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PERISTALTIC FLOW OF A THIRD-GRADE FLUID IN A PLANAR CHANNEL

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Abstract - The problem of peristaltic transport of a non-Newtonian fluid represented by the constituve equation for a third grade fluid was analysed for the case of planar channel with harmonically undulating extensible wall, under zero Reynolds number and long wavelength approximation. New exact analytical solution of the non-linear equation resulting from the momentum equation was given when $\gamma_2+\gamma_3$(which are the dimensionless material constants)>0 and under some conditions when $\gamma_2+\gamma_3<0$. Also, the exact range of validity of the perturbation analysis was obtained using the Girolamo Cardano formulas and binomial theorem for both $\gamma_2+\gamma_3>0$ and $\gamma_2+\gamma_3<0$. Finally, we have shown that pumping rate of a third-grade fluid can be greater or less than that for a Newtonian fluid having a shear viscosity same as the lower-limiting viscosity of non-Newtonian material depending on the value of the material constants ,amplitude ratio and flow rate.

INTRODUCTION

From Peristalsis is now well known to physiologists to be one of the major mechanism for fluid transport in many biological systems. In particular peristaltic mechanism may be involved in urine transport kidney to bladder through the ureter, movement of chyme in the gastro-intestinal tract.

The mechanics of peristalsis in both mechanical and physiological situations has been examined by a number of investigators. Since the first investigations of Latham[1] All such investigations seem to differ in various details. The outline of main investigations can be found in [2]-[3]. From this work, it is known that there many works on the transport of Newtonian and second-grade fluids. Although second-grade fluid model is able to predict the normal stress differences which are characteristic of non-Newtonian fluids, it does not take into account shear thinning and thickening phenomena that many show. The third-grade model represents a further, although inconclusive, attempt toward a comprehensive description of the properties of viscoelastic fluids.

Hence, present work has been undertaken in order to investigate the peristaltic motion of the third-grade fluid. Due to the complexity of the non-linear equation of motion, we only consider the case: planar flow; a symmetric, harmonic, infinite wave train having a wavelength that is large relative to the gap between the walls (long wavelength approximation); transverse displacement only.

Siddiqui and Schwarz[4] were the first authors to address this problem and obtained the perturbation series solution(up to second order) in terms of a variant of the Deborah number. However, they did not consider whether it is possible to solve the problem analytically or not and could not find the range of validity of their perturbation analysis.

This provide the motivation for present work where we present exact analytical solution for stream function along with the numerical results. Finally, the exact range of validity of perturbation analysis was derived using the Girolamo Cardano formulas and binomial theorem.

BASIC EQUATIONS

An incompressible simple fluid is defined as a material whose state of present stress is determined by the history of the deformation gradient without a preferred reference configuration[5]. Its constitutive equation can be written in the form of a functional

$$\overline{\mathbf{T}}(x,t) = -\overline{p}\mathbf{I} + \Im_{s=0}^{\infty}(\overline{\mathbf{F}}_t^t(s)) \tag{2.1}$$

where $\overline{p}\mathbf{I}$ is the undetermined part of the stress tensor and $\overline{\mathbf{F}}$ is the deformation gradient.

Truesdell and Noll [5] defined the incompressible fluid of differential type of grade n as the simple fluid obeying the constitutive equation

$$\overline{\mathbf{T}} = -\overline{p} + \sum_{i=1}^{n} \overline{\mathbf{S}}_j \tag{2.2}$$

obtained by asymptotic expansion of the functional in (2.1) through a retardation parameter α. If n=3 the first three tensors are given by

$$\overline{\mathbf{S}}_1 = \mu\overline{\mathbf{A}}_1, \tag{2.3}$$

$$\overline{\mathbf{S}}_2 = \alpha_1\overline{\mathbf{A}}_2 + \alpha_2\overline{\mathbf{A}}_1^2, \tag{2.4}$$

$$\overline{\mathbf{S}}_3 = \beta_1\overline{\mathbf{A}}_3 + \beta_2\left(\overline{\mathbf{A}}_1\overline{\mathbf{A}}_2 + \overline{\mathbf{A}}_2\overline{\mathbf{A}}_1\right), + \beta_3\left(tr\overline{\mathbf{A}}_1^2\right)\overline{\mathbf{A}}_1. \tag{2.5}$$

