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Invariant Subspaces and Exact Solutions for Nonlinear Variable-Coefficient Beam-Type Equations

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Abstract

This paper investigates a class of generalized nonlinear variable-coefficient beam-type evolution equations motivated by the modeling of nonhomogeneous elastic structures with nonlinear effects. The invariant subspace method is employed to construct exact solutions by identifying appropriate finite-dimensional invariant subspaces associated with the nonlinear operators. This approach reduces the original partial differential equations to systems of nonlinear ordinary differential equations. Several representative nonlinear beam-type equations are studied, leading to polynomial, trigonometric, hyperbolic, and negative-power invariant subspaces together with their corresponding exact solutions. The results demonstrate the effectiveness of the invariant subspace method for constructing explicit exact solutions of nonlinear variable-coefficient fourth-order beam-type equations and highlight its applicability to a broad class of nonlinear beam models.

Keywords: invariant subspace; beam-type equations; exact solutions

MSC: 35C05; 35B06; 35G20

1. Introduction

Nonlinear fourth-order evolution equations constitute an important class of partial differential equations arising in applied mathematics, mechanics, and engineering. Among these equations, nonlinear beam-type models have attracted considerable attention because they combine important physical applications with challenging mathematical problems involving higher-order differential operators, variable coefficients, and nonlinear effects. Such equations arise naturally in the study of elastic structures, vibration theory, suspension bridges, composite materials, and other systems in which bending plays a dominant role [1–4].

A classical example is the Euler–Bernoulli beam equation, whose variable-coefficient form is

$$\rho(x)u_{tt} + (EI(x)u_{xx})_{xx} = q(x,t), \tag{1}$$

where $u(x,t)$ denotes the transverse displacement of the beam, $\rho(x)$ is the mass density, and $EI(x)$ is the flexural rigidity. This equation describes the transverse vibration of nonhomogeneous elastic beams and illustrates how the fourth-order operator naturally arises from the bending stiffness of the material. Motivated by this classical model, we consider the generalized nonlinear variable-coefficient beam-type equation

$$u_{tt} = (a(x)u_{xx})_{xx} + f(x,u,u_x,u_{xx}), \tag{2}$$

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where $a(x)$ is a spatially varying coefficient and $f(x, u, u_x, u_{xx})$ is a nonlinear term depending on the displacement and its lower-order spatial derivatives. Equation (2) serves as a unified mathematical framework for studying nonlinear beam equations with variable coefficients. The fourth-order operator models the effect of variable bending stiffness, whereas the nonlinear term allows a broad range of nonlinear interactions involving the displacement and its lower-order derivatives to be incorporated into the model.

Because of their mathematical and engineering significance, nonlinear beam equations have been extensively investigated using analytical, asymptotic, numerical, and symmetry-based methods. These models arise in the study of beams on nonlinear elastic foundations, suspension bridges, composite structures, and other flexible systems that exhibit resonance, bifurcation, instability, and large-amplitude oscillations. Representative contributions include [5–8]. More recently, Huang, Li, and Yu [9] applied Lie symmetry analysis to a class of generalized nonlinear beam equations and obtained several exact invariant solutions, demonstrating that analytical reduction methods remain effective tools for investigating nonlinear fourth-order beam equations.

From a mathematical point of view, nonlinear variable-coefficient beam equations are considerably more difficult to analyze than their second-order counterparts. The higher-order spatial operator, together with variable coefficients and nonlinear lower-order terms, creates substantial analytical challenges in the study of existence, uniqueness, regularity, stability, and exact solutions. General results on higher-order elliptic and evolution equations can be found in [10,11].

In recent years, several important analytical results have been established for nonlinear Euler–Bernoulli beam equations with variable coefficients. Wei and Ji [12] investigated periodic solutions using functional analytic methods. Wei [13] proved the existence of infinitely many periodic solutions for related beam models. Chen, Gao, and Li [14] studied periodic solutions using the Lyapunov–Schmidt reduction together with a differentiable Nash–Moser iteration. More recently, Deng, Zheng, and Zhu [15] investigated the well-posedness and stability of nonlinear Euler–Bernoulli beam equations. These works have significantly advanced the theoretical understanding of nonlinear beam equations with variable coefficients. Nevertheless, they mainly focus on the existence, multiplicity, periodicity, stability, and qualitative behavior of solutions, whereas comparatively fewer studies are devoted to the construction of explicit exact solutions.

Exact solutions remain valuable because they provide insight into the qualitative behavior of nonlinear models, serve as benchmark solutions for validating numerical algorithms, and reveal the influence of variable coefficients and nonlinear terms. Consequently, the development of analytical methods capable of constructing explicit exact solutions continues to be an important research direction.

Among the available analytical techniques, the invariant subspace method has proved to be particularly effective for constructing exact solutions of nonlinear evolution equations. The method was introduced by Galaktionov [16,17] and systematically developed by Galaktionov and Svirshchevskii in their monograph [18]. Since then, it has been refined and applied to various classes of nonlinear evolution equations by several authors, including King [19] and Ma [20]. More recently, the invariant subspace method has been extended to higher-dimensional and nonlinear evolution equations, demonstrating its effectiveness for constructing finite-dimensional invariant subspaces and explicit exact solutions [21,22]. These recent developments indicate that the invariant subspace method remains an active and evolving analytical tool.

