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Asymmetric Cross-Iterate Ćirić–Reich–Rus Contraction: Existence, Uniqueness, and Comparative Analysis

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Abstract

We introduce a novel generalization of the classical Ćirić–Reich–Rus (CRR) contraction, termed the Asymmetric Cross-Iterate CRR (ACI-CRR). The contraction condition involves distinct iterate orders $p \neq q$ in an asymmetric cross-coupled form. We establish the existence and uniqueness of a fixed point under the assumption of continuity. We then analyze the symmetric case $p = q$, demonstrating that continuity is indispensable for $p > 1$, and contrast this with the classical $p = q = 1$ case where continuity is unnecessary. We conclude with a structural example and a conjecture regarding the necessity of continuity in higher-iterate asymmetric contractions.

Keywords: fixed point theory; Ćirić–Reich–Rus (CRR) contraction; Asymmetric Cross-Iterate CRR (ACI-CRR); higher-iterate asymmetric contractions; complete metric spaces; comparative analysis; nonlinear analysis

MSC: 47H10; 54H25; 47H09

1. Introduction

Fixed-point theory stands as one of the fundamental pillars of modern mathematical analysis, playing a pivotal role in studying the existence and stability of solutions for a wide range of differential, integral, and partial differential equations. Since Banach [1] introduced his celebrated principle in 1922, which states that every contraction mapping on a complete metric space possesses a unique fixed point, the theory has undergone remarkable development over the decades. Numerous mathematicians have contributed to extending the scope of this principle to encompass broader classes of mappings, among them Ćirić [2], Reich [3], and Rus [4], who introduced a significant generalization now known as the Ćirić–Reich–Rus (CRR) contraction, which incorporates additional terms involving images of the points themselves.

In this context, this research paper aims to introduce an innovative generalization of the classical CRR contraction concept, which we designate as the “Asymmetric Cross-Iterate Ćirić–Reich–Rus contraction” (ACI-CRR). This novel generalization is characterized by a contraction condition that relies on distinct iterative orders $p \neq q$ in an asymmetric



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cross-coupled form, thereby breaking the traditional symmetry and introducing directional dependence in the contraction mechanism. This innovation enables the modeling of systems with multi-level memory, distributed delays, and hierarchical feedback—phenomena commonly encountered in numerous scientific and engineering applications.

The paper systematically investigates this new class of contractions. First, we establish the existence and uniqueness of the fixed point under the assumption of continuity of the mapping T when $p \neq q$ (Theorem 1). We then analyze the symmetric case $p = q$, demonstrating that continuity is indispensable when $p > 1$, while it can be dispensed with in the classical case $p = q = 1$ (Theorem 2). This subtle distinction reveals a significant structural threshold in the theory of higher-iterate contractions.

Furthermore, we provide a Ulam–Hyers stability analysis [5–7] for this new framework (Theorem 3), with an explicit computation of the stability constant that depends on the iteration depths p and q , illustrating how stability deteriorates linearly with the iterative depth q and the weight c .

Additionally, we strengthen the theoretical foundations of the proposed framework by embedding it within the general theory of retraction–displacement conditions [8]. This approach provides a unified and elegant proof of Ulam–Hyers stability, establishes Tykhonov well-posedness of the fixed-point problem, and offers deep insight into the structural role of continuity in higher-iterate contractions. Specifically, we demonstrate that the retraction–displacement condition holds for the ACI–CRR contraction in both the classical symmetric case $p = q = 1$ (without any continuity assumption) and the asymmetric/higher-iterate case under continuity and the strengthened condition $a + bp + cq < 1$. This analysis not only confirms the Ulam–Hyers stability results obtained in Section 5 but also reveals why continuity is indispensable when $p, q > 1$ or $p \neq q$, while being unnecessary in the classical setting $p = q = 1$.

A comparative analysis between the Ulam–Hyers stability and retraction–displacement approaches is also provided, highlighting their complementary roles and the relative sharpness of their respective stability constants.

We also apply the theoretical results to integral equations of Volterra, Hammerstein, and perturbed types (Theorems 5–7), demonstrating the practical importance of the proposed framework.

We conclude the study with a comparative analysis highlighting the advantages of the new framework relative to the classical and symmetric cases, accompanied by a structural example illustrating the ability of asymmetric contraction to enforce strong contraction even in the presence of discontinuous mappings. Finally, we propose a conjecture (Section 10) regarding the necessity of continuity in the case $p \neq q$ without continuity, opening new avenues for research into weakened continuity conditions.

The paper is organized as follows: Section 2 presents the fundamental definitions and preliminaries. Section 3 addresses the first main result concerning existence and uniqueness of the fixed point with continuity. Section 4 discusses the symmetric case $p = q$ and establishes the necessity of continuity. Section 5 is devoted to the Ulam–Hyers stability analysis. Section 6 presents a retraction–displacement analysis of the ACI–CRR framework, establishing Tykhonov well-posedness and providing deeper insight into the role of continuity, along with a comparative analysis between the two stability approaches. Section 7 presents applications to integral equations. Section 8 discusses the theoretical implications and advantages of the new framework. Section 9 presents a comparative example, while Section 10 presents a conjecture. Section 11 concludes with findings and future research directions.

2. Preliminaries

Let (X, d) be a metric space and $T : X \rightarrow X$ a self-mapping. The classical Ćirić–Reich–Rus theorem [2–4,9,10] extends the Banach contraction principle [1] by incorporating terms involving the images of the points themselves. In this work, we propose a strictly more general framework where the contraction condition couples the points with *different iterate orders* in an asymmetric manner.

Definition 1 (Asymmetric Cross-Iterate CRR Contraction). *A mapping $T : X \rightarrow X$ is called an Asymmetric Cross-Iterate Ćirić–Reich–Rus (ACI-CRR) contraction if there exist constants $a, b, c \geq 0$ with $a + b + c < 1$, and integers $p, q \geq 1$ with $p \neq q$, such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq a d(x, y) + b d(x, T^p y) + c d(y, T^q x), \quad (1)$$

where T^k denotes the k -fold composition of T .

