mathematics

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Six-Step Dynamics and Explicit Solutions for a Three-Dimensional Rational Difference System with Two-Periodic Coefficients

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Abstract

This paper investigates a three-dimensional nonlinear rational difference system with two-periodic coefficients, incorporating nonlinear rational interactions and periodically varying affine terms. An algebraic transformation is employed to convert the original system into an auxiliary cyclic inverse system, which is then reduced to a set of independent sixth-order recurrence relations. This reduction provides explicit solution formulas for solutions satisfying the nonvanishing conditions required for the original system and the auxiliary transformation. Several special cases are subsequently derived under additional assumptions on the periodic coefficients. Furthermore, the behavior of the multiplicative factors appearing in the auxiliary solution formulas is analyzed to characterize their dependence on the periodic coefficients. Finally, numerical examples are presented to illustrate the theoretical results for selected parameter configurations.

Keywords: rational difference systems; linear difference equations; three-dimensional discrete systems; two-periodic coefficients; closed-form solutions; algebraic transformation; periodic recurrence relations

MSC: 39A10; 39A20

1. Introduction

Difference equations play an important role in modeling discrete-time phenomena encountered in many scientific fields, including biology, engineering, economics, computer science, and time series analysis. Their ability to describe nonlinear interactions, delayed effects, and the relationships among different components of dynamical systems has made them a valuable tool in both theoretical and applied research. As a result, difference equation models have been widely used to study nonlinear dynamical systems, biological rhythms, neural networks [1,2], and stochastic time series [3–6], where discrete-time formulations naturally arise. From a mathematical point of view, difference equations and their associated systems exhibit a wide range of qualitative behaviors, including boundedness, persistence, oscillation, periodicity, convergence, and stability. Understanding these properties, as well as obtaining explicit and asymptotic descriptions of their solutions, continues to be an active area of research in the theory of discrete dynamical systems.

Among the various topics in the theory of difference equations, the derivation of explicit solution formulas has attracted considerable attention, as it provides a valuable

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Academic Editors: Mihaela Neamțu, Eva Kaslik, Anca Rădulescu and Luigi Fortuna

Received: 7 September 2026
Revised: 12 November 2026
Accepted: 16 November 2026
Published: 18 November 2026

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analytical framework for understanding the behavior of nonlinear discrete systems. Although general methods for solving nonlinear difference equations are often unavailable, certain classes of such equations can be transformed into simpler recursive forms through appropriate algebraic techniques, thereby making explicit solutions accessible and facilitating the investigation of their qualitative properties. In particular, these approaches have been employed to study the boundedness, stability, periodicity, and global behavior of solutions [7–9]. Further contributions have focused on obtaining explicit representations of solution sequences and analyzing their asymptotic properties, thereby illustrating the usefulness of closed-form expressions in the qualitative study of nonlinear recurrences [10,11]. In addition, explicit solution techniques have been extended to higher- order and multidimensional difference equations, including the derivation of closed-form expressions and the characterization of forbidden sets [12]. More recent investigations have explored connections between nonlinear difference systems and special sequences, particularly generalized Fibonacci sequences [13], as well as algebraic transformations leading to explicit solutions of multidimensional systems [14]. These developments have also extended to three-dimensional cyclic delayed rational difference systems, further demonstrating the potential of algebraic methods in the analysis of higher-dimensional nonlinear discrete dynamics [15]. Collectively, these studies highlight the continuing importance of explicit solvability as a means of understanding the algebraic structure and qualitative behavior of nonlinear difference systems.

An early contribution to this line of research was made by Elabbasy et al. [16], who studied the qualitative behavior of the higher-order rational difference equation

$$s_{k+1} = \frac{\varepsilon_1 s_k s_{k-1}}{\lambda_1 s_k - \mu_1} + \mu_2, \; k \geq 0.$$

They established several qualitative properties of its solutions and showed that suitable algebraic transformations can simplify the analysis of nonlinear recursive equations and, in some cases, lead to explicit solution formulas. Their results encouraged further studies on nonlinear rational difference equations for which closed-form solutions can be derived. Building on this line of research, Haddad et al. [17] considered the following coupled second-order rational difference system:

$$s_{k+1} = \frac{\varepsilon_1 s_k r_{k-1}}{r_k - \mu_1} + \mu_2, \; r_{k+1} = \frac{\varepsilon_2 r_k t_{k-1}}{t_k - \mu_2} + \mu_1, \; k \geq 0.$$

Using suitable algebraic transformations, they reduced the nonlinear system to a family of first-order linear difference equations, from which explicit solution formulas were obtained. They also established conditions for the existence of well-defined solutions and investigated the periodic and asymptotic behavior of the resulting sequences. This study showed that algebraic transformation techniques can also be applied effectively to coupled nonlinear rational systems. Yazlık and Kara [18] extended this framework by introducing period-two coefficients into the coupled rational difference system:

$$s_{k+1} = \frac{\varepsilon_k^{(1)} s_k r_{k-1}}{r_k - \mu_k^{(1)}} + \mu_k^{(2)}, \; r_{k+1} = \frac{\varepsilon_k^{(2)} r_k t_{k-1}}{t_k - \mu_k^{(2)}} + \mu_k^{(1)}, \; k \geq 0.$$

The resulting model is non-autonomous, with coefficients that vary periodically. Nevertheless, the authors derived explicit solution formulas and investigated the qualitative behavior of the corresponding solutions. Their results showed that the introduction of periodic coefficients leads to more diverse solution behavior while preserving the analytical tractability of the system. More recently, Ghezal et al. [19] investigated a class of

three-dimensional second-order rational systems featuring cyclic interactions among three dependent variables:

$$\begin{aligned} s_{k+1} &= \frac{\varepsilon_1 s_k r_{k-1}}{r_k - \lambda_1 r_{k-1} - \mu_1} + \lambda_3 s_k + \mu_3, \\ r_{k+1} &= \frac{\varepsilon_2 r_k t_{k-1}}{t_k - \lambda_2 t_{k-1} - \mu_2} + \lambda_1 r_k + \mu_1, \\ t_{k+1} &= \frac{\varepsilon_3 t_k s_{k-1}}{s_k - \lambda_3 s_{k-1} - \mu_3} + \lambda_2 t_k + \mu_2, k \geq 0. \end{aligned}$$

Using suitable auxiliary sequences and algebraic transformations, they reduced the nonlinear system to an equivalent higher-order periodic recurrence relation, which allowed them to derive explicit solution formulas. They also established conditions for the existence of well-defined solutions and examined the resulting dynamical behavior through numerical examples. This study extended previous analytical results from coupled two-component models to a three-component rational system with cyclic interactions.