For Equation (2), we define the nonlinear operator

$$F[u] = (a(x)u_{xx})_{xx} + f(x, u, u_x, u_{xx}),$$

acting on sufficiently smooth functions, and seek a finite-dimensional linear space,

$$W_n = \text{span}\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x)\},$$

that satisfies the invariance condition

$$F(W_n) \subseteq W_n.$$

Whenever this condition is satisfied, solutions of Equation (2) may be sought in the form

$$u(x,t) = \sum_{j=1}^{n} y_j(t)\phi_j(x),$$

where the unknown coefficients $y_j(t)$ satisfy a finite system of ordinary differential equations. Consequently, the original nonlinear partial differential equation is reduced to a finite-dimensional dynamical system while preserving its nonlinear structure. This reduction frequently makes it possible to construct explicit exact solutions.

Although the invariant subspace method has been successfully applied to many nonlinear evolution equations, its application to nonlinear variable-coefficient beam-type equations remains relatively limited. Existing studies on nonlinear beam equations mainly emphasize existence theory, periodic solutions, stability analysis, numerical approximation, or symmetry methods. In contrast, the systematic construction of finite-dimensional invariant subspaces leading to explicit exact solutions for nonlinear variable-coefficient beam-type equations has received comparatively less attention. This observation motivates the present work.

The purpose of this paper is not to classify all nonlinear beam equations admitting invariant subspaces but to develop a general analytical framework and demonstrate its applicability through several representative classes of nonlinear variable-coefficient beam equations. Specifically, we identify suitable finite-dimensional invariant subspaces for representative classes of nonlinear variable-coefficient beam equations, reduce the governing partial differential equations to systems of ordinary differential equations, and construct explicit exact solutions. The examples considered in this paper were selected because the corresponding variable coefficients and nonlinear terms satisfy compatibility conditions that permit invariant subspaces to be constructed explicitly. They illustrate different invariant structures and demonstrate the flexibility of the proposed framework rather than provide a complete classification of nonlinear beam equations.

The remainder of this paper is organized as follows. In Section 2, we review the invariant subspace method and establish the general framework used throughout the paper. In Section 3, we construct invariant subspaces for several nonlinear variable-coefficient beam equations, derive the corresponding reduced systems of ordinary differential equations, and obtain explicit exact solutions. Finally, Section 4 concludes the paper and discusses possible directions for future research.

2. The Invariant Subspace Method

This section briefly reviews the invariant subspace method that will be used throughout the paper. The method provides an effective analytical technique for constructing exact solutions of nonlinear evolution equations by reducing partial differential equations to finite-dimensional systems of ordinary differential equations. Since its introduction by Galaktionov [16,17] and its systematic development by Galaktionov and Svirshchevskii [18], the invariant subspace method has been successfully applied to a wide variety of nonlinear evolution equations; see also [19,20].

Consider the general nonlinear evolution equation

$$u_{tt} = F[u], \tag{3}$$

where

$$F[u] = F(x, u, u_x, u_{xx}, \ldots, u_{x^k}), \qquad k \in \mathbb{N}_0,$$

is a nonlinear differential operator acting on sufficiently smooth functions and depending on the spatial variable, the unknown function, and its spatial derivatives.

The main idea of the invariant subspace method is to identify a finite-dimensional linear space that is invariant under the nonlinear operator F. Once such a space has been identified, the original partial differential equation can be reduced to a finite-dimensional system of ordinary differential equations governing the unknown time-dependent coefficients. This reduction frequently makes it possible to construct explicit exact solutions.

Definition 1. *Let $f_1, f_2, \ldots, f_n$ be linearly independent functions and define*

$$W_n = \mathrm{span}\{f_1, f_2, \ldots, f_n\}.$$

The space W_n is said to be invariant under the nonlinear operator F if

$$F[u] \in W_n, \qquad \text{for every } u \in W_n.$$

Proposition 1 ([18])**.** *Let $f_1, \ldots, f_n$ be a fundamental system of solutions of the linear ordinary differential equation*

$$L[y] := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y = 0.$$

Let

$$F[y] = F(x, y, y', \ldots, y^{(k)})$$

be a differential operator of order $k \leq n-1$. Then the subspace

$$W_n = \mathrm{span}\{f_1, \ldots, f_n\}$$

is invariant under F if and only if

$$L[F[y]] = 0$$

for every function satisfying

$$L[y] = 0.$$

Proposition 2 (Reduction Principle)**.** *Assume that the finite-dimensional subspace*

$$W_n = \mathrm{span}\{f_1, f_2, \ldots, f_n\}$$

is invariant under the nonlinear operator F. Then a function of the form

$$u(t, x) = \sum_{i=1}^{n} u_i(t) f_i(x)$$

is a solution of Equation (3) if and only if the coefficient functions

$$u_1(t), u_2(t), \ldots, u_n(t)$$

satisfy the finite-dimensional system of ordinary differential equations

$$\begin{cases} u_1'' = F_1(u_1,\dots,u_n), \\ u_2'' = F_2(u_1,\dots,u_n), \\ \quad\vdots \\ u_n'' = F_n(u_1,\dots,u_n), \end{cases}$$

where the functions F_i are determined by the representation

$$F\left(\sum_{i=1}^{n} c_i f_i(x)\right) = \sum_{i=1}^{n} F_i(c_1,\dots,c_n) f_i(x).$$

Conversely, every solution of the reduced system generates an exact solution of Equation (3).

3. Applications

In this section, we apply the invariant subspace method developed in Section 2 to several representative nonlinear variable-coefficient beam-type equations. The selected examples were chosen because their variable coefficients and nonlinear terms satisfy compatibility conditions that allow finite-dimensional invariant subspaces to be constructed explicitly. They illustrate different invariant structures, including polynomial, trigonometric, hyperbolic, and negative-power invariant spaces, and demonstrate the flexibility of the proposed approach. The purpose of these examples is not to provide a complete classification of nonlinear beam-type equations but to show how the invariant subspace method can be systematically applied to construct explicit exact solutions.

