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A Rocq-Based Formalization of Hilbert's Geometry: Building a Reusable Foundation for 3D Perpendicularity Theory and Verification

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Abstract

Hilbert's axiom system for geometry is a landmark in formal methods. This paper presents a complete formalization of spatial perpendicularity—a theory not fully developed in Hilbert's original work—using the Rocq proof assistant. We systematically defined the relations of perpendicularity between lines and planes based solely on Hilbert's primitive notions and axioms. Within this framework, we mechanized the proof of a significant spatial congruence theorem that Hilbert stated without proof—a theorem that fundamentally reveals the relationship between congruence and motion. The formal proof of this theorem demonstrates the intrinsic completeness of Hilbert's system for three-dimensional (3D) space. Crucially, no additional spatial congruence axioms are needed, as all properties are derived rigorously from the original planar axioms. All proofs are mechanically verified by Rocq, ensuring logical correctness. This work completes Hilbert's geometry system. It also delivers a reusable Rocq library that offers a rigorous foundation for verifying geometric reasoning in safety-critical software systems.

Keywords: Hilbert's axiom system; formalization; Rocq; perpendicularity theory; congruence; 3D space

1. Introduction

Euclid's *Elements* is widely regarded as the origin of mathematical axiomatization, laying the foundation for geometry [1]. Even today, it continues to influence modern research in areas such as formal systems and verification [2,3]. Euclid's major contribution was his organization of geometric knowledge through a system of axioms and deductive reasoning, thereby establishing a structured approach to geometry. However, from a strict logical perspective, Euclid's work contained several limitations: it relied on undefined concepts, included an insufficient set of axioms, and invoked the controversial Fifth Postulate (the Parallel Postulate) [2]. These issues motivated later mathematicians to seek a more logically rigorous axiomatic system for geometry. Among these efforts, Hilbert's *Foundations of Geometry* [4] stands out as a landmark achievement.

Hilbert's *Foundations of Geometry* has significantly influenced modern mathematics and formal methods. In this work, Hilbert proposed a complete and concise axiom system from which all of Euclidean geometry could be derived without relying on external intuition or assumptions. The achievement marked the birth of the modern axiomatic method [5–7]. Hilbert's axiom system began with three primitive geometric objects—points, lines, and



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planes—and three basic relations: incidence, betweenness, and congruence. Unlike Euclid, who provided descriptive definitions, Hilbert treated points, lines, and planes as purely abstract entities, emphasizing that their interpretation could be arbitrary—famously stating that one could replace “points, lines, and planes” with “tables, chairs, and beer mugs” without affecting the logical structure [7]. Furthermore, Hilbert reorganized Euclid’s axioms into five logically independent groups: incidence, order, congruence, parallels, and continuity. This clear classification clarified the underlying logical architecture of geometry and provided a robust foundation for further mathematical development.

The formalization of mathematics, which involves encoding mathematical proofs into a computer, has become an essential endeavor in modern computational mathematics and artificial intelligence [8]. This process, known as formalization, relies on programs called proof assistants to verify the correctness of such encoded reasoning [9]. By establishing rigorous mathematical and logical foundations, researchers can advance their work with greater confidence and efficiency [10,11]. Prominent proof assistants currently in use include Rocq, which was formerly known as Coq [12–14], Isabelle/HOL [15], HOL Light [16], and Lean [17,18], among others. Each has its own theoretical foundation, distinctive features, and target application scenarios. Rocq, for instance, is an interactive theorem prover based on the Calculus of Inductive Constructions (CIC). It provides highly expressive higher-order logic, enables logical derivation through imperative commands, and supports automated reasoning using previously proven propositions. Its system of inductive types extends traditional type definitions found in functional programming languages, analogous to recursive type constructions [12–14]. With the help of proof assistants like Rocq, formalized mathematics may emerge as a new standard of mathematical rigor [19]. Wiedijk describes the current formalization movement as the third mathematical revolution, following Euclid’s introduction of the concept of proof and the development of rigorous methods in the 19th century [9]. Similarly, Buzzard emphasized in an invited lecture at the 2022 International Congress of Mathematicians that mathematicians should employ machines for proof verification, allowing them to concentrate on deeper theoretical problems.

There are many notable achievements in the formalization of complex mathematical theorems. In 2005, Gonthier and Werner used Rocq to provide a computer-aided proof of the Four Color Theorem [20]. Gonthier’s team later verified the Odd Order Theorem in 2012, a process involving approximately 4300 definitions and 15,000 theorems, totaling around 170,000 lines of Rocq code [21]. This achievement has earned Rocq widespread recognition in academia. In 2015, Hales and his team verified the Kepler Conjecture using HOL Light and Isabelle/HOL [22]. In June 2021, Fields Medalist Peter Scholze announced that Lean had verified core parts of his “condensed mathematics” theory [23]. Wiedijk has highlighted in his work that global research teams have completed or plan to complete formal proofs of 100 famous mathematical theorems, including Gödel’s First Incompleteness Theorem, the Jordan Curve Theorem, the Prime Number Theorem, and Fermat’s Last Theorem [9]. These accomplishments have significantly increased the academic influence of proof assistants.

There are also many achievements in the formalization of geometry. In the domain of projective geometry, Magaud, Narboux, and Schreck introduced alternative axiom systems for plane and spatial projective geometry based on the concept of ranks, as a substitute for the traditional framework [24]. They formally verified in Coq that Desargues’ property holds within their system [25]. Subsequently, Braun, Magaud, and Schreck completed a formal proof in Coq demonstrating the mutual interpretability between this system and the traditional axiom sets [26]. Coghetto has also made significant contributions to projective geometry formalization in Mizar, including proofs of Pascal’s theorem in the real projective plane [27]. Furthermore, he developed analytical approaches to duality notions in the real projective plane, enabling concise proofs of dual theorems such as the converses of

Desargues' and Pappus' theorems through systematic dual definitions [28]. Formalization has also reached complex geometry, notably through Marić and Petrović's work [29] on the extended complex plane and Möbius transformations. Additionally, the intersection of geometry with non-standard analysis has been explored in Isabelle by Fleuriot, who mechanized geometric propositions from Newton [30] and Kepler [31]. Discrete models of the continuum, such as the Hartong–Reeb line, have also been implemented in both Coq [32] and Isabelle [33], demonstrating the versatility of proof assistants.

Researchers have also shown significant interest in the formalization of Hilbert's *Foundations of Geometry*. Dehlinger, Dufourd, and Schreck focused on formalizing the first two groups of axioms, along with several theorems, within the intuitionistic setting of Rocq [34]. Simultaneously, Meikle and Iwama employed Isabelle/HOL to formalize the planar portion of the first three groups of axioms and their associated theorems [35–37]. Scott continued Meikle's work, rectifying some errors and emphasizing code readability [38]. His contribution extended to verifying Theorem 9—the Polygonal Jordan Curve Theorem on the plane—from *Foundations of Geometry* [39]. Richter also employed HOL Light to formalize Hilbert's axioms for plane geometry, along with numerous theorems including the Crossbar Theorem and parallelogram properties, in 3300 lines of code, providing a rigorous treatment suitable for high school students [40]. In a parallel effort, Braun, Boutry, and Narboux concentrated primarily on Tarski's system [41,42]. They confirmed the equivalence between Hilbert's axioms and Tarski's axioms in the realm of plane geometry, except for the continuity axiom [43–45]. Beeson and Wos explored automated theorem proving in Tarskian geometry, developing methods to find proofs within Tarski's axiom system and deriving Hilbert's 1899 axioms from Tarski's axioms [46]. Grabowski and Coghetto focused on formalizing Tarski's in Mizar, developing advanced techniques with structures, inheritance, and automated theorem proving, while proving that the Euclidean plane satisfies all eleven of Tarski's axioms and demonstrating the parallel postulate's independence [47]. Their comprehensive series spans five parts, systematically formalizing Schwabhäuser, Szmielw, and Tarski's foundational work on geometry [48–51].

This collective endeavor has largely completed the formalization of planar geometry within Hilbert's system. However, a transition to three-dimensional space introduces formidable new challenges. Formalizing spatial geometry, particularly concepts like perpendicularity between lines and planes, demands managing a vastly increased complexity of incidence relations and geometric configurations that have no direct analog in the plane. Proving spatial theorems often requires novel constructive methods and a rigorous chain of derived lemmas that cannot be straightforwardly extrapolated from their planar counterparts. Consequently, while the planar foundation is robust, the spatial theory of perpendicularity—a cornerstone of solid geometry—remains a significant gap in the formalization of Hilbert's work.

Our work addresses this gap by presenting the complete formalization of 3D perpendicularity theory within the Rocq proof assistant, which is publicly available at: <https://github.com/a-qmeng/Formal-Hilbert-Geometry> (accessed on 14 October 2025). We begin with a full encoding of Hilbert's primitive notions and axiom groups, then systematically develop the theory of spatial perpendicularity with machine-verified proofs, and finally apply this framework to prove Hilbert's previously unproven spatial congruence theorem. The paper is structured as follows: Section 2 details our formalization of Hilbert's first three axiom groups (incidence, order, and congruence), which serve as the basis for all subsequent work. In Section 3, we formalize basic geometry propositions derivable from these axioms, creating the necessary groundwork for spatial theory. Section 4 presents our core contribution: the systematic construction of spatial perpendicularity theory, including formal definitions and machine-verified properties. Section 5 demonstrates the power of

this theory by applying it to the computer-verified proof of Hilbert’s spatial congruence theorem. The paper concludes in Section 7 with a summary of contributions and future research directions.

2. Hilbert’s Axioms and Formalizations

In this section, we delve into the formalization of Hilbert’s first three groups of axioms (incidence, order, and congruence), which provide the necessary foundation for all subsequent works.

2.1. Group I

There are 8 axioms in Group I that establish the incidence relations between three basic entities: points, lines, and planes. These axioms articulate fundamental geometric truths, such as the determinations of a line by two points, a plane by three points, and the existence of non-collinear and non-coplanar points. The details of these axioms are available in [4], we focus here on the formalization process.