Although considerable progress has been made in the analysis of rational difference systems, the extension of periodic coefficients to three-dimensional cyclic models has received relatively little attention. Existing three-dimensional cyclic systems are typically formulated with constant parameters, whereas periodic coefficients have mainly been studied in lower-dimensional settings. Extending these models to the non-autonomous case is not merely a direct generalization. The combination of cyclic interactions, nonlinear rational terms, and periodically varying coefficients leads to a more involved recursive structure, making the analytical reduction of the system and the derivation of explicit solution formulas substantially more difficult. This naturally raises the question of whether the explicit solvability established for autonomous three-dimensional models can be maintained when periodic coefficients are introduced. To address this problem, we introduce a new three-dimensional nonlinear rational difference system with periodic coefficients. The proposed model extends the periodic two-dimensional system studied by Yazlık and Kara [18] and the autonomous three-dimensional cyclic system developed by Ghezal et al. [19], combining periodic coefficients with cyclic interactions among three dependent variables:

$$\begin{aligned} s_{k+1} &= \frac{\varepsilon_k^{(1)} s_k r_{k-1}}{r_k - \lambda_{k-1}^{(1)} r_{k-1} - \mu_{k-1}^{(1)}} + \lambda_k^{(3)} s_k + \mu_k^{(3)}, \\ r_{k+1} &= \frac{\varepsilon_k^{(2)} r_k t_{k-1}}{t_k - \lambda_{k-1}^{(2)} t_{k-1} - \mu_{k-1}^{(2)}} + \lambda_k^{(1)} r_k + \mu_k^{(1)}, \\ t_{k+1} &= \frac{\varepsilon_k^{(3)} t_k s_{k-1}}{s_k - \lambda_{k-1}^{(3)} s_{k-1} - \mu_{k-1}^{(3)}} + \lambda_k^{(2)} t_k + \mu_k^{(2)}, k \geq 0. \end{aligned} \tag{1}$$

The introduction of periodic coefficients together with cyclic interactions leads to a broader class of nonlinear rational systems while retaining an algebraic structure that permits explicit analytical treatment. The present work is closely related to [19], but considers a genuinely non-autonomous extension in which the coefficients are two-periodic. The objective is not to introduce a fundamentally new algebraic technique, but to determine whether the explicit-solvability mechanism established for the autonomous three-dimensional setting can be preserved when periodic coefficients are incorporated into the cyclic rational structure. The periodic extension is nontrivial because the coefficient sequences enter the auxiliary recurrences with shifted indices, producing periodic multiplicative factors that affect the resulting sixth-order recurrences and their explicit solution formulas. In particular, the present analysis derives closed-form expressions for the auxiliary sequences in each residue class modulo six and identifies the balance parameter governing the attenuation, critical, and amplification regimes. Thus, the contribution of the present paper lies in extending the explicit-solvability framework of [19] from constant coefficients to a two-periodic

coefficient setting and in deriving the corresponding solution structure in closed form. The underlying algebraic reduction is related to the approach in [19], whereas the periodic coefficient structure leads to different recurrence coefficients, multiplicative factors, and parameter-dependent asymptotic behavior.

The aim of this paper is to establish the explicit solvability of the proposed two-periodic system through a suitable algebraic transformation. We derive the associated auxiliary recurrences, obtain closed-form expressions for all well-defined solutions, and characterize the attenuation, critical, and amplification regimes induced by the periodic coefficient structure. In this way, the paper provides a precise non-autonomous extension of the solvability results available for the corresponding autonomous three-dimensional system.

The remainder of this paper is organized as follows. Section 2 presents the theoretical analysis of the proposed three-dimensional rational difference system and provides representative examples illustrating its dynamical behavior under different parameter settings. Section 3 concludes the paper by summarizing the main results and outlining possible directions for future research.

2. Algebraic Reduction and Closed-Form Solutions

This section establishes explicit solution formulas for system (1). The analysis is based on an algebraic transformation that converts the original nonlinear rational system into an equivalent auxiliary system. The auxiliary system is solved explicitly, and the resulting formulas are combined with the inverse transformation and Lemma 1 to obtain explicit representations of the original solution sequences.

Remark 1. *Throughout* this paper, we consider only well-defined solutions of system (1). Accordingly, we assume that the initial data generate a solution satisfying the nonvanishing conditions $r_k - \lambda^{(1)}_{k-1} r_{k-1} - \mu^{(1)}_{k-1} \neq 0$, $t_k - \lambda^{(2)}_{k-1} t_{k-1} - \mu^{(2)}_{k-1} \neq 0$, $s_k - \lambda^{(3)}_{k-1} s_{k-1} - \mu^{(3)}_{k-1} \neq 0$ and $s_{k-1} r_{k-1} t_{k-1} \neq 0$, for all $k \geq 0$. The first three conditions ensure that all rational expressions in the original system are well-defined, while the latter condition guarantees that divisions by (s_k), (r_k), and (t_k) are valid in the definition of the auxiliary sequences. Consequently, these conditions ensure that the auxiliary transformation is well-defined and that the original system can be equivalently reformulated in terms of the auxiliary variables along every admissible solution.

Before deriving the explicit solution formulas for the proposed nonlinear system, we recall a standard result for first-order linear difference equations. This result is used throughout the subsequent analysis after the nonlinear system is transformed into linear recurrence relations by algebraic transformations. The corresponding statement is given in the following lemma.