For each example, we verify the invariance of the associated nonlinear operator, derive the corresponding reduced system of ordinary differential equations, and obtain explicit exact invariant solutions.

Example 1. *Consider the nonlinear fourth-order evolution equation*

$$u_{tt} = (x u_{xx})_{xx} + u^2 - \frac{1}{2}x^2 u\, u_{xx}, \qquad (t,x) \in (0,\infty) \times \mathbb{R}. \tag{4}$$

Define the associated nonlinear operator

$$F[u] = (x u_{xx})_{xx} + u^2 - \frac{1}{2}x^2 u\, u_{xx}.$$

We verify that the finite-dimensional space

$$W = \mathrm{span}\{1, x^2\}$$

is invariant under the nonlinear operator F.

Verification of invariance.

Let

$$u(t,x) = y_0(t) + y_1(t)x^2.$$

Direct substitution into the operator F gives

$$F[u] = y_0^2 + y_0 y_1 x^2.$$

Therefore,

$$F[u] \in W,$$

and hence, W is invariant under the nonlinear operator F.

Reduced dynamical system.

Applying Proposition 2, Equation (4) admits invariant solutions of the form

$$u(t,x) = y_0(t) + y_1(t)x^2,$$

where the coefficient functions satisfy

$$y_0'' = y_0^2, \qquad y_1'' = y_0 y_1. \tag{5}$$

Construction of an exact solution.

A particular solution of the first equation in (5) is

$$y_0(t) = \frac{6}{(t-c)^2},$$

where c is an arbitrary constant.

Substituting this expression into the second equation gives

$$y_1'' = \frac{6}{(t-c)^2} y_1.$$

Introducing the change of variable

$$z = t - c,$$

we obtain

$$\frac{d^2 y_1}{dz^2} = \frac{6}{z^2} y_1, \qquad z \neq 0.$$

Assuming a solution of the form

$$y_1 = z^m,$$

leads to the characteristic equation

$$m(m-1) = 6,$$

whose roots are

$$m = 3, \qquad m = -2.$$

Therefore,

$$y_1(t) = C_1(t-c)^3 + C_2(t-c)^{-2},$$

where C_1 and C_2 are arbitrary constants.

Exact invariant solution.

Consequently,

$$u(t,x) = \frac{6}{(t-c)^2} + \Big(C_1(t-c)^3 + C_2(t-c)^{-2}\Big)x^2, \qquad t \neq c. \tag{6}$$

A direct substitution verifies that the exact solution (6) satisfies Equation (4).

Figure 1 presents the exact invariant solution (6) for the parameter values $c = 0$, $C_1 = 0.02$, and $C_2 = 1$. The figure illustrates the temporal evolution of the polynomial invariant solution obtained through the invariant subspace reduction.

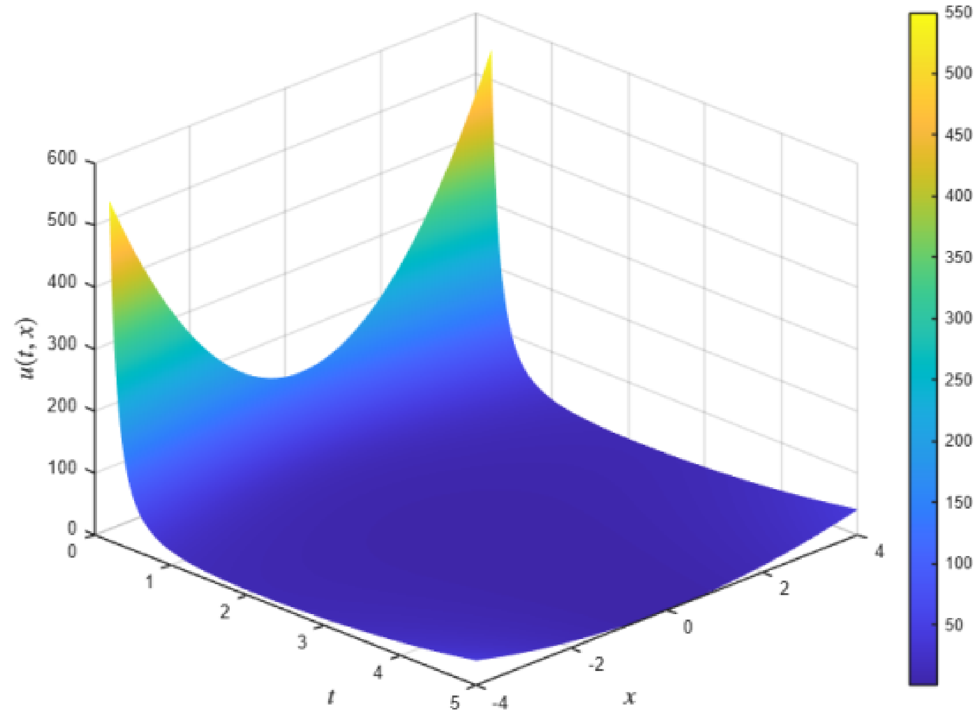

Figure 1. Three-dimensional surface corresponding to the exact invariant solution (6) of Equation (4) for $c = 0$, $C_1 = 0.02$, and $C_2 = 1$.