Before formalizing the axioms, we first establish the fundamental geometric entities and their incidence relations in Rocq. Using the *Parameter* declaration (see Appendix A for an introduction to Rocq syntax), we define points, lines, and planes as distinct types, along with their basic incidence relations:

```
Parameter Point Line Plane : Set.
Parameter Incid : Point -> Line -> Prop. (* Point-line incidence *)
Parameter Incid_P_F : Point -> Plane -> Prop. (* Point-plane incidence *)
(* Line-plane incidence: a line lies on a plane if all its points do *)
Definition Incid_L_F (l : Line) (f : Plane) : Prop :=
  forall A, Incid A l -> Incid_P_F A f.
```

Based on these definitions, we further formalize collinearity (three points on a line) and coplanarity (four points on a plane):

```
Definition ColH (A B C : Point) : Prop :=
  exists l, Incid A l /\ Incid B l /\ Incid C l.
Definition Cof A B C D := exists f, Incid_P_F A f /\
  Incid_P_F B f /\ Incid_P_F C f /\ Incid_P_F D f.
```

Now we can formalize the axioms of Group I as follows:

```
Axiom Incid_1 : forall A B, A<>B -> exists l, Incid A l /\ Incid B l.
Axiom Incid_2 : forall A B l1 l2, A<>B ->
  Incid A l1 -> Incid B l1 -> Incid A l2 -> Incid B l2 -> l1=l2.
Axiom Incid_3_1 : forall l, exists A B, A<>B /\ Incid A l /\ Incid B l.
Axiom Incid_3_2 : exists A B C, ~ ColH A B C.
Axiom Incid_4_1 : forall A B C, ~ ColH A B C -> exists! f,
  Incid_P_F A f /\ Incid_P_F B f /\ Incid_P_F C f.
Axiom Incid_4_2 : forall f, exists A, Incid_P_F A f.
Axiom Incid_6 : forall l f A B, A<>B -> Incid A l -> Incid B l ->
  Incid_P_F A f -> Incid_P_F B f -> Incid_L_F l f.
Axiom Incid_7 : forall A f1 f2, f1<>f2 -> Incid_P_F A f1 -> Incid_P_F A f2 ->
  exists B, A<>B /\ Incid_P_F B f1 /\ Incid_P_F B f2.
Axiom Incid_8 : exists A B C D, ~ Cof A B C D.
```

2.2. Group II

The second group of axioms defines the concept of “betweenness” (denoted *BetH*), which establishes an order for points on lines, in planes, and in space. We begin by defining the betweenness relation. The first three axioms establish its basic geometric properties.

```

Parameter BetH : Point -> Point -> Point -> Prop. (* B is between A and C *)
Axiom BetH_1_1 : forall A B C, BetH A B C -> ColH A B C.
Axiom BetH_1_2 : forall A B C, BetH A B C -> A <> C.
Axiom BetH_1_3 : forall A B C, BetH A B C -> BetH C B A.
Axiom BetH_2 : forall A B, A <> B -> exists C, BetH A B C.
Axiom BetH_3 : forall A B C, BetH A B C -> ~ BetH B C A.

```

Axiom II, 4 (Pasch's Axiom) states that if a line enters the interior of a triangle, it must exit again. We first formalize the key concept of a line cutting a segment.

```

(* line l intersects segment AB *)
Definition cut (l : Line) (A B : Point) :=
  ~ Incid A l /\ ~ Incid B l /\ exists I, Incid I l /\ BetH A I B.
(* Pasch's Axiom *)
Axiom BetH_4: forall A B C l (H: ~ ColH A B C),
  (forall D, Incid D l -> Cof A B C D) ->
  ~ Incid C l -> cut l A B -> cut l A C /\ cut l B C.

```

Removing the condition that the line l lies on the plane ABC in Pasch's Axiom restricts the space to two dimensions. Dehlinger and Iwama's work omits this condition from their formal description [34–36], while Scott and Boutry include it by introducing a plane f and specifying four conditions involving points A , B , C , and line l within that plane [39,43]. Our approach uses the assumption $(\text{forall } D, \text{Incid } D \text{ } l \rightarrow \text{Cof } A \text{ } B \text{ } C \text{ } D)$ to avoid introducing a plane f . Notably, our formalization does not complicate subsequent verification efforts.

Building upon the primitive notion of betweenness ($BetH$) and the intersection relation (cut), we can also introduce key derived geometric concepts essential for further development. These definitions formalize the intuitive notions of points lying on the “same side” or “opposite side” relative to a point, a line, or a plane.

```

(* Points A and A' are on the same side of Point O *)
Definition Ipsilateral_P A A' O :=
  A <> O /\ A' <> O /\ ColH A O A' /\ ~ BetH A O A'.
(* Points A and B are on the different sides of Point O *)
Definition Heteromeral_P A B O := BetH A O B.
(* Points A and A' are on the same side of line l *)
Definition Ipsilateral_L A A' l := exists P, cut l A P /\ cut l A' P.
(* Points A and B are on the different sides of line l *)
Definition Heteromeral_L A B l := cut l A B.
(* Points C and D are on the same side of line AB *)
Definition Ipsilateral_S A B C D := A<>B /\
  forall l, Incid A l -> Incid B l -> Ipsilateral_L C D l.
(* Points C and D are on the different sides of line AB *)
Definition Heteromeral_S A B C D := A<>B /\
  forall l, Incid A l -> Incid B l -> cut l C D.
(* Points A and B are on the same side of flat f *)
Definition Ipsilateral_f A A' f := ~ Incid_P_F A f /\ ~ Incid_P_F A' f /\
  ~ (exists P, BetH A P A' /\ Incid_P_F P f).
(* Points A and B are on the different sides of flat f *)
Definition Heteromeral_f A B f := ~ Incid_P_F A f /\ ~ Incid_P_F B f /\
  (exists P, BetH A P B /\ Incid_P_F P f).

```

A significant design choice in our formalization concerns the definition of two points being on the same side of a line. Hilbert's original definition, which can be formalized as Ipl_S_Hil , is plane-dependent: it requires specifying a plane f containing the points and the line, and states that the segment AA' does not intersect l [4].

```

Definition Ipl_S_Hil A A' l f := Incid_P_F A f /\ Incid_P_F A' f /\

```

$\text{Incid_L_F } l \text{ f } \wedge \sim \text{Incid } A \text{ l } \wedge \sim \text{Incid } A' \text{ l } \wedge \sim \text{cut } l \text{ A } A'.$

In contrast, our definition *Ipsilateral_L*, inspired by Braun's description of Tarski's system [44], is intrinsic and operates without reference to an ambient plane. It states that points A and A' are on the same side of l if there exists a point P such that both segments AP and $A'P$ are intersected by l . The equivalence of *Ipsilateral_L* and *Ipl_S_Hil* has been proven in our prior work [52]. Other researchers, like Scott and Iwama, have followed Hilbert's original description by using *Ipl_S_Hil*. However, we chose *Ipsilateral_L* because it does not include plane parameters, thereby simplifying and streamlining subsequent formalization.

2.3. Group III

Hilbert's third group of axioms introduces the concept of congruence for segments and angles, which is fundamental to Euclidean geometry. This group consists of five axioms: three governing segment congruence, one for angle congruence, and one connecting the two via the triangle congruence condition (SAS). Our formalization captures these notions Rocq as follows.

2.3.1. Congruence of Segments

We begin by declaring a primitive relation "*CongH A B C D*", meaning segment AB and segment CD are congruent, which can be denoted as $AB \equiv CD$.

Parameter CongH : Point -> Point -> Point -> Point -> Prop.

The axioms about segment congruence can be formally described as follows:

Axiom CongH_1 : forall A B A' P, A<>B -> A'<>P ->
exists B', Ipsilateral_P B' P A' /\ CongH A B A' B'.
Axiom CongH_2 : forall A B C D E F, A<>B -> C<>D -> E<>F ->
CongH A B E F -> CongH C D E F -> CongH A B C D.
Axiom CongH_3 : forall A B C A' B' C', Beth A B C ->
Beth A' B' C' -> CongH A B A' B' -> CongH B C B' C' -> CongH A C A' C'.

From these axioms, we derive three fundamental corollaries that establish key properties of segment congruence.

(* Reflexivity *)
Corollary CongH_self : forall A B, A<>B -> CongH A B A B.
(* Symmetry *)
Corollary CongH_symmetry : forall A B C D, A<>B -> C<>D ->
CongH A B C D -> CongH C D A B.
(* Transitivity *)
Corollary CongH_transitivity : forall A B C D E F, A<>B ->
C<>D -> E<>F -> CongH A B C D -> CongH C D E F -> CongH A B E F.

Hilbert emphasizes that congruence is a property of the segment itself, independent of the order of its endpoints. To capture this, we add the following axiom, from which all other permutations can be derived.

Axiom CongH_exc34 : forall A B C D, A<>B -> C<>D ->
CongH A B C D -> CongH A B D C.

2.3.2. Congruence of Angles

We now introduce angle congruence, denoted "*CongaH A B C D E F*" for $\angle ABC \equiv \angle DEF$. The primary axiom for angles (Axiom III.4) states that an angle can be uniquely transferred to a given ray in a specified half-plane.

```

Parameter CongaH : Point -> Point -> Point -> Point -> Point -> Point -> Prop.

Axiom CongH_4_exists : forall A B C P' A' B', ~ ColH A B C -> ~ ColH A' B' P' ->
  exists C', Ipsilateral_S A' B' C' P' /\ CongaH A B C A' B' C'.
Axiom CongH_4_unique : forall A B C P' A' B' C' D', ~ ColH A B C ->
  ~ ColH A' B' P' -> Ipsilateral_S A' B' C' P' -> Ipsilateral_S A' B' D' P' ->
  CongaH A B C A' B' C' -> CongaH A B C A' B' D' -> Ipsilateral_P C' D' B'.
Axiom CongH_4_self : forall A B C, ~ ColH A B C -> CongaH A B C A B C.

```

Similar to segments, the congruence of angles is independent of the order of the rays. We formalize this with an exchange axiom and an additional axiom stating that congruent angles remain congruent when their sides are extended.

```

Axiom CongH_4_exc : forall A B C D E F, CongaH A B C D E F ->
  CongaH A B C F E D /\ CongaH C B A D E F /\ CongaH C B A F E D.
(* Congruence is preserved under extension of the angle's sides *)
Axiom CongH_4_CongaH : forall A B C D E F A' C' D' F', ~ ColH A B C ->
  ~ ColH D E F -> CongaH A B C D E F -> Ipsilateral_P A A' B ->
  Ipsilateral_P C C' B -> Ipsilateral_P D D' E ->
  Ipsilateral_P F F' E -> CongaH A' B C' D' E F'.

```

2.3.3. The Side-Angle-Side (SAS) Axiom

Axiom III, 5 (SAS Axiom) plays a crucial role in Group III. It gives the connection between the congruence of segments and the congruence of angles. It posits that if two sides and the included angle of one triangle are congruent to those of another, then the corresponding angles opposite the first congruent side are also congruent.

```

Axiom CongH_5 : forall A B C A' B' C', ~ ColH A B C ->
  ~ ColH A' B' C' -> CongH A B A' B' -> CongH A C A' C' ->
  CongaH B A C B' A' C' -> CongaH A B C A' B' C'.