Lemma 1. *Let* $\{s_k, k \geq 0\}$ *be a solution of the first-order difference equation* $s_{k+1} = \theta_k s_k + \eta_k$, $k \geq 0$, *where* $(\theta_k)_{k\geq 0}$ *and* $(\eta_k)_{k\geq 0}$ *are given sequences of real numbers. Then,* $s_k = \left\{\prod_{l=0}^{k-1} \theta_l\right\} s_0 + \sum_{s=0}^{k-1} \left\{\prod_{l=s+1}^{k-1} \theta_l\right\} \eta_s$, $k \geq 0$, *where the empty product is taken to be* 1 *and the empty sum is taken to be* 0.

Throughout this paper, we consider solutions of system (1) for which the original system and the auxiliary transformation are well-defined. Specifically, for every $k \geq 0$, we require

$$r_k - \lambda^{(1)}_{k-1} r_{k-1} - \mu^{(1)}_{k-1} \neq 0, t_k - \lambda^{(2)}_{k-1} t_{k-1} - \mu^{(2)}_{k-1} \neq 0, s_k - \lambda^{(3)}_{k-1} s_{k-1} - \mu^{(3)}_{k-1} \neq 0,$$

together with

$$s_{k-1}r_{k-1}t_{k-1} \neq 0.$$

The first three conditions ensure that the rational expressions in system (1) are well-defined, whereas the latter condition allows the auxiliary variables to be introduced. Accordingly, the explicit solution formulas derived below apply to solutions satisfying these nonvanishing conditions throughout the interval under consideration. Combining the algebraic transformation with the linear recurrence results established in Lemma 1 yields explicit solution formulas for the original nonlinear system. The main result is stated in the following theorem.

Theorem 1. *Let* $\{(s_k, r_k, t_k), k \geq -1\}$ *be a solution of system (1), satisfying the nonvanishing conditions stated above, and suppose that all coefficient sequences are two-periodic. Then, for all* $k \geq 0$ *and* $l = 1, ..., 6$, *the sequences* (s_k), (r_k) *and* (t_k) *are given explicitly by*

$$\begin{aligned} s_{6k+l} &= \left\{\prod_{i=1}^{6k+l}\left(S_i + \lambda_{i-1}^{(3)}\right)\right\}s_0 + \sum_{s=0}^{6k+l-1}\left\{\prod_{i=s+2}^{6k+l}\left(S_i + \lambda_{i-1}^{(3)}\right)\right\}\mu_s^{(3)}, \\ r_{6k+l} &= \left\{\prod_{i=1}^{6k+l}\left(R_i + \lambda_{i-1}^{(1)}\right)\right\}r_0 + \sum_{s=0}^{6k+l-1}\left\{\prod_{i=s+2}^{6k+l}\left(R_i + \lambda_{i-1}^{(1)}\right)\right\}\mu_s^{(1)}, \\ t_{6k+l} &= \left\{\prod_{i=1}^{6k+l}\left(T_i + \lambda_{i-1}^{(2)}\right)\right\}t_0 + \sum_{s=0}^{6k+l-1}\left\{\prod_{i=s+2}^{6k+l}\left(T_i + \lambda_{i-1}^{(2)}\right)\right\}\mu_s^{(2)}, \end{aligned} \tag{2}$$

where $\{(S_k, R_k, T_k), k \geq 1\}$ *is the solution of system (3) with initial values*

$$S_1 = \frac{s_1 - \lambda_0^{(3)}s_0 - \mu_0^{(3)}}{s_0},\ R_1 = \frac{r_1 - \lambda_0^{(1)}r_0 - \mu_0^{(1)}}{r_0},\ T_1 = \frac{t_1 - \lambda_0^{(2)}t_0 - \mu_0^{(2)}}{t_0}.$$

Proof. The proof proceeds by applying a sequence of algebraic transformations that reduce the original nonlinear system to a family of linear recurrence relations. The nonlinear rational coupling among the three dependent variables is handled by first separating the fractional terms from the affine components. To this end, each equation of system (1) is rewritten in the following equivalent form:

$$\begin{aligned} \frac{s_{k+1} - \lambda_k^{(3)}s_k - \mu_k^{(3)}}{s_k} &= \frac{\varepsilon_k^{(1)}r_{k-1}}{r_k - \lambda_{k-1}^{(1)}r_{k-1} - \mu_{k-1}^{(1)}}, \\ \frac{r_{k+1} - \lambda_k^{(1)}r_k - \mu_k^{(1)}}{r_k} &= \frac{\varepsilon_k^{(2)}t_{k-1}}{t_k - \lambda_{k-1}^{(2)}t_{k-1} - \mu_{k-1}^{(2)}}, \\ \frac{t_{k+1} - \lambda_k^{(2)}t_k - \mu_k^{(2)}}{t_k} &= \frac{\varepsilon_k^{(3)}s_{k-1}}{s_k - \lambda_{k-1}^{(3)}s_{k-1} - \mu_{k-1}^{(3)}}, k \geq 0. \end{aligned}$$

Since only well-defined solutions are considered, all denominators remain nonzero. The resulting algebraic identities motivate the introduction of the following auxiliary sequences:

$$S_{k+1} = \frac{s_{k+1} - \lambda_k^{(3)}s_k - \mu_k^{(3)}}{s_k},\ R_{k+1} = \frac{r_{k+1} - \lambda_k^{(1)}r_k - \mu_k^{(1)}}{r_k},\ T_{k+1} = \frac{t_{k+1} - \lambda_k^{(2)}t_k - \mu_k^{(2)}}{t_k}, k \geq 0.$$

Using these auxiliary sequences, system (1) can be rewritten as the following cyclic inverse system:

$$S_{k+1} = \varepsilon_k^{(1)}/R_k,\ R_{k+1} = \varepsilon_k^{(2)}/T_k \text{ and } T_{k+1} = \varepsilon_k^{(3)}/S_k, k \geq 0.$$

