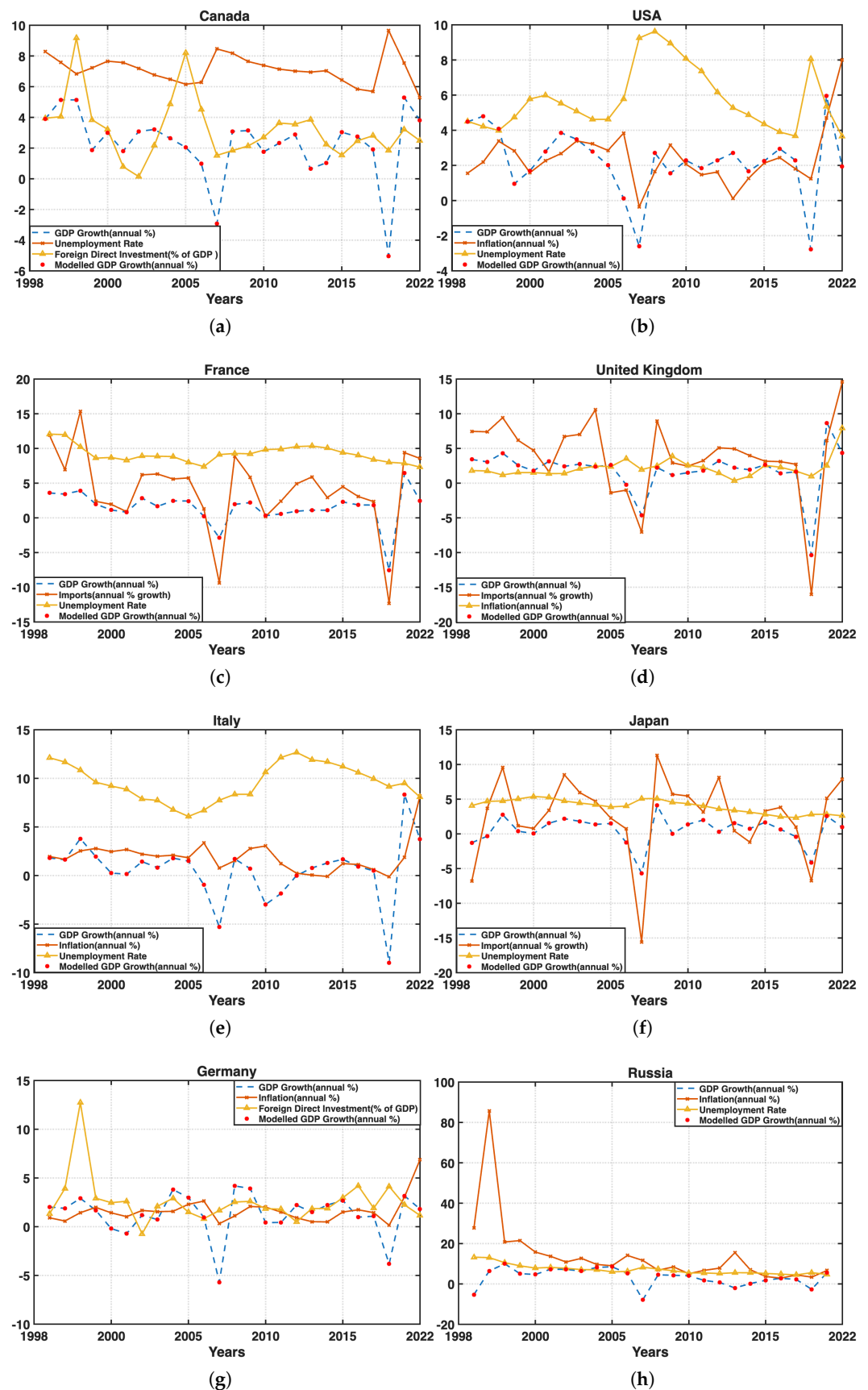


Figure 1. Cont.

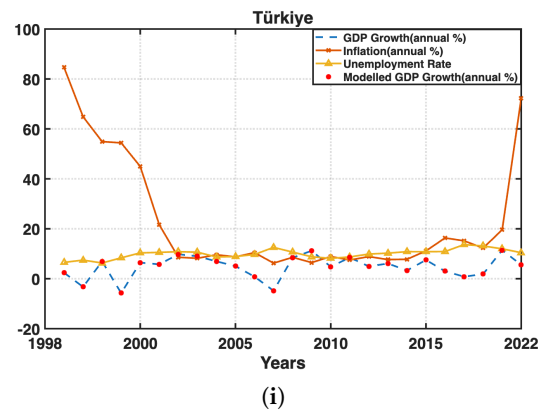


Figure 1. Combinations yielding the highest MAPE results [36]. (a) Unemployment rate–foreign direct investment for Canada; (b) Inflation–unemployment rate for USA; (c) Import–unemployment rate for France; (d) Import–inflation for United Kingdom; (e) Inflation–unemployment rate for Italy; (f) Import–unemployment rate for Japan; (g) Inflation–foreign direct investment for Germany; (h) Inflation–unemployment rate for Russia; (i) Inflation–unemployment rate for Türkiye.

The results reveal that trade-related variables play a dominant role in explaining GDP growth across the countries considered. The lowest MAPE values are generally obtained from combinations involving imports, exports, inflation, and foreign direct investment. In particular, the imports–exports combination provides the best performance for Canada, Russia, the United States, and Türkiye. This finding is consistent with open-economy growth theories, which emphasize the importance of international trade, external demand, and global economic integration as major drivers of economic growth.

