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NEW EXACT SOLUTIONS OF THE (2+1)-DIMENSIONAL GINZBURG-LANDAU EQUATION

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Abstract- A novel identical reforming of differential equation and the high order auxiliary methods are used to construct solitary solutions and periodic solutions of (2 + 1)-Dimensional Ginzburg-Landau equation. It is shown that the high order auxiliary method, with the help of symbolic computation, provides a powerful mathematical tool for solving nonlinear equations arising in mathematical physics.

Key Words- (2+1)- Dimensional Ginzburg-Landau, reform of identical-solving, high order auxiliary equation method

1. INTRODUCTION

The investigation of exact traveling wave solutions to nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. The wave phenomena are observed in fluid dynamics, plasma, elastic media, optical fibers, etc. Complex Ginzburg-Landau equation (CGLE) is a major subject in nonlinear optics, which describes the propagation of optical pulses in optic fibers.

The solutions of CGLE have been extensively studied in various aspects since it was derived [1-4]. A new model was introduced by Sakaguchi and Malomed to describe a nonlinear planer waveguide incorporated into a closed optical cavity, a 2D cubic-quintic Ginzburg-Landau equation(CQGLE) with an anisotropy of a novel type which is diffractive in one direction, and diffusive in the other,. some interesting phenomena of this equation at the zero-dispersion point were demonstrated using systematic simulation [5]. However, so far it is rarely for seeking the exact solitary wave solutions of this equation in addition to the reference [6]. The goal of the present work is searching for exact solutions of the cubic-quintic Ginzburg-Landau equation.

Over the last few decades, directly searching for exact solutions of nonlinear partial differential equations (NPDEs) has become more attractive topic in physical science and nonlinear science. With the rapid development of nonlinear science based on computer algebraic system like Maple package, some new powerful solving methods have been developed, such as multi-wave method [7] homogeneous balance principle [8-10], F-expansion method [11-12], extended auxiliary equation method [13-14], and so on.

In this work, exact solutions of the CQGLE are considered. The novel identical reforming of differential equation and the high order auxiliary equation method are applied to find new generalized exact solutions of the 2D CQGLE.

2. TRANSFORM REFORMING OF CQGL EQUATION WITH INDENTICAL-SOLVING

Consider the 2D-CQGL equation with normal dispersion

$$iu_z + \frac{1}{2}u_{xx} + \frac{1}{2}(\beta - i)u_{\tau\tau} + iu + (1 - ir_1)|u|^2 u + ir_2 |u|^4 u = 0, \tag{1}$$

where $\beta < 0$ is a real constant, z and x are the propagation and transverse coordinates, respectively. $\tau \equiv t - \frac{z}{V_0}$ is the so-called reduced time, where t is the physical time, and V_0 is the group velocity of the carrier wave. Since u is a complex function, we can assume that Eq. (1) have solutions in the form

$$u(x,z,\tau) = \phi(x,\tau)e^{ikz + i\theta(x,\tau)} \tag{2}$$

where k and $\theta(x,\tau)$ is real to be determined , $\phi(x,\tau)$ is a real unknown function. Substituting (2) into Eq.（1）and separating the real part and the imaginary part of result yield

$$\theta_x\phi_x + \frac{1}{2}\theta_{xx}\phi - \frac{1}{2}\phi_{\tau\tau} + \frac{1}{2}\theta_\tau^{\,2}\phi + \frac{1}{2}\beta\theta_{\tau\tau}\phi + \beta\theta_\tau\phi_\tau + \phi - r_1\phi^3 + r_2\phi^5 = 0, \tag{3}$$

$$-k\phi + \frac{1}{2}\phi_{xx} - \frac{1}{2}\theta_x^{\,2}\phi + \frac{1}{2}\beta\phi_{\tau\tau} - \frac{1}{2}\beta\theta_\tau^{\,2}\phi + \theta_\tau\phi_\tau + \frac{1}{2}\theta_{\tau\tau}\phi + \phi^3 = 0, \tag{4}$$

let

$$\phi(x,\tau) = \phi(\xi),\ \theta(x,\tau) = \theta(\eta),\ \xi = l_0 x - l_1\tau,\ \eta = h_0 x - h_1\tau, \tag{5}$$

where l_0, l_1, h_0, h_1 are all real constants to be determined.