The auxiliary variables remain cyclically coupled. Eliminating the intermediate variables yields the following third-step recurrence relations:

$$\begin{aligned} S_{k+1} &= \left(\varepsilon_k^{(1)}\varepsilon_{k-2}^{(3)}/\varepsilon_{k-1}^{(2)}\right)/S_{k-2}, \\ R_{k+1} &= \left(\varepsilon_k^{(2)}\varepsilon_{k-2}^{(1)}/\varepsilon_{k-1}^{(3)}\right)/R_{k-2}, \\ T_{k+1} &= \left(\varepsilon_k^{(3)}\varepsilon_{k-2}^{(2)}/\varepsilon_{k-1}^{(1)}\right)/T_{k-2}, k \geq 3. \end{aligned}$$

Further elimination of the intermediate variables leads to the following sixth-order recurrence relations:

$$\begin{aligned} S_{k+1} &= \left(\varepsilon_k^{(1)}\varepsilon_{k-2}^{(3)}\varepsilon_{k-4}^{(2)}/\varepsilon_{k-1}^{(2)}\varepsilon_{k-3}^{(1)}\varepsilon_{k-5}^{(3)}\right)S_{k-5}, \\ R_{k+1} &= \left(\varepsilon_k^{(2)}\varepsilon_{k-2}^{(1)}\varepsilon_{k-4}^{(3)}/\varepsilon_{k-1}^{(3)}\varepsilon_{k-3}^{(2)}\varepsilon_{k-5}^{(1)}\right)R_{k-5}, \\ T_{k+1} &= \left(\varepsilon_k^{(3)}\varepsilon_{k-2}^{(2)}\varepsilon_{k-4}^{(1)}/\varepsilon_{k-1}^{(1)}\varepsilon_{k-3}^{(3)}\varepsilon_{k-5}^{(2)}\right)T_{k-5}, k \geq 6. \end{aligned}$$

The preceding recurrence relations evolve independently within each residue class modulo six. Successive iteration then yields the following explicit expressions:

$$\begin{aligned} S_{6k+l} &= f_{l,k}\left(\varepsilon^{(1)},\varepsilon^{(2)},\varepsilon^{(3)}\right)g_l^{(1,3,2)}(\varepsilon)h_l^{(1,2,3)}(S_1,R_1,T_1), \\ R_{6k+l} &= f_{l,k}\left(\varepsilon^{(2)},\varepsilon^{(3)},\varepsilon^{(1)}\right)g_l^{(2,1,3)}(\varepsilon)h_l^{(3,2,1)}(S_1,R_1,T_1), \\ T_{6k+l} &= f_{l,k}\left(\varepsilon^{(3)},\varepsilon^{(1)},\varepsilon^{(2)}\right)g_l^{(3,2,1)}(\varepsilon)h_l^{(2,1,3)}(S_1,R_1,T_1), \end{aligned} \tag{3}$$

where the factors $f_{l,k}(.,.,.), l = 1...,6$, represent the cumulative effect of the periodic coefficients over each six-step cycle:

$$f_{k,l}(x,y,z) = \prod_{j=0}^{k-1} \frac{x_{6(k-j)+l-1}z_{6(k-j)+l-3}y_{6(k-j)+l-5}}{y_{6(k-j)+l-2}x_{6(k-j)+l-4}z_{6(k-j)+l-6}},$$

$$g_l^{(i,j,t)}(\varepsilon) = \sum_{u=1}^{6} \mathbf{1}_{\{l=u\}} \frac{\left(\varepsilon_{u-1}^{(i)}\right)^{1-\mathbf{1}_{\{l=1\}}}\left(\varepsilon_{u-3}^{(t)}\right)^{1-\mathbf{1}_{\{l=1,2,3\}}}\left(\varepsilon_{u-5}^{(j)}\right)^{\mathbf{1}_{\{l=6\}}}}{\left(\varepsilon_{u-2}^{(j)}\right)^{1-\mathbf{1}_{\{l=1,2\}}}\left(\varepsilon_{u-4}^{(i)}\right)^{\mathbf{1}_{\{l=4,5\}}}},$$

$$\mathbf{1}_{\{l=u\}} = \begin{cases} 1, & l = u \\ 0, & l \neq u \end{cases},$$

$$h_l^{(i,j,t)}(S_1,R_1,T_1) = S_1^{I_i(l)}R_1^{I_j(l)}T_1^{I_t(l)},\ i,j,t \in \{1,2,3\},$$

$$I_1(l) = \left\{\prod_{u\in\{1,\ldots,6\}\setminus\{1,4\}}(l-u)\right\}\frac{1-\lfloor l/2\rfloor}{4(13-3l)},$$

$$I_2(l) = \left\{\prod_{u\in\{1,\ldots,6\}\setminus\{3,6\}}(l-u)\right\}\frac{2-\lfloor l/2\rfloor}{4(3l-8)},$$

$$I_3(l) = \left\{\prod_{u\in\{1,\ldots,6\}\setminus\{3,6\}}(l-u)\right\}\frac{\lfloor (l-1)/2\rfloor-1}{8},$$

where $\lfloor . \rfloor$ denotes the floor function. Using the 2-periodicity of the coefficient sequences,

$$\varepsilon_{2k+s}^{(i)} = \varepsilon_s^{(i)} \text{ for } k \geq 0,\ i = 1,2,3, \text{ and } j = 0,1,$$

the above expressions simplify to

$$f_{k,1}(x,y,z) = f_{k,3}(x,y,z) = f_{k,5}(x,y,z) = \left(\frac{x_0z_0y_0}{y_1x_1z_1}\right)^k,$$

$$f_{k,2}(x,y,z) = f_{k,4}(x,y,z) = f_{k,6}(x,y,z) = \left(\frac{x_1z_1y_1}{y_0x_0z_0}\right)^k.$$

The original variables are recovered from the auxiliary sequences through the inverse transformation. Hence,

$$\begin{aligned} s_{k+1} &= \left(S_{k+1} + \lambda_k^{(3)}\right)s_k + \mu_k^{(3)}, \\ r_{k+1} &= \left(R_{k+1} + \lambda_k^{(1)}\right)r_k + \mu_k^{(1)}, \\ t_{k+1} &= \left(T_{k+1} + \lambda_k^{(2)}\right)t_k + \mu_k^{(2)}. \end{aligned}$$

Each equation is therefore transformed into a first-order nonhomogeneous linear recurrence. Applying Lemma 1 to these recurrences and substituting the explicit formulas from (3) yields the closed-form expressions for s_k, r_k and t_k. This completes the proof. □

Theorem 1 admits a simpler form under the condition $\rho := \prod_{t=1}^{3} \varepsilon_0^{(t)} \left(\varepsilon_1^{(t)}\right)^{-1} = 1$. In this case, the multiplicative factors in the auxiliary solution formulas simplify, leading to explicit representations in a more compact form. The corresponding result is given in the following corollary.

