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A NEW ALGORITHM TO OBTAIN δ-OPERATOR BASED TRANSFER FUNCTION FROM ITS CONTINUOUS TIME COUNTERPART

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ABSTRACT

This paper discusses the advantages of the δ-operator over the forward shift operator q in implementation of discrete-time systems. The δ-operator gives better coefficient representation and less round-off noise in many cases. As an application, a new algorithm is introduced here to obtain δ-operator based transfer functions from continuous time counterparts. Finally, a fourth-order Butterworth low-pass filter is taken as an example to compare coefficient sensitivity effects of q- and δ-operator systems.

Keywords: δ-Transform, coefficient sensitivity, numerical instability, discretization, stability

INTRODUCTION

A widely used operator in digital control is the forward shift-operator q. However, q-operator based algorithms are ill-conditioned in high speed applications. As an alternative, δ-operator-based implementation of discrete-time systems has become the center of recent research activities because of their

a) superior finite word length coefficient representations and reduced round-off noise
b) convergence to continuous-time counterparts as the sampling time approaches to zero.

Therefore, better results can be obtained if the shift operator is replaced by an incremental difference operator, namely δ-operator:

$$\delta = \frac{q-1}{T} \tag{1}$$

where T is the sampling time [1-4].

As an application, the counterpart of Groutage algorithm [5] is introduced here to show how to obtain the δ-operator equivalent of a continuous time transfer function by means of the bilinear transformation. The counterpart of the bilinear transformation

$$s = \frac{2\delta}{2+T\delta} \tag{2}$$

where T is the sampling time, is frequently used to obtain an approximate discrete equivalent H(δ) from a continuous transfer function H(s) [1]. Given the continuous time transfer function of linear time invariant, finite-dimensional system

$$H(s) = \frac{A_n s^n + A_{n-1}s^{n-1} + \ldots + A_0}{B_n s^n + B_{n-1}s^{n-1} + \ldots + B_0} \tag{3}$$

where A_n and B_n are non zero, the function H(δ) will be

$$H(\delta) = \frac{\sum_{i=0}^{n} A_{n-i} 2^{n-i} \delta^{n-i} (2+T\delta)^i}{\sum_{i=0}^{n} B_{n-i} 2^{n-i} \delta^{n-i} (2+T\delta)^i} \tag{4}$$

H(δ) can also be written as the ratio of two n-th degree polynomials:

$$H(\delta) = \frac{A'_n \delta^n + A'_{n-1} \delta^{n-1} + \ldots + A'_0}{B'_n \delta^n + B'_{n-1} \delta^{n-1} + \ldots + B'_0} \tag{5}$$

The problem becomes finding the coefficients A'_n and B'_n of the polynomials in equation (5). To solve this, one equates the respective numerators and denominators of (4) and (5)

$$A'_n \delta^n + A'_{n-1} \delta^{n-1} + \ldots + A'_0 = \sum_{i=0}^{n} A_{n-i} 2^{n-i} \delta^{n-i} (2+T\delta)^i \tag{6}$$

$$B'_n \delta^n + B'_{n-1} \delta^{n-1} + \ldots + B'_0 = \sum_{i=0}^{n} B_{n-i} 2^{n-i} \delta^{n-i} (2+T\delta)^i \tag{7}$$

Let δ take (n+1) values of r_k's on the circumference of the shifted circle of radius 1/T, which is located at (-1/T,0). The values of the roots can be calculated using the formula

$r_k = \frac{1}{T}(-1+e^{i\theta k})$ for k=0,1,...,n. When these values are substituted into the equations (6) and (7), the resulting (n+1) equations can be solved for (n+1) coefficients A'$_n$ and B'$_n$. The set of equations in the matrix form is given as

$$F_n a_n = \Sigma_a \tag{8}$$

where F$_n$ is the (n+1)x(n+1) complex matrix of coefficients

$$F_n = \begin{bmatrix} r_0^n & r_0^{n-1} & \ldots & r_0^0 \\ r_1^n & r_1^{n-1} & \ldots & r_1^0 \\ & \ldots & & \\ r_n^n & r_n^{n-1} & \ldots & r_n^0 \end{bmatrix} \tag{9}$$

a$_n$ is the (n+1)x1 complex vector of unknowns

$$a_n = [A'_n \ A'_{n-1} \ \ldots \ A'_0]^T \tag{10}$$

and Σ_n is the (n+1)x1 complex vector of function evaluations

$$\Sigma_a = \begin{bmatrix} \sum_{i=0}^{n} A_{n-i} 2^{n-i} \delta^{n-i} (2+T\delta)^i \Big|_{\delta=r_0} \\ \ldots \\ \sum_{i=0}^{n} A_{n-i} 2^{n-i} \delta^{n-i} (2+T\delta)^i \Big|_{\delta=r_n} \end{bmatrix} \tag{11}$$

The solution to this system is given by

$$a_n = F_n^{-1} \Sigma_a \tag{12}$$

However, since r_0's are all zero, each element at the first row of the matrix F$_n$ is zero. Therefore, it is not possible to invert this matrix to get a solution, but if the system is carefully investigated it can be seen that it is not actually necessary to solve the equations for A'$_0$ and B'$_0$, because they can easily be figured out from equations (5) and (6) that $A'_0 = 2^n A_0$ and $B'_0 = 2^n B_0$ when i=n. Thus, the system turns out to be a nxn matrix system, and as is expected the complex term in the solution is zero. Similarly, the coefficients B'$_n$ of the denominator can be evaluated from $b_n = F_n^{-1}\Sigma_b$ in the same way.

As an example, step responses for q- and δ-operator implementations of a fourth order Butterworth low-pass filter [1] are drawn in the figure to show the finite word length effect and the convergence of the δ system to its continuous counterpart as the sampling rate is increased.

Conclusion

The superiority of the δ-operator over forward shift operator q is mainly due to better numerical results obtained by the computers that allow only a certain number of binary bits on each mantissa. Since the δ-operator gives better coefficient representation and converges to its continuous-time counterpart as the sampling rate is increased, it leads to an implementation of shorter wordlength and a unified treatment of both continuous and discrete time systems.

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Fig. Unit step responses for analog, q- and delta-operators