where, μ is the coefficient of shear viscosity; and $\alpha_1, \alpha_2, \beta_1, \beta_2$ and β_3 are material constants. The Revlin Ericksen tensors $\overline{\mathbf{A}}_n$ are defined by the recursion relation:

$$\overline{\mathbf{A}}_n = \overline{\mathbf{A}}_{n-1} + \overline{\mathbf{A}}_{n-1}(Grad\overline{V}) + (Grad\overline{V})^T\overline{\mathbf{A}}_{n-1},$$

$$\overline{\mathbf{A}}_1 = (Grad\overline{V}) + (Grad\overline{V})^T, n > 1 \tag{2.6}$$

where the superposed dot signifies material differentiation with respect to time. The dimensional basic field equations governing the flow of an incompressible fluid, neglecting thermal effects, are

$$div\overline{\mathbf{V}} = 0 \tag{2.7}$$

$$div\overline{\mathbf{T}} + \rho\overline{\mathbf{f}} = \rho\overline{\mathbf{V}} \tag{2.8}$$

where ρ is the density, $\overline{\mathbf{V}}$ is the velocity vector $\overline{\mathbf{f}}$ is the bady-force vector per unit mass.

Now, we specialise the above equations for unsteady two-dimensional flows. Introducing the
voracity

$$\Omega = \frac{\partial \overline{V}}{\partial \overline{X}} - \frac{\partial \overline{U}}{\partial \overline{Y}} \tag{2.9}$$

(2.8) becomes

$$\rho[\frac{\partial\Omega}{\partial t}+(\overline{U}\frac{\partial}{\partial\overline{X}}+\overline{V}\frac{\partial}{\partial\overline{Y}})\Omega]=[\frac{\partial^2}{\partial\overline{X}^2}-\frac{\partial^2}{\partial\overline{Y}^2}\overline{S}_{12})+[\frac{\partial^2}{\partial\overline{XY}}(S_{22}-S_{11})] \quad (2.10)$$

FORMULATION OF THE PROBLEM AND THE FLOW EQUATIONS

Consider the peristaltic flow of a third-grade fluid in a infinitive channel having width 2a. We assume an infinitive wave train travelling with velocity c along the walls. We choose a rectangular coordinate system for the channel with $\overline{X}$ along the centreline in the direction of wave propagation, and $\overline{Y}$ transverse to it. The geometry of the wall is defined as:

$$\overline{h}(\overline{X},t)=a+b\sin[\frac{2\pi}{\lambda}(\overline{X}-ct) \quad (2.11)$$

where b is the wave amplitude, and λ is the wavelength. we also assume the wall to have only a transverse motion.

In the laboratory frame $(\overline{X},\overline{Y})$, the flow in the channel is unsteady, but if we choose moving coordinates $(\overline{x},\overline{y})$ which travel in the positive $\overline{X}$-direction with the same speed as the wave, then the flow can be treated as steady. This coordinate system is known as the *wave frame*. The coordinate frames are related through

$$\overline{u}(\overline{x},\overline{y})=\overline{U}(\overline{X}-ct,\overline{Y})-c, \qquad \overline{v}(\overline{x},\overline{y})=\overline{V}(\overline{X}-ct,\overline{Y}) \quad (2.12)$$

where $\overline{u}$, $\overline{v}$ are the velocity components in the direction of $\overline{x}$, $\overline{y}$ respectively.
we find that continuity equation, after defining the dimensionless stream function $\psi(x,y)$ by

$$u=\frac{\partial\psi}{\partial y}, \qquad v=-\delta\frac{\partial\psi}{\partial y} \quad (2.13)$$

is satisfied identically and the equation of motion become

$$\delta\operatorname{Re}[(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y})(\frac{\partial\psi}{\partial y})]+\frac{\partial p}{\partial x}=\delta\frac{\partial S_{11}}{\partial x}+\frac{\partial S_{12}}{\partial y} \quad (2.14)$$

$$\delta^3\operatorname{Re}[(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y})(\frac{\partial\psi}{\partial y})]+\frac{\partial p}{\partial y}=\delta^2\frac{\partial S_{12}}{\partial x}+\delta\frac{\partial S_{22}}{\partial y} \quad (2.15)$$

