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Exploring Nonlinear Dynamics and Chaos in the Modified Korteweg–de Vries–Zakharov–Kuznetsov Equation with NARX Neural Networks

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Abstract

This work examines the nonlinear dynamics of a generalized Korteweg–de Vries–Zakharov–Kuznetsov equation, a model that appears in plasma physics, shallow water flows, and nonlinear wave propagation. By applying a solitary-wave transformation, the governing partial differential equation is reduced to an autonomous dynamical system, enabling a direct study of its phase portraits and equilibrium behavior. Stability of the fixed points is assessed through Jacobian matrices and eigenvalue classification, revealing parameter regimes that admit saddle states, centers, and oscillatory structures. The system's richer behavior is explored by varying key parameters, with phase-space trajectories exhibiting periodic, quasiperiodic, and irregular wave patterns. To probe the onset of complexity, we employ several diagnostic tools, including time-series evolution, Lyapunov exponents, bifurcation analysis, sensitivity tests, and Poincaré sections, which together indicate transitions to chaotic motion. The resulting dynamics are further captured using a nonlinear autoregressive neural network, which accurately reproduces the observed trajectories. The combination of analytical and computational perspectives provides a clear framework for understanding this generalized equation and offers a practical approach for investigating other nonlinear systems with a similar structure.

Keywords: dynamical system modeling; modified Korteweg–de Vries–Zakharov–Kuznetsov; phase portrait; chaos and bifurcation; Poincaré Map; NARX neural network modeling



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1. Introduction

Nonlinear partial differential equations (PDEs) provide a fundamental mathematical framework for describing complex processes in which several interacting components evolve under nonlinear effects. Such equations arise naturally in many branches of science and engineering, including physics [1,2], engineering [3], biology [4], chemistry [5,6], and economics [7]. They are especially important in the study of wave propagation, pattern formation, instabilities, oscillatory motion, and chaotic dynamics. Because nonlinear PDEs often involve strong coupling between nonlinearity and dispersion, their analytical treatment is usually challenging, and their qualitative and numerical investigation remains an active area of research.

A variety of analytical and computational techniques have been developed to understand the behavior of nonlinear PDEs. In some cases, exact solutions can be obtained

through methods such as the simple equation method [8], the extended Jacobi elliptic function method [9,10], and F-expansion techniques [11]. However, for many nonlinear models, closed-form solutions are either unavailable or difficult to derive in a tractable form. In such situations, numerical simulations and dynamical system-based approaches are essential for revealing the underlying structure of the solution space and understanding the system's long-term behavior [12].

Dynamical systems theory offers a particularly effective framework for studying nonlinear PDEs following suitable reductions. Instead of relying only on explicit solutions, this approach focuses on the qualitative features of the reduced system, including equilibria, stability, phase portraits, bifurcations, and transitions between different dynamical regimes [13,14]. Stability analysis and phase space methods help describe how trajectories evolve under different initial conditions. At the same time, bifurcation theory identifies parameter values at which the system's qualitative behavior changes [15–18]. When these changes lead to aperiodic, highly sensitive behavior, chaos theory provides tools to characterize the resulting unpredictability and its dependence on initial conditions [19,20].

Alongside classical dynamical system methods, artificial neural networks (ANNs) have become powerful tools for modeling nonlinear and chaotic systems [21]. Their ability to approximate complex nonlinear mappings makes them useful for reproducing and predicting complicated temporal dynamics. Several studies have demonstrated the effectiveness of neural network-based models in simulating chaotic and hyperchaotic systems. For instance, Awais et al. [22], Malik et al. [23], Bashir et al. [24], and Hussain et al. [25] employed radial basis function neural networks (RBFNNs) to model nonlinear dynamical behavior. Similarly, Akhtar et al. [26] extended neural network-based modeling to fractional-order chaotic systems and assessed the performance through statistical error measures.

Although RBFNNs are attractive for their relatively simple structure and fast training, they may be less effective for highly nonlinear systems with strong temporal dependence. Recurrent neural networks are better suited to such problems because their feedback connections allow them to capture memory effects and time-dependent patterns. Among these models, the nonlinear autoregressive neural network with exogenous inputs (NARXNN), a type of recurrent neural network, is particularly useful for nonlinear dynamical systems because it incorporates both present and past values of inputs and outputs [27,28]. For this reason, an NARXNN is employed in the present work to learn and reproduce the chaotic temporal dynamics generated by the considered nonlinear wave model.