The asymmetry $p \neq q$ breaks the reciprocal iterate coupling found in standard generalizations and introduces directional dependence in the contraction mechanism.

3. Main Result I: Existence and Uniqueness with Continuity ($p \neq q$)

When the mapping T is assumed to be continuous, the verification of the fixed-point property becomes direct, as the limit of the Picard sequence commutes with T .

Theorem 1. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ a continuous ACI-CRR contraction with $p \neq q$. Then T possesses a unique fixed point $z \in X$. Moreover, for any $x_0 \in X$, the Picard iteration sequence $x_{n+1} = Tx_n$ converges to z .*

Proof. *Step 1: Uniqueness*

Assume $u, v \in X$ are fixed points. Substituting them into (1) yields

$$d(u, v) \leq a d(u, v) + b d(u, T^p v) + c d(v, T^q u) = (a + b + c)d(u, v).$$

Given $a + b + c < 1$, we must have $d(u, v) = 0$; hence $u = v$.

Step 2: Cauchyness of the Picard Sequence

Fix $x_0 \in X$ and define $x_{n+1} = Tx_n$. Let

$$D = \limsup_{n, m \rightarrow \infty} d(x_n, x_m).$$

We will show that $D = 0$, which implies that D is finite. Applying (1) to (x_n, x_m) gives

$$d(x_{n+1}, x_{m+1}) \leq a d(x_n, x_m) + b d(x_n, x_{m+p}) + c d(x_m, x_{n+q}).$$

Taking $\limsup_{n, m \rightarrow \infty}$ and using the shift invariance of $\limsup$ for sequences in metric spaces yields

$$D \leq (a + b + c)D.$$

Since $a + b + c < 1$, this inequality forces $D = 0$. Thus D is finite. Consequently, $\{x_n\}$ is Cauchy and converges to some $z \in X$.

Step 3: Verification that z is a Fixed Point

Since $x_n \rightarrow z$ and T is continuous, $Tx_n \rightarrow Tz$. But $Tx_n = x_{n+1} \rightarrow z$. By uniqueness of limits in metric spaces, $Tz = z$. $\square$

4. Main Result II: The Symmetric Case $p = q$ Without Continuity

When $p = q$, the cross-terms are symmetric. However, as identified in the analysis of higher-iterate contractions, continuity is strictly necessary for $p > 1$ to guarantee $Tz = z$. The only case where continuity can be dropped is the classical baseline $p = q = 1$. We present both to highlight the structural threshold.

Theorem 2 recalls the classical Ćirić–Reich–Rus theorem (with $p = q = 1$) [2–4], which is included here to establish a baseline for comparison with the higher-iterate and asymmetric cases, and to highlight the structural threshold at which continuity becomes essential.

Theorem 2. Let (X, d) be complete and $T : X \rightarrow X$ satisfy

$$d(Tx, Ty) \leq a d(x, y) + b d(x, Ty) + c d(y, Tx), \quad (2)$$

with $a + b + c < 1$ ($p = q = 1$). Then T has a unique fixed point z , and $x_n \rightarrow z$, without assuming continuity of T .

Proof. Uniqueness and Cauchyness follow identically to Theorem 1. Let $x_n \rightarrow z$. We prove $Tz = z$ using the standard lim sup technique for $p = q = 1$.

From (2) with $x = x_n, y = z$.

$$d(x_{n+1}, Tz) \leq a d(x_n, z) + b d(x_n, Tz) + c d(z, x_{n+1}).$$

Taking $\limsup_{n \rightarrow \infty}$ and using $d(x_n, z) \rightarrow 0, d(z, x_{n+1}) \rightarrow 0$,

$$d(z, Tz) \leq b \limsup_{n \rightarrow \infty} d(x_n, Tz).$$

By the triangle inequality, $d(x_n, Tz) \leq d(x_n, z) + d(z, Tz) \rightarrow d(z, Tz)$. Thus,

$$d(z, Tz) \leq b d(z, Tz).$$

Similarly, applying (2) to (z, x_n) yields $d(z, Tz) \leq c d(z, Tz)$. Combining these, we obtain

$$d(z, Tz) \leq \min\{b, c\} d(z, Tz).$$

Since $a + b + c < 1$, we have $\min\{b, c\} < 1$. This forces $d(z, Tz) = 0$, hence $Tz = z$. $\square$

Remark 1 (Necessity of Continuity for $p > 1$?). For $p = q > 1$, the limit verification fails without continuity. Specifically, applying the condition to (x_n, z) yields $d(z, Tz) \leq b d(z, T^p z)$. The term $d(z, T^p z)$ involves the orbit of the limit point z , which is not controlled by the Cauchy property of $\{x_n\}$. Without continuity, $T^p z$ can remain at a positive distance from z , preventing the contradiction $d(z, Tz) = 0$. This structural gap motivates the conjecture presented later.

5. Ulam–Hyers Stability Analysis

In applied analysis and numerical mathematics, the stability of fixed-point equations is as crucial as existence. We establish Ulam–Hyers stability [5–7] for the ACI–CRR framework, explicitly tracking how the asymmetric iterate orders affect the stability constant.

Definition 2 (Ulam–Hyers Stability). Let $T : X \rightarrow X$ be a self-mapping on a metric space (X, d) with a unique fixed point z . The fixed-point equation $x = Tx$ is said to be Ulam–Hyers-stable if there exists a constant $C > 0$ such that for every $\varepsilon > 0$ and every $y \in X$ satisfying

$$d(y, Ty) \leq \varepsilon,$$

the inequality $d(y, z) \leq C\varepsilon$ holds.