For France and Japan, the exports–inflation combination yields the lowest MAPE values. These results suggest that, in addition to export performance, price dynamics play an important role in explaining growth patterns in these economies. In particular, Japan’s long-standing experience with low inflation and deflationary pressures may help explain the strong interaction between export performance and inflation dynamics.

For the United Kingdom and Italy, the exports–FDI combination produces the best results. This finding highlights the importance of international capital flows, investment activity, and export competitiveness in supporting economic growth. Similarly, the imports–inflation combination provides the best performance for Germany, indicating the joint importance of trade and price stability in the German economy.

By contrast, combinations involving unemployment generally yield the highest MAPE values. Although unemployment is an important macroeconomic indicator, it does not directly represent the production, trade, investment, and capital accumulation processes that drive economic growth. Consequently, unemployment-based combinations appear less effective in explaining GDP growth dynamics than trade- and investment-related variables.

The estimated fractional orders are generally different from unity, indicating that the relationships among the variables may not be fully represented by classical integer-order structures. In the fractional-calculus literature, fractional orders are often associated with memory and hereditary effects. Therefore, the obtained results may suggest that GDP growth dynamics involve cumulative and delayed interactions among macroeconomic variables. However, in the present study, the fractional orders are not interpreted as direct quantitative measures of economic memory. Rather, they are viewed as parameters that enhance the flexibility of the proposed functional structure and improve its ability to represent complex relationships among variables. From an economic perspective, the fractional-order framework provides additional flexibility for capturing the influence of past economic conditions, policy decisions, and structural adjustments on current growth performance.

Table 2. GDP growth modeling results for G8 countries and Türkiye (1998–2022) with $M = 7$, including estimated fractional orders (α , γ) and MAPE (%) values.

Countries	Results	Parameters (x-y)									
		Imports–Export	Imports–FDI	Imports–Unemp. Rate	Imports–Inflation	Exports–Inflation	Exports–FDI	Exports–Unemp. Rate	Inflation–FDI	Inflation–Unemp. Rate	Unemp. Rate–FDI
Canada	α	0.44	0.73	0.71	0.73	0.65	0.57	0.45	0.51	0.62	0.94
	γ	0.57	0.43	0.96	0.54	0.53	0.43	0.91	0.73	0.97	0.90
	MAPE	1.1×10^{-7}	3.6×10^{-6}	3.2×10^{-4}	2.0×10^{-6}	1.1×10^{-5}	2.9×10^{-4}	3.0×10^{-3}	2.0×10^{-4}	6.7×10^{-3}	4.4×10^{-2}
France	α	0.41	0.57	0.76	0.79	0.67	0.85	0.65	0.49	0.87	0.97
	γ	0.67	0.90	0.93	0.48	0.65	0.65	0.94	0.95	0.99	0.88
	MAPE	6.3×10^{-4}	8.7×10^{-6}	4.1×10^{-2}	4.3×10^{-6}	3.2×10^{-6}	6.1×10^{-6}	4.1×10^{-3}	3.7×10^{-4}	3.5×10^{-3}	8.8×10^{-3}
Russia	α	0.76	0.72	0.75	0.86	0.54	0.74	0.72	0.33	0.27	0.95
	γ	0.44	0.72	0.76	0.49	0.27	0.80	0.77	0.78	0.94	0.97
	MAPE	7.8×10^{-8}	9.3×10^{-6}	3.1×10^{-6}	9.8×10^{-6}	5.2×10^{-5}	7.6×10^{-7}	1.0×10^{-3}	1.2×10^{-4}	3.7×10^{-1}	4.9×10^{-5}
United States	α	0.42	0.48	0.48	0.58	0.84	0.89	0.63	0.79	0.96	0.97
	γ	0.65	0.75	0.98	0.53	0.47	0.99	0.98	0.98	0.85	1.00
	MAPE	3.2×10^{-7}	6.2×10^{-6}	1.8×10^{-5}	8.5×10^{-6}	8.7×10^{-6}	4.9×10^{-6}	2.0×10^{-4}	7.2×10^{-5}	6.4×10^{-4}	4.3×10^{-5}
Germany	α	0.61	0.73	0.67	1.00	0.65	0.40	0.47	0.64	0.78	0.90
	γ	0.64	0.39	0.97	0.68	0.60	0.35	0.95	0.54	0.93	0.32
	MAPE	1.3×10^{-5}	7.3×10^{-4}	1.3×10^{-5}	9.3×10^{-6}	2.6×10^{-5}	4.9×10^{-3}	3.9×10^{-5}	1.1×10^{-2}	1.8×10^{-5}	3.5×10^{-4}
United Kingdom	α	0.50	0.66	1.00	0.89	0.64	0.37	0.62	0.72	0.71	0.93
	γ	0.61	0.58	0.95	0.42	0.76	0.59	1.00	0.50	0.95	0.64
	MAPE	7.7×10^{-5}	8.3×10^{-7}	3.1×10^{-4}	1.2×10^{-3}	2.6×10^{-4}	3.0×10^{-7}	9.9×10^{-5}	9.8×10^{-6}	4.4×10^{-4}	2.2×10^{-5}
Italy	α	0.40	0.65	0.55	0.67	0.58	0.59	0.58	0.45	0.53	0.99
	γ	0.55	0.60	0.93	0.38	0.33	0.80	0.92	0.57	0.97	0.74
	MAPE	1.9×10^{-5}	9.5×10^{-7}	1.0×10^{-5}	1.9×10^{-6}	4.8×10^{-6}	6.9×10^{-7}	3.1×10^{-4}	5.6×10^{-4}	5.9×10^{-4}	1.1×10^{-4}
Japan	α	0.42	0.77	0.68	0.67	0.58	0.60	0.38	0.48	0.51	0.99
	γ	0.45	0.42	0.89	0.29	0.35	0.55	0.72	0.32	0.99	0.60
	MAPE	1.4×10^{-6}	1.3×10^{-6}	1.0×10^{-4}	1.3×10^{-5}	1.0×10^{-8}	8.1×10^{-7}	1.4×10^{-5}	7.2×10^{-7}	1.5×10^{-5}	1.3×10^{-5}
Türkiye	α	0.62	0.46	0.64	0.62	0.67	0.51	0.78	0.70	0.50	0.88
	γ	0.67	0.58	1.00	0.36	0.44	0.81	1.00	0.81	0.93	0.79
	MAPE	2.5×10^{-8}	9.6×10^{-7}	1.9×10^{-6}	1.7×10^{-5}	1.0×10^{-4}	7.0×10^{-6}	7.7×10^{-3}	5.4×10^{-4}	5.5×10^{-2}	1.1×10^{-3}