```

This completes the formalization of Hilbert's congruence axioms, providing the foundation for all subsequent results about congruent triangles and geometric equality.

3. Formalization of Elementary Consequences

Having established Hilbert's axiom system in Rocq, we now present a suite of fundamental geometric theories formally derived from these axioms. The development of these theories is crucial, as they form the bedrock for our subsequent construction of the spatial perpendicularity theory.

3.1. Key Lemmas About Order and Congruence

The initial step involves deriving essential, often non-trivial, lemmas that extend the axiomatic foundation. These results, while not explicitly stated in Hilbert's original work, are indispensable for more complex proofs.

3.1.1. Original Lemmas for Spatial Order

We first introduce two original lemmas concerning point-plane relationships that are critical for 3D reasoning. These lemmas address the geometric configuration when a point lies outside a given line or plane.

```

Lemma exists_He : forall A B C P, Cof A B C P -> ~ ColH A B C ->
  ~ ColH A B P -> exists P', ColH A C P' /\ Heteromeral_S A B P P'.
Lemma exists_uncof : forall f A B D P, ~ Incid_P_F D f
  -> ~ Incid_P_F P f -> Incid_P_F A f -> ~ Cof A B D P
  -> exists Q, Cof A B D Q /\ Heteromeral_f Q P f.

```

exists_He: Given a point P on the plane ABC , where the P does not lie on the line AB , there exists a distinct point P' along the line AC , ensuring that P and P' are on opposite sides of the line AB .

exists_uncof: Given two different planes, f and ABD , which share a common point A , and a point P not lying on either plane, there exists a point Q situated on plane ABD such that P and Q are on opposite sides of plane f .

3.1.2. Strengthening the Congruence Axioms

We further establish stronger properties of segment congruence, such as the uniqueness of segment transfer and a variant for segment subtraction, which are implied by but not explicit in Hilbert's axioms.

Corollary CongH_1_unique : $\text{forall } A B A' P B' B'', A \langle \rangle B \rightarrow A' \langle \rangle P \rightarrow$
 $\text{Ipsilateral_P } B' P A' \rightarrow \text{Ipsilateral_P } B'' P A' \rightarrow$
 $\text{CongH } A B A' B' \rightarrow \text{CongH } A B A' B'' \rightarrow B' = B''.$

Corollary CongH_3_BetH : $\text{forall } A B C A' B' C', \text{BetH } A B C \rightarrow \text{BetH } A' B' C' \rightarrow$
 $\text{CongH } A C A' C' \rightarrow \text{CongH } A B A' B' \rightarrow \text{CongH } B C B' C'.$

Corollary CongH_3_exists_Ips : $\text{forall } A B C A' B', \text{Ipsilateral_P } B C A \rightarrow$
 $B \langle \rangle C \rightarrow A' \langle \rangle B' \rightarrow \text{CongH } A B A' B' \rightarrow \text{exists } C', \text{Ipsilateral_P } B' C' A' \wedge$
 $\text{CongH } A C A' C' \wedge \text{CongH } B C B' C' \wedge C' \langle \rangle B'.$

CongH_1_unique establishes the uniqueness of segment transfer, derived from the uniqueness of angle transfer and Axiom III, 5. *CongH_3_BetH* is a variant of *CongH_3*, requiring the subtraction of segments.

Using *CongH_1_unique* and *CongH_3*, we can derive *CongH_3_exists_Ips*: Given three points A, B , and C on a line, with B and C on the same side of A , we can assert that for any segment $A'B'$ congruent to AB , there exists a point C' on the line $A'B'$ ensuring that B' and C' are on the same side of A' , and $AC \equiv A'C', BC \equiv B'C'$.

3.2. Hilbert's Key Theorems

In this subsection, we present several important theorems from Hilbert's *Foundations of Geometry*.

3.2.1. Theorems on Triangle Congruence

We define triangle congruence (*CongtH*) and prove the three classic criteria: SAS (*Theorem12*), ASA (*Theorem13*), and SSS (*Theorem18*). Their machine-verified proofs validate the logical soundness of Hilbert's system.

Definition CongtH $A B C A' B' C' := \sim \text{ColH } A B C \wedge \sim \text{ColH } A' B' C' \wedge$
 $\text{CongH } A B A' B' \wedge \text{CongH } A C A' C' \wedge$
 $\text{CongH } B C B' C' \wedge \text{CongH } B A C B' A' C' \wedge$
 $\text{CongH } A B C A' B' C' \wedge \text{CongH } A C B A' C' B'.$

Theorem Theorem12 : $\text{forall } A B C A' B' C', \sim \text{ColH } A B C \rightarrow \sim \text{ColH } A' B' C' \rightarrow$
 $\text{CongH } A B A' B' \rightarrow \text{CongH } A C A' C' \rightarrow$
 $\text{CongH } B A C B' A' C' \rightarrow \text{CongtH } A B C A' B' C'.$

Theorem Theorem13 : $\text{forall } A B C A' B' C', \sim \text{ColH } A B C \rightarrow \sim \text{ColH } A' B' C' \rightarrow$
 $\text{CongH } A B A' B' \rightarrow \text{CongH } B A C B' A' C' \rightarrow$
 $\text{CongH } A B C A' B' C' \rightarrow \text{CongtH } A B C A' B' C'.$

Theorem Theorem18 : $\text{forall } A B C A' B' C', \sim \text{ColH } A B C \rightarrow \sim \text{ColH } A' B' C' \rightarrow$
 $\text{CongH } A B A' B' \rightarrow \text{CongH } A C A' C' \rightarrow$
 $\text{CongH } B C B' C' \rightarrow \text{CongtH } A B C A' B' C'.$

3.2.2. Theorem on Angles and Midpoints

We begin by formalizing the concept of a right angle, defined as an angle congruent to its supplement.

Definition $\text{Right_angle } A B C := \sim \text{ColH } A B C \wedge$
 $\text{forall } C', \text{BetH } C B C' \rightarrow \text{CongaH } A B C A B C'.$

The significance of right angles is established through the following key results, with (Theorem21) being particularly fundamental.

Theorem Theorem21 : $\text{forall } A B C A' B' C', \text{Right_angle } A B C \rightarrow$
 $\text{Right_angle } A' B' C' \rightarrow \text{CongaH } A B C A' B' C'.$
Corollary Cor_Theorem21 : $\text{forall } A B C A' B' C', \sim \text{ColH } A' B' C' \rightarrow$
 $\text{CongaH } A B C A' B' C' \rightarrow \text{Right_angle } A B C \rightarrow \text{Right_angle } A' B' C'.$

(Theorem14) establishes the critical behavior of angles with respect to supplementation, from which important corollaries about vertical angles and the existence of right angles follow.

Theorem Theorem14 : $\text{forall } A B C D A' B' C' D', \sim \text{ColH } A B C \rightarrow$
 $\sim \text{ColH } A' B' C' \rightarrow \text{BetH } A B D \rightarrow \text{BetH } A' B' D' \rightarrow$
 $\text{CongaH } A B C A' B' C' \rightarrow \text{CongaH } C B D C' B' D'.$
Corollary Cor1_Theorem14 : $\text{forall } A B C A' C', \sim \text{ColH } A B C \rightarrow$
 $\text{BetH } A B A' \rightarrow \text{BetH } C B C' \rightarrow \text{CongaH } C' B A' A B C.$
Corollary Cor2_Theorem14 : $\text{exists } A B C, \text{Right_angle } A B C.$

(Theorem19) establishes that angle congruence is an equivalence relation, a property not directly assumed by Hilbert's axioms but derived from them.

Theorem Theorem19 : $\text{forall } H O K H' O' K' H'' O'' K'', \sim \text{ColH } H O K \rightarrow$
 $\sim \text{ColH } H' O' K' \rightarrow \sim \text{ColH } H'' O'' K'' \rightarrow \text{CongaH } H' O' K' H O K \rightarrow$
 $\text{CongaH } H'' O'' K'' H O K \rightarrow \text{CongaH } H' O' K' H'' O'' K''.$
Corollary CongaH_symmetry : $\text{forall } A B C A' B' C', \sim \text{ColH } A B C \rightarrow$
 $\sim \text{ColH } A' B' C' \rightarrow \text{CongaH } A B C A' B' C' \rightarrow \text{CongaH } A' B' C' A B C.$
Corollary CongaH_trans : $\text{forall } A B C A' B' C' A'' B'' C'', \sim \text{ColH } A B C \rightarrow$
 $\sim \text{ColH } A' B' C' \rightarrow \sim \text{ColH } A'' B'' C'' \rightarrow \text{CongaH } A B C A' B' C' \rightarrow$
 $\text{CongaH } A' B' C' A'' B'' C'' \rightarrow \text{CongaH } A B C A'' B'' C''.$

(Theorem26) resolves the fundamental geometric construction of finding the midpoint of a segment, relying critically on the properties of congruence and betweenness.

Definition $\text{MidPoint } C A B := \text{BetH } A C B \wedge \text{CongH } A C B C.$
Theorem Theorem26 : $\text{forall } A B, A <> B \rightarrow \text{exists } C, \text{MidPoint } C A B.$

3.3. Advanced Lemmas on Congruence and Configuration

The final set of lemmas establishes profound connections between congruence and geometric configurations.

The following lemmas establish relationships between congruent segments and their corresponding collinear and non-collinear configurations.

Definition $\text{diff_th } (A B C : \text{Point}) := A < B \wedge A < C \wedge B < C.$
Lemma Cong_colh : $\text{forall } A B C A' B' C', \text{diff_th } A B C \rightarrow \text{CongH } A B A' B' \rightarrow$
 $\text{CongH } A C A' C' \rightarrow \text{CongH } B C B' C' \rightarrow \text{ColH } A B C \rightarrow \text{ColH } A' B' C'.$
Lemma Cong_uncolh : $\text{forall } A B C A' B' C', \text{diff_th } A' B' C' \rightarrow \text{CongH } A B A' B' \rightarrow$
 $\text{CongH } A C A' C' \rightarrow \text{CongH } B C B' C' \rightarrow \sim \text{ColH } A B C \rightarrow \sim \text{ColH } A' B' C'.$
Lemma Cong_bet : $\text{forall } A B C A' B' C', A' < B' \rightarrow A' < C' \rightarrow \text{CongH } A B A' B' \rightarrow$
 $\text{CongH } A C A' C' \rightarrow \text{CongH } B C B' C' \rightarrow \text{BetH } A B C \rightarrow \text{BetH } A' B' C'.$
Lemma Cong_Ips : $\text{forall } A B C A' B' C', A < B \rightarrow A' < B' \rightarrow$

$$\begin{aligned} A' < C' &\rightarrow B' < C' \rightarrow \text{CongH } A \ B \ A' \ B' \rightarrow \text{CongH } A \ C \ A' \ C' \rightarrow \\ \text{CongH } B \ C \ B' \ C' &\rightarrow \text{Ipsilateral_P } A \ B \ C \rightarrow \text{Ipsilateral_P } A' \ B' \ C'. \end{aligned}$$

These lemmas collectively state that if two sets of points are pairwise congruent ($AB \equiv A'B'$, $AC \equiv A'C'$, $BC \equiv B'C'$) then the configuration is preserved. Specifically, if points A , B , and C are collinear, points A' , B' , and C' are also collinear, with the same order as A , B , and C . Conversely, if points A , B , and C are non-collinear, then points A' , B' , and C' are likewise non-collinear.