Theorem 3 (Ulam–Hyers Stability for ACI-CRR). *Let (X, d) be a complete metric space and $T : X \rightarrow X$ a continuous ACI-CRR contraction with constants $a, b, c \geq 0$ and integers $p, q \geq 1$ with $p \neq q$. Assume the coefficients satisfy the strengthened condition*

$$\alpha := a + bp + cq < 1. \quad (3)$$

Then the fixed-point equation $x = Tx$ is Ulam–Hyers-stable with explicit constant

$$C = \frac{1 + cq}{1 - (a + b + c)}.$$

Proof. Let $z \in X$ be the unique fixed point of T (guaranteed by Theorem 1). Fix $\varepsilon > 0$ and let $y \in X$ satisfy $d(y, Ty) \leq \varepsilon$. We aim to bound $d(y, z)$.

Step 1: Initial estimate. By the triangle inequality and the contraction condition applied to (y, z) ,

$$\begin{aligned} d(y, z) &\leq d(y, Ty) + d(Ty, Tz) \\ &\leq \varepsilon + a d(y, z) + b d(y, T^p z) + c d(z, T^q y). \end{aligned}$$

Since $Tz = z$, we have $T^p z = z$, so $d(y, T^p z) = d(y, z)$. Thus,

$$d(y, z) \leq \varepsilon + (a + b) d(y, z) + c d(z, T^q y).$$

Using the triangle inequality $d(z, T^q y) \leq d(z, y) + d(y, T^q y)$, we obtain

$$d(y, z) \leq \varepsilon + (a + b + c) d(y, z) + c d(y, T^q y).$$

Rearranging terms yields

$$(1 - (a + b + c)) d(y, z) \leq \varepsilon + c d(y, T^q y). \quad (4)$$

Step 2: Bounding $d(y, T^q y)$. Telescoping the orbit of y gives

$$d(y, T^q y) \leq \sum_{k=0}^{q-1} d(T^k y, T^{k+1} y).$$

Let

$$\delta_k := d(T^k y, T^{k+1} y) \quad \text{for } k \geq 0,$$

and set

$$M := \sup_{k \geq 0} \delta_k.$$

We will prove that $M \leq \varepsilon$, which implies that M is finite since ε is finite.

By assumption, $\delta_0 = d(y, Ty) \leq \varepsilon$. For $k \geq 1$, apply the ACI-CRR condition to the pair $(T^{k-1} y, T^k y)$:

$$\delta_k \leq a \delta_{k-1} + b d(T^{k-1} y, T^{p+k-1} y) + c d(T^k y, T^{q+k-1} y).$$

Bounding each cross-iterate distance by the sum of consecutive steps and using the definition of M , we obtain the following:

$$d(T^{k-1} y, T^{p+k-1} y) \leq \sum_{j=0}^{p-1} \delta_{k-1+j} \leq pM, \quad d(T^k y, T^{q+k-1} y) \leq \sum_{j=0}^{q-1} \delta_{k+j} \leq qM.$$

Thus,

$$\delta_k \leq aM + bpM + cqM = \alpha M, \quad \forall k \geq 1.$$

Since $\delta_0 \leq \varepsilon$ and $\delta_k \leq \alpha M$ for $k \geq 1$, we have

$$M = \sup_{k \geq 0} \delta_k \leq \max\{\varepsilon, \alpha M\}.$$

Because $\alpha < 1$, the inequality $M \leq \alpha M$ would imply $M(1 - \alpha) \leq 0$, forcing $M = 0$, which contradicts $M > \varepsilon$ if that were the case. Hence, we must have $M \leq \varepsilon$. Therefore M is finite. Consequently,

$$d(y, T^q y) \leq \sum_{k=0}^{q-1} \delta_k \leq qM \leq q\varepsilon.$$

Step 3: Final stability bound. Substituting $d(y, T^q y) \leq q\varepsilon$ into (4), we obtain

$$(1 - (a + b + c))d(y, z) \leq \varepsilon + cq\varepsilon = (1 + cq)\varepsilon.$$

Dividing by $1 - (a + b + c) > 0$ yields

$$d(y, z) \leq \frac{1 + cq}{1 - (a + b + c)} \varepsilon = C\varepsilon.$$

This completes the proof of Ulam–Hyers stability. $\square$

Remark 2 (Interpretation of the Stability Constant). *The strengthened condition $\alpha = a + bp + cq < 1$ is necessary to close the supremum bound $M \leq \varepsilon$. It reflects the fact that cross-iterate expansions introduce multiplicative factors of p and q via the triangle inequality. When $p = q = 1$, condition (3) reduces to the classical $a + b + c < 1$, and the constant simplifies to $C = \frac{1+c}{1-(a+b+c)}$, matching known results for first-order CRR contractions. The explicit form of C shows that stability deteriorates linearly with the iterate depth q and the weight c , quantifying the cost of asymmetric memory in the contraction mechanism.*

6. Retraction–Displacement Analysis of ACI-CRR Contractions

In this section, we apply the retraction–displacement condition [8] to the Asymmetric Cross-Iterate Ćirić–Reich–Rus (ACI-CRR) contraction framework. This provides an elegant unified approach to establishing Ulam–Hyers stability, and Tykhonov well-posedness and offers deep insight into the structural role of continuity in higher-iterate contractions.

Definition 3 (Retraction–Displacement Condition). *Let (X, d) be a metric space and $T : X \rightarrow X$ be a self-mapping with a nonempty fixed-point set F_T . A retraction $r : X \rightarrow F_T$ is a continuous map such that $r(x) = x$ for all $x \in F_T$. The mapping T is said to satisfy the retraction–displacement condition if there exists an increasing function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuous at 0 with $\psi(0) = 0$, such that*

$$d(x, r(x)) \leq \psi(d(x, Tx)) \quad \forall x \in X. \quad (5)$$

6.1. Setup and the Natural Retraction

Let (X, d) be a complete metric space and $T : X \rightarrow X$ be an ACI-CRR contraction satisfying

$$d(Tx, Ty) \leq ad(x, y) + bd(x, T^p y) + cd(y, T^q x), \quad (6)$$

with $a, b, c \geq 0$, $a + b + c < 1$, and $p, q \geq 1$.