The present study focuses on the $(3 + 1)$ -dimensional modified Korteweg–de Vries–Zakharov–Kuznetsov (mKdV-ZK) equation with cubic nonlinearity. This model is relevant to multidimensional nonlinear wave propagation and arises in physical settings such as plasma physics, shallow-water waves, nonlinear optics, and dispersive media. The cubic nonlinear term strengthens the nonlinear interaction and can give rise to rich dynamical behavior, including solitary waves, oscillatory structures, bifurcations, and chaotic motion.

In this study, we develop a hybrid analytical–computational framework for studying the nonlinear dynamics of the mKdV-ZK equation. The governing equation is first reduced by means of a traveling-wave transformation and then reformulated as an autonomous dynamical system. The resulting reduced system is examined through equilibrium analysis, stability classification, Hamiltonian structure, phase portraits, and nonlinear wave trajectories. This analysis reveals several characteristic dynamical patterns, including periodic, quasi-periodic, homoclinic, heteroclinic, and escaping trajectories.

To explore more complex behavior, the analysis is further extended to a higher-order dynamical formulation. The resulting system is studied using numerical simulations and standard chaos detection tools, including time-series analysis, Lyapunov exponents, bifurcation maps, sensitivity analysis, and Poincaré sections. These results confirm the ef-

fectiveness of the analytical–computational setup proposed for detecting chaotic structures, providing further insight into the complex dynamics of the mKdV-ZK system.

The analytical and numerical investigation is then complemented by neural network-based modeling. An NARX neural network is trained using the simulated dynamics of the mKdV-ZK system and is tested for its ability to reproduce the observed chaotic trajectories. Network performance is evaluated using statistical error indicators, regression analysis, residual correlation plots, and comparisons between predicted and target responses. In this way, the proposed framework combines the interpretability of dynamical systems theory with the predictive capability of machine learning.

The main objectives of this study are summarized as follows:

- To systematically convert the nonlinear mKdV-ZK equation into a dynamical system by use of traveling wave and Galilean transformations in order to analyze equilibrium points, stability, and several nonlinear wave phenomena.
- To precisely determine and diagnose chaotic behavior by employing Lyapunov exponents, bifurcation diagrams, sensitivity to initial values, time series, and Poincaré section.
- To construct and validate a data-driven modeling algorithm with an NARXNN, which will learn and predict well the chaotic time dynamics of the mKdV-ZK model.

The major outcomes of this investigation are summarized as follows:

- Delivery of a comprehensive classification of equilibrium states and uncovered critical bifurcations: Fold, Fold–Hopf, and Triplet, underpinning the onset of chaotic behavior.
- Confirmation of the presence of chaotic attractors through a combination of numerical, geometric, and spectral analyses, highlighting the system’s extreme sensitivity and dynamical richness.
- Demonstration of the effectiveness of a trained NARXNN in accurately capturing and reproducing the chaotic dynamics of the mKdV-ZK equation, establishing a powerful hybrid analytical–computational framework.

The remainder of the paper is structured as follows: Section 2 presents the mathematical formulation of the classical $(3 + 1)$ -dimensional Korteweg–de Vries–Zakharov–Kuznetsov (KdV-ZK) equation and the modified mKdV-ZK equation, followed by the traveling-wave reduction and the derivation of the reduced ordinary differential equation (ODE). Section 3 extends the analysis to a higher-order dynamical system and investigates bifurcation and chaotic behavior using numerical and qualitative tools. Section 4 describes the construction, training, and validation of the NARX neural network model. Finally, Section 5 summarizes the main findings and outlines possible directions for future research.