Building upon the previous results, the next lemmas handle more complex scenarios involving a fourth point, demonstrating that congruence can be extended consistently.

Lemma CongH_colh : $\text{forall } A \ B \ C \ P \ A' \ B' \ C' \ P', P < C \rightarrow B < P \rightarrow$
 $B' < P' \rightarrow A < C \rightarrow B < C \rightarrow A' < C' \rightarrow B' < C' \rightarrow$
 $\text{Ipsilateral_P } B \ P \ A \rightarrow \text{Ipsilateral_P } B' \ P' \ A' \rightarrow$
 $\text{ColH } A \ B \ C \rightarrow \text{CongH } C \ A \ C' \ A' \rightarrow \text{CongH } C \ B \ C' \ B' \rightarrow$
 $\text{CongH } A \ B \ A' \ B' \rightarrow \text{CongH } A \ P \ A' \ P' \rightarrow$
 $\text{CongH } B \ P \ B' \ P' \rightarrow \text{CongH } C \ P \ C' \ P'$

Lemma CongH_col : $\text{forall } A \ B \ C \ P \ A' \ B' \ C' \ P',$
 $\sim \text{ColH } A \ B \ C \rightarrow \text{diff_th } A' \ B' \ C' \rightarrow$
 $\text{CongH } A \ B \ A' \ B' \rightarrow \text{CongH } A \ C \ A' \ C' \rightarrow \text{CongH } B \ C \ B' \ C' \rightarrow$
 $\text{ColH } A \ B \ P \rightarrow A < P \rightarrow B < P \rightarrow A' < P' \rightarrow B' < P' \rightarrow$
 $\text{CongH } A \ P \ A' \ P' \rightarrow \text{CongH } B \ P \ B' \ P' \rightarrow \text{CongH } C \ P \ C' \ P'.$

CongH_colh and CongH_col establish congruence relationships based on the congruence of three segments. Specifically, if segments AB , AC , and BC are congruent to segments $A'B'$, $A'C'$, and $B'C'$ respectively, and points P and P' are chosen on lines AB and $A'B'$ in such a way that $AP \equiv A'P'$ and $BP \equiv B'P'$, then $CP \equiv C'P'$ holds, regardless of whether points A , B , and C share a common line.

The final lemmas leverage the concept of midpoints to establish deep spatial relationships, which are pivotal for our later work on perpendicularity and symmetry.

Lemma Midpoint_Conga : $\text{forall } A \ A' \ B \ B' \ C \ C' \ M, \sim \text{ColH } A \ B \ C \rightarrow$
 $\text{MidPoint } M \ A \ A' \rightarrow \text{MidPoint } M \ B \ B' \rightarrow$
 $\text{MidPoint } M \ C \ C' \rightarrow \text{CongaH } A \ B \ C \ A' \ B' \ C'.$

Lemma Mid_Cong_BetH : $\text{forall } A \ O \ B \ A' \ B' \ P \ Q, \sim \text{ColH } A \ O \ B \rightarrow$
 $\text{BetH } A \ O \ A' \rightarrow \text{BetH } B \ O \ B' \rightarrow$
 $\text{CongH } O \ A \ O \ B \rightarrow \text{CongH } O \ A' \ O \ B' \rightarrow$
 $\text{MidPoint } P \ A \ B \rightarrow \text{MidPoint } Q \ A' \ B' \rightarrow \text{BetH } P \ O \ Q.$

Midpoint_Conga states that if point M is the midpoint of segments AA' , BB' , and CC' , then $\angle ABC \equiv \angle A'B'C'$. Mid_Cong_BetH states that if segments AA' and BB' intersect at point O , with $OA \equiv OB$, $OA' \equiv OB'$, and if P and Q are the midpoints of segments AB and $A'B'$ respectively, then point O lies between P and Q .

4. Establishment of Perpendicularity Theory in Hilbert's System

A significant gap in Hilbert's *Foundations of Geometry* and its subsequent formalizations is the lack of a rigorous theory of perpendicularity for lines and planes in three-dimensional space. Although Hilbert defines the concept of a right angle, he does not systematically extend it to establish fundamental properties of spatial perpendicularity, such as the existence and uniqueness of perpendiculars from a point to a line or a plane.

The primary objective of this section is to demonstrate how, starting strictly from Hilbert's axioms, one can rigorously prove the essential properties concerning perpendicularity between lines and planes. We construct this theory from the ground up, formalizing and machine-verifying it in Rocq.

4.1. Foundation Definitions and Planar Propositions

Based on Hilbert's system, perpendicularity can be understood as follows:

- **Line–Line Perpendicularity:** Given a right angle PBA , we can say that the line PB is perpendicular to the line AB , with B as the intersection point. PB is also referred to as the perpendicular to AB .
- **Line–Plane Perpendicularity:** If point A lies outside the plane f , and B is a point on f , such that for any other point C on the plane f , angle ABC is a right angle, we say that the line AB is perpendicular to the plane f , with B as the foot of the perpendicular. The line AB is also called the perpendicular to f .

With these definitions in place, we begin the first stage of our theory, establishing critical planar properties that serve as the building blocks for spatial reasoning.

Proposition 1. *Given points A, B, C , and P , where P is outside line AB , and B is the midpoint of segment AC , if $PA \equiv PC$, then angle PBA is a right angle.*

Proposition 2. *Given points A, B, C , and P , where B is the midpoint of segment AC and angle PBA is a right angle, then $PA \equiv PC$.*

The above two propositions are inverse relations that link perpendicularity to symmetric distances, proven using triangle congruence theorems (*Theorem18, Theorem12*).

Having established a basic characterisation of perpendicularity, we now prove the following uniqueness property: in a given plane, there is only one perpendicular direction to a line through a point on it.

Proposition 3. *Given two angles, $\angle ABC$ and $\angle ABD$, on a plane, if both are right angles, then points B, C , and D must be collinear.*

Proof. Points C and D lie either on the same side or on opposite sides of line AB .

When C and D are on the same side of the line AB , both ABC and ABD are right angles. According to *Theorem21*, $\angle ABC \equiv \angle ABD$. Then, by the uniqueness of angle transfer, points C and D are positioned on the same side of B , making B, C , and D collinear.

When C and D reside on opposite sides of line AB , we consider another point C' on line BC such that C and C' are positioned on different sides of point B . Thus, C' and D share the same side of line AB . According to the definition of a right angle, $\angle ABC \equiv \angle ABC'$. Since angle ABC is a right angle, by *Cor_Theorem21*, angle ABC' is also a right angle. From the above proof, it follows that points B, C' , and D are collinear. Hence, D lies on line BC' . Moreover, since C lies on line BC' , B, C , and D are also situated on the same line. $\square$

The natural progression from uniqueness is to establish existence. The next proposition guarantees that a perpendicular line can always be constructed from a point on a given line.

Proposition 4. *Given three points A, B , and P not lying on the same line, there must exist a point C such that the angle ABC forms a right angle, and C and P are positioned on the same side of the line AB . In other words, a perpendicular can be drawn from a point on the given line.*

Proof. According to *Cor2_Theorem14*, a right angle exists, denoted as angle $A'B'C'$. According to Hilbert's Axiom III, 4 (*CongH_4_exists*), an angle ABC , congruent to angle $A'B'C'$, can be constructed with ray BA serving as one of its sides, and C lies on the side of line AB as point P . According to *Cor_Theorem21*, angle ABC is also a right angle. $\square$

4.2. Key Properties About Line–Line Perpendicularity

Next, we address the fundamental problem of dropping a perpendicular from an external point to a line. This is a much more complex construction than the previous propositions.

Proposition 5. *Given a line l and a point C that does not lie on l , there exists a point X on l such that, for any other point X' also on the line l , the angle CXX' forms a right angle. In other words, through a point C external to line l , a perpendicular can always be drawn to l .*

Proof. By *Incid_3_1*, two distinct points, A and B , lie on the line l (see Figure 1). A point Y is chosen on line l such that Y and B are situated on opposite sides of point A and $AC \equiv AY$.

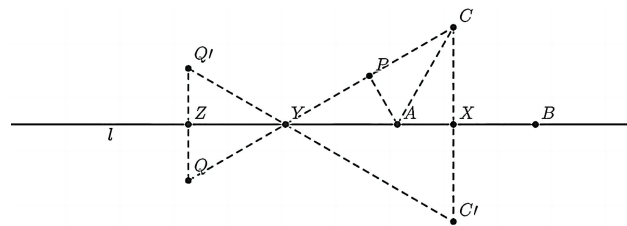


Figure 1. Proposition 5.

The midpoint of segment YC is denoted as P (*Theorem26*). A point Z is selected on line l such that Z and A are on opposite sides of point Y and $YP \equiv YZ$. A point Q is chosen on line YP , ensuring that points Q and P lie on opposite sides of point Y and $YA \equiv YQ$. Thus, Q and C are also situated on opposite sides of Y . A point Q' is chosen on line ZQ in such a way that Q' and Q are on opposite sides of point Z and $QZ \equiv Q'Z$. In other words, Z serves as the midpoint of segment QQ' . A point C' is chosen on line YQ' such that C' and Q' are situated on opposite sides of point Y and $YC \equiv YC'$. The midpoint of segment CC' is denoted as X .

We assert that X is the desired point.

According to *Theorem18*, triangle YPA is congruent to triangle CPA . Thus, $\angle YPA \equiv \angle CPA$. And P lies between Y and C . By the definition of a right angle, angle YPA is a right angle. In triangles YZQ and YPA , according to *Cor1_Theorem14*, we have $\angle ZYQ \equiv \angle PYA$ and $YA \equiv YQ, YZ \equiv YP$. According to *Theorem12*, these two triangles are congruent, so $\angle YZQ \equiv \angle YPA$. Thus, YZQ is also a right angle (*Cor_Theorem21*). By the definition of a right angle, angle YZQ is congruent to its supplementary angle YZQ' , and we have $QZ \equiv Q'Z, ZY \equiv ZY$. According to *Theorem12*, $YQ \equiv YQ'$. And according to *Mid_Cong_BetH*, Y lies between Z and X , meaning X lies on line l . According to *Theorem18*, triangle CYX is congruent to $C'YX$. Thus, angle CXY is congruent to angle $C'XY$, with X lying between C and C' . By the definition of a right angle, angle CXY is a right angle.