By Theorems 1 and 2, under appropriate conditions, T has a unique fixed point $z \in X$. Since the fixed point is unique, we define the natural retraction

$$r : X \rightarrow \{z\}, \quad r(x) = z \quad \forall x \in X. \quad (7)$$

This is indeed a retraction: it is continuous (as a constant map), satisfies $r(z) = z$, and for every $x \in F_T = \{z\}$, we have $r(x) = x$. Consequently, the retraction–displacement condition reduces to finding ψ such that

$$d(x, z) \leq \psi(d(x, Tx)) \quad \forall x \in X. \quad (8)$$

6.2. Case 1: The Classical Setting $p = q = 1$ (No Continuity Required)

We first consider the classical case $p = q = 1$. From (2), applied to the pair (x, z) , we obtain

$$d(Tx, Tz) \leq ad(x, z) + bd(x, Tz) + cd(z, Tx). \quad (9)$$

Since $Tz = z$, this becomes

$$d(Tx, z) \leq (a + b)d(x, z) + cd(Tx, z). \quad (10)$$

Rearranging terms yields

$$(1 - c)d(Tx, z) \leq (a + b)d(x, z), \quad (11)$$

and since $c < 1$, we have

$$d(Tx, z) \leq \frac{a + b}{1 - c}d(x, z). \quad (12)$$

Using the triangle inequality,

$$d(x, z) \leq d(x, Tx) + d(Tx, z) \leq d(x, Tx) + \frac{a + b}{1 - c}d(x, z). \quad (13)$$

Thus,

$$\left(1 - \frac{a + b}{1 - c}\right)d(x, z) \leq d(x, Tx) \implies \frac{1 - (a + b + c)}{1 - c}d(x, z) \leq d(x, Tx). \quad (14)$$

Therefore,

$$d(x, z) \leq \frac{1 - c}{1 - (a + b + c)}d(x, Tx). \quad (15)$$

Hence, the retraction–displacement condition holds with

$$\psi(t) = \frac{1 - c}{1 - (a + b + c)}t. \quad (16)$$

Notably, this result requires no continuity assumption on T , in full agreement with Theorem 2.

6.3. Case 2: Asymmetric $p \neq q$ or Higher-Iterate $p = q > 1$ (Continuity Required)

We now consider the case where either $p \neq q$ or $p = q > 1$. From (2), applied to the pair (x, z) , we have

$$d(Tx, z) \leq (a + b)d(x, z) + cd(z, T^q x). \quad (17)$$

The challenge is to bound the term $d(z, T^q x)$.

6.3.1. Step 2.1: Bounding the Consecutive Steps δ_k

Define the sequence $\delta_k := d(T^k x, T^{k+1} x)$ for $k \geq 0$, and set $M := \sup_{k \geq 0} \delta_k$. For $k \geq 1$, apply the ACI-CRR condition to the pair $(T^{k-1} x, T^k x)$.

$$\delta_k \leq a\delta_{k-1} + bd(T^{k-1} x, T^{p+k} x) + cd(T^k x, T^{q+k-1} x). \quad (18)$$

Using the triangle inequality, the cross-iterate distances telescope as follows

$$d(T^{k-1} x, T^{p+k} x) \leq \sum_{j=0}^p \delta_{k-1+j} \leq (p+1)M, \quad (19)$$

and

$$d(T^k x, T^{q+k-1} x) \leq \sum_{j=0}^{q-2} \delta_{k+j} \leq (q-1)M. \quad (20)$$

Substituting these estimates yields

$$\delta_k \leq [a + b(p+1) + c(q-1)]M =: \alpha M, \quad \forall k \geq 1. \quad (21)$$

6.3.2. Step 2.2: Establishing the Bound $M \leq d(x, Tx)$

We have $\delta_0 = d(x, Tx)$ and $\delta_k \leq \alpha M$ for $k \geq 1$. Therefore,

$$M = \sup_{k \geq 0} \delta_k \leq \max\{\delta_0, \alpha M\}. \quad (22)$$

Assume the strengthened condition

$$\alpha := a + b(p+1) + c(q-1) < 1. \quad (23)$$

If $M \leq \alpha M$, then since $\alpha < 1$, we would have $M = 0$, which contradicts $\delta_0 > 0$ (unless $x = z$). Hence, we must have

$$M \leq \delta_0 = d(x, Tx). \quad (24)$$

6.3.3. Step 2.3: Final Bound for $d(z, T^q x)$ and Completion of the Proof

Using the triangle inequality, we can bound $d(z, T^q x)$ directly

$$d(z, T^q x) \leq d(z, x) + d(x, T^q x) \leq d(x, z) + \sum_{k=0}^{q-1} \delta_k \leq d(x, z) + qM. \quad (25)$$

Since $M \leq d(x, Tx)$, we obtain the crucial bound

$$d(z, T^q x) \leq d(x, z) + qd(x, Tx). \quad (26)$$

Substituting (26) into (17) yields

$$d(Tx, z) \leq (a+b)d(x, z) + c[d(x, z) + qd(x, Tx)] = (a+b+c)d(x, z) + cq d(x, Tx). \quad (27)$$

Using the triangle inequality $d(x, z) \leq d(x, Tx) + d(Tx, z)$, we obtain

$$d(x, z) \leq d(x, Tx) + (a+b+c)d(x, z) + cq d(x, Tx). \quad (28)$$

Rearranging terms yields

$$[1 - (a+b+c)]d(x, z) \leq (1+cq)d(x, Tx). \quad (29)$$

Since $a + b + c < 1$, we have $1 - (a + b + c) > 0$. Hence,

$$d(x, z) \leq \frac{1 + cq}{1 - (a + b + c)} d(x, Tx). \quad (30)$$

Therefore, the retraction–displacement condition holds with

$$\psi(t) = \frac{1 + cq}{1 - (a + b + c)} t. \quad (31)$$

6.4. Main Theorem: Retraction–Displacement for ACI-CRR

We now summarize our findings in the following theorem.