2. Mathematical Model and Problem Formulation

This section presents the mathematical model considered in this study and derives its reduced ODE using a traveling-wave transformation. We begin with the classical KdV-ZK equation, first established by Verheest et al. [29], which is written as

$$\mathbb{U}_t + \kappa \mathbb{U} \mathbb{U}_x + \mathbb{U}_{xxx} + \mathbb{U}_{xyy} + \mathbb{U}_{xzz} = 0. \quad (1)$$

Here, $\mathbb{U} = \mathbb{U}(x, y, z, t)$ denotes the wave field, where x , y , and z are spatial variables and t represents time. The parameter κ is a real constant that controls the strength of the nonlinear interaction. In Equation (1), the term $\mathbb{U}_t$ describes the temporal evolution of the wave field, the nonlinear term $\kappa \mathbb{U} \mathbb{U}_x$ represents quadratic nonlinear steepening, and the dispersive terms $\mathbb{U}_{xxx}$, $\mathbb{U}_{xyy}$, and $\mathbb{U}_{xzz}$ describe longitudinal and transverse dispersion effects.

In the present work, we consider the modified KdV-ZK equation [30–32], in which a cubic nonlinear interaction replaces the quadratic nonlinear term. The governing equation is

$$\mathbb{U}_t + \kappa \mathbb{U}^2 \mathbb{U}_x + \mathbb{U}_{xxx} + \mathbb{U}_{xyy} + \mathbb{U}_{xzz} = 0. \quad (2)$$

Here, the cubic term $\kappa \mathbb{U}^2 \mathbb{U}_x$ represents nonlinear self-interaction, where the square-wave amplitude modulates the spatial variation of the wave in the x -direction. Compared with the classical KdV-ZK model, this modification introduces stronger nonlinear effects, thereby allowing richer dynamical structures, including solitary waves, oscillatory structures, bifurcations, and chaotic responses. The dispersive terms in Equation (2) retain the multidimensional dispersion mechanism of the classical model and account for wave spreading along the longitudinal and transverse directions.

To analyze the qualitative behavior of Equation (2), we reduce it to an ODE by using the traveling-wave transformation

$$\mathbb{U}(x, y, z, t) = \mathbb{V}(\xi), \quad \xi = px + qy + rz - \omega t, \quad (3)$$

where p, q, r , and ω are nonzero real constants. Under this transformation, the partial derivatives appearing in Equation (2) become

$$\mathbb{U}_t = -\omega \mathbb{V}', \quad \mathbb{U}_x = p \mathbb{V}', \quad \mathbb{U}_{xxx} = p^3 \mathbb{V}''', \quad \mathbb{U}_{xyy} = pq^2 \mathbb{V}''', \quad \mathbb{U}_{xzz} = pr^2 \mathbb{V}'''. \quad (4)$$

Here, the prime denotes differentiation with respect to the traveling variable ξ . Substituting Equation (4) into Equation (2) gives

$$-\omega \mathbb{V}' + p\kappa \mathbb{V}^2 \mathbb{V}' + p^3 \mathbb{V}''' + pq^2 \mathbb{V}''' + pr^2 \mathbb{V}''' = 0. \quad (5)$$

After collecting the dispersive terms, Equation (5) can be expressed more compactly as

$$-\omega \mathbb{V}' + p\kappa \mathbb{V}^2 \mathbb{V}' + (p^3 + pq^2 + pr^2) \mathbb{V}''' = 0. \quad (6)$$

Integrating Equation (6) once with respect to ξ and taking the integration constant to be zero, we obtain

$$-\omega \mathbb{V} + \frac{1}{3} p\kappa \mathbb{V}^3 + (p^3 + pq^2 + pr^2) \mathbb{V}'' = 0. \quad (7)$$

Solving Equation (7) for $\mathbb{V}''$ yields the second-order nonlinear ODE

$$\mathbb{V}'' = \epsilon_1 \mathbb{V} - \epsilon_2 \mathbb{V}^3, \quad (8)$$

where the parameters ϵ_1 and ϵ_2 are defined as

$$\epsilon_1 = \frac{\omega}{p^3 + pq^2 + pr^2}, \quad \text{and} \quad \epsilon_2 = \frac{p\kappa}{3(p^3 + pq^2 + pr^2)}. \quad (9)$$

Equation (8) is a second-order nonlinear ODE that lays the foundation for the subsequent phase-space and stability analysis in the following subsections. This simplification provides the background for investigating the system's qualitative behavior.

2.1. Dynamical Analysis of the Governing Equation

To analyze the qualitative dynamics of the system described by Equation (8), such as its stability and nonlinear wave dynamics across different parameter regimes, it is important to rewrite the equation in a more analytically convenient form. One effective way to do this

is to rewrite the second-order nonlinear ODE as a planar autonomous system. This allows analysis of phase spaces and, thus, qualitative analysis of the system's evolution.