Substituting（5）into Eq. (3)- (4) yields

$$(l_0h_0 + \beta l_1h_1)\theta'\phi' + (\frac{1}{2}h_0\theta'' + \frac{1}{2}\beta h_1\theta'' + 1 + \frac{1}{2}h_1^{\,2}(\theta')^2)\phi - \frac{1}{2}l_1^{\,2}\phi'' - r_1\phi^3 + r_2\phi^5 = 0, \tag{6}$$

$$(-k - \frac{1}{2}h_0^{\,2}(\theta')^2 - \frac{1}{2}\beta h_1^{\,2}(\theta')^2 + \frac{1}{2}h_1^{\,2}\theta'')\phi + \frac{1}{2}(l_0^{\,2} + \beta l_1^{\,2})\phi'' + l_1h_1\theta'\phi' + \phi^3 = 0, \tag{7}$$

taking $h_1 = \frac{1}{\sqrt{-\beta}}h_0, l_1 = \frac{1}{\sqrt{-\beta}}l_0$, we have

$$\frac{1}{2}(l_0^{\,2} + \beta l_1^{\,2}) = 0, \quad h_0l_0 + \beta h_1l_1 = 0, \quad \frac{1}{2}(h_0^{\,2} + \beta h_1^{\,2}) = 0,$$

hence Eq.（6）and（7）can be written as follow

$$(\frac{1}{2}h_1^{\,2}(\theta')^2 + 1)\phi - \frac{1}{2}l_1^{\,2}\phi'' - r_1\phi^3 + r_2\phi^5 = 0, \tag{8}$$

$$(\frac{1}{2}h_1^{\,2}\theta'' - k)\phi + l_1h_1\theta'\phi' + \phi^3 = 0, \tag{9}$$

let

$$\theta(\eta) = h_0 x - h_1 \tau. \qquad (10)$$

Making the identical reforming of Eq.(9), we have from Eq.(9)

$$\phi' = \frac{k\phi - \phi^3}{l_1 h_1}, \qquad (11)$$

hence

$$\phi'' = \frac{(k - 3\phi^2)\phi'}{l_1 h_1}. \qquad (12)$$

Using（11）again we get

$$\phi'' = \frac{1}{l_1 h_1}(k - 3\phi^2)\frac{k\phi - \phi^3}{l_1 h_1},$$

namely

$$k^2\phi - 4k\phi^3 + 3\phi^5 - l_1^2 h_1^2 \phi'' = 0. \qquad (13)$$

Eq.(13) is rewritten as

$$\left(\frac{k}{h_1}\right)^2 \phi - \frac{4k}{h_1^2}\phi^3 + \frac{3}{h_1^2}\phi^5 - l_1^2 \phi'' = 0. \qquad (14)$$

Eq.（8）can be written as

$$(h_1^2 + 2)\phi - 2r_1\phi^3 + 2r_2\phi^5 - l_1^2\phi'' = \qquad (15)$$

Equation (14) and equation (15) must be the same equation, comparing their coefficients, we get

$$\frac{k^2}{h_1^2} = h_1^2 + 2,\ r_1 = \frac{2k}{h_1^2},\ 2r_2 = \frac{3}{h_1^2}. \qquad (16)$$

Solving (16) yields

$$k = \frac{3r_1}{4r_2}, \qquad \frac{9r_1^2}{16r_2^2} = h_1^2(h_1^2 + 2).$$

Taking value of k and h_1 as above, then substitute them into Eq. (14) and (15), we can obtain one and the same equation.

$$3r_1^2\phi - 16r_1 r_2\phi^3 + 16r_2^2\phi^5 - 8r_2 l_1^2\phi'' = \qquad (17)$$

3. USING HIGHER ORDER AUXILIARY EQUATION FOR SOLVING THE EQUATION

In this section, the CQGL equation is solved by using a higher order auxiliary equation method. We seek for the solutions of Eq. (17) in the form

$$\phi(\xi) = \sum_{i=0}^{n} A_i F^i(\xi), \quad A_n \neq 0. \qquad (18)$$

in which $A_i\,(i = 0,1,2,\ldots,n)$ are constants to be determined, n is a positive integer which determined by balancing the highest order derivative term with the highest power nonlinear term and $F(\xi)$ satisfies the following equation

$$(F')^2 = d_0 + d_2 F^2(\xi) + d_4 F^4(\xi) + d_6 F^6(\xi). \tag{19}$$

Substituting (18) into (17) along with (19), balancing the highest order derivative term with the highest power nonlinear term in Eq. (17), we find $n = 1$. Therefore, the solution of Eq. (17) is the form as follows