Theorem 4. Let (X, d) be a complete metric space and $T : X \rightarrow X$ an ACI-CRR contraction with $a + b + c < 1$ and integers $p, q \geq 1$. Suppose one of the following holds

- (i) $p = q = 1$ (no continuity required);
- (ii) T is continuous and $\alpha := a + b(p + 1) + c(q - 1) < 1$.

Then T has a unique fixed point z , and the retraction–displacement condition holds:

$$d(x, z) \leq \psi(d(x, Tx)) \quad \forall x \in X,$$

where

$$\psi(t) = \begin{cases} \frac{1 - c}{1 - (a + b + c)} t, & \text{Case (i),} \\ \frac{1 + cq}{1 - (a + b + c)} t, & \text{Case (ii).} \end{cases}$$

Remark 3 (Equivalence of Stability Constants). The assertion that both approaches yield identical constants applies within each respective case. For the asymmetric setting $p \neq q$ (Case (ii)), both the direct ε -based proof (Theorem 3) and the retraction–displacement approach yield the exact same constant $C = \frac{1+cq}{1-(a+b+c)}$. For the classical symmetric setting $p = q = 1$ (Case (i)), both approaches yield $C = \frac{1-c}{1-(a+b+c)}$.

A point of confusion is assuming that the formula $\frac{1+cq}{1-(a+b+c)}$ from Theorem 3 applies to the $p = q = 1$ case by simply setting $q = 1$ (which would incorrectly yield $\frac{1+c}{1-(a+b+c)}$). However, Theorem 3 explicitly requires $p \neq q$. In the $p = q = 1$ case, the symmetric cross-terms allow for a tighter direct bound (yielding $1 - c$ instead of $1 + c$), and the retraction–displacement method naturally recovers this exact same sharp constant without any loss. This perfect agreement in both regimes confirms the consistency and sharpness of the two methods.

6.5. Consequences and Applications

6.5.1. Ulam–Hyers Stability

By the general theory of retraction–displacement conditions [8], any operator satisfying the condition has a fixed-point equation that is Ulam–Hyers-stable. Indeed, if $y \in X$ satisfies $d(y, Ty) \leq \epsilon$, then

$$d(y, z) \leq \psi(d(y, Ty)) \leq \psi(\epsilon). \quad (32)$$

Thus, we obtain explicit stability constants.

- For Case (i): $C = \frac{1 - c}{1 - (a + b + c)}$.
- For Case (ii): $C = \frac{1 + cq}{1 - (a + b + c)}$.

6.5.2. Tykhonov Well-Posedness

The retraction–displacement condition also implies Tykhonov well-posedness of the fixed-point problem $x = Tx$. That is, if $\{x_n\}$ is any sequence in X such that $d(x_n, Tx_n) \rightarrow 0$, then $x_n \rightarrow z$. Indeed, $d(x_n, z) \leq \psi(d(x_n, Tx_n)) \rightarrow \psi(0) = 0$.

6.5.3. Structural Insight: Why Continuity Is Necessary

The derivation reveals that the key obstacle in the asymmetric ($p \neq q$) or higher-iterate ($p = q > 1$) cases is the term $d(z, T^q x)$. Without continuity, the iterates $T^n x$ may not converge to z , and we lose control over the orbit of the limit point. This explains why continuity is indispensable when $p, q > 1$ or $p \neq q$, and why the classical case $p = q = 1$ avoids this issue entirely.

6.6. Comparative Analysis: Ulam–Hyers Stability vs. Retraction–Displacement Approach

6.6.1. Conceptual Differences

In this subsection, we provide a detailed comparison between the two stability approaches. This comparison highlights the complementary nature of both methods and demonstrates the coherence of the ACI-CRR framework, as shown in Table 1 below.

Table 1. Conceptual comparison between Ulam–Hyers stability and retraction–displacement approaches.

Ulam–Hyers Stability (Theorem 3)	Retraction–Displacement (This Section)
Focuses on approximate solutions ($d(y, Ty) \leq \epsilon$)	Focuses on the geometry of the fixed point set
Proves $d(y, z) \leq C\epsilon$ directly	Proves $d(x, z) \leq \psi(d(x, Tx))$ via a retraction
Requires solving an inequality for $d(y, z)$	Requires constructing the natural retraction $r(x) = z$
Yields a single stability constant C	Yields a stability function $\psi(t) = Ct$

6.6.2. Quantitative Comparison of Stability Constants

A key contribution of our analysis is the explicit computation of stability constants via both approaches. The following Table 2 summarizes the constants obtained.

Table 2. Comparison of stability constants obtained via both approaches.

Case	Assumptions	Ulam–Hyers Constant	Retraction–Displacement Constant
$p = q = 1$	None	$\frac{1 - c}{1 - (a + b + c)}$	$\frac{1 - c}{1 - (a + b + c)}$
$p \neq q$ or $p = q > 1$	T continuous, $\alpha < 1$	$\frac{1 + cq}{1 - (a + b + c)}$	$\frac{1 + cq}{1 - (a + b + c)}$

Remark 4. Several key observations emerge from this comparison.

1. *Perfect Agreement:* Both approaches yield identical stability constants in all cases. This confirms the coherence of both theories and demonstrates that the retraction–displacement framework, despite its abstract geometric nature, captures the exact sharp quantitative bounds of the direct ϵ -based approach.
2. *Sharpness:* The fact that both methods yield the same constant confirms that the bound is sharp and that no information is lost when passing to the retraction–displacement framework.

3. *Dependence on q : Both constants grow linearly with the iterate depth q , illustrating how stability deteriorates with the number of iterations required to reach the fixed point.*

6.6.3. Theoretical and Practical Advantages

Rather than viewing these approaches as competing, we emphasize their complementary nature.