Abbreviations: MAPE, mean absolute percentage error; FDI, foreign direct investment; Unemp., unemployment.

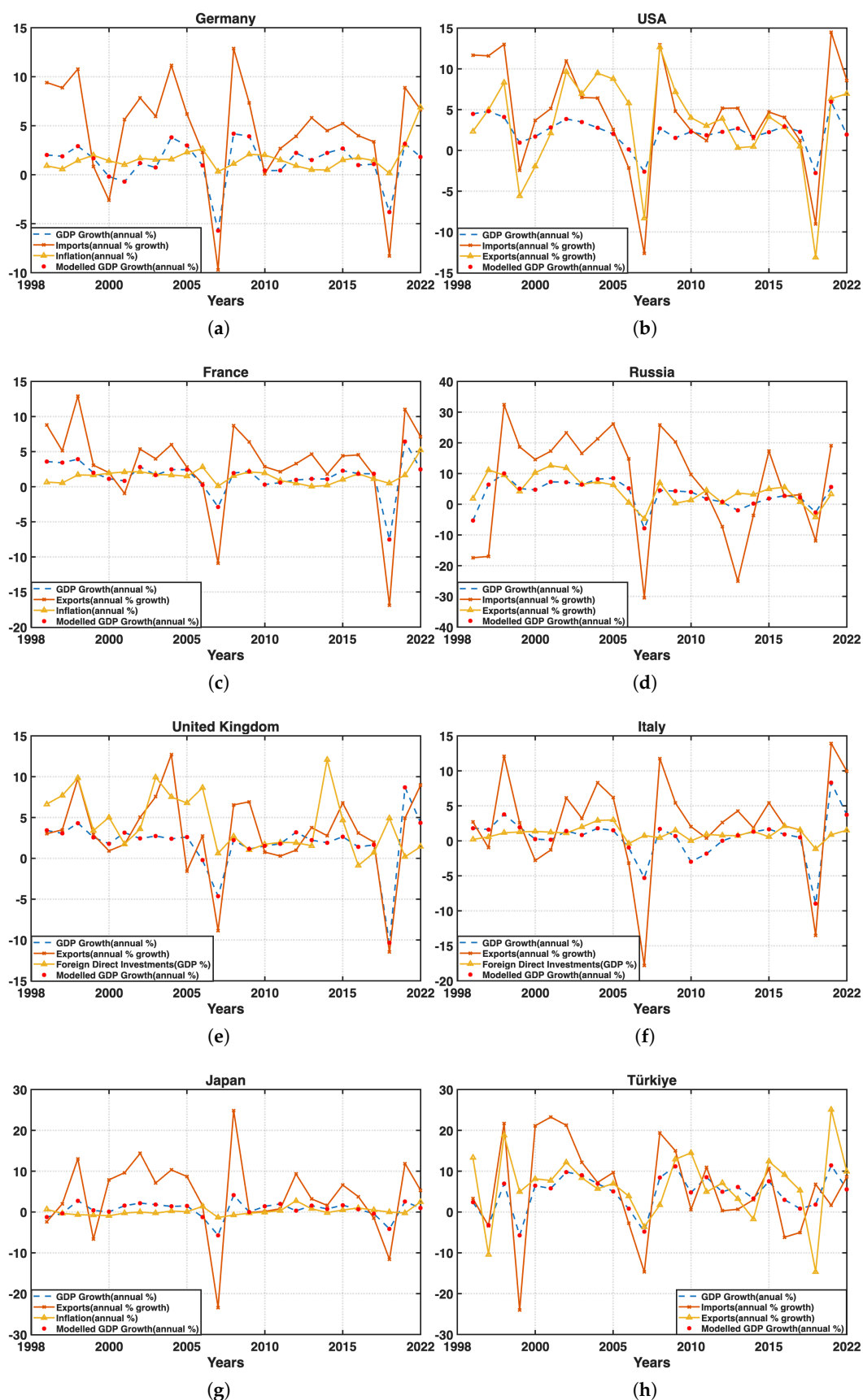


Figure 2. Cont.

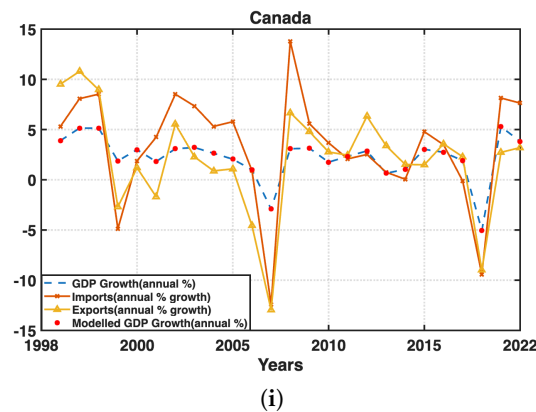


Figure 2. Combinations yielding the lowest MAPE results [36]. (a) Import–inflation for Germany; (b) Import–export for USA; (c) Export–inflation for France; (d) Import–export for Russia; (e) Export–foreign direct investment for United Kingdom; (f) Export–foreign direct investment for Italy; (g) Export–inflation for Japan; (h) Import–export for Türkiye; (i) Import–export for Canada.