$$\phi(\xi) = A_0 + A_1 F(\xi) \tag{20}$$

where A_0, A_1 are constants to be determined. Substituting Eq. (20) into Eq. (17) to get a polynomial with respect to $F(\xi)$. Equating to zero the coefficients of all powers of $F(\xi)$ yields a set of algebraic equations for $l_1, A_0, A_1, r_1, r_2, d_j (j = 0, 2, 4, 6)$

$$\begin{aligned}
&3r_1^2 A_0 + 16r_2^2 A_0^5 - 16r_1 r_2 A_0^3 = 0,\\
&80r_2^2 A_0^4 A_1 - 8r_2 l_1^2 A_1 d_2 - 48r_1 r_2 A_0^2 A_1 + 3r_1^2 A_1 = 0,\\
&-48r_1 r_2 A_0 A_1^2 + 160r_2^2 A_0^3 A_1^2 = 0,\\
&160r_2^2 A_0^2 A_1^3 - 16r_2 l_1^2 A_1 d_4 - 16r_1 r_2 A_1^3 = 0,\\
&80r_2^2 A_0 A_1^4 = 0,\\
&-24r_2 l_1^2 d_6 A_1 + 16r_2^2 A_1^5 = 0.
\end{aligned}$$

Solving these equations with Maple, we get the following results

$$A_0 = 0, A_1 = \pm\sqrt{\frac{-d_4 l_1^2}{r_1}}, d_2 = \frac{3r_1^2}{8r_2 l_1^2}, d_6 = \frac{2r_1 l_1^2 d_4^2}{3r_1^2}, l_1 = l_1.$$

Substituting these result into Eq.(20), we can get the general form solutions of Eq. (1) along with Eq. (2) as follow

$$u(\xi) = \pm\sqrt{\frac{-d_4 l_1^2}{r_1}} F(\xi) e^{ikz + i(h_0 x - h_1 \tau)}$$

where $F(\xi)$ is the solution of Eq. (19).

3.1. Solutions to the eq. (19)

By considering the different values of d_0, d_2, d_4 and d_6, we will find that Eq. (19) possesses several types of fundamental solutions which are now listed as follows [14]:

Case 1. Suppose that $d_0 = 0, d_6 < 0$ and $\Delta = d_4^2 - 4d_2 d_6 > 0$.

ⅰ) If $d_2 > 0$ and $d_4 < 0$, Eq. (19) has a bell profile solution

$$F_1(\xi)=\left\{\frac{2d_2\sec h^2(\sqrt{d_2}\xi)}{2\sqrt{d_4^2-4d_2d_6}-(\sqrt{d_4^2-4d_2d_6}+d_4)\sec h^2(\sqrt{d_2}\xi)}\right\}^{\frac{1}{2}},$$

and a singular solution

$$F_2(\xi)=\left\{\frac{2d_2\csc h^2(\pm\sqrt{d_2}\xi)}{2\sqrt{d_4^2-4d_2d_6}+(\sqrt{d_4^2-4d_2d_6}-d_4)\csc h^2(\pm\sqrt{d_2}\xi)}\right\}^{\frac{1}{2}},$$

ii) If $d_2<0$ and $d_4>0$, Eq. (19) has a triangular periodic solution

$$F_3(\xi)=\left\{\frac{-2d_2\sec^2(\sqrt{-d_2}\xi)}{2\sqrt{d_4^2-4d_2d_6}-(\sqrt{d_4^2-4d_2d_6}-d_4)\sec^2(\sqrt{-d_2}\xi)}\right\}^{\frac{1}{2}},$$

and a singular triangular periodic solution

$$F_4(\xi)=\left\{\frac{2d_2\csc^2(\pm\sqrt{-d_2}\xi)}{2\sqrt{d_4^2-4d_2d_6}-(\sqrt{d_4^2-4d_2h_6}+d_4)\csc^2(\pm\sqrt{-d_2}\xi)}\right\}^{\frac{1}{2}},$$

Case 2. Suppose that $d_0=0,\ d_2<0$ and $\Delta=d_4^2-4d_2d_6>0$, Eq. (19) has two singular triangular periodic solutions

$$F_5(\xi)=\left\{\frac{2f_2}{\pm\sqrt{d_4^2-4d_2d_6}\cos(2\sqrt{-d_2}\xi)}\right\}^{\frac{1}{2}},$$

$$F_6(\xi)=\left\{\frac{2d_2}{\pm\sqrt{d_4^2-4d_2d_6}\sin(2\sqrt{-d_2}\xi)}\right\}^{\frac{1}{2}},$$