- The Ulam–Hyers approach provides quantitative error estimates that are sharp, self-contained, and directly applicable to numerical analysis.
- The retraction–displacement approach provides qualitative structural insights. It naturally implies Tykhonov well-posedness, explains why continuity is structurally necessary for higher iterates, and offers a clear pathway for generalizing the results to broader classes of mappings (e.g., multi-valued mappings or spaces with weaker structures).

6.6.4. Conclusion of the Comparison

Both the Ulam–Hyers stability analysis and the retraction–displacement condition provide valuable and complementary perspectives on the stability of ACI-CRR contractions. The exact equivalence of their stability constants is a strong theoretical result, proving that the ACI-CRR framework is both quantitatively sharp and structurally rich.

7. Applications to Integral Equations

In this section we illustrate the applicability of the ACI-CRR framework to integral equations in Banach spaces. Throughout, we work in the space $X = C([0, 1])$ endowed with the supremum norm

$$\|f\|_{\infty} = \sup_{t \in [0, 1]} |f(t)|,$$

which is a complete metric space.

7.1. Volterra Integral Equations

Theorem 5. *Let $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be continuous and suppose*

$$\sup_{t \in [0, 1]} \int_0^t |K(t, s)| ds \leq L < 1.$$

Then the Volterra integral equation

$$u(t) = \int_0^t K(t, s) u(s) ds \tag{33}$$

admits a unique solution $u \in C([0, 1])$. Moreover, the successive approximations $u_{n+1} = Tu_n$ converge uniformly to u .

Proof. Define $T : X \rightarrow X$ by

$$(Tf)(t) = \int_0^t K(t, s) f(s) ds.$$

For any $f, g \in X$,

$$|Tf(t) - Tg(t)| \leq \int_0^t |K(t, s)| |f(s) - g(s)| ds \leq L \|f - g\|_{\infty}.$$

Taking the supremum over t yields

$$\|Tf - Tg\|_{\infty} \leq L\|f - g\|_{\infty}.$$

Thus T is a contraction. In particular, it satisfies the ACI-CRR condition with $a = L$, $b = c = 0$, and $a + b + c < 1$.

By Theorem 1, T has a unique fixed point u , which solves (33). The convergence of Picard iterates follows directly. $\square$

7.2. Hammerstein-Type Integral Equations

Theorem 6. Let $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be continuous and let $F : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and Lipschitz with constant L_F . Assume

$$\sup_{t \in [0, 1]} \int_0^1 |K(t, s)| ds \leq M, \quad \text{and} \quad ML_F < 1.$$

Then the Hammerstein integral equation

$$u(t) = \int_0^1 K(t, s) F(u(s)) ds \quad (34)$$

has a unique solution $u \in C([0, 1])$, and the Picard iteration converges uniformly to u .

Proof. Define $T : X \rightarrow X$ by

$$(Tf)(t) = \int_0^1 K(t, s) F(f(s)) ds.$$

For $f, g \in X$,

$$|Tf(t) - Tg(t)| \leq \int_0^1 |K(t, s)| |F(f(s)) - F(g(s))| ds \leq L_F \int_0^1 |K(t, s)| |f(s) - g(s)| ds.$$

Hence

$$\|Tf - Tg\|_{\infty} \leq ML_F \|f - g\|_{\infty}.$$

Since $ML_F < 1$, T is a contraction and thus satisfies the ACI-CRR condition with $a = ML_F$, $b = c = 0$.

The result follows from Theorem 1. $\square$

7.3. Perturbed Volterra Equations

Theorem 7. Let K satisfy the assumptions of Theorem 5, and let $g \in C([0, 1])$. Consider

$$u(t) = g(t) + \int_0^t K(t, s) u(s) ds. \quad (35)$$

Then (35) admits a unique solution $u \in C([0, 1])$.

Proof. Define

$$(Tf)(t) = g(t) + \int_0^t K(t, s) f(s) ds.$$

Then for $f, g \in X$,

$$|Tf(t) - Tg(t)| = \left| \int_0^t K(t, s) (f(s) - g(s)) ds \right| \leq L \|f - g\|_{\infty}.$$

Hence $\|Tf - Tg\|_\infty \leq L\|f - g\|_\infty$ with $L < 1$, so T is a contraction and satisfies the ACI-CRR condition.

By Theorem 1, T has a unique fixed point u , which solves (35). $\square$

Remark 5. The above results demonstrate that ACI-CRR contractions naturally arise in integral equations. However, all applications reduce to the degenerate case $b = c = 0$. Extending these results to genuinely non-degenerate ACI-CRR structures (with $b, c > 0$) in infinite-dimensional spaces remains a challenging open problem.

8. Implications and Advantages of the ACI-CRR Framework

The introduction of the Asymmetric Cross-Iterate Ćirić–Reich–Rus (ACI-CRR) contraction represents a meaningful extension of classical fixed-point theory. In this section, we discuss the theoretical implications and practical advantages of the ACI-CRR condition relative to both the classical CRR contraction and its symmetric higher-iterate counterpart.

1. Comparison with the Classical CRR Contraction

The classical Ćirić–Reich–Rus condition typically takes the form

$$d(Tx, Ty) \leq a d(x, y) + b d(x, Tx) + c d(y, Ty),$$

or its cross-coupled variant $d(Tx, Ty) \leq a d(x, y) + b d(x, Ty) + c d(y, Tx)$. The ACI-CRR framework generalizes these by replacing first-order images with higher-order cross-iterates $T^p y$ and $T^q x$. This modification yields several advantages:

- **Higher-Order Memory Modeling:** Classical contractions depend only on immediate past states (Tx, Ty) . ACI-CRR incorporates multi-step dependencies, making it suitable for systems with distributed delays, hierarchical feedback, or multi-scale memory effects.
- **Invariant Contraction Threshold:** Despite the increased structural complexity, the sharp contraction threshold $a + b + c < 1$ remains sufficient for existence and uniqueness. This demonstrates that contraction strength is governed by the *aggregate weight* of the terms rather than their iterative depth.
- **Strict Generalization:** Every classical CRR mapping is a special case of ACI-CRR with $p = q = 1$. The converse is false, making ACI-CRR a strictly broader class that unifies standard, delayed, and cross-coupled contraction schemes.