Corollary 1. *Let $\{(s_k, r_k, t_k), k \geq -1\}$ be a solution of system (1). Assume that $\rho = 1$. Then the explicit solution formulas given in Theorem 1 remain valid for all $k \geq 0$ and $l = 1, ..., 6$. Moreover, the corresponding auxiliary sequences simplify to (2) where*

$$\begin{aligned} S_{6k+l} &= g_l^{(1,3,2)}(\varepsilon) h_l^{(1,2,3)}(S_1, R_1, T_1), \\ R_{6k+l} &= g_l^{(2,1,3)}(\varepsilon) h_l^{(3,2,1)}(S_1, R_1, T_1), \\ T_{6k+l} &= g_l^{(3,2,1)}(\varepsilon) h_l^{(2,1,3)}(S_1, R_1, T_1). \end{aligned}$$

Remark 2. *The asymptotic behavior of the multiplicative factors appearing in the auxiliary solution formulas is governed by the balance parameter ρ. Under the assumptions of Theorem 1, this parameter is well-defined and nonzero. In particular,*

$$f_{k,1} = f_{k,3} = f_{k,5} = \rho^k, \qquad f_{k,2} = f_{k,4} = f_{k,6} = \rho^{-k}.$$

Consequently, if $0 < |\rho| < 1$, the magnitudes of the factors associated with odd values of ℓ tend to zero, whereas those associated with even values of ℓ tend to infinity. If $|\rho| > 1$, the opposite behavior holds. For negative values of ρ, the signs of these factors alternate with k. These conclusions describe the asymptotic behavior of the multiplicative factors only. They do not, by themselves, determine the asymptotic behavior of the auxiliary sequences or the original solution components, which also depend on the remaining terms in the explicit formulas and on the initial data. In particular, unbounded growth of the original solution components cannot be concluded solely from the divergence of the factors ρ^{-k}.

Remark 3. *A special case of system (1) is obtained by setting $\varepsilon_k^{(i)} = 0$ for all k and $i = 1, 2, 3$. Under this assumption, the rational terms vanish, and system (1) reduces to the periodic linear nonhomogeneous system*

$$s_{k+1} = \lambda_k^{(3)} s_k + \mu_k^{(3)}, r_{k+1} = \lambda_k^{(1)} r_k + \mu_k^{(1)}, t_{k+1} = \lambda_k^{(2)} t_k + \mu_k^{(2)}, k \geq 0.$$

The three equations can be solved independently, and their explicit solution formulas follow from Lemma 1 by applying it to each equation with the corresponding coefficient sequence and forcing term. This degenerate linear case is a special case of the original model (1), but it is not covered by the auxiliary solvability theory developed in Theorem 1. Indeed, when all the $\varepsilon_k^{(i)}$ vanish, the auxiliary variables satisfy $S_{k+1} = R_{k+1} = T_{k+1} = 0$. Consequently, the cyclic inverse relations used in the auxiliary transformation cannot be inverted. The reduced linear system must therefore be treated separately.

Next, we present numerical examples illustrating the theoretical results established in this section. The simulations are performed for several sets of two-periodic coefficient sequences and different initial values. The three examples correspond to the cases $\rho = 1$, $0 < \rho < 1$, and $\rho > 1$, respectively, and illustrate the results of Theorem 1, Corollary 1, and Remark 2. In each figure, the upper panel shows the trajectories of the three solution components, whereas the remaining six panels display the subsequences associated with the six residue classes modulo six, illustrating the six-step decomposition described by the explicit solution formulas.

Example 1. *Consider system (1) with the following two-periodic coefficient sequences:*

$$\begin{array}{llllll} \varepsilon_{2k}^{(1)} & = 0.30, & \lambda_{2k}^{(1)} & = 0.30, & \mu_{2k}^{(1)} & = 0.10, \\ \varepsilon_{2k+1}^{(1)} & = 0.25, & \lambda_{2k-1}^{(1)} & = -0.20, & \mu_{2k-1}^{(1)} & = 1.10, \\ \varepsilon_{2k}^{(2)} & = 0.28, & \lambda_{2k}^{(2)} & = -0.25, & \mu_{2k}^{(2)} & = 0.00, \\ \varepsilon_{2k+1}^{(2)} & = 0.42, & \lambda_{2k-1}^{(2)} & = -0.15, & \mu_{2k-1}^{(2)} & = 1.00, \\ \varepsilon_{2k}^{(3)} & = 0.35, & \lambda_{2k}^{(3)} & = -1.35, & \mu_{2k}^{(3)} & = 0.05, \\ \varepsilon_{2k+1}^{(3)} & = 0.28, & \lambda_{2k-1}^{(3)} & = 0.25, & \mu_{2k-1}^{(3)} & = 1.05, \end{array}$$

for $k \geq 0$. The initial values are

$$\begin{array}{llllll} s_{-1} & = 1.20, & r_{-1} & = -0.90, & t_{-1} & = 0.25, \\ s_0 & = 1.00, & r_0 & = 0.10, & t_0 & = 0.95. \end{array}$$

For these parameter values, $\rho = 1$. Therefore, Corollary 1 applies. The numerical solutions are shown in Figure 1. The numerical results are consistent with the theoretical analysis and illustrate the six-step decomposition described by the explicit solution formulas.