Eliminating the pressure between this equations, we obtain the following compatibility equation:

$$\delta\operatorname{Re}[(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y})(\Delta^2\psi)]=[(\frac{\partial^2}{\partial y^2}-\delta^2\frac{\partial^2}{\partial x^2})S_{12}+\delta[\frac{\partial^2}{\partial xy}(S_{11}-S_{22}] \quad (2.16)$$

where above, δ(dimensionless wave number) and Re(Reynolds number) are the dimensionless parameters as described [4] and

$$\Delta^2=(\delta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}) \quad (2.17)$$

METHOD OF SOLUTION

In this communication, we are interested in the case of inertia -free, infinitive wave length case. This can be achieved by making the assumption that the wavenumber(Shapiro et al. [6]):

$$\delta \equiv \frac{2\pi a}{\lambda} = 0 \tag{2.18}$$

then the equations (2.14)-(2.16) reduce to;

$$\frac{\partial^2}{\partial y^2}\left[\frac{\partial^2 \psi}{\partial y^2} + \Gamma\left(\frac{\partial^2 \psi}{\partial y^2}\right)^3\right]=0, \tag{2.19}$$

where $\Gamma = 2(\gamma_2 + \gamma_3)$

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left[\frac{\partial^2 \psi}{\partial y^2} + \Gamma\left(\frac{\partial^2 \psi}{\partial y^2}\right)^3\right], \tag{2.20}$$

$$\frac{\partial p}{\partial y} = 0. \tag{2.21}$$

with the boundary conditions:

$$\psi = 0, \quad \frac{\partial^2 \psi}{\partial y^2} = 0, \quad \text{on } y = 0$$

$$\frac{\partial \psi}{\partial y} = -1, \quad \psi = 0 \quad \text{for } y = h = 1 + \phi \sin(x) \tag{2.22}$$

In[4], Siddiqui and Schwarz obtained the perturbation solution by assuming Γ to be small up to the second-order. But, following[7], it is possible to obtain exact solution of above system of equations .

Exact solution

Let us assume time being $\Gamma > 0$,then equation(2.19) can be solved formally[7], the only real solution being

$$\frac{\partial^2 \psi}{\partial y^2} = \sqrt[3]{\frac{c(x)y}{2\Gamma} + \sqrt{\left(\frac{c(x)y}{2\Gamma}\right)^2 + \frac{1}{27\Gamma^3}}} + \sqrt[3]{\frac{c(x)y}{2\Gamma} - \sqrt{\left(\frac{c(x)y}{2\Gamma}\right)^2 + \frac{1}{27\Gamma^3}}} \tag{2.23}$$

we find, after lengthy but straightforward calculations, that

$$\psi(x,y) = c_1(x)y + c_3(x) \quad -\Gamma^2\left[\left(\sqrt{\frac{c(x)y}{2\Gamma}} + \sqrt{\left(\frac{c(x)y}{2\Gamma}\right)^2 + \frac{1}{27\Gamma^3}}\right)\right)^{3/7} 3/7$$

$$+\frac{1}{1215}\Gamma^{-6}\left(\sqrt{\frac{c(x)y}{2\Gamma}} + \sqrt{\left(\frac{c(x)y}{2\Gamma}\right)^2 + \frac{1}{27\Gamma^3}}\right)\Bigg] c(x)^{-2}$$

$$-\Gamma^{2}\left[\left(\sqrt{\frac{c(x)y}{2\Gamma}-\sqrt{\left(\frac{c(x)y}{2\Gamma}\right)^{2}+\frac{1}{27\Gamma^{3}}})}\right)^{3/7} 3/7\right.$$

$$\left.+\frac{1}{1215}\Gamma^{-6}\left(\sqrt{\frac{c(x)y}{2\Gamma}}-\sqrt{\left(\frac{c(x)y}{2\Gamma}\right)^{2}+\frac{1}{27\Gamma^{3}}}\right)\right]c(x)^{-2}$$