Example 2. *Consider the nonlinear beam-type equation*

$$u_{tt} = (x^2 u_{xx})_{xx} + u - x u_x + \frac{1}{4} u_x^2, \qquad (t, x) \in (0, \infty) \times \mathbb{R}. \tag{7}$$

Define the associated nonlinear operator

$$F[u] = (x^2 u_{xx})_{xx} + u - x u_x + \frac{1}{4} u_x^2.$$

We verify that the finite-dimensional space

$$W = \operatorname{span}\{1, x, x^2\}$$

is invariant under the nonlinear operator F.

Verification of invariance.

Let

$$u(t, x) = y_0(t) + y_1(t) x + y_2(t) x^2.$$

Direct substitution into the operator F gives

$$F[u] = \left(4y_2 + y_0 + \frac{1}{4} y_1^2\right) + y_1 y_2 x + \left(-y_2 + y_2^2\right) x^2.$$

Since $F[u] \in W$, the space W is invariant under the nonlinear operator F.

Reduced dynamical system.

Applying Proposition 2, Equation (7) admits invariant solutions of the form

$$u(t, x) = y_0(t) + y_1(t) x + y_2(t) x^2,$$

where the coefficient functions satisfy

$$\begin{cases} y_0'' = 4y_2 + y_0 + \frac{1}{4}y_1^2, \\ y_1'' = y_1 y_2, \\ y_2'' = -y_2 + y_2^2. \end{cases} \tag{8}$$

The reduced system admits several families of exact solutions. Three representative cases are presented below.

Case 1. Constant solution **($y_2 = 0$).**

Then

$$y_1(t) = A_1 t + A_2,$$

and

$$y_0'' - y_0 = \frac{1}{4}(A_1 t + A_2)^2.$$

Solving the above equation yields

$$y_0(t) = B_1 e^t + B_2 e^{-t} - \frac{A_1^2}{4}t^2 - \frac{A_1 A_2}{2}t - \frac{A_1^2}{2} - \frac{A_2^2}{4}.$$

Hence,

$$\begin{aligned} u(t,x) = B_1 e^t + B_2 e^{-t} - \frac{A_1^2}{4}t^2 - \frac{A_1 A_2}{2}t - \frac{A_1^2}{2} - \frac{A_2^2}{4} \\ + (A_1 t + A_2)x. \end{aligned} \tag{9}$$

Case 2. Constant solution **($y_2 = 1$).**

Then

$$y_1(t) = A_1 e^t + A_2 e^{-t},$$

and

$$y_0'' - y_0 = 4 + \frac{1}{4}(A_1 e^t + A_2 e^{-t})^2.$$

Solving this equation gives

$$y_0(t) = B_1 e^t + B_2 e^{-t} - 4 - \frac{A_1 A_2}{2} + \frac{A_1^2}{12}e^{2t} + \frac{A_2^2}{12}e^{-2t}.$$

Hence,

$$\begin{aligned} u(t,x) = B_1 e^t + B_2 e^{-t} - 4 - \frac{A_1 A_2}{2} + \frac{A_1^2}{12}e^{2t} + \frac{A_2^2}{12}e^{-2t} \\ + \left(A_1 e^t + A_2 e^{-t}\right)x + x^2. \end{aligned} \tag{10}$$

Case 3. Nonconstant solution.

The third equation in (8) *admits the nonconstant solution*

$$y_2(t) = 1 - \frac{3}{2}\operatorname{sech}^2\left(\frac{t-C}{2}\right).$$

The corresponding solution of the second equation is

$$y_1(t) = A\operatorname{sech}^2\left(\frac{t-C}{2}\right).$$

Consequently, the first equation becomes

$$y_0'' - y_0 = 4 - 6\,\mathrm{sech}^2\left(\frac{t-C}{2}\right) + \frac{A^2}{4}\,\mathrm{sech}^4\left(\frac{t-C}{2}\right).$$

Setting

$$E = e^{t-C},$$

a particular solution is given by

$$\begin{aligned}\Phi(E) := \frac{2}{3E(1+E)^2}\Big[&2A^2E + A^2 - 18E^4\ln E + 18E^4\ln(1+E)\\ &- 36E^3\ln E + 36E^3\ln(1+E) - 24E^3\\ &- 18E^2\ln E + 36E^2\ln(1+E) - 30E^2\\ &+ 36E\ln(1+E) + 12E + 18\ln(1+E) + 18\Big].\end{aligned}$$

The expression for $\Phi(E)$ was obtained by symbolic computation and verified by direct substitution into the reduced ordinary differential equation.

Therefore,

$$y_0(t) = B_1e^t + B_2e^{-t} + \Phi(e^{t-C}).$$

Hence,

$$\begin{aligned}u(t,x) = {} & B_1e^t + B_2e^{-t} + \Phi(e^{t-C})\\ & + A\,\mathrm{sech}^2\left(\frac{t-C}{2}\right)x\\ & + \left(1 - \frac{3}{2}\,\mathrm{sech}^2\left(\frac{t-C}{2}\right)\right)x^2.\end{aligned} \tag{11}$$

A direct substitution verifies that the exact solutions (9), (10), and (11) satisfy Equation (7).

Figure 2 presents the three families of exact invariant solutions corresponding to Cases 1–3. Panels (a), (b), and (c) illustrate the different qualitative behaviors generated by distinct solutions of the reduced dynamical system while preserving the same polynomial invariant space.