Table 3. Overall assessment of the results (MAPE, %).

Countries	Combination Yielding the Highest MAPE (%)	Combination Yielding the Lowest MAPE (%)
Canada	Unemployment Rate–Foreign Direct Investment	Imports–Exports
France	Imports–Unemployment Rate	Exports–Inflation
Russia	Inflation–Unemployment Rate	Imports–Exports
Germany	Inflation–Foreign Direct Investment	Imports–Inflation
Japan	Imports–Unemployment Rate	Exports–Inflation
Italy	Inflation–Unemployment Rate	Exports–Foreign Direct Investment
United Kingdom	Imports–Inflation	Exports–Foreign Direct Investment
Türkiye	Inflation–Unemployment Rate	Imports–Exports
United States	Inflation–Unemployment Rate	Imports–Exports

Abbreviation: MAPE, mean absolute percentage error.

4. Conclusions

This study proposed a novel two-variable fractional calculus framework for modeling GDP growth rates using macroeconomic indicators. The proposed methodology extends classical integer-order formulations by incorporating fractional orders and provides a flexible mathematical structure capable of representing complex nonlinear relationships between economic variables. The model was applied to G8 countries and Türkiye using combinations of imports, exports, inflation, unemployment, and foreign direct investment variables.

The empirical results demonstrated that the proposed framework achieved very low approximation errors across all countries. The findings revealed that trade-related variables generally provided the most successful models. In particular, the import–export combination produced the lowest MAPE values for the United States, Canada, Russia,

and Türkiye, highlighting the importance of international trade, external demand, and economic integration in explaining economic growth. For Japan and France, export–inflation combinations yielded the best performance, suggesting that both external competitiveness and price dynamics play important roles in the growth process. In contrast, combinations involving unemployment frequently produced higher errors, indicating that labor market indicators alone may not capture GDP growth dynamics as effectively as trade- and investment-related variables.

The estimated fractional orders were generally different from unity, indicating that the relationships between macroeconomic variables cannot always be adequately represented by classical integer-order formulations. Rather than interpreting the fractional orders as direct quantitative measures of economic memory, they may be viewed as parameters controlling the flexibility of the functional structure and the degree of departure from the corresponding integer-order model. From an economic perspective, the obtained results suggest that GDP growth dynamics may involve cumulative and delayed interactions among macroeconomic variables, which can be represented more effectively through a fractional-order framework.

The results are broadly consistent with established macroeconomic growth theories emphasizing the roles of international trade, investment, productivity, and capital accumulation in economic development. The findings further suggest that country-specific economic structures influence which variable combinations are most effective in explaining growth dynamics. Consequently, policymakers may benefit from considering the dominant growth drivers specific to each economy when designing economic development strategies.

Overall, the proposed fractional-order framework provides a flexible and effective approach for modeling GDP growth dynamics and offers a promising foundation for future studies involving economic forecasting, policy analysis, and multidimensional macroeconomic modeling.

Author Contributions: The contribution of each author is listed as follows. E.K. and N.Ö.Ö.T. played an important role in supervision, methodology, editing, and validation. V.T. played an important role in resources, supervision, and validation. Ş.B. was the key person in investigation, writing, conceptualization, data curation, and visualization. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Scientific Research Projects Department of Istanbul Technical University. Project Number: THD-2026-48510.

Data Availability Statement: The original contributions presented in this study are included in Appendix A of the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest. The funders played no role in the design of the study, in the collection, analyses, or interpretation of data, in the writing of the manuscript, or in the decision to publish the results.

Abbreviations

The following abbreviations are used in this manuscript:

FDI	Foreign Direct Investment
G8	Group of Eight
GDP	Gross Domestic Product
MAPE	Mean Absolute Percentage Error
USA	United State of America

Appendix A

Table A1. Canada economic factors data for the period 1998–2022 [35].

Years	GDP Growth (Annual %)	Inflation, Consumer Prices (Annual %)	Exports of Goods and Services (Annual % Growth)	Imports of Goods and Services (Annual % Growth)	Unemployment, Total (% of Total Labor Force)	Foreign Direct Investment, Net Inflows (% of GDP)
1998	3.89288653	0.995942457	9.519735558	5.309811948	8.28	3.939938217
1999	5.141003	1.734843	10.79228	8.076368	7.58	4.047104
2000	5.138539	2.71944	8.965365	8.52418	6.83	9.171024
2001	1.875098	2.52512	−2.69394	−4.87961	7.22	3.841921
2002	2.999255	2.258394	1.200489	1.882049	7.66	3.219148
2003	1.806385	2.758563	−1.69176	4.241937	7.57	0.782968
2004	3.092364	1.857259	5.558726	8.520537	7.19	0.141466
2005	3.210454	2.213552	2.280533	7.340141	6.76	2.178164
2006	2.637944	2.002025	0.886689	5.316006	6.48	4.874101
2007	2.049905	2.138384	1.083612	5.783604	6.16	8.200559
2008	0.995406	2.370271	−4.55393	0.912584	6.28	4.515137
2009	−2.91509	0.299467	−12.9503	−12.4122	8.46	1.524131
2010	3.090806	1.776872	6.675848	13.78412	8.18	1.837256
2011	3.137194	2.912135	4.797254	5.592304	7.64	2.137833
2012	1.755661	1.515678	2.764197	3.677597	7.39	2.700169
2013	2.325814	0.938292	2.453414	2.081262	7.14	3.629804
2014	2.873467	1.906636	6.319079	2.51653	7.02	3.553903
2015	0.649971	1.125241	3.398837	0.759808	6.95	3.853895
2016	1.038551	1.42876	1.525246	0.052145	7.04	2.23835
2017	3.033835	1.596884	1.490838	4.781136	6.43	1.537521
2018	2.742963	2.268226	3.574122	3.497829	5.84	2.469312
2019	1.908432	1.949269	2.286556	−0.10816	5.69	2.806767
2020	−5.03823	0.717	−8.98088	−9.41716	9.66	1.835165
2021	5.286957	3.395193	2.729318	8.137415	7.53	3.224305
2022	3.819866	6.802801	3.200737	7.638415	5.28	2.484647