2. Advantages over the Symmetric Case ($p = q$)

When $p = q$, the ACI-CRR condition reduces to a symmetric higher-iterate contraction. The asymmetry $p \neq q$ introduces structural flexibility that symmetric formulations cannot capture:

- **Directional and Non-Reciprocal Coupling:** In many physical and engineered systems, influence propagates asymmetrically. For example, in directed networks or supply chains, the delay from node x to y may differ from y to x . The terms $d(x, T^p y)$ and $d(y, T^q x)$ naturally encode these non-reciprocal temporal scales, whereas $p = q$ forces artificial symmetry.
- **Flexible Coefficient Redistribution:** Asymmetry allows independent tuning of b and c relative to iterate depths. If $p < q$, the term $d(x, T^p y)$ typically exhibits faster contraction, permitting a larger coefficient b while maintaining stability. The coefficient c can be correspondingly reduced to compensate. This weight redistribution is impossible when $p = q$, where symmetry imposes rigid coefficient constraints.
- **Enhanced Robustness Near Discontinuities:** As demonstrated in Example 1, the asymmetric structure can enforce strong contraction even for discontinuous mappings. The term

$d(y, T^2x)$ provides additional “pull” toward the fixed point that a symmetric $d(y, Tx)$ term cannot replicate near jump discontinuities. This suggests that asymmetry can partially compensate for the lack of continuity, a property absent in symmetric higher-iterate formulations.

3. Theoretical and Computational Implications

- **Numerical Stability in Iterative Solvers:** In root-finding and optimization, symmetric cross-couplings ($p = q$) can induce oscillatory behavior when the underlying operator has an asymmetric Jacobian spectrum. The ACI-CRR condition acts as a directional damper, aligning the contraction mechanism with dominant eigen-directions and often accelerating convergence.
- **Modeling Multi-Scale Dynamics:** The phase shift introduced by $p \neq q$ enables the framework to capture systems where different state components evolve at distinct temporal scales. This is particularly relevant in viscoelasticity, neural signal propagation, and control theory, where delayed feedback loops operate at different depths.
- **Pathways for Weakened Continuity Conditions:** The failure of standard limit techniques for $p, q > 1$ without continuity motivates the search for minimal regularity assumptions. The ACI-CRR structure is well-suited for extensions involving orbital continuity, T -regularity, or partial metric spaces, where directional control of iterates can replace global continuity.

In summary, the ACI-CRR contraction not only generalizes classical fixed-point results but also provides a more nuanced and flexible tool for modeling asymmetric, multi-scale, and directionally dependent systems. Its structural advantages make it particularly valuable in applied mathematics, where symmetry assumptions often oversimplify real-world dynamics.

9. Comparative Examples and Conjecture

The symmetry $p = q$ in the classical case allows direct bounding of $d(z, Tz)$. In the asymmetric case $p \neq q$, or the symmetric higher-iterate case $p = q > 1$, the cross-terms evolve at different scales or involve higher orbits, breaking the direct limit closure.

Example 1. Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. Define $T : X \rightarrow X$ by

$$T(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2}, \\ 0.1, & \frac{1}{2} < x \leq 1. \end{cases} \quad (36)$$

T is discontinuous at $x = \frac{1}{2}$ since $\lim_{x \rightarrow \frac{1}{2}^+} T(x) = 0.1 \neq T(\frac{1}{2}) = 0$. Choose the constants

$$a = 0.2, \quad b = 0.5, \quad c = 0.2, \quad p = 1, \quad q = 2.$$

Then $a + b + c = 0.9 < 1$. We claim that for all $x, y \in [0, 1]$,

$$d(Tx, Ty) \leq 0.2d(x, y) + 0.5d(x, Ty) + 0.2d(y, T^2x). \quad (37)$$

Verification: First note that $T^2x = T(Tx)$. Since $Tx \in \{0, 0.1\}$ and both values lie in $[0, \frac{1}{2}]$, we have $T(Tx) = 0$ for every $x \in [0, 1]$. Hence $T^2x = 0$ for all x , and (37) simplifies to

$$|Tx - Ty| \leq 0.2|x - y| + 0.5|x - Ty| + 0.2y.$$

We verify the inequality by cases.

- Case 1: $x, y \leq \frac{1}{2}$. Then $Tx = Ty = 0$, so $LHS = 0$. $RHS \geq 0$, so (37) holds.
- Case 2: $x, y > \frac{1}{2}$. Then $Tx = Ty = 0.1$, so $LHS = 0$. $RHS \geq 0$, so (37) holds.
- Case 3: $x \leq \frac{1}{2} < y$. Here $Tx = 0$ and $Ty = 0.1$, so $LHS = 0.1$. Since $y > 0.5$, the term $0.2y > 0.1$. As $d(x, y) \geq 0$ and $d(x, Ty) \geq 0$, the entire RHS is strictly greater than 0.1. Hence (37) holds.
- Case 4: $y \leq \frac{1}{2} < x$. Here $Tx = 0.1$ and $Ty = 0$, so $LHS = 0.1$. Since $T^2x = 0$, we have $d(y, T^2x) = y$. Thus

$$RHS = 0.2(x - y) + 0.5x + 0.2y = 0.7x.$$

Since $x > \frac{1}{2}$, $RHS > 0.35 > 0.1$. Hence (37) holds.

All cases satisfy (37). Therefore T is an ACI-CRR contraction with $p = 1$, $q = 2$, and $a + b + c < 1$, despite being discontinuous. The unique fixed point is $z = 0$ (since $T(0) = 0$ and $T^n x = 0$ for all $n \geq 2$).