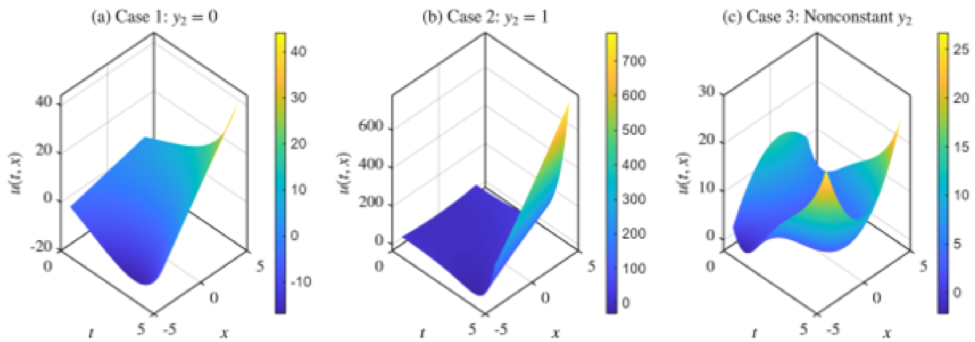

Figure 2. Surface plots of the exact invariant solutions corresponding to Cases 1–3 of Equation (7).

Example 3. *Consider the nonlinear fourth-order equation*

$$u_{tt} = (u_{xx})_{xx} + u + \sigma u_{xx} + \left(u + \sigma u_{xx}\right)^2, \qquad (t,x) \in (0,\infty) \times \mathbb{R}, \tag{12}$$

where $\sigma \in \{-1, 1\}$.

Define the associated nonlinear operator

$$F[u] = (u_{xx})_{xx} + u + \sigma u_{xx} + \left(u + \sigma u_{xx}\right)^2.$$

Verification of invariance.

For the case $\sigma = 1$, consider the finite-dimensional space

$$W_1 = \operatorname{span}\{1, \sin x, \cos x\}.$$

Let

$$u(t,x) = y_0(t) + y_1(t)\sin x + y_2(t)\cos x.$$

Direct substitution into the operator F gives

$$F[u] = y_0 + y_0^2 + y_1 \sin x + y_2 \cos x.$$

Therefore,

$$F[u] \in W_1,$$

and hence, W_1 is invariant under the nonlinear operator F.

Proceeding in the same manner, for the case $\sigma = -1$, consider

$$W_2 = \operatorname{span}\{1, \sinh x, \cosh x\}.$$

Let

$$u(t,x) = y_0(t) + y_1(t)\sinh x + y_2(t)\cosh x.$$

Then

$$F[u] = y_0 + y_0^2 + y_1 \sinh x + y_2 \cosh x,$$

which implies that

$$F[u] \in W_2.$$

Hence, W_2 is also invariant under the nonlinear operator F.

Reduced dynamical system.

Applying Proposition 2, the corresponding invariant solutions satisfy

$$\begin{cases} y_0'' = y_0 + y_0^2, \\ y_1'' = y_1, \\ y_2'' = y_2. \end{cases} \tag{13}$$

The last two equations have the general solutions

$$y_1(t) = C_1 e^t + C_2 e^{-t}, \qquad y_2(t) = C_3 e^t + C_4 e^{-t},$$

whereas the first equation admits the explicit solutions

$$y_0(t) = 0, \qquad y_0(t) = -1,$$

and

$$y_0(t) = -\frac{3}{2}\operatorname{sech}^2\left(\frac{t - C}{2}\right).$$

Exact invariant solutions.

Combining the above results, for $\sigma = 1$ we obtain

$$u(t,x) = y_0(t) + \left(C_1 e^t + C_2 e^{-t}\right)\sin x + \left(C_3 e^t + C_4 e^{-t}\right)\cos x, \tag{14}$$

where

$$y_0(t) \in \left\{0,\ -1,\ -\frac{3}{2}\operatorname{sech}^2\left(\frac{t-C}{2}\right)\right\}.$$

Similarly, for $\sigma = -1$,

$$u(t,x) = y_0(t) + \left(C_1 e^t + C_2 e^{-t}\right)\sinh x + \left(C_3 e^t + C_4 e^{-t}\right)\cosh x, \tag{15}$$

where $y_0(t)$ belongs to the same family of solutions.

A direct substitution confirms that the exact solutions (14) and (15) satisfy Equation (12).

Figure 3 presents representative exact invariant solutions corresponding to the trigonometric and hyperbolic invariant spaces. Panel (a) illustrates the solution associated with $\sigma = 1$, whereas panel (b) corresponds to $\sigma = -1$, demonstrating that the same nonlinear beam equation admits two different invariant structures.

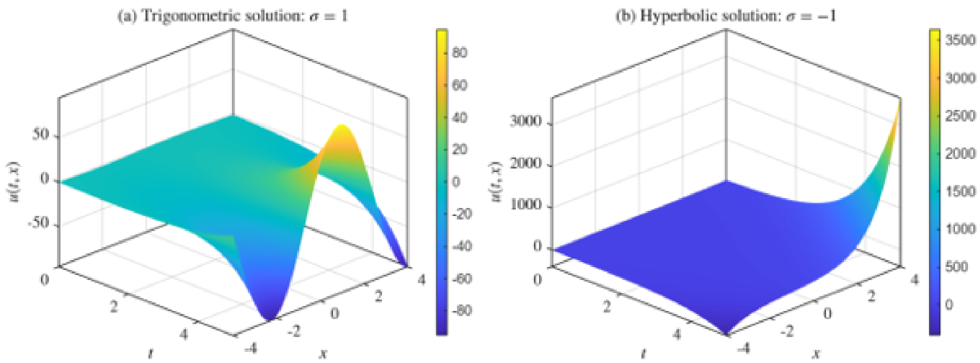

Figure 3. Surface plots of the exact invariant solutions of (12). (**a**) Trigonometric invariant solution corresponding to $\sigma = 1$. (**b**) Hyperbolic invariant solution corresponding to $\sigma = -1$.