Table A2. France economic factors data for the period of 1998–2022 [35].

Years	GDP Growth (Annual %)	Inflation, Consumer Prices (Annual %)	Exports of Goods and Services (Annual % Growth)	Imports of Goods and Services (Annual % Growth)	Unemployment, Total (% of Total Labor Force)	Foreign Direct Investment, Net Inflows (% of GDP)
1998	3.588659425	0.651126868	8.812540704	11.90040808	12.07	1.963825255
1999	3.421373799	0.537141639	5.111921499	6.950599802	11.98	3.078804070
2000	3.923669227	1.675959887	12.88554421	15.35348874	10.22	3.030746972
2001	1.983721419	1.634780795	3.072397094	2.381767891	8.61	3.639660784
2002	1.135531482	1.923412287	1.997254116	1.945585270	8.70	3.433472871
2003	0.823160757	2.098472191	−0.958891771	0.893617042	8.31	2.296002833
2004	2.829752929	2.142089646	5.358050642	6.204036885	8.91	1.679938591
2005	1.663219980	1.745869364	3.975619735	6.320066914	8.88	3.875247351
2006	2.449323601	1.675124496	5.997733577	5.600584028	8.83	3.402534856
2007	2.424736243	1.487998060	2.791189943	5.755701820	8.01	3.148833873
2008	0.254945960	2.812861949	0.440686648	1.295002635	7.39	2.320491389
2009	−2.873313828	0.087620478	−10.89804241	−9.354562790	9.12	0.682106492
2010	1.949437623	1.531122704	8.695173397	8.892522592	9.28	1.470195727
2011	2.192700633	2.111597952	6.363889724	5.837368345	9.23	1.542875408
2012	0.313134751	1.954195316	2.842135544	0.200223647	9.84	1.227596621
2013	0.576326675	0.863715498	2.140711692	2.418481061	9.91	1.123423424
2014	0.956183052	0.507758823	3.272421028	4.898686390	10.27	0.203254563
2015	1.112912341	0.037514381	4.645919204	5.898053683	10.35	1.755723741
2016	1.095464404	0.183334861	1.775862602	2.928765377	10.06	1.326491698
2017	2.291419994	1.032282751	4.385892502	4.486842018	9.41	1.382120390
2018	1.865066071	1.850815083	4.526985173	3.079156125	9.02	2.776578144
2019	1.842971814	1.108254923	1.618733498	2.344320626	8.41	1.960494002
2020	−7.540459162	0.476498853	−16.90380968	−12.32449670	8.01	0.731611887
2021	6.435209621	1.642331410	11.03185559	9.389210589	7.86	3.239382005
2022	2.454758177	5.222367484	7.114197818	8.558675308	7.31	3.793503383

Table A3. Germany economic factors data for the period of 1998–2022 [35].

Years	GDP Growth (Annual %)	Inflation, Consumer Prices (Annual %)	Exports of Goods and Services (Annual % Growth)	Imports of Goods and Services (Annual % Growth)	Unemployment, Total (% of Total Labor Force)	Foreign Direct Investment, Net Inflows (% of GDP)
1998	2.013932785	0.911183535	7.678197834	9.413686746	9.79	1.318741891
1999	1.887261154	0.585433064	5.281090214	8.881298817	8.85	3.919717995
2000	2.912502964	1.440268387	13.82339190	10.78431373	7.92	12.73150359
2001	1.681468481	1.983857313	5.666125509	0.849557820	7.77	2.926755399
2002	−0.197973835	1.420806156	4.209109950	−2.614953492	8.48	2.466615193
2003	−0.700116686	1.034222186	1.899669092	5.640656311	9.78	2.614345224
2004	1.175088132	1.665736967	11.52941141	7.830093001	10.73	−0.725154707
2005	0.731707164	1.546911148	6.686140019	5.964245866	11.17	2.101793081
2006	3.816441913	1.577426428	12.29084333	11.15258258	10.25	2.919960360
2007	2.976455131	2.298344007	8.886480278	6.205506826	8.66	1.484338647
2008	0.959879134	2.628379816	1.940781854	2.301757728	7.52	0.826503411
2009	−5.693836336	0.312739007	−14.27874119	−9.692174183	7.74	1.662198034
2010	4.179882499	1.103810378	14.40774522	12.86789870	6.97	2.530762023
2011	3.925192705	2.075172837	8.349925120	7.325651503	5.82	2.601419304
2012	0.418497594	2.008488848	2.905708724	0.101706978	5.38	1.855413408
2013	0.437591303	1.504723303	1.004463968	2.675547333	5.23	1.799764604
2014	2.209543431	0.906794000	4.795580079	3.925233460	4.98	0.502227193
2015	1.491931528	0.514426137	5.440742334	5.797714957	4.62	1.860729182
2016	2.229999868	0.491747008	2.470000127	4.489999664	4.12	1.865906917
2017	2.680231114	1.509494851	4.898994739	5.225380758	3.75	2.966416856
2018	0.981232606	1.732168798	2.223462424	3.992723660	3.38	4.199721800
2019	1.075471636	1.445659769	2.275209323	3.358405209	3.14	1.902059228
2020	−3.826768716	0.144877926	−9.254315480	−8.283974014	3.86	4.136109564
2021	3.163819942	3.066666667	9.668562345	8.884583631	3.64	2.241530947
2022	1.806208967	6.872574386	3.308297443	6.634468886	3.14	1.160447552

Table A4. Japan economic factors data for the period of 1998–2022 [35].