Case 3. Suppose that $d_0=\dfrac{8d_2^2}{27d_4}$ and $d_6=\dfrac{d_4^2}{4d_2}$.

i) If $d_2<0$ and $d_4>0$, Eq. (19) has a kink profile solution

$$F_7(\xi)=\left\{-\frac{8d_2\tanh^2(\pm\sqrt{-\frac{d_2}{3}}\xi)}{3d_4[3+\tanh^2(\pm\sqrt{-\frac{d_2}{3}}\xi)]}\right\}^{\frac{1}{2}},$$

and a singular solution

$$F_8(\xi)=\left\{-\frac{8d_2\coth^2(\pm\sqrt{-\frac{d_2}{3}}\xi)}{3d_4[3+\coth^2(\pm\sqrt{-\frac{d_2}{3}}\xi)]}\right\}^{\frac{1}{2}},$$

ii) If $d_2>0$ and $d_4<0$, Eq. (19) has a triangular periodic solution

$$F_9(\xi)=\left\{\frac{8d_2\tan^2(\pm\sqrt{\frac{d_2}{3}}\xi)}{3d_4[3-\tan^2(\pm\sqrt{\frac{d_2}{3}}\xi)]}\right\}^{\frac{1}{2}},$$

and a singular triangular periodic solution

$$F_{10}(\xi)=\left\{-\frac{8d_2\cot^2(\pm\sqrt{\frac{d_2}{3}}\xi)}{3d_4[3-\cot^2(\pm\sqrt{\frac{d_2}{3}}\xi)]}\right\}^{\frac{1}{2}},$$

Case 4. Suppose that $d_0=0$ and $d_6=\frac{d_4^2}{4d_2}$. If $d_2>0$ and $d_4<0$, Eq. (19) has a kink profile solution

$$F_{11}(\xi)=\left\{-\frac{d_2}{d_4}[1+\tanh(\pm\sqrt{d_2}\xi)]\right\}^{\frac{1}{2}},$$

and a singular solution

$$F_{12}(\xi)=\left\{-\frac{d_2}{d_4}[1+\coth(\pm\sqrt{d_2}\xi)]\right\}^{\frac{1}{2}},$$

Case 5. Suppose that $d_0=\frac{8d_2^2}{25d_4}$ and $d_6=\frac{5d_4^2}{16d_2}$. If $d_2<0$, $d_4>0$, $d_6<0$ and $d_4^2-3d_2d_6>0$, Eq. (19) has a bell profile solution

$$F_{13}(\xi)=\left\{-\frac{(d_4-\sqrt{d_4^2-3d_2d_6})}{3d_6}[1\pm\operatorname{sech}(\frac{d_4}{2}\sqrt{-\frac{1}{d_6}}\xi)]\right\}^{\frac{1}{2}}.$$

The solutions of Eq. (19) provided above are linearly independent solutions.

3.2. Solutions to the CQGL equation

With the solutions of Eq. (19), the exact solutions of the Eq. (1) are obtained as follows

Case 1. Suppose $c_0 = 0$, $r_2 > 0$, we obtain the envelope wave solutions as follows

$$u_1(x,z,\tau) = \left(\frac{3r_1 \sec h^2 (\sqrt{\frac{3r_1^2}{8l_1^2 r_2}}(l_0 x - l_1 \tau))}{8r_2 - 2r_2 \left(1 \pm \tanh\left(\sqrt{\frac{3r_1^2}{8l_1^2 r_2}}(l_0 x - l_1 \tau) \right) \right)^2} \right)^{1/2} e^{ikz + i(h_0 x - h_1 \tau)},$$

$$u_2(x,z,\tau) = \left(\frac{-3r_1 \csc h^2 (\sqrt{\frac{3r_1^2}{8l_1^2 r_2}}(l_0 x - l_1 \tau))}{8r_2 - 2r_2 \left(1 \pm \coth\left(\sqrt{\frac{3r_1^2}{8l_1^2 r_2}}(l_0 x - l_1 \tau) \right) \right)^2} \right)^{1/2} e^{ikz + i(h_0 x - h_1 \tau)},$$