To this end, we will use a Galilean-type transformation by defining the following state variables:

$$x_1 = \mathbb{V}, \quad x_2 = \mathbb{V}',$$

which shows that the second derivative can be computed as $\mathbb{V}'' = \frac{dx_2}{d\xi}$.

Substituting these variables into Equation (8), the governing equation is recast into the following two-dimensional autonomous system:

$$\begin{cases} \frac{dx_1}{d\xi} = x_2, \\ \frac{dx_2}{d\xi} = \epsilon_1 x_1 - \epsilon_2 x_1^3. \end{cases} \quad (10)$$

where $\epsilon_1 = \frac{\omega}{p^3 + pq^2 + pr^2}$, and $\epsilon_2 = \frac{\frac{1}{3}p\kappa}{p^3 + pq^2 + pr^2}$. System (10) is an unperturbed autonomous dynamical model that forms the foundation for phase portrait analysis and stability investigation.

2.2. Equilibrium Points and Stability Analysis

To investigate the local stability properties of the system, we begin by identifying its equilibrium points. At equilibrium, the system reaches a steady state in which the derivatives of all state variables vanish, implying no further evolution along the trajectory. For the autonomous system defined in Equation (10), the equilibrium conditions are obtained by setting the right-hand side of the system to zero:

$$\begin{cases} x_2 = 0, \\ \epsilon_1 x_1 - \epsilon_2 x_1^3 = 0, \end{cases} \quad (11)$$

where the parameters ϵ_1 and ϵ_2 are defined by:

$$\epsilon_1 = \frac{\omega}{p^3 + pq^2 + pr^2}, \quad \text{and} \quad \epsilon_2 = \frac{\frac{1}{3}p\kappa}{p^3 + pq^2 + pr^2}.$$

Solving this system involves finding all pairs (x_1, x_2) that satisfy the above conditions simultaneously. From the first equation, we immediately obtain $x_2 = 0$. Substituting this into the second equation yields:

$$\epsilon_1 x_1 - \epsilon_2 x_1^3 = 0 \quad \Rightarrow \quad x_1(\epsilon_1 - \epsilon_2 x_1^2) = 0,$$

which leads to three distinct equilibrium points:

$$\begin{cases} \mathbb{E}_0(x_1, x_2) = (0, 0), \\ \mathbb{E}_{1,2}(x_1, x_2) = (\pm\sqrt{\epsilon_1/\epsilon_2}, 0). \end{cases} \quad (12)$$

For the autonomous system described by Equation (10), the Jacobian matrix is given by:

$$\mathbb{J}(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ \epsilon_1 - 3\epsilon_2 x_1^2 & 0 \end{bmatrix}. \quad (13)$$

The characteristic polynomial corresponding to the Jacobian matrix is expressed as:

$$\lambda^2 - (\epsilon_1 - 3\epsilon_2 x_1^2) = 0. \quad (14)$$

Solving the characteristic equation yields the eigenvalues:

$$\lambda_{1,2} = \pm \sqrt{\epsilon_1 - 3\epsilon_2 x_1^2}. \quad (15)$$

2.3. Hamiltonian Dynamics

Inspired by the structure of Hamilton's canonical equations in classical mechanics, a dynamical system of the form

$$\dot{x}_1 = \mathbb{A}(x_1, x_2), \quad \dot{x}_2 = \mathbb{B}(x_1, x_2), \quad (16)$$

where $(\cdot)$ denotes the time derivative, is referred to as a Hamiltonian system if there exists a scalar function $\mathbb{H}(x_1, x_2)$, called the Hamiltonian function, such that

$$\mathbb{A} = \frac{\partial \mathbb{H}}{\partial x_2}, \quad \text{and} \quad \mathbb{B} = -\frac{\partial \mathbb{H}}{\partial x_1}. \quad (17)$$

Then $\mathbb{H}$ is called the Hamiltonian function for the given system [33].

Definition 1. The necessary and sufficient condition for the dynamical System (16) to be Hamiltonian is that the functions $\mathbb{A}$ and $\mathbb{B}$ satisfy the following partial differential equation:

$$\frac