For any other point X' on line l , angle CXX' either coincides with angle CXY or is complementary to angle CXY . In either case, angle CXX' is a right angle. Therefore, X is the desired point. $\square$

We then prove a powerful spatial congruence theorem that is essential for transferring angle measures between different perpendicular setups.

Proposition 6. *Given six points A, B, C, A', B' , and C' in space (see Figure 2), where A, B , and C are not on the same line, points B and B' lie on the same side of the line AA' , and points C and C' also lie on the same side of the line AA' , if the lines AB and AC are perpendicular to the line AA' at point A , and the lines $A'B'$ and $A'C'$ are perpendicular to the line AA' at point A' , then we have*

$$\angle BAC \equiv \angle B'A'C'.$$

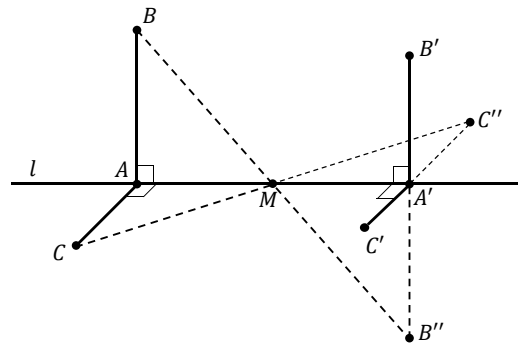


Figure 2. The nonplanar configuration of Proposition 6.

Proof. M denotes the midpoint of segment AA' . Points B'' and C'' are chosen on lines BM and CM , respectively, such that M is the midpoint of segments BB'' and CC'' . According to *Midpoint_Conga*, $\angle BAC \equiv \angle B''A'C''$. In triangles BMA and $B''MA'$, by *Cor1_Theorem14*, $\angle BMA \equiv \angle B''MA'$. Moreover, $AM \equiv A'M$, $BM \equiv B''M$. Thus, by *Hil_Theorem12*, $\angle BAM \equiv \angle B''A'M$.

Since $\angle BAM$ coincides with $\angle BAA'$, $\angle BAM$ is a right angle. Therefore, by *Cor_Theorem21*, $\angle B''A'M$ is also a right angle. Similarly, $\angle B''A'M$ coincides with $\angle B''A'A$, indicating that $\angle B''A'A$ is a right angle. By Proposition 3, points B' , A' , and B'' are collinear, and B' and B'' are positioned on opposite sides of A' . Similarly, it is easy to prove that C' , A' , and C'' are collinear, with C' and C'' on opposite sides of A' .

Angles $B'A'C'$ and $B''A'C''$ are vertical angles, so $\angle B'A'C' \equiv \angle B''A'C''$.

Therefore, both $\angle BAC$ and $\angle B'A'C'$ are congruent to $\angle B''A'C''$. According to *Theorem19*, $\angle BAC \equiv \angle B'A'C'$. $\square$

4.3. Key Properties About Line–Plane Perpendicularity

Next, we investigate the properties of perpendicularity between lines and planes.

Proposition 7. Given four points A, B, C, D in space. If angles ABC and ABD are right angles and points B, C, D are not collinear, then for any other point E on the plane BCD , angle ABE is also a right angle. In other words, a line perpendicular to two intersecting lines in a plane is perpendicular to that plane.

Proof. A point A' is chosen on the line AB in such a way that B is the midpoint of the segment AA' . According to Proposition 2, we have: $AC \equiv A'C$, $AD \equiv A'D$. Furthermore, since $AB \equiv A'B$, by *uncol_cong*, for any point E on the plane BCD , $AE \equiv A'E$. According to Proposition 1, angle ABE is a right angle. $\square$

Proposition 8. Given points P, Q, P', Q' and a plane ABC , where PQ is perpendicular to the plane ABC , with Q as the intersection point, and point Q' lies on the plane ABC while P' lies on the plane PQQ' . If angle $P'Q'Q$ is a right angle, then $P'Q'$ is perpendicular to plane ABC .

Proof. P and P' are either on opposite sides or situated on the same side of the line QQ' .

When P and P' are situated on the same side of the line QQ' , a perpendicular $C'Q$ is drawn through point Q on the plane ABC , forming a right angle $C'QQ'$. Then, through point Q' , another perpendicular $C''Q'$ is drawn on the plane ABC , also forming a right angle $C''Q'Q$ and ensuring that points C' and C'' are on the same side of the line QQ' .

Since PQ is the perpendicular from the plane ABC , angles PQQ' and PQC' are also right angles. According to Proposition 6, $\angle P'Q'C'' \equiv \angle PQC'$. Therefore, by *Cor_Theorem21*, angle $P'Q'C''$ is also a right angle. According to Proposition 7, the line $P'Q'$ is perpendicular to the plane ABC .

When P and P' are located on opposite sides of the line QQ' , we select a point P'' lying on the line $P'Q'$ in such a way that Q' lies between P' and P'' . This ensures that P and P'' are situated on the same side of QQ' . $P'Q'Q$ forms a right angle. According to the definition of a right angle, angle $P''Q'Q$ is also a right angle. Following the previous proof, the line $P''Q'$ is perpendicular to the plane ABC , and since P' lies on the line $P''Q'$, it follows that $P'Q'$ is also perpendicular to the plane ABC . $\square$

Proposition 9. *For any plane ABC and a point P outside the plane, There exists a specific point D on the plane such that for any other point E on the plane, the angle EDP is a right angle. In other words, a perpendicular can be drawn from a point outside a plane to the plane.*

Proof. Draw a line l within the plane ABC . A perpendicular line is drawn from point P to line l , and its foot is denoted as D_0 . For any other point Q on line l , the angle PD_0Q is a right angle. On the plane ABC , construct the perpendicular line D_0D_1 to the given line l . If PD_0 is perpendicular to D_0D_1 , then according to Proposition 7, PD_0 is perpendicular to the plane ABC , and D_0 is the desired point.

If PD_0 is not perpendicular to D_0D_1 , a perpendicular line is drawn from point P to line D_0D_1 , and its foot is denoted as D . We assert that D is the desired point.

Angle PDD_0 is already a right angle. If PD is also perpendicular to line DQ , meaning angle PDQ is a right angle, then according to Proposition 7, PD is perpendicular to the plane ABC .

On the lines PD , D_0D , and QD , points P' , D'_0 , and Q' are chosen respectively, in such a way that D serves as the midpoint of the segments PP' , $D_0D'_0$, and QQ' . According to *Midpoint_Conga*, angle PD'_0Q' is congruent to angle $P'D_0Q$.

Since angles PD_0Q and DD_0Q are right angles, and according to Proposition 7, line D_0Q is perpendicular to plane PDD_0 . Since point P' lies on a plane PDD_0 , angle $P'D_0Q$ is a right angle. According to *Cor_Theorem21*, angle PD'_0Q' is also a right angle. Therefore, by *Theorem21*, $\angle PD_0Q \equiv \angle PD'_0Q'$.

In triangles DD_0Q and DD'_0Q' ,

$$DQ \equiv DQ', DD_0 \equiv DD'_0, \angle D_0DQ \equiv \angle D'_0DQ'.$$

By *Hil_T12*, triangles DD_0Q and DD'_0Q' are congruent, so $D_0Q \equiv D'_0Q'$. And by Proposition 2, $D_0P \equiv D'_0P$. Therefore, by *Theorem12*, triangles PD_0Q and PD'_0Q' are congruent, so $PQ \equiv PQ'$. Since D is the midpoint of QQ' , according to Proposition 1, PDQ is a right angle. $\square$

4.4. Formalization Summary

The Rocq formalization code of the above propositions is as follows:

```

Proposition Per_Mid : forall A B C P, ~ ColH P B A ->
  MidPoint B A C -> CongH P A P C -> Right_angle P B A.
Proposition Mid_Per : forall A B C P, MidPoint B A C ->
  Right_angle P B A -> CongH P A P C.
Proposition ColH_Right : forall A B C D, Cof A B C D ->
  Right_angle A B C -> Right_angle A B D -> ColH B C D.
Proposition Right_exists : forall A B P, ~ ColH A B P ->
  exists C, Right_angle A B C /\ Ipsilateral_S A B C P.
Proposition Right_exists_to_line : forall l C, ~ Incid C l ->
  exists X, Incid X l /\
  (forall X', X' <> X -> Incid X' l -> Right_angle X' X C).
Proposition Conga_flat : forall A B C A' B' C', ~ ColH B A C ->
  Ipsilateral_S A A' B B' -> Ipsilateral_S A A' C C' ->

```

```

Right_angle B A A' -> Right_angle A A' B' ->
Right_angle C A A' -> Right_angle A A' C' ->
CongraH B A C B' A' C'.
Proposition Perp_flat : forall A B C D E, Right_angle A B C ->
  Right_angle A B D -> ~ ColH B C D -> B<>E ->
  Cof B C D E -> Right_angle A B E.
Proposition Per_move : forall A B C P Q P' Q', ~ ColH A B C ->
  Cof A B C Q -> Cof A B C Q' -> Cof Q Q' P P' ->
  (forall D, D<>Q -> Cof A B C D -> Right_angle P Q D) ->
  Right_angle P' Q' Q ->
  (forall D, D<>Q' -> Cof A B C D -> Right_angle P' Q' D).
Proposition Per_flat_exists : forall A B C P, ~ Cof A B C P ->
  exists D, Cof A B C D /\ (forall E, Cof A B C E ->
  E<>D -> Right_angle E D P).

```

5. Applications of the Perpendicularity Theory

This section demonstrates a central application of the perpendicularity theory formalized in the previous section: proving a theorem on the congruence of geometric figures that was posited by Hilbert but never proven. We begin by defining geometric figures and their congruence relations in Rocq. Next, we formalize and prove the theorem for the restricted case of plane figures. Finally, building upon the perpendicularity theory established in the previous section, we extend the result to its full generality in three-dimensional space, completing the proof. The entire development is accompanied by its machine verification in Rocq.