Years	GDP Growth (Annual %)	Inflation, Consumer Prices (Annual %)	Exports of Goods and Services (Annual % Growth)	Imports of Goods and Services (Annual % Growth)	Unemployment, Total (% of Total Labor Force)	Foreign Direct Investment, Net Inflows (% of GDP)
1998	−1.270330495	0.661974191	−2.423660920	−6.803898397	4.08	0.061293304
1999	−0.333929958	−0.341296928	2.000582052	3.702456236	4.68	0.318821542
2000	2.764647551	−0.676578684	13.02389320	9.562400674	4.75	0.215124715
2001	0.386103426	−0.740055504	−6.633823934	1.162947380	5.02	0.112602474
2002	0.041962499	−0.923494027	7.892537607	0.787279420	5.39	0.276304070
2003	1.535125499	−0.256541816	9.629038995	3.393965213	5.25	0.194079344
2004	2.186115694	−0.008573388	14.37086551	8.504060150	4.73	0.153847736
2005	1.803900872	−0.282946069	7.119734240	5.945933211	4.45	0.113001254
2006	1.372350128	0.249355116	10.30604918	4.704135626	4.19	−0.052087901
2007	1.483969412	0.060039454	8.703067736	2.270819174	3.89	0.472322727
2008	−1.224289001	1.380078862	1.587887261	0.732843712	4.00	0.482208589
2009	−5.693236359	−1.352836730	−23.38277052	−15.56032021	5.07	0.231146375
2010	4.097917919	−0.728243208	24.86073710	11.28551363	5.10	0.129204490
2011	0.023809524	−0.272455616	−0.119600974	5.727995086	4.55	−0.013648274
2012	1.374750999	−0.044064510	0.145733293	5.453191658	4.36	0.008720202
2013	2.005100177	0.335037912	0.810553199	3.169043430	4.04	0.204293384
2014	0.296205514	2.759226714	9.341806747	8.124892011	3.59	0.403354543
2015	1.560626697	0.795279631	3.211234756	0.439964097	3.39	0.118161988
2016	0.753826746	−0.127258844	1.618284418	−1.175364557	3.13	0.818481615
2017	1.675331752	0.484199796	6.619072394	3.306787319	2.82	0.381319638
2018	0.643391023	0.989094598	3.759599700	3.813561895	2.47	0.501685482
2019	−0.402169201	0.468776159	−1.458490573	0.988306807	2.35	0.780785313
2020	−4.147118899	−0.024995834	−11.59467703	−6.751026751	2.81	1.237931782
2021	2.559320239	−0.233352779	11.87217693	5.142160374	2.83	0.688616709
2022	0.954736938	2.497702782	5.283076875	7.885840298	2.60	1.116490156

Table A5. Türkiye economic factors data for the period of 1998–2022 [35].

Years	GDP Growth (Annual %)	Inflation, Consumer Prices (Annual %)	Exports of Goods and Services (Annual % growth)	Imports of Goods and Services (Annual % Growth)	Unemployment, Total (% of Total Labor Force)	Foreign Direct Investment, Net Inflows (% of GDP)
1998	2.404150257	84.64134348	13.33234128	3.286715668	6.55	0.340651308
1999	−3.263168406	64.86748764	−10.38817601	−3.408019795	7.45	0.305387060
2000	6.933239705	54.91537058	18.80470507	21.70923810	6.30	0.358009205
2001	−5.750006555	54.40018876	4.931829189	−24.03027802	8.37	1.661436481
2002	6.447722047	44.96412095	8.141235321	21.13008473	10.35	0.450365944
2003	5.763206067	21.60243845	7.708244162	23.24930677	10.55	0.541012065
2004	9.795936389	8.598261681	12.16385797	21.23146662	10.83	0.681153209
2005	8.992304936	8.179160368	8.331541186	12.14211626	10.63	1.981178830
2006	6.947988086	9.597242123	5.701051915	7.239166693	8.72	3.623382498
2007	5.043507932	8.756180910	6.976941575	9.641389265	8.87	3.235919042
2008	0.815024573	10.44412838	3.883589792	−2.760215764	9.71	2.576549074
2009	−4.823153953	6.250976631	−3.616906527	−14.66170090	12.55	1.322214870
2010	8.427104323	8.566444206	1.721515212	19.36516604	10.66	1.171091807
2011	11.20011058	6.471879671	12.92105453	14.97768739	8.80	1.929218383
2012	4.788492711	8.891569965	14.49775413	0.618577567	8.15	1.560832223
2013	8.485816997	7.493090305	4.932653541	10.92074377	8.73	1.416058933
2014	4.939715161	8.854572714	7.077929125	0.299314890	9.88	1.420439706
2015	6.084486904	7.670853648	3.238798440	0.662400830	10.24	2.228704409
2016	3.323084208	7.775134153	−1.741888790	2.995930839	10.84	1.590809741
2017	7.501997489	11.14431108	12.40929144	10.63507619	10.82	1.302694983
2018	3.013179908	16.33246390	9.090606721	−6.172328997	10.89	1.598259861
2019	0.818525936	15.17682157	5.278520453	−5.011971152	13.67	1.254786515
2020	1.859839690	12.27895745	−14.62375264	6.774326834	13.11	1.068941896
2021	11.43938474	19.59649269	25.05632475	1.652617480	11.98	1.625267072
2022	5.533469266	72.30883599	9.932072045	8.567890749	10.43	1.443471931

Table A6. Italy economic factors data for the period of 1998–2022 [35].