To assess the accuracy of the explicit solution formulas derived in Theorem 1, we compare the numerical solutions obtained by direct iteration of system (1) with those computed using the closed-form expressions given in (2)–(3). The comparison is carried out at selected indices k ranging from 1 to 120 for all three solution components, (s_k), (r_k), and (t_k). The absolute error for each component is calculated as the absolute difference between the value obtained by direct numerical iteration and the corresponding value computed using the explicit solution formula. These errors quantify the discrepancies between the two computational approaches and provide a measure of the numerical accuracy of the derived closed-form solutions. The results of this comparison, including the direct numerical solutions, the corresponding closed-form solutions, and their absolute errors, are presented in Table 1.

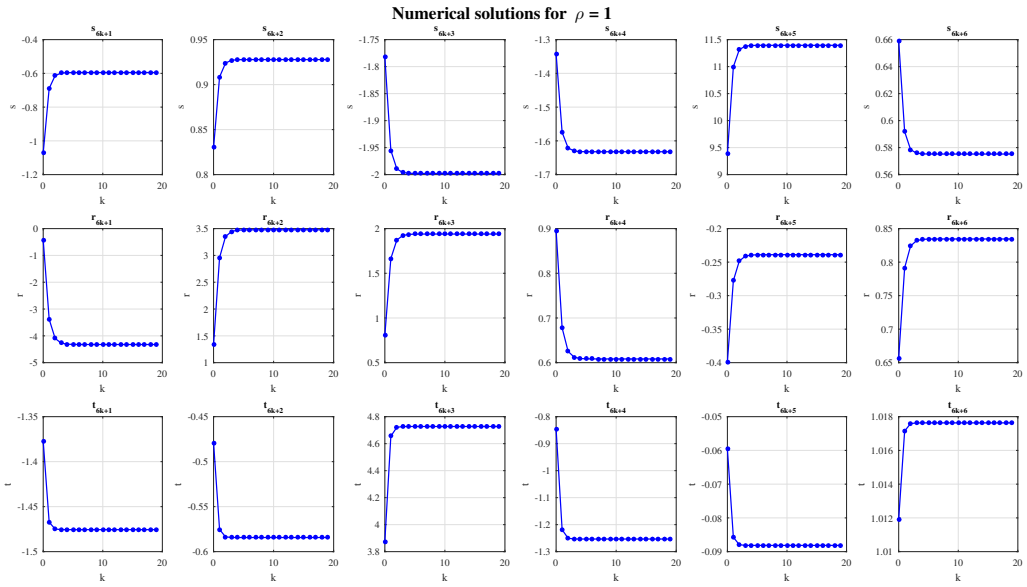


Figure 1. Six-step dynamics for the critical case $\rho = 1$.

Table 1. Comparison of the direct numerical and closed-form solutions, together with their absolute errors.

k	s_{direct}	s_{closed}	$error_s$	r_{direct}	r_{closed}	$error_r$	t_{direct}	t_{closed}	$error_t$
1	−1.0711	−1.0711	0.0000	−4.2999	−4.2999	0.0000	−1.3775	−1.3775	0.0000
2	0.8300	0.8300	0.0000	1.3365	1.3365	0.0000	−0.4790	−0.4790	2.2204×10^{-16}
3	−1.7820	−1.7820	0.0000	0.8068	0.8068	0.0000	3.8753	3.8753	1.7764×10^{-15}
6	0.6592	0.6592	1.1102×10^{-16}	0.6568	0.6568	1.1102×10^{-16}	1.0119	1.0119	0.0000
7	−0.6890	−0.6890	2.2204×10^{-16}	−3.3810	−3.3810	8.6597×10^{-14}	−1.4673	−1.4673	2.2204×10^{-16}
12	0.5918	0.5918	8.5487×10^{-15}	0.7915	0.7915	2.6645×10^{-15}	1.0172	1.0172	6.66133×10^{-16}
18	0.5785	0.5785	4.1078×10^{-15}	0.8241	0.8241	2.1094×10^{-15}	1.0176	1.0176	4.4409×10^{-16}
30	0.5754	0.5754	5.2180×10^{-15}	0.8339	0.8339	1.1102×10^{-15}	1.0176	1.0176	2.2204×10^{-16}
60	0.5752	0.57523	4.5852×10^{-14}	0.8345	0.8345	1.9207×10^{-14}	1.0176	1.0176	2.8866×10^{-15}
120	0.5752	0.57523	3.2196×10^{-15}	0.8345	0.8345	9.7700×10^{-15}	1.0176	1.0176	4.4409×10^{-16}

As shown in Table 1, the solutions obtained by direct iteration agree closely with those computed using the closed-form expressions for all three components. The absolute errors are generally very small, with most values ranging from 10^{-16} *to* 10^{-14}*. Even at* $k = 120$*, the discrepancies remain negligible, confirming the numerical agreement between the two approaches over the tested range. The slightly larger error observed in the r-component at* $k = 7$*, approximately* 8.66×10^{-14}*, remains consistent with floating-point round-off effects. Overall, these numerical results provide strong numerical evidence supporting the correctness of the explicit solution formulas derived in Theorem 1 for the parameter configuration considered in this example.*

Example 2. *Consider system (1) with the following two-periodic coefficient sequences:*

$$\begin{array}{llllll} \varepsilon_{2k}^{(1)} & = 0.70, & \lambda_{2k}^{(1)} & = 0.15, & \mu_{2k}^{(1)} & = 0.02, \\ \varepsilon_{2k+1}^{(1)} & = 0.40, & \lambda_{2k-1}^{(1)} & = 1.10, & \mu_{2k-1}^{(1)} & = 0.23, \\ \varepsilon_{2k}^{(2)} & = 0.30, & \lambda_{2k}^{(2)} & = -0.08, & \mu_{2k}^{(2)} & = -1.01, \\ \varepsilon_{2k+1}^{(2)} & = 0.60, & \lambda_{2k-1}^{(2)} & = -0.05, & \mu_{2k-1}^{(2)} & = -0.02, \\ \varepsilon_{2k}^{(3)} & = 0.25, & \lambda_{2k}^{(3)} & = 0.72, & \mu_{2k}^{(3)} & = 0.13, \\ \varepsilon_{2k+1}^{(3)} & = 0.80, & \lambda_{2k-1}^{(3)} & = -0.10, & \mu_{2k-1}^{(3)} & = 0.12, \end{array}$$

for $k \geq 0$*. The initial values are*

$$\begin{array}{llllll} s_{-1} & = 1.10, & r_{-1} & = 0.30, & t_{-1} & = 1.00, \\ s_0 & = 0.25, & r_0 & = 1.05, & t_0 & = 0.12. \end{array}$$