5.1. The Formalization of the Definitions

Hilbert defines a geometric figure and the congruence of two figures as follows:

Definition 1. *Any finite number of points is referred to as a figure. If all the points of a figure are situated within a plane, it is called a plane figure. Two figures are considered congruent when their points can be paired in such a way that all corresponding segments and angles are equivalent.*

Figures can be formalized using Rocq's *List* structure with elements of type *Point*. Importing the *Lists.List* library introduces this structure. Additionally, basic linear arithmetic operations on natural numbers can be facilitated by directly importing the *micromega.Lia* library [53]. The following code defines a type *graph_f*, which represents a plane figure.

```

Require Import micromega.Lia.
Require Export Coq.Lists.List.

Definition dP : Point.
Proof.
...
Defined.

Class graph_f :={
  gr_fun : list Point;
  gr_Cof : exists f, forall A, In A gr_fun -> Incid_P_F A f;
  gr_noteq : forall n m, n<>m -> n < length gr_fun ->
    m < length gr_fun -> (nth n gr_fun dP) <> (nth m gr_fun dP)
}

```

The type *graph_f* includes three constructors:

- *gr_fun*: Represents a list of points, forming a finite set of points $A_1, A_2, A_3, \dots, A_n$ in a plane figure.
- *gr_Cof*: Declares the existence of a plane f , on which all points in the list *gr_fun* lie. The function *In*, defined in Rocq's *Lists.List* library, is used to denote that elements are present in a list.
- *gr_noteq*: Indicates that the points in *gr_fun* are pairwise distinct.

Together, these three constructors inductively define a plane figure.

In the following code, *CongH_L* defines congruence between two sequences of points, *L1* and *L2*, indicating that *L1* and *L2* have equal lengths and that corresponding segments and angles are congruent. When applied to sequences of points, *CongH_L* denotes congruence between two plane figures, which is formalized as *CongH_gf*.

```
Definition CongH_L (L1 L2 : list Point) : Prop := length L1 = length L2 /\
  (forall m1 m2, m1 <> m2 -> m1 < length L1 -> m2 < length L1 ->
    CongH (nth m1 L1 dP) (nth m2 L1 dP) (nth m1 L2 dP) (nth m2 L2 dP)) /\
  (forall m0 m1 m2, m0 < length L1 -> m1 < length L1 -> m2 < length L1 ->
    ~ ColH (nth m0 L1 dP) (nth m1 L1 dP) (nth m2 L1 dP) ->
    CongaH (nth m0 L1 dP) (nth m1 L1 dP) (nth m2 L1 dP)
    (nth m0 L2 dP) (nth m1 L2 dP) (nth m2 L2 dP)).
```

```
Definition CongH_gf x y: Prop := CongH_L (@gr_fun x) (@gr_fun y).
```

Similarly, we can define a type *graph* representing figures, and *CongH_g* defines congruence between two figures.

```
Class graph := {
  g_fun : list Point;
  g_noteq : forall n m, n <> m -> n < length g_fun -> m < length g_fun ->
    (nth n g_fun dP) <> (nth m g_fun dP)
}.
```

```
Definition CongH_g x y: Prop := CongH_L (@g_fun x) (@g_fun y).
```

5.2. The Congruence Theorem for Plane Figures

Theorem 1. *If $(A_1, A_2, A_3, \dots, A_n)$ and $(A'_1, A'_2, A'_3, \dots, A'_n)$ are congruent plane figures and P denotes a point within the plane of the first, then a point P' can be found within the plane of the second figure, in such a way that $(A_1, A_2, A_3, \dots, A_n, P)$ and $(A'_1, A'_2, A'_3, \dots, A'_n, P')$ are again congruent figures. If the figure $(A_1, A_2, A_3, \dots, A_n)$ contains at least three points that are not collinear, then the construction of P' is possible in only one way.*

We formalize the theorem as *Hil_T28*. *Hil_T28* is a formal proof of unique existence which is divided into proving existence and uniqueness separately, where the assumption that the figure $(A_1, A_2, A_3, \dots, A_n)$ contains at least three points that are not collinear is only used in the uniqueness proof.

```
Theorem Hil_T28 : forall x y P, CongH_gf x y ->
  (forall A B C, In A (@gr_fun x) -> In B (@gr_fun x) ->
    In C (@gr_fun x) -> Cof A B C P) ->
  (forall Q, In Q (@gr_fun x) -> P <> Q) ->
  (exists A B C, In A (@gr_fun x) /\ In B (@gr_fun x) /\
    In C (@gr_fun x) /\ ~ ColH A B C) ->
  exists! P',
  (exists f, forall A', In A' ((@gr_fun y)++[P']) -> Incid_P_F A' f) /\
  (forall n m, n <> m -> n < length ((@gr_fun y)++[P']) ->
    m < length ((@gr_fun y)++[P']) ->
```

```

nth n ((@gr_fun y)++[P']) dP <> nth m ((@gr_fun y)++[P']) dP /\
CongH_L ((@gr_fun x)++[P]) ((@gr_fun y)++[P']).

```

The mathematical proof of this plane theorem has been provided in the annotations of the Russian translation of the 7th edition of *Grundlagen der Geometrie*. We will provide a more detailed version of the proof and formalize it in Rocq.

Depending on the location of point P relative to the figure $(A_1, A_2, A_3, \dots, A_n)$, three cases can be distinguished for the proof.

- **Case 1:** There are two points A and B in the figure which are collinear with P .
- **Case 2:** P does not lie on the same line with any two points in the figure $(A_1, A_2, A_3, \dots, A_n)$, and $A_1, A_2, A_3, \dots, A_n$ are not all collinear.
- **Case 3:** All points of the figure $(A_1, A_2, A_3, \dots, A_n)$ lie on a line, and P is outside this line.

5.2.1. Proof and Formalization for Case 1 in Theorem 1

Suppose point P is collinear with two points A and B from the figure $(A_1, A_2, A_3, \dots, A_n)$.

If B and P are on the same side of A , then according to *CongH_3_exists_Ips*, there is a point P' lying on the line $A'B'$ and the order of A', B', P' corresponds exactly to the order of A, B, P , and $AP \equiv A'P', BP \equiv B'P'$.

Now prove that P' is indeed the desired point.

Existence: First, P' is not a point in the figure $(A'_1, A'_2, A'_3, \dots, A'_n)$, otherwise, according to the uniqueness of segment transfer (*CongH_1_unique*), P would also be a point in the figure $(A_1, A_2, A_3, \dots, A_n)$.

Let C be a point in the figure $(A_1, A_2, A_3, \dots, A_n)$, and let C' be the corresponding point in the figure $(A'_1, A'_2, A'_3, \dots, A'_n)$. If A, B , and C are collinear, then according to *CongH_coll*, $CP \equiv C'P'$. If A, B , and C are not collinear, according to *CongH_col*, $CP \equiv C'P'$.

In summary, in the extended figures with points P and P' , all corresponding segments remain congruent. According to *Theorem18*, corresponding angles are also congruent.

Uniqueness: Suppose there exists another point Q' such that the extended figures with points P and Q' remain congruent. According to *Cong_Ips*, Q' lies on the line $A'B'$, and the order of A', B', Q' is also identical to the order of A, B, P . By the uniqueness of segment transfer, Q' must be equal to P' . Therefore, if the point P' exists, it will be uniquely determined by the method specified in *CongH_3_exists_Ips*.

If B and P are on opposite sides of A , simply interchange A and B in the above proof process.

Case 1 of Theorem 1 can be formalized as follows:

```

Lemma L1_T28 : forall x y P, CongH_gf x y ->
  (forall Q, In Q (@gr_fun x) -> P<>Q) ->
  (exists A B, In A (@gr_fun x) /\ In B (@gr_fun x) /\
    A<>B /\ ColH A B P) ->
  exists! P',
  (exists f, forall A', In A' ((@gr_fun y)++[P']) -> Incid_P_F A' f) /\
  (forall n m, n<>m -> n < length ((@gr_fun y)++[P']) ->
    m < length ((@gr_fun y)++[P']) ->
    nth n ((@gr_fun y)++[P']) dP <> nth m ((@gr_fun y)++[P']) dP /\
  CongH_L ((@gr_fun x)++[P]) ((@gr_fun y)++[P']).

```

5.2.2. Proof and Formalization for Case 2 in Theorem 1

Suppose P does not lie on the same line with any two points in the figure $(A_1, A_2, A_3, \dots, A_n)$, and $A_1, A_2, A_3, \dots, A_n$ are not all collinear.

In this case, three non-collinear points, such as A , B , and C , can be identified in the figure. According to *exists_He*, a point P_1 can be selected on the line AC such that points P and P_1 lie on opposite sides of the line AB . Thus, the segment PP_1 intersects AB at a point denoted as P_2 .

Consequently, point P lies on the line P_1P_2 , where P_1 is chosen on line AC and P_2 is chosen on line AB .

Following the proof of Case 1, points P_1' and P_2' can be uniquely identified on the lines $A'C'$ and $A'B'$, respectively, such that the extended figures $(A_1, A_2, A_3, \dots, A_n, P_1, P_2)$ and $(A_1', A_2', A_3', \dots, A_n', P_1', P_2')$ are congruent.

Similarly, P lies on the line P_1P_2 and a point P' can be identified on the line $P_1'P_2'$ such that figures $(A_1, A_2, A_3, \dots, A_n, P_1, P_2, P)$ and $(A_1', A_2', A_3', \dots, A_n', P_1', P_2', P')$ are congruent.

Let's prove that the point P' is uniquely determined. Suppose there is another point P'' such that the figures $(A_1, A_2, A_3, \dots, A_n, P)$ and $(A_1', A_2', A_3', \dots, A_n', P'')$ are congruent. Then, according to the conclusion of Case 1, points P_1'' and P_2'' can be found such that $(A_1, A_2, A_3, \dots, A_n, P, P_1, P_2)$ and $(A_1', A_2', A_3', \dots, A_n', P'', P_1'', P_2'')$ are congruent. If we eliminate P and P'' from these two figures, the resulting figures should still be congruent. However, points P_1' and P_2' have already been uniquely determined. Therefore, P_1'' coincides with P_1' , and P_2'' coincides with P_2' .

In conclusion, the figures

$$(A_1, A_n, A_3, \dots, A_n, P_1, P_2, P)$$

and

$$(A_1', A_2', A_3', \dots, A_n', P_1', P_2', P'')$$

are congruent. Based on the uniqueness of the construction of P' , comparing these two congruent figures shows that P' is equal to P'' . The proof is complete.

In the proof of Case 2, we assume that the figures $(A_1, A_2, A_3, \dots, A_n, P, P_1, P_2)$ and $(A_1, A_2, A_3, \dots, A_n, P_1, P_2, P)$ are the same. Consequently, if $(A_1, A_2, A_3, \dots, A_n, P, P_1, P_2)$ is congruent to another figure, then $(A_1, A_2, A_3, \dots, A_n, P_1, P_2, P)$ must also be congruent to that figure. In essence, for figures, rearranging or removing points does not affect their congruence.