Years	GDP Growth (Annual %)	Inflation, Consumer Prices (Annual %)	Exports of Goods and Services (Annual % Growth)	Imports of Goods and Services (Annual % Growth)	Unemployment, Total (% of Total Labor Force)	Foreign Direct Investment, Net Inflows (% of GDP)
1998	1.810615162	1.955085577	2.694132503	9.035771109	12.12	0.207870545
1999	1.625727599	1.663459950	−0.978272835	4.494825141	11.68	0.554256596
2000	3.786955143	2.537685321	12.08464520	10.99081820	10.83	1.148796315
2001	1.951371556	2.785165427	2.571156434	2.430980883	9.60	1.273783228
2002	0.253942999	2.465323192	−2.804626466	0.562751835	9.21	1.350232927
2003	0.138626891	2.672555528	−1.298363609	0.739811925	8.87	1.240657798
2004	1.423594158	2.206736614	6.129574660	4.573083886	7.87	1.112663427
2005	0.817848974	1.985292985	3.216572869	3.440307948	7.73	1.978351837
2006	1.790639681	2.090843910	8.310327372	7.958543290	6.78	2.923517470
2007	1.487072980	1.829741122	6.197332434	5.454208410	6.08	2.981133303
2008	−0.962012841	3.347832584	−3.169132053	−4.024321844	6.72	−0.394476496
2009	−5.280937208	0.774768131	−17.81641254	−13.11816502	7.75	0.754896991
2010	1.713295839	1.525516021	11.76156395	12.16685880	8.36	0.464889359
2011	0.707333347	2.780632729	5.419871476	0.563230920	8.36	1.501767646
2012	−2.980905768	3.041363332	2.032309504	−8.119492766	10.65	0.001672340
2013	−1.841065451	1.219993423	0.374187156	−2.702130672	12.15	0.911862937
2014	−0.004547542	0.241047430	2.615222878	3.452286711	12.68	0.787819995
2015	0.778304351	0.038790400	4.274025708	6.463675291	11.90	0.724336604
2016	1.293462732	−0.094016657	1.865502485	3.927074150	11.69	1.366845175
2017	1.667859041	1.226533166	5.436440978	6.091371221	11.21	0.567766927
2018	0.925810941	1.137487636	2.141987139	3.438160066	10.61	2.115255482
2019	0.483198317	0.611246944	1.573533126	−0.699962528	9.95	1.550498698
2020	−8.974192121	−0.137707574	−13.50762273	−12.12805201	9.16	−1.164236567
2021	8.313760318	1.873783258	13.91546899	15.13264056	9.50	0.877417111
2022	3.724549463	8.201289912	9.935450135	12.44548685	8.07	1.523516081

Table A7. United Kingdom economic factors data for the period of 1998–2022 [35].

Years	GDP Growth (Annual %)	Inflation, Consumer Prices (Annual %)	Exports of Goods and Services (Annual % Growth)	Imports of Goods and Services (Annual % Growth)	Unemployment, Total (% of Total Labor Force)	Foreign Direct Investment, Net Inflows (% of GDP)
1998	3.414269874	1.820561637	3.016896441	7.460098679	6.20	6.640741936
1999	3.066176125	1.752950801	3.496443230	7.396796096	6.04	7.695772086
2000	4.318727058	1.182956242	9.758078589	9.433623040	5.56	9.854512069
2001	2.578741046	1.532349603	2.888452579	6.180363116	4.70	3.399805910
2002	1.791985616	1.520402459	0.920754256	4.719181809	5.04	5.026548637
2003	3.146572528	1.376500385	1.706870740	1.646840042	4.81	1.752864088
2004	2.426496549	1.390397567	5.037683145	6.714329421	4.59	3.595239298
2005	2.738842404	2.089136490	7.555619161	7.029342871	4.75	9.934546400
2006	2.386275368	2.455661664	12.72099798	10.56340018	5.35	7.518583129
2007	2.620331222	2.386561508	−1.565916453	−1.374963211	5.26	6.779300045
2008	−0.225726975	3.521408563	2.733352595	−1.008840487	5.62	8.652048202
2009	−4.614957402	1.961731736	−8.840087550	−7.024467627	7.54	0.602903981
2010	2.238688634	2.492654725	6.545463701	8.963955836	7.79	2.684973590
2011	1.145101541	3.856112447	6.928212749	2.928354202	8.04	1.014039752
2012	1.512936749	2.573234797	0.760115582	2.454003218	7.88	1.726975994
2013	1.792322216	2.291666667	0.284523361	3.243248021	7.52	1.956051740
2014	3.195782466	1.451120163	1.004984746	5.095061173	6.11	1.921567748
2015	2.220258322	0.368046842	3.773421854	4.955662331	5.30	1.548321685
2016	1.921063414	1.008417368	2.774381153	3.976550236	4.81	12.07885281
2017	2.655069668	2.557755776	6.765555799	3.164551301	4.33	4.677309144
2018	1.403792086	2.292839903	3.086065102	3.098049365	4.00	−0.872604334
2019	1.641610996	1.738104601	1.996940273	2.718045825	3.74	0.694070008
2020	−10.35990082	0.989486704	−11.46971253	−16.00264011	4.472	4.910508973
2021	8.674904114	2.518371096	4.941180142	6.142373380	4.826	0.188516351
2022	4.346561401	7.922048831	8.970320917	14.60775279	3.73	1.428483802

Table A8. United States economic factors data for the period of 1998–2022 [35].