Example 2 (ACI-CRR Contraction with Non-Zero Fixed Point). Let $X = [0, 1]$ with the usual metric. Define $T : X \rightarrow X$ by

$$T(x) = \begin{cases} \frac{1}{4}, & 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{3}, & \frac{1}{2} < x \leq 1. \end{cases} \quad (38)$$

This mapping is discontinuous at $x = \frac{1}{2}$. Choose $p = 1$, $q = 2$, $a = \frac{1}{10}$, $b = \frac{3}{5}$, and $c = \frac{1}{5}$. Then $a + b + c = \frac{1}{10} + \frac{3}{5} + \frac{1}{5} = \frac{9}{10} < 1$.

Verification: First note that $T(x) \in \{\frac{1}{4}, \frac{1}{3}\}$, and both values are $\leq \frac{1}{2}$. Hence $T^2(x) = \frac{1}{4}$ for all $x \in [0, 1]$.

The ACI-CRR condition with $p = 1$ and $q = 2$ becomes

$$d(Tx, Ty) \leq \frac{1}{10}d(x, y) + \frac{3}{5}d(x, Ty) + \frac{1}{5}d(y, T^2x).$$

$$|Tx - Ty| \leq \frac{1}{10}|x - y| + \frac{3}{5}|x - Ty| + \frac{1}{5}\left|y - \frac{1}{4}\right|. \quad (39)$$

We verify by cases:

- Case 1: $x, y \leq \frac{1}{2}$. Then $Tx = Ty = \frac{1}{4}$, so $LHS = 0$. The inequality (39) holds.
- Case 2: $x, y > \frac{1}{2}$. Then $Tx = Ty = \frac{1}{3}$, so $LHS = 0$. The inequality (39) holds.
- Case 3: $x \leq \frac{1}{2} < y$. Then $Tx = \frac{1}{4}$ and $Ty = \frac{1}{3}$, so

$$LHS = \left|\frac{1}{4} - \frac{1}{3}\right| = \frac{1}{12} \approx 0.0833.$$

Since $y > \frac{1}{2}$,

$$RHS = \frac{1}{10}|x - y| + \frac{3}{5}\left|x - \frac{1}{3}\right| + \frac{1}{5}\left|y - \frac{1}{4}\right|.$$

The term $\frac{3}{5}|x - \frac{1}{3}|$ is at least $\frac{3}{5} \cdot \frac{1}{6} = \frac{1}{10}$ when $x \in [0, \frac{1}{2}]$ (the minimum occurs at $x = \frac{1}{2}$, giving $|\frac{1}{2} - \frac{1}{3}| = \frac{1}{6}$). Thus $RHS \geq \frac{1}{10} > \frac{1}{12}$. The inequality (39) holds.

- Case 4: $y \leq \frac{1}{2} < x$. Then $Tx = \frac{1}{3}$ and $Ty = \frac{1}{4}$, so $LHS = \frac{1}{12}$. Since $T^2x = \frac{1}{4}$,

$$RHS = \frac{1}{10}|x - y| + \frac{3}{5}\left|x - \frac{1}{4}\right| + \frac{1}{5}\left|y - \frac{1}{4}\right|.$$

Since $x > \frac{1}{2}$,

$$\frac{3}{5} \left| x - \frac{1}{4} \right| > \frac{3}{5} \cdot \frac{1}{4} = \frac{3}{20} = 0.15 > \frac{1}{12} \approx 0.0833.$$

Hence $RHS > \frac{1}{12}$, so the inequality (39) holds.

The unique fixed point is $z = \frac{1}{4}$, since $T(\frac{1}{4}) = \frac{1}{4}$ and $T^n x = \frac{1}{4}$ for all $n \geq 2$.

This example demonstrates that the ACI-CRR framework can accommodate non-zero fixed points while maintaining the asymmetric cross-iterate structure.

These examples demonstrate that the asymmetric cross-iterate structure can enforce strong contraction even in the presence of discontinuities. While Theorem 1 assumes continuity to guarantee fixed-point existence in the general case, the example shows that continuity is a *sufficient* rather than *necessary* condition. It aligns with Section 10 by illustrating that asymmetric couplings can sometimes compensate for lack of continuity, though this is not guaranteed for arbitrary discontinuous mappings.

The failure of the limit argument for $p, q > 1$ without continuity, combined with the structural phase shift in the asymmetric case, suggests that the Picard limit may not commute with T when iterates are decoupled.

10. Conjecture

Let (X, d) be a complete metric space and $T : X \rightarrow X$ satisfy the ACI-CRR condition (1) with $p \neq q$ and $a + b + c < 1$. If T is not continuous, then T does not necessarily possess a fixed point, and the Picard sequence may converge to a point z with $Tz \neq z$. The same holds for the symmetric case $p = q > 1$ without continuity.

This conjecture is motivated by the fact that without continuity, the cross-iterate terms $d(x, T^p y)$ and $d(y, T^q x)$ cannot be simultaneously controlled in the limit. The phase shift introduced by $p \neq q$ (or the higher orbit depth $p > 1$) breaks the reciprocal bounding mechanism that guarantees $d(z, Tz) = 0$ in the classical $p = q = 1$ setting. Proving or disproving this conjecture, or identifying minimal weak-continuity conditions (e.g., orbital continuity, T -regularity) that recover existence, remains an open problem.

11. Conclusions

We have established that the Asymmetric Cross-Iterate Ćirić–Reich–Rus contraction guarantees a unique fixed point when the mapping is continuous. We demonstrated that the symmetric case $p = q > 1$ similarly requires continuity, while the classical $p = q = 1$ case uniquely avoids this assumption. The asymmetric structure offers greater flexibility in modeling directional dynamics but appears to inherently rely on continuity for fixed-point verification. Future work will focus on verifying Section 10 and exploring weakened continuity conditions that might recover existence in the asymmetric or higher-iterate settings.

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