For these parameter values, $\rho \approx 0.27344 \in (0, 1)$*. The numerical solutions are shown in Figure 2.*

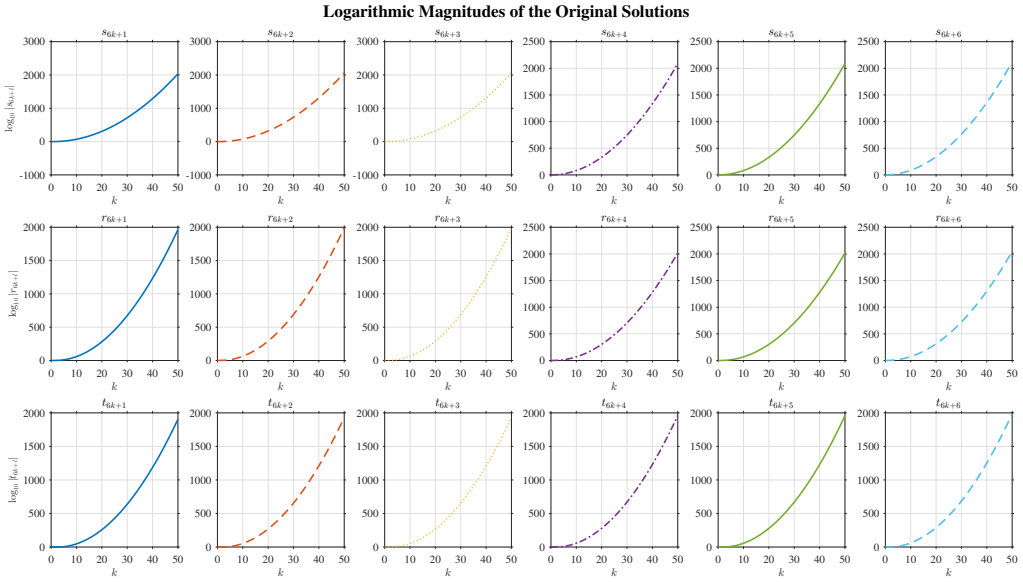


Figure 2. Asymmetric six-step dynamics for $0 < \rho < 1$.

Example 3. *Consider system (1) with the following two-periodic coefficient sequences:*

$$\begin{array}{llllll} \varepsilon_{2k}^{(1)} & = 0.80, & \lambda_{2k}^{(1)} & = 0.30, & \mu_{2k}^{(1)} & = 0.10, \\ \varepsilon_{2k+1}^{(1)} & = 0.70, & \lambda_{2k-1}^{(1)} & = -0.20, & \mu_{2k-1}^{(1)} & = -1.10, \\ \varepsilon_{2k}^{(2)} & = 0.65, & \lambda_{2k}^{(2)} & = -0.25, & \mu_{2k}^{(2)} & = 0.05, \\ \varepsilon_{2k+1}^{(2)} & = 0.40, & \lambda_{2k-1}^{(2)} & = -0.15, & \mu_{2k-1}^{(2)} & = -1.00, \\ \varepsilon_{2k}^{(3)} & = 0.50, & \lambda_{2k}^{(3)} & = -1.35, & \mu_{2k}^{(3)} & = 0.05, \\ \varepsilon_{2k+1}^{(3)} & = 0.60, & \lambda_{2k-1}^{(3)} & = 0.25, & \mu_{2k-1}^{(3)} & = 1.05, \end{array}$$

for $k \geq 0$. The initial values are

$$\begin{array}{llllll} s_{-1} & = 1.20, & r_{-1} & = -0.90, & t_{-1} & = -0.25, \\ s_0 & = 1.00, & r_0 & = 0.10, & t_0 & = -0.95. \end{array}$$

For these parameter values, $\rho \approx 1.54762 > 1$. The numerical solutions are shown in Figure 3.

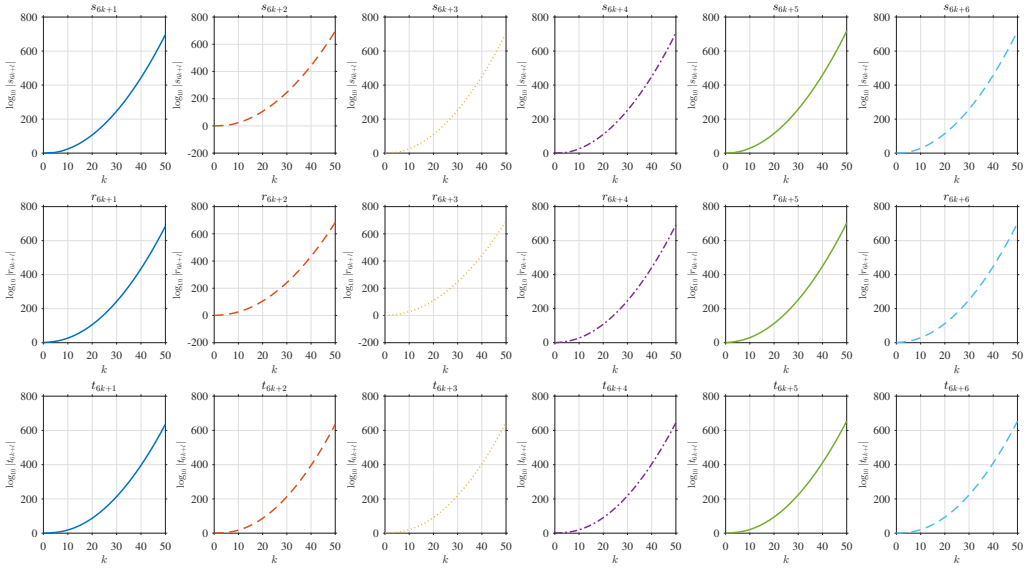


Figure 3. Six-step dynamics in the expansive regime $\rho > 1$.