Example 4. *Consider the nonlinear fourth-order evolution equation*

$$u_{tt} = \left(x^4 u_{xx}\right)_{xx} + xu^2 - \frac{1}{4}x^3 u_x^2, \qquad (t,x) \in (0,\infty) \times (\mathbb{R} \setminus \{0\}). \tag{16}$$

Define the associated nonlinear operator

$$F[u] = \left(x^4 u_{xx}\right)_{xx} + xu^2 - \frac{1}{4}x^3 u_x^2.$$

We verify that the finite-dimensional space

$$W = \operatorname{span}\left\{\frac{1}{x}, \frac{1}{x^2}\right\}$$

is invariant under the nonlinear operator F.

Verification of invariance.

Let

$$u(t,x) = \frac{y_1(t)}{x} + \frac{y_2(t)}{x^2},$$

where y_1 and y_2 are sufficiently smooth functions of t. Then

$$u_x = -\frac{y_1}{x^2} - \frac{2y_2}{x^3}, \qquad u_{xx} = \frac{2y_1}{x^3} + \frac{6y_2}{x^4}.$$

Therefore,

$$x^4 u_{xx} = 2y_1 x + 6y_2, \qquad \left(x^4 u_{xx}\right)_{xx} = 0.$$

Furthermore,

$$xu^2 = \frac{y_1^2}{x} + \frac{2y_1y_2}{x^2} + \frac{y_2^2}{x^3},$$

and

$$\frac{1}{4}x^3u_x^2 = \frac{1}{4}\frac{y_1^2}{x} + \frac{y_1y_2}{x^2} + \frac{y_2^2}{x^3}.$$

Hence,

$$F[u] = \frac{3}{4}\frac{y_1^2}{x} + \frac{y_1y_2}{x^2},$$

which belongs to W. Therefore, W is invariant under the nonlinear operator F.

Reduced dynamical system.

Applying Proposition 2, Equation (16) admits invariant solutions of the form

$$u(t,x) = \frac{y_1(t)}{x} + \frac{y_2(t)}{x^2},$$

where the coefficient functions satisfy

$$\begin{cases} y_1'' = \dfrac{3}{4}y_1^2, \\ y_2'' = y_1y_2. \end{cases} \tag{17}$$

Construction of an exact solution.

A particular solution of the first equation in (17) is

$$y_1(t) = \frac{8}{(t-C)^2}, \qquad t \neq C,$$

since

$$y_1'' = \frac{48}{(t-C)^4} = \frac{3}{4}y_1^2.$$

Substituting this expression into the second equation gives

$$y_2'' = \frac{8}{(t-C)^2}y_2.$$

Introducing the change of variable $z = t - C$, we obtain

$$\frac{d^2y_2}{dz^2} = \frac{8}{z^2}y_2, \qquad z \neq 0.$$

Assuming a solution of the form

$$y_2 = z^m$$

leads to the characteristic equation

$$m(m-1) = 8.$$

Its roots are

$$m_1 = \frac{1+\sqrt{33}}{2}, \qquad m_2 = \frac{1-\sqrt{33}}{2}.$$

Therefore,

$$y_2(t) = C_1(t-C)^{\frac{1+\sqrt{33}}{2}} + C_2(t-C)^{\frac{1-\sqrt{33}}{2}},$$

where C_1 and C_2 are arbitrary constants.

Exact invariant solution.

Consequently, Equation (16) possesses the exact invariant solution

$$u(t,x) = \frac{8}{x(t-C)^2} + \frac{C_1(t-C)^{\frac{1+\sqrt{33}}{2}} + C_2(t-C)^{\frac{1-\sqrt{33}}{2}}}{x^2}, \qquad x \neq 0, \quad t \neq C. \tag{18}$$

A direct substitution confirms that the exact solution (18) satisfies Equation (16).

Figure 4 presents the exact invariant solution (18). The solution exhibits a spatial singularity at $x = 0$, which is consistent with the negative-power basis functions defining the invariant space.

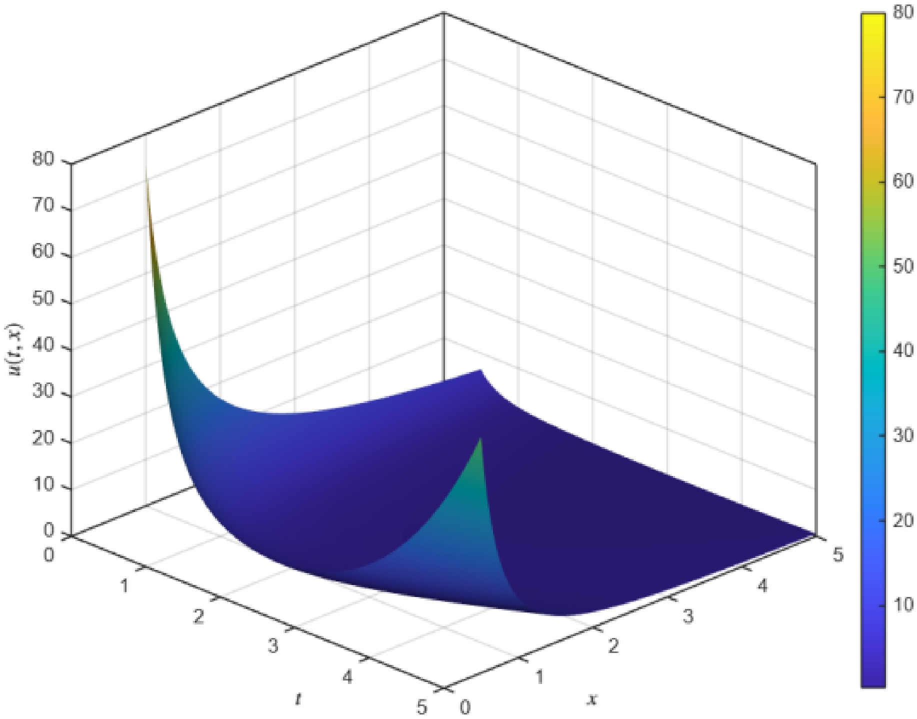

Figure 4. Surface plot of the exact invariant solution (18) of Equation (16).