$$+y\left[\Gamma\left(\sqrt{\frac{c(x)y}{2\Gamma}+\sqrt{\left(\frac{c(x)y}{2\Gamma}\right)^{2}+\frac{1}{27\Gamma^{3}}})}\right)^{4/3} 3/4\right.$$

$$\left.-\frac{1}{18}\Gamma^{-3}\left(\left(\sqrt{\frac{c(x)y}{2\Gamma}+\sqrt{\left(\frac{c(x)y}{2\Gamma}\right)^{2}+\frac{1}{27\Gamma^{3}}}}\right)^{-2/3}\right)\right]c(x)^{-1}$$

$$+y\left[\Gamma\left(\sqrt{\frac{c(x)y}{2\Gamma}-\sqrt{\left(\frac{c(x)y}{2\Gamma}\right)^{2}+\frac{1}{27\Gamma^{3}}})}\right)^{4/3} 3/4\right.$$

$$\left.-\frac{1}{18}\Gamma^{-3}\left(\left(\sqrt{\frac{c(x)y}{2\Gamma}-\sqrt{\left(\frac{c(x)y}{2\Gamma}\right)^{2}+\frac{1}{27\Gamma^{3}}}}\right)^{-2/3}\right)\right]c(x)^{-1} \qquad (2.24)$$

Since $\psi = 0$ on y=0, we find $c_2(x)$ and applying the other boundary conditions(2.22) to the equation(2.24), give the non-linear algebraic equations for $c(x)$ and $c_1(x)$. Since the resulting equations are non-linear, the values of $c(x)$ and $c_1(x)$ can be found only for a given x as shown in Table II. The Quasi-Newton method was used to solve the resulting non-linear algebraic equations and available in the NAG library.

The exact solution for $\Gamma < 0$ when $\frac{c(x)^2 y^2}{4\Gamma^2}+\frac{1}{27\Gamma^3} > 0$ can be obtained exactly the same way as above, but it is impossible to obtain the exact solution when both and $\Gamma < 0$ and $\frac{c(x)^2 y^2}{4\Gamma^2}+\frac{1}{27\Gamma^3} < 0$. From (2.21), it is clear that the transverse pressure gradient is zero and from the (2.22) the longitudinal pressure gradient is $\frac{\partial p}{\partial x} = c(x)$.

it is also of some interest to calculate the pressure rise over a wavelength Δp_λ in the longitudinal direction on the axis (y=0), the pressure rise per wavelength in dimensionless form is given as:

$$\Delta p_\lambda = \int_0^{2\pi} \frac{\partial p}{\partial x} dx \qquad (2.25)$$

Hence, $c(x)$ ones known for a given x, the pressure rise per wavelength can be calculated from the equation numerically. For comparison, the exact and approximate values [6] of $c(x)$ are given in Table 2

Table 2. Exact and Approximate values of $c(x)$

$(\gamma_2+\gamma_3)$	x	ϕ	Exact	Approximate	$(\gamma_2+\gamma_3)$	x	ϕ	Exact	Approximate
0.02	0	0.2	-0.300725	-0.300647	0.02	0	0.8	-0.300725	0.300647
0.02	$\pi/6$	0.2	-0.452098	-0.453432	0.02	$\pi/6$	0.8	-0.550577	-0.554262
0.02	$\pi/4$	0.2	-0.489074	-0.496063	0.02	$\pi/4$	0.8	-0.524587	-0.52854
0.02	$\pi/2$	0.2	-0.523394	-0.525694	0.02	$\pi/2$	0.8	-0.46693	-0.470605
0.02	$-\pi/6$	0.2	0.000055	$-0.88.10^{-10}$	0.02	$-\pi/6$	0.8	4.468397	4.73804
0.02	$-\pi/4$	0.2	0.19738	-0.496063	0.02	$-\pi/4$	0.8	26.82	22.614
0.02	$-\pi/2$	0.2	0.587506	-0.525694	0.02	$-\pi/2$	0.8	7094.5	- 638746