However, in the formalization process, we represent figures as lists, where the order of elements matters. As a result, rearranging elements in a list L_1 produces a different list L_2 , but this does not affect the conclusion that the two lists are congruent. The same principle applies when elements are removed from a list.

To reconcile the differences between manual and formal proofs, we need certain lemmas to account for these distinctions.

```

Lemma exc_Cong : forall L1 L2 P P1 P' P1',
  (forall Q, In Q L1 -> P1 <> Q) ->
  (forall Q, In Q (L1++[P1]) -> P <> Q) ->
  (forall n m, n<>m -> n < length ((L2++[P1'])++[P']) ->
    m < length ((L2++[P1'])++[P']) ->
    nth n ((L2++[P1'])++[P']) dP <> nth m ((L2++[P1'])++[P']) dP) ->
  CongH_L ((L1++[P1])++[P]) ((L2++[P1'])++[P']) ->
  CongH_L ((L1++[P])++[P1]) ((L2++[P'])++[P1']).
Lemma remove_Cong_simpl : forall L1 L2 P P1,
  CongH_L (L1++[P]) (L2++[P']) ->
  CongH_L L1 L2.

```

Case 2 in Theorem 1 can be formalized as the following lemma.

```

Lemma L2_T28: forall x y P, CongH_gf x y ->
  (forall Q, In Q (@gr_fun x) -> P<>Q) ->
  (exists A B C, In A (@gr_fun x) /\ In B (@gr_fun x) /\
    In C (@gr_fun x) /\ ~ ColH A B C /\ Cof A B C P) ->
  (forall A B, In A (@gr_fun x) -> In B (@gr_fun x) ->
    A<>B -> ~ ColH A B P) ->
  exists! P',
  (exists f, forall A', In A' ((@gr_fun y)++[P']) -> Incid_P_F A' f) /\
  (forall n m, n<>m -> n < length ((@gr_fun y)++[P']) ->
    m < length ((@gr_fun y)++[P']) ->
    nth n ((@gr_fun y)++[P']) dP <> nth m ((@gr_fun y)++[P']) dP) /\
  CongH_L ((@gr_fun x)++[P]) ((@gr_fun y)++[P']).

```

5.2.3. Proof and Formalization for Case 3 in Theorem 1

Suppose all points of the figure $(A_1, A_2, A_3, \dots, A_n)$ lie on a line, and P is outside this line.

According to *Cong_colh* and *Cong_bet*, points $A'_1, A'_2, A'_3, \dots, A'_n$ are also collinear and they have the same order as $A_1, A_2, A_3, \dots, A_n$.

Take any two points A and B from $(A_1, A_2, A_3, \dots, A_n)$, and their corresponding points A' and B' from $(A'_1, A'_2, A'_3, \dots, A'_n)$. Next, construct an angle $Q'A'B'$ on the ray $A'B'$ such that angle $Q'A'B'$ is congruent to angle PAB , and on the ray $A'Q'$, construct segment $A'P'$ congruent to AP . Consequently, we have:

$$\angle PAB \equiv \angle P'A'B', AP \equiv A'P'.$$

We conclude that P' is indeed the desired point. Indeed, consider any point C in the first figure. Either $\angle PAC$ and $\angle PAB$ coincide or they are supplementary angles. In the former case, angles $P'A'C'$ and $P'A'B'$ coincide, and in the latter case, they are supplementary angles (*Cong_Ips*).

Since $\angle PAB \equiv \angle P'A'B'$, it follows that in either case, $\angle PAC \equiv \angle P'A'C'$. Furthermore, $AP \equiv A'P'$, $AC \equiv A'C'$. From *Theorem12*, it follows that $PC \equiv P'C'$.

Therefore, $(A_1, A_2, A_3, \dots, A_n, P)$ and $(A'_1, A'_2, A'_3, \dots, A'_n, P')$ are congruent figures. It is worth noting that in the considered scenario, point P' is not uniquely determined, because it can be replaced by a point that is symmetric to it relative to the line $A'_1A'_2A'_3 \dots A'_n$.

The formalization of this case is also straightforward.

```

Lemma L3_T28 : forall x y P, CongH_gf x y ->
  (forall Q, In Q (@gr_fun x) -> P<>Q) ->
  (forall A B C, In A (@gr_fun x) -> In B (@gr_fun x) ->
    In C (@gr_fun x) -> ColH A B C) ->
  (forall A B, In A (@gr_fun x) -> In B (@gr_fun x) ->
    A<>B -> ~ ColH A B P) ->
  exists P',
  (exists f, forall A', In A' ((@gr_fun y)++[P']) ->
    Incid_P_F A' f) /\
  (forall n m, n<>m -> n < length ((@gr_fun y)++[P']) ->
    m < length ((@gr_fun y)++[P']) ->
    nth n ((@gr_fun y)++[P']) dP <> nth m ((@gr_fun y)++[P']) dP) /\
  CongH_L ((@gr_fun x)++[P]) ((@gr_fun y)++[P']).

```

5.3. The Congruence Theorem for Space Figures

Theorem 2. Given two congruent geometric figures, $(A_1, A_2, A_3, \dots, A_n)$ and $(A'_1, A'_2, A'_3, \dots, A'_n)$, any point P can be corresponded to a point P' such that the extended figures $(A_1, A_2, A_3, \dots, A_n, P)$

and $(A'_1, A'_2, A'_3, \dots, A'_n, P')$ remain congruent. If the figure $(A_1, A_2, A_3, \dots, A_n)$ contains at least four non-coplanar points, then the construction of P' is possible in only one way.

The formalization of Theorem 2 is as follows.

```
Theorem Hil_T29 : forall x y P, CongH_g x y ->
  (forall Q, In Q (@g_fun x) -> P<>Q) ->
  (exists A B C D, In A (@g_fun x) /\ In B (@g_fun x) /\
    In C (@g_fun x) /\ In D (@g_fun x) /\ ~ Cof A B C D) ->
  exists! P',
  (forall n m, n<>m -> n < length ((@g_fun y)++[P']) ->
    m < length ((@g_fun y)++[P']) ->
    nth n ((@g_fun y)++[P']) dP <> nth m ((@g_fun y)++[P']) dP) /\
  CongH_L ((@g_fun x)++[P]) ((@g_fun y)++[P']).
```

Proof. Still based on the position of point P , the proof is divided into four cases.

1. Suppose P and three non-collinear points from $(A_1, A_2, A_3, \dots, A_n)$, such as A , B , and C , lie on the same plane.

In this case, if P lies on the same line as any two points of the first figure, according to the proof of Case 1 in Theorem 1, it is known that a unique point P' exists such that the figures $(A_1, A_2, A_3, \dots, A_n, P)$ and $(A'_1, A'_2, A'_3, \dots, A'_n, P')$ are congruent.

When P does not share a common line with any two points of the figure $(A_1, A_2, A_3, \dots, A_n)$, according to Case 2 of Theorem 1, Theorem 2 can also be proved.

The formalization of this case is as follows.

```
Lemma L1_T29 : forall x y P, CongH_g x y ->
  (forall Q, In Q (@g_fun x) -> P<>Q) ->
  (exists A B C, In A (@g_fun x) /\ In B (@g_fun x) /\
    In C (@g_fun x) /\ ~ ColH A B C /\ Cof A B C P) ->
  exists! P',
  (forall n m, n<>m -> n < length ((@g_fun y)++[P']) ->
    m < length ((@g_fun y)++[P']) ->
    nth n ((@g_fun y)++[P']) dP <> nth m ((@g_fun y)++[P']) dP) /\
  CongH_L ((@g_fun x)++[P]) ((@g_fun y)++[P']).
```

2. Suppose P does not lie in the same plane as any three non-collinear points of the figure $(A_1, A_2, A_3, \dots, A_n)$, and $A_1, A_2, A_3, \dots, A_n$ are not all coplanar.

In this case, we can identify four non-coplanar points in the first figure, such as A , B , C , D . By *exists_uncof*, a point P_1 can be chosen in the plane ABD such that P and P_1 lie on opposite sides of the plane ABC . Consequently, the segment PP_1 intersects the plane ABC at a certain point P_2 .

Thus, point P lies on the plane AP_1P_2 , where P_1 is chosen in the plane ABD and P_2 is chosen in the plane ABC .

Following a similar proof method as in Case 2 of Theorem 1, a unique point P' can be found such that the figures $(A_1, A_2, A_3, \dots, A_n, P)$ and $(A'_1, A'_2, A'_3, \dots, A'_n, P')$ are congruent.

We can also directly give the formalization of this case.

```
Lemma L2_T29 : forall x y P, CongH_g x y ->
  (forall Q, In Q (@g_fun x) -> P<>Q) ->
  ~ (exists A B C, In A (@g_fun x) /\ In B (@g_fun x) /\
    In C (@g_fun x) /\ ~ ColH A B C /\ Cof A B C P) ->
  (exists A B C D, In A (@g_fun x) /\ In B (@g_fun x) /\
    In C (@g_fun x) /\ In D (@g_fun x) /\ ~ Cof A B C D) ->
  exists! P',
```

```

(forall n m, n < m -> n < length ((@g_fun y)++[P']) ->
  m < length ((@g_fun y)++[P']) ->
  nth n ((@g_fun y)++[P']) dP < nth m ((@g_fun y)++[P']) dP /\
  CongH_L ((@g_fun x)++[P]) ((@g_fun y)++[P']).

```

3. Suppose all points of the figure $(A_1, A_2, A_3, \dots, A_n)$ lie on a plane, and P is outside this plane, but the points $(A_1, A_2, A_3, \dots, A_n)$ are not collinear.

Before beginning the proof, let's introduce the following lemma:

```

Lemma Cof_ConHg : forall x y m0 m1 m2 m3, CongH_g x y ->
  m0 < length (@g_fun x) -> m1 < length (@g_fun x) ->
  m2 < length (@g_fun x) -> m3 < length (@g_fun x) ->
  Cof (nth m0 (@g_fun x) dP) (nth m1 (@g_fun x) dP)
  (nth m2 (@g_fun x) dP) (nth m3 (@g_fun x) dP) ->
  Cof (nth m0 (@g_fun y) dP) (nth m1 (@g_fun y) dP)
  (nth m2 (@g_fun y) dP) (nth m3 (@g_fun y) dP).

```

Cof_ConHg: Given two congruent figures $(A_1, A_2, A_3, \dots, A_n)$ and $(A'_1, A'_2, A'_3, \dots, A'_n)$, if any four points such as A, B, C, D in $(A_1, A_2, A_3, \dots, A_n)$ are coplanar, then the corresponding points A', B', C', D' in $(A'_1, A'_2, A'_3, \dots, A'_n)$ are also coplanar.