Example 5. *Consider the nonlinear fourth-order equation*

$$u_{tt} = (x^2 u_{xx})_{xx} + u + u_x u_{xx}, \qquad (t,x) \in (0,\infty) \times \mathbb{R}. \tag{19}$$

Define the associated nonlinear operator

$$F[u] = (x^2 u_{xx})_{xx} + u + u_x u_{xx}.$$

We verify that the finite-dimensional space

$$W = \text{span}\{1, x, x^2\}$$

is invariant under the nonlinear operator F.

Verification of invariance.

Let

$$u(t,x) = y_0(t) + y_1(t)x + y_2(t)x^2.$$

Direct substitution into the operator F gives

$$F[u] = (4y_2 + y_0 + 2y_1y_2) + (y_1 + 4y_2^2)x + y_2x^2,$$

which belongs to the invariant space W. Therefore,

$$F[u] \in W,$$

and hence W is invariant under the nonlinear operator F.
Reduced dynamical system.
Applying Proposition 2, Equation (19) admits invariant solutions of the form

$$u(t,x) = y_0(t) + y_1(t)x + y_2(t)x^2,$$

where the coefficient functions satisfy

$$\begin{cases} y_0'' = 4y_2 + y_0 + 2y_1y_2, \\ y_1'' = y_1 + 4y_2^2, \\ y_2'' = y_2. \end{cases} \tag{20}$$

Construction of an exact solution.
The third equation has the general solution

$$y_2(t) = Ae^t + Be^{-t}.$$

Substituting this expression into the second equation gives

$$y_1'' - y_1 = 4(Ae^t + Be^{-t})^2,$$

whose solution is

$$y_1(t) = Ce^t + De^{-t} + \frac{4}{3}A^2e^{2t} - 8AB + \frac{4}{3}B^2e^{-2t}.$$

The first equation becomes

$$y_0'' - y_0 = 4y_2 + 2y_1y_2.$$

A particular solution is

$$\begin{aligned} y_0^{(p)}(t) = &\frac{2}{3}ACe^{2t} + \frac{2}{3}BDe^{-2t} - (2AD + 2BC) \\ &+ \frac{1}{3}A^3e^{3t} + \frac{1}{3}B^3e^{-3t} \\ &+ \left(2A - \frac{20}{3}A^2B\right)te^t - \left(2B - \frac{20}{3}AB^2\right)te^{-t}. \end{aligned}$$

Hence,

$$y_0(t) = Ee^t + Fe^{-t} + y_0^{(p)}(t).$$

Exact invariant solution.
Consequently,

$$u(t,x) = y_0(t) + y_1(t)x + y_2(t)x^2, \tag{21}$$

where

$$y_2(t) = Ae^t + Be^{-t},$$
$$y_1(t) = Ce^t + De^{-t} + \frac{4}{3}A^2e^{2t} - 8AB + \frac{4}{3}B^2e^{-2t},$$

and

$$y_0(t) = Ee^t + Fe^{-t} + y_0^{(p)}(t).$$

A direct substitution confirms that the exact solution (21) satisfies Equation (19).

Figure 5 presents the exact invariant solution (21). The figure illustrates the interaction between the polynomial invariant structure and the derivative-dependent nonlinear term, demonstrating that the invariant subspace method remains effective for nonlinear beam-type equations involving derivative interactions.

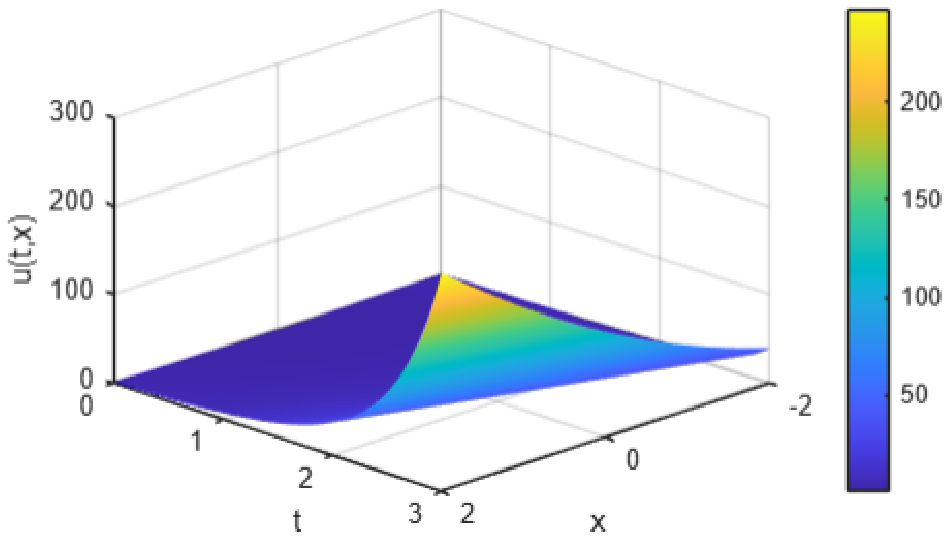

Figure 5. Surface plot of the exact invariant solution (21).