Years	GDP Growth (Annual %)	Inflation, Consumer Prices (Annual %)	Exports of Goods and Services (Annual % Growth)	Imports of Goods and Services (Annual % Growth)	Unemployment, Total (% of Total Labor Force)	Foreign Direct Investment, Net Inflows (% of GDP)
1998	4.481394432	1.552279099	2.335725411	11.68874025	4.51	2.329849229
1999	4.794499045	2.188027197	4.979719763	11.59421977	4.22	3.244142242
2000	4.077159479	3.376857271	8.313501816	12.99954878	3.99	3.405782569
2001	0.954338729	2.826171119	−5.582093417	−2.461316877	4.73	1.630099648
2002	1.695942923	1.586031627	−1.953701476	3.669427333	5.78	1.016139191
2003	2.796209101	2.270094973	2.106482227	5.124490105	5.99	1.022184720
2004	3.852552601	2.677236693	9.643873800	10.98146759	5.53	1.748691345
2005	3.483220320	3.392746845	6.947477023	6.491207075	5.08	1.091662133
2006	2.782810626	3.225944101	9.475888558	6.420254879	4.62	2.160335287
2007	2.010507552	2.852672482	8.761360384	2.563232756	4.62	2.394690938
2008	0.122188443	3.839100297	5.789743535	−2.147047704	5.78	2.309379021
2009	−2.599888351	−0.355546266	−8.312533832	−12.62753166	9.25	1.112600342
2010	2.708856694	1.640043442	12.64900205	12.96908794	9.63	1.754532686
2011	1.549894950	3.156841569	7.178158379	4.816092973	8.95	1.689112771
2012	2.280687603	2.069337265	4.018170766	2.448369650	8.07	1.540208119
2013	1.841875395	1.464832656	3.010343739	1.213003744	7.37	1.710667534
2014	2.287775933	1.622222977	3.902322561	5.160977388	6.17	1.435021307
2015	2.706369582	0.118627136	0.319626118	5.190803661	5.28	2.809147629
2016	1.667472076	1.261583206	0.474057153	1.456628045	4.87	2.537497659
2017	2.241921216	2.130110004	4.114380824	4.710078144	4.36	1.955210863
2018	2.945384831	2.442583297	2.854086253	4.037844308	3.90	1.045703992
2019	2.294439078	1.812210075	0.511353288	1.197293599	3.67	1.477874528
2020	−2.767802511	1.233584396	−13.12944921	−9.014446535	8.05	0.656979527
2021	5.945484761	4.697858864	6.343282895	14.47246506	5.35	2.114875815
2022	1.935634592	8.002799821	6.959662795	8.584107930	3.65	1.525481826

Table A9. Russia economic factors data for the period of 1998–2022 [35].

Years	GDP Growth (Annual %)	Inflation, Consumer Prices (Annual %)	Exports of Goods and Services (Annual % Growth)	Imports of Goods and Services (Annual % Growth)	Unemployment, Total (% of Total Labor Force)	Foreign Direct Investment, Net Inflows (% of GDP)
1998	−5.299961627	27.68567981	1.899743465	−17.40023183	13.26	1.019119425
1999	6.399914691	85.74649410	11.19956454	−17.00012028	13.04	1.662282546
2000	10.00006681	20.79876066	9.499479899	32.39940100	10.58	1.031161116
2001	5.100051226	21.47700721	4.200311071	18.69892002	8.98	0.928663004
2002	4.699991909	15.78873079	10.30004558	14.60055943	7.88	1.005535945
2003	7.299952345	13.66329302	12.59977148	17.30010997	8.21	1.842378885
2004	7.199947870	10.88861573	11.80013818	23.29933917	7.76	2.606185281
2005	6.399965448	12.68530395	6.500035399	16.59974964	7.12	2.029807038
2006	8.200068255	9.668654548	7.300046543	21.29991173	7.06	3.797712094
2007	8.499977769	9.007298689	6.299815807	26.19992001	6.00	4.298955948
2008	5.199969264	14.11076778	0.599852686	14.79994181	6.21	4.502694249
2009	−7.799993913	11.64732958	−4.700150125	−30.40007602	8.30	2.992125520
2010	4.499999999	6.849392303	7.000254252	25.79971913	7.37	2.830828698
2011	4.300029186	8.440464859	0.300123460	20.30028167	6.54	2.692361181
2012	4.024086157	5.074743008	1.360421664	9.681343576	5.44	2.290798699
2013	1.755422149	6.753710262	4.561727799	3.539025298	5.46	3.019402114
2014	0.736267222	7.823411839	0.510432046	−7.260442493	5.16	1.069876410
2015	−1.972719226	15.53440505	3.673262303	−25.03877488	5.57	0.502607961
2016	0.193690071	7.042447630	3.168854896	−3.672042601	5.56	2.548499988
2017	1.825790064	3.683329444	5.013643181	17.31519269	5.21	1.814092975
2018	2.807245411	2.878297236	5.554217178	2.656747628	4.85	0.530060791
2019	2.198075713	4.470366608	0.732544576	3.130357973	4.50	1.888517316
2020	−2.653654501	3.381659372	−4.170377317	−11.88112458	5.59	0.634851184
2021	5.614290376	6.694458920	3.280495435	19.11134579	4.72	2.202088546
2022	−2.069711525	−	−13.88677765	−14.99000977	3.867	−1.787319191

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