Since all the points $(A_1, A_2, A_3, \dots, A_n)$ lie in one plane, by *Cof_ConHg* and *Incid_4_1*, all the points of the figure $(A'_1, A'_2, A'_3, \dots, A'_n)$ also lie in one plane.

In the figure $(A_1, A_2, A_3, \dots, A_n)$, we have three non-collinear points, such as A, B, C . Consequently, the corresponding points A', B' and C' are also non-collinear (*Cong_uncollh*).

According to Proposition 9, a perpendicular from P to the plane ABC can be constructed, with the intersection point denoted as M . Similarly, for a point Q outside the plane $A'B'C'$, we can construct a perpendicular from Q to the plane $A'B'C'$, with the intersection point referred to as Q' . By Case 1 of Theorem 2, a point M' can be found such that the figures $(A_1, A_2, A_3, \dots, A_n, M)$ and $(A'_1, A'_2, A'_3, \dots, A'_n, M')$ are congruent, with M' lying on the plane $A'B'C'$.

According to Proposition 4, there exists a point P_0 passing through M' such that $M'P_0$ is perpendicular to $M'Q'$, with P_0 and Q located on the same side of $Q'M'$. On the ray $M'P_0$, a segment $M'P'$ is constructed congruent to MP . We assert that point P' is the desired point.

Angle $P_0M'Q'$ is a right angle, and since it coincides with angle $P'M'Q'$, angle $P'M'Q'$ must also be a right angle. Since $Q'Q$ is perpendicular to plane $A'B'C'$, Proposition 8 implies $P'M'$ is also perpendicular to plane $A'B'C'$ with M' being the point where the perpendicular meets the plane.

Let C be any point in the figure $(A_1, A_2, A_3, \dots, A_n)$, and let C' be its corresponding point.

In triangles PMC and $P'M'C'$, angles PMC and $P'M'C'$ are both right angles. According to *Hil_T21*, these two angles are congruent. Also,

$$MP \equiv M'P', MC \equiv M'C'.$$

By *Theorem12*,

$$PC \equiv P'C'.$$

In conclusion, in the figures extended by points P and P' , all corresponding segments remain congruent. According to *Theorem18*, corresponding angles are also congruent.

This scenario extends Case 3 of Theorem 1 into three dimensions, but the proof cannot be achieved as simply as in Theorem 1 through straightforward angle and segment transfer. Constructing point P' requires additional theories of perpendicularity.

The formalization of this case is also straightforward.

```

Lemma L3_T29 : forall x y P f, CongH_g x y ->
  (forall Q, In Q (@g_fun x) -> P<>Q) ->
  (forall A, In A (@g_fun x) -> Incid_P_F A f) ->
  ~ Incid_P_F P f ->
  (exists A B C, In A (@g_fun x) /\ In B (@g_fun x) /\
    In C (@g_fun x) /\ ~ColH A B C) ->
  exists P',
  (forall n m, n<>m -> n < length ((@g_fun y)++[P']) ->
    m < length ((@g_fun y)++[P']) ->
    nth n ((@g_fun y)++[P']) dP <> nth m ((@g_fun y)++[P']) dP) /\
  CongH_L ((@g_fun x)++[P]) ((@g_fun y)++[P']).

```

4. When the figure degenerates into a straight line, we have already completed the proof in Theorem 1. $\square$

6. Technical Comparison with GeoCoq

This section provides a substantive technical comparison between our Hilbert-native formalization and the Tarski-based GeoCoq library. We focus on the extension of dimensionality, the architectural differences, and the resulting impact on proof density.

6.1. Dimensional Extension Beyond Planar Constraints

While Braun and Narboux established that Hilbert’s and Tarski’s systems are mutually interpretable [44,45], their formal verification is strictly confined to two-dimensional geometry. This dimensional constraint is reflected in the fundamental structure of their framework: it omits the spatial axioms (Group I, 4–8) and operates on a reduced sort hierarchy that includes only points and lines, omitting the primitive notion of a plane. Furthermore, their implementation employs the planar version of Pasch’s Axiom, which is insufficient for capturing the intersection properties inherent in three-dimensional configurations. Our work addresses these structural limitations by providing a native 3D formalization with a complete multi-sorted hierarchy and the full suite of spatial axioms, thereby extending the verified Hilbert’s framework beyond the planar scope of prior studies.

The technical necessity of this 3D extension is best demonstrated by the mechanization of *Hil_T29* in Section 5. Unlike purely planar theorems, the verification of spatial congruence requires the rigorous management of incidence between lines and planes—structures that are structurally absent in the Hilbert’s interfaces of GeoCoq.

6.2. Architectural Distinctions

The most fundamental technical divergence between the two libraries lies in their underlying sort hierarchy. GeoCoq is based on Tarski’s axiomatization, which utilizes a single-sorted first-order logic where “Point” is the exclusive primitive entity. In their framework, higher-order geometric structures such as lines and planes are not first-class types. Conversely, our implementation natively incorporates Hilbert’s multi-sorted logic, elevating Points, Lines, and Planes to distinct primitive types.

On the one hand, our design internalizes dimensional constraints within the type system. Sort-inconsistent statements are rejected during static type-checking, shifting the verification burden from the proof script to the Rocq kernel. On the other hand, our framework reduces the complexity of managing non-degeneracy hypotheses in 3D configurations by avoiding the recurrent unfolding of point relations.

6.3. Semantic Fidelity and Definitional Primitivity

A key technical distinction between the two libraries lies in the methodology of geometric definitions. GeoCoq often adopts constructive definitions that facilitate Tarski’s

quantifier-based reasoning. In contrast, our work adheres to Hilbert’s synthetic approach, where definitions are derived from more primitive axiomatic notions. This leads to a higher degree of semantic fidelity relative to classical geometric intuition.

The formalization of a Right Angle serves as a representative case study. Following Hilbert’s original prose, we define a right angle as an angle that is congruent to its adjacent supplementary angle:

Definition $\text{Right_angle } A B C := \sim \text{ColH } A B C \wedge$
 $\text{forall } C', \text{BetH } C B C' \rightarrow \text{CongaH } A B C A B C'.$

This definition relies solely on the primitive notions of betweenness (BetH) and angle congruence (CongaH), capturing the intrinsic symmetry of perpendicularity without invoking auxiliary constructions. GeoCoq employs a constructive definition (Per), which characterizes a right angle via the existence of a midpoint and equidistant segments:

Definition $\text{Per } A B C := \text{exists } C', \text{Midpoint } B C C' \wedge \text{Cong } A C A C'.$

While these two definitions are logically equivalent in Euclidean geometry, our work demonstrates that the property used by GeoCoq as a definition is, in fact, a derived theorem in the Hilbert’s framework. Specifically, we have mechanized the proof of Mid_Per , which establishes that Hilbert’s definition implies Tarski’s definition.

Proposition $\text{Mid_Per} : \text{forall } A B C P, \text{MidPoint } B A C \rightarrow$
 $\text{Right_angle } P B A \rightarrow \text{CongH } P A P C.$

7. Conclusions

This paper has successfully addressed a significant gap in the formalization of Hilbert’s *Foundations of Geometry* by establishing a theory of spatial perpendicularity based on the first three groups of Hilbert’s axioms (incidence, order, and congruence) and proving the congruence theorem for three-dimensional figures. The theory is independent of the parallel postulate and continuity axioms, and thus holds in both Euclidean and non-Euclidean settings. Our work demonstrates that Hilbert’s axiom system is intrinsically complete for spatial geometry, requiring no additional spatial congruence axioms beyond the original set. The entire development has been rigorously formalized and machine-verified in the Rocq proof assistant, ensuring absolute logical correctness.

Our research makes three primary contributions:

1. **Axiom Formalization and Foundational Theories:** We began with a complete and careful formalization of Hilbert’s first three groups of axioms (Incidence, Order, and Congruence) in Rocq. Beyond merely restating the axioms, we derived an extensive library of fundamental geometric theories, including the classic triangle congruence theorems (SAS, ASA, SSS) and key properties of angles and midpoints. This library serves as a reusable foundation for future work in formalized geometry.
2. **Theory of Spatial Perpendicularity:** The core of our contribution lies in systematically constructing the theory of perpendicularity for lines and planes in space—a theory not fully developed by Hilbert himself. We introduced nine interconnected propositions, progressing from planar properties to sophisticated spatial theorems, such as the existence and uniqueness of perpendiculars from a point to a line or a plane. This theory filled a critical conceptual gap and provided the necessary tools for spatial reasoning.
3. **Mechanized Proof of the Figures Congruence Theorem:** Leveraging the perpendicularity theory, we achieved our ultimate goal: the formal proof of the congruence theorem for figures. The successful machine-checked proof conclusively demonstrates that all spatial congruence properties can be derived from Hilbert’s original, planar axioms, confirming the self-sufficiency of his system for three-dimensional space.

The significance of this work extends beyond mere theorem proving, as it serves as a valuable complement to the formalization of Hilbert’s geometry. Unlike libraries based on Tarski’s system, such as GeoCoq, our research is not intended to replace existing systems. Instead, it provides a reusable Rocq library for Hilbert’s geometry by creating a new foundation that enables subsequent researchers to perform formalizations directly based on Hilbert’s axioms. This library includes complete axiom encoding, derived theories, and perpendicularity theory, supporting the direct verification of Hilbert’s original propositions and lowering the barrier for conducting advanced geometric reasoning within the Hilbert framework. In the future, this resource can promote the application of Hilbert’s geometry in education, historical research, and the verification of safety-critical systems, thereby enriching the diversity of formalized geometry.

Our future work will focus on advancing the thorough formalization of Hilbert’s axiom system. And then we plan to investigate the compatibility and independence of the axioms.

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Abbreviations

The following abbreviation is used in this manuscript:

CIC Calculus of Inductive Constructions

Appendix A. Introduction to Rocq Syntax

Parameter	Declare axioms or parameters
Set	Represents types
Prop	Represents the proposition type
Definition	Define new concepts
Axiom	Declare an axiom
forall	Universal quantifier, meaning “for all”
exists	Existential quantifier, meaning “there exists”
->	Implication, meaning “if... then”
\/	Conjunction, meaning “and”
\/	Disjunction, meaning “or”
~	Negation, meaning “not”
Theorem	Declare a theorem
Lemma	Declare a lemma
Corollary	Declare a corollary
Proposition	Declare a proposition
Proof ... Qed	Proof start and end markers

Fixpoint	Declare a fixpoint function
Fact	Declare a fact
Hypothesis	Declare a hypothesis
Let	Declare a let-in expression

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