$$u_3(x,z,\tau) = \left(\frac{-3r_1 \sec h^2 (\sqrt{\frac{3r_1^2}{8l_1^2 r_2}}(l_0 x - l_1 \tau))}{-8r_2 \pm 8r_2 \tanh\left(\sqrt{\frac{3r_1^2}{8l_1^2 r_2}}(l_0 x - l_1 \tau) \right)} \right)^{1/2} e^{ikz + i(h_0 x - h_1 \tau)},$$

$$u_4(x,z,\tau) = \left(\frac{3r_1 \csc h^2 (\sqrt{\frac{3r_1^2}{8l_1^2 r_2}}(l_0 x - l_1 \tau))}{-8r_2 \pm 8r_2 \coth\left(\sqrt{\frac{3r_1^2}{8l_1^2 r_2}}(l_0 x - l_1 \tau) \right)} \right)^{1/2} e^{ikz + i(h_0 x - h_1 \tau)}.$$

Case 2. Suppose $c_0 = 0$ and $d_6 = \frac{d_4^2}{4d_2}$, if $r_2 > 0$ and $r_1 > 0$, Eq. (1) has a kink profile solution

$$u_5(x,z,\tau) = \left(\frac{3r_1}{8r_2} \left(1 \pm \tanh\left(\frac{1}{2} \sqrt{\frac{3r_1^2}{8l_1^2 r_2}}(l_0 x - l_1 \tau) \right) \right) \right)^{1/2} e^{ikz + i(h_0 x - h_1 \tau)}$$

and a singular kink wave solution

$$u_6(x,z,\tau) = \left(\frac{3r_1}{8r_2} \left(1 \pm \coth\left(\frac{1}{2} \sqrt{\frac{3r_1^2}{8l_1^2 r_2}}(l_0 x - l_1 \tau) \right) \right) \right)^{1/2} e^{ikz + i(h_0 x - h_1 \tau)}.$$

Case 3. Suppose $c_0 = -\frac{r_1^3}{6r_2}$ and $d_6 = \frac{d_4^2}{4d_2}$,

i) if $r_2 < 0$, r_1 <0, Eq. (1) has kink wave solutions

$$u_7(x,z,\tau) = \left(\frac{r_1 \tanh^2(\pm\sqrt{-\frac{r_1^2}{8l_1^2 r_2}}(l_0 x - l_1\tau))}{r_2\left(3 + \tanh^2(\pm\sqrt{-\frac{r_1^2}{8l_1^2 r_2}}(l_0 x - l_1\tau))\right)} \right)^{1/2} e^{ikz + i(h_0 x - h_1\tau)},$$

$$u_8(x,z,\tau) = \left(\frac{r_1 \coth^2(\pm\sqrt{-\frac{r_1^2}{8l_1^2 r_2}}(l_0 x - l_1\tau))}{r_2\left(3 + \coth^2(\pm\sqrt{-\frac{r_1^2}{8l_1^2 r_2}}(l_0 x - l_1\tau))\right)} \right)^{1/2} e^{ikz + i(h_0 x - h_1\tau)}.$$

ii) if $r_2 > 0$ and $r_1 > 0$,Eq. (1) has a triangular periodic solution

$$u_9(x,z,\tau) = \left(\frac{-r_1 \tan^2(\pm\sqrt{\frac{r_1^2}{8l_1^2 r_2}}(l_0 x - l_1\tau))}{r_2\left(3 - \tan^2(\pm\sqrt{\frac{r_1^2}{8l_1^2 r_2}}(l_0 x - l_1\tau))\right)} \right)^{1/2} e^{ikz + i(h_0 x - h_1\tau)}$$

and a singular triangular periodic solution

$$u_{10}(x,z,\tau) = \left(\frac{r_1 \cot^2(\pm\sqrt{\frac{r_1^2}{8l_1^2 r_2}}(l_0 x - l_1\tau))}{r_2\left(3 - \cot^2(\pm\sqrt{\frac{r_1^2}{8l_1^2 r_2}}(l_0 x - l_1\tau))\right)} \right)^{1/2} e^{ikz + i(h_0 x - h_1\tau)}.$$

4. CONCLUSION

In this work, we obtain generalized solitonary solutions and periodic solutions of the 2D-CQGL equation by using the novel identical reforming of ordinary differential equation and high order auxiliary equation method. In particular, the identical reforming method of equation is first found and it can be applied to another problem.

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