2.1. Discussion

Figure 1 illustrates the numerical dynamics of the system in the critical case $\rho = 1$. Over the displayed range, the six subsequences associated with each solution component exhibit an initial transient followed by apparent stabilization at different levels. This behavior illustrates the six-step decomposition of the solutions and shows that subsequences belonging to different residue classes modulo six may exhibit distinct numerical patterns. Figures 2 and 3 display different behaviors for the parameter regimes $0 < \rho < 1$ and $\rho > 1$, respectively. In both cases, the logarithmic plots reveal substantial changes in the magnitudes of several subsequences over the computed range. These observations illustrate the sensitivity of the numerical dynamics to the parameter configuration and the initial conditions. However, the observed behavior over a finite interval does not, by itself, establish convergence or unboundedness of the original solution components. As clarified in Remark 2, the asymptotic predictions based on ρ concern the multiplicative factors appearing in the explicit solution formulas and cannot be directly transferred to the original solution components without further analysis.

To assess the admissibility of the computed solutions, the denominators in system (1) were checked at each iteration over the computed interval for Examples 1–3. The calculations were also examined for zero solution components, as required by the assumptions underlying the auxiliary transformation. The resulting numerical trajectories are therefore interpreted only over the intervals on which the relevant denominators and solution components satisfy the required nonvanishing conditions. These numerical checks support the validity of the computed trajectories on the intervals considered, but do not establish the admissibility of the solutions for all future indices.

Theoretical Predictions vs. Numerical Observations

Remark 2 describes the asymptotic behavior of the multiplicative factors appearing in the explicit solution formulas, rather than that of the auxiliary sequences or the original solution components themselves. For Example 2, where $\rho = 0.2734375$, the factors ρ^k tend to zero, whereas ρ^{-k} diverge to infinity as $k \to \infty$. For Example 3, where $\rho > 1$, these behaviors are reversed. These conclusions follow analytically from the explicit expressions for the multiplicative factors.

Figures 2 and 3 display the logarithmic magnitudes of the six subsequences $s_{6k+\ell}$, $r_{6k+\ell}$, and $t_{6k+\ell}$, where $\ell = 1, \ldots, 6$, over the finite range $0 \le k \le 50$. The different subsequences are distinguished by their respective colors or line styles, allowing their numerical behavior to be compared over the computed range. The plots illustrate the magnitude variations of the original solution components for the chosen parameter values and initial conditions. However, these finite-range numerical observations do not, by themselves, establish that the original solution components are asymptotically unbounded or convergent.

Consequently, the numerical experiments illustrate the behavior of the original solution components over the considered range, whereas Remark 2 provides analytical predictions only for the multiplicative factors. The predicted behavior of these factors is established analytically; it should not be interpreted as a proof of the corresponding asymptotic behavior of the original solution components. Establishing such behavior for the latter would require a separate asymptotic analysis of the complete explicit solution formulas.

3. Conclusions

In this paper, we investigated a three- dimensional nonlinear rational difference system with two-periodic coefficients. By introducing a suitable algebraic transformation, the original nonlinear system was reformulated as an auxiliary cyclic inverse system, which was subsequently reduced to explicitly solvable recurrence relations. This approach enabled us

to derive closed-form expressions for all well-defined solutions, subject to the nonvanishing conditions required by the original system and the auxiliary transformation. We also established a simplified representation under a particular balance condition on the periodic coefficients and analyzed the asymptotic behavior of the multiplicative factors appearing in the auxiliary solution formulas. Finally, three representative numerical examples were presented to illustrate the theoretical results under different parameter configurations. Beyond its theoretical implications, the analytical approach developed in this paper may offer a useful perspective on the study of discrete-time components arising in the modeling of complex dynamical systems. In particular, the explicit solution formulas provide a means of examining how periodic variations in the coefficients influence the evolution of coupled state variables. This feature may be relevant to broader numerical simulation settings in which time-dependent inputs play an important role. An illustrative example is provided by the study of Vicari et al. [20], where time-varying lava discharge rates, estimated from thermal infrared satellite imagery, were used to drive the MAGFLOW cellular automaton model in simulations of the 2006 Mount Etna eruption. While the present system does not account for the spatial propagation of lava or the underlying physical and thermal processes, its analytical framework may provide a starting point for investigating reduced discrete-time descriptions or auxiliary dynamical components within more comprehensive simulation models. Such an extension would, however, require a physically meaningful interpretation of the state variables and model coefficients, as well as appropriate parameter estimation and validation against observational data. Moreover, the two-periodic coefficient structure assumed in the present study should be viewed as a mathematical idealization rather than as a direct representation of the temporal variability of actual lava discharge rates. Several directions remain open for future research. These include extending the present analysis to higher-dimensional periodically forced rational difference systems, considering more general classes of periodic and non-periodic coefficients, and investigating nonlinear systems with delays. Another promising direction is the incorporation of uncertainty through fuzzy-valued coefficients and initial conditions, using an appropriate α-cut representation (see [21–23]). Further work could also investigate the coupling of analytically tractable discrete-time subsystems with spatially distributed numerical simulation models arising in applied sciences.

Author Contributions: Methodology, A.G., A.A.A.G. and H.J.A.S.; Software, H.J.A.S.; Validation, A.A.A.G.; Formal Analysis, A.A.A.G. and H.J.A.S.; Writing—Original Draft, A.G., A.A.A.G. and H.J.A.S.; Writing—Review and Editing, A.G., A.A.A.G. and H.J.A.S. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. KFU266724].

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

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