It is clear that the approximate results are relevant if the condition in [8] is satisfied.
In the pioneering work of Fostik and Rajagopal[9] have shown that equation (2.2) for n=3 to be compatible with thermodynamics and the free energy to be minimum when the fluid is at rest, the material constants should satisfy the relations

$$\mu \geq 0, \qquad \alpha_1 \geq 0, \qquad \beta_1 = \beta_2 = 0,$$
$$\beta_3 \geq 0, \qquad -\sqrt{24\mu\beta_3} \leq \alpha_1 + \alpha_2 \leq -\sqrt{24\mu\beta_3}$$

Hence, the stress tensor for an incompressible homogeneous fluid of third-grade simplifies to

$$\overline{\mathbf{T}} = -\overline{p}\mathbf{I} + \mu\overline{\mathbf{A}}_1 + \alpha_1\overline{\mathbf{A}}_2 + \alpha_2\overline{\mathbf{A}}^2{}_1 + \beta_3(tr\overline{\mathbf{A}}^2{}_1)\overline{\mathbf{A}}_1 \qquad (2.26)$$

In this case, it is easy to see that exact and approximate solution of compatible equation for our flow problem is obtained from the equation (2.21) with substituting $\Gamma = 2\gamma_3$. Hence, in this paper, we shall consider the case only $\Gamma > 0$.

RESULTS AND CONCLUSIONS

For a mild occlusion, we observed that $\Gamma \neq 0$, the pumping curves are non-linear and above the Newtonian case. But, this case is not always true as shown in Fig.1. Similar results were also observed for $\Gamma < 0$. Therefore, for the same adverse pressure gradient, the pumping rate of a third-grade fluid is greater or less then a Newtonian fluid that has the same lower-limiting viscosity depending on the parameters Γ, ϕ, and θ. This behaviour of this fluid for the present problem has not been noted before.
The effect of the non-Newtonian parameter on Γ pumping are presented in Fig.2. We note that as Γ increases and all other parameters are held fixed, the rate of pumping is increased for low occlusion. However, similar results were not observed for the high values of ϕ.

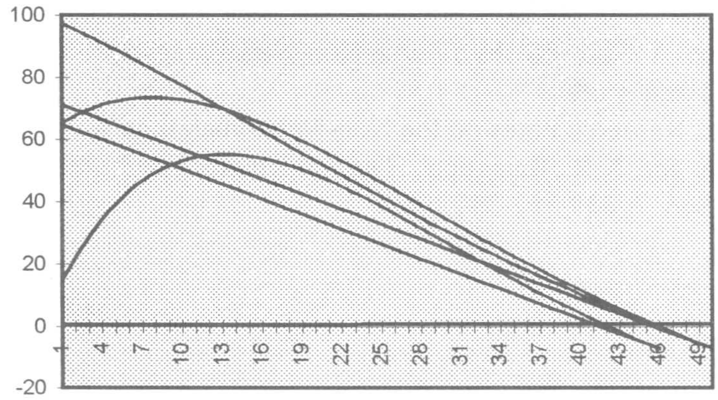

Figure 1. Graph of the pressure rise Per wawelength versus flow rate for a high occulusion (ϕ=0.8) and with Γ=0.0, 0.01, 0.002, 0.003.

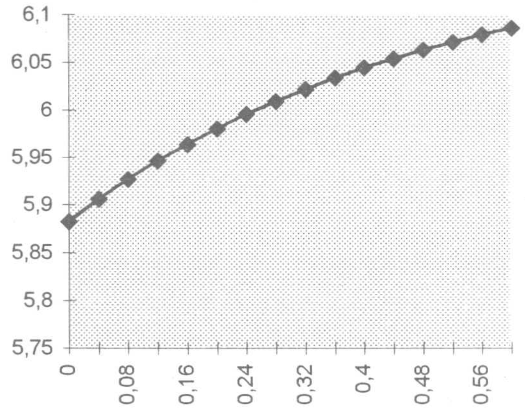

Figure 2. Graph of θ obtained $\Delta P_{\lambda} = 0$ versus Γ, the non-Newtonian parameter.

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