4. Conclusions

In this paper, we applied the invariant subspace method to study a class of nonlinear variable-coefficient beam-type equations. By constructing suitable finite-dimensional invariant spaces for the associated nonlinear operators, we reduced the original fourth-order nonlinear partial differential equations to finite systems of ordinary differential equations and obtained several families of explicit exact solutions.

The examples presented in this work demonstrate that the invariant subspace method is an effective tool for constructing exact solutions of nonlinear beam-type equations with different nonlinear structures. In particular, polynomial, trigonometric, hyperbolic, and negative-power invariant spaces were identified, leading to a variety of exact invariant solutions. These results show that the method remains applicable to beam equations with variable coefficients as well as derivative-dependent nonlinearities.

The explicit solutions obtained in this paper provide additional analytical results for nonlinear beam-type equations and may serve as useful benchmark solutions for future theoretical and numerical investigations. More generally, the present work illustrates the versatility of the invariant subspace method for studying higher-order nonlinear evolution equations.

Future work will focus on extending the present approach to broader classes of nonlinear beam models, including fractional and coupled systems, as well as investigating more general invariant spaces for higher-order nonlinear evolution equations.

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References

1. Timoshenko, S.P.; Gere, J.M. *Theory of Elastic Stability*, 2nd ed.; McGraw–Hill: New York, NY, USA, 1961.
2. Nayfeh, A.H.; Pai, P.F. *Linear and Nonlinear Structural Mechanics*; Wiley: Hoboken, NJ, USA, 2004.
3. Timoshenko, S.P.; Young, D.H. *Vibration Problems in Engineering*, 4th ed.; Wiley: New York, NY, USA, 1974.
4. Landau, L.D.; Lifshitz, E.M. *Theory of Elasticity*, 3rd ed.; Butterworth–Heinemann: Oxford, UK, 1986.
5. Pellicano, F.; Mastroddi, F. Nonlinear dynamics of beams on elastic foundation. *Nonlinear Dyn.* **1997**, *14*, 335–355. [CrossRef]
6. Nayfeh, A.H. *Nonlinear Interactions: Analytical, Computational, and Experimental Methods*; Wiley: New York, NY, USA, 2000.
7. Lazer, A.C.; McKenna, P.J. Large-amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis. *SIAM Rev.* **1990**, *32*, 537–578. [CrossRef]
8. Gazzola, F. *Mathematical Models for Suspension Bridges*; MS&A Vol. 15; Springer: New York, NY, USA, 2015.
9. Huang, D.; Li, X.; Yu, S. Lie Symmetry Classification of the Generalized Nonlinear Beam Equation. *Symmetry* **2017**, *9*, 115. [CrossRef]
10. Gazzola, F.; Grunau, H.C.; Sweers, G. *Polyharmonic Boundary Value Problems*; Lecture Notes in Mathematics 1991; Springer: Berlin, Germany, 2010.
11. Pausader, B. Global well-posedness for energy-critical fourth-order Schrödinger equations in the radial case. *Dyn. Partial Differ. Equ.* **2007**, *4*, 197–225. [CrossRef]
12. Wei, H.; Ji, S. Periodic solutions of a semilinear Euler–Bernoulli beam equation with variable coefficients. *J. Math. Anal. Appl.* **2020**, *489*, 124150.
13. Wei, H. Infinitely many periodic solutions for a semilinear Euler–Bernoulli beam equation with variable coefficients. *Commun. Nonlinear Sci. Numer. Simul.* **2021**, *97*, 105756. [CrossRef]
14. Chen, B.; Gao, Y.; Li, Y. Periodic solutions to nonlinear Euler–Bernoulli beam equations. *Commun. Math. Sci.* **2019**, *17*, 2005–2034. [CrossRef]
15. Deng, P.; Zheng, J.; Zhu, G. Well-posedness and stability for a nonlinear Euler–Bernoulli beam equation. *Commun. Anal. Mech.* **2024**, *16*, 193–216. [CrossRef]
16. Galaktionov, V.A. Invariant subspaces and new explicit solutions to evolution equations with quadratic nonlinearities. *Proc. R. Soc. Edinb. Sect. A Math.* **1995**, *125*, 225–246. [CrossRef]
17. Galaktionov, V.A. On new exact blow-up solutions for nonlinear heat conduction equations with source and applications. *Differ. Integral Equ.* **1990**, *3*, 863–874. [CrossRef]
18. Galaktionov, V.A.; Svirshchevskii, S.R. *Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations*; Chapman & Hall/CRC: Boca Raton, FL, USA, 2007.
19. King, J.R. Exact polynomial solutions to some nonlinear diffusion equations. *J. Phys. A Math. Gen.* **1993**, *26*, 5079–5090.
20. Ma, W.X. A refined invariant subspace method and applications to evolution equations. *Sci. China Math.* **2012**, *55*, 1769–1778. [CrossRef]
21. Prakash, P.; Priyendhu, K.S.; Lakshmanan, M. Invariant subspace method for $(m+1)$-dimensional non-linear time-fractional partial differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **2022**, *111*, 106436. [CrossRef]
22. Qu, G.; Wang, M.; Shen, S. Applications of the invariant subspace method on searching explicit solutions to certain special-type non-linear evolution equations. *Front. Phys.* **2023**, *11*, 1160391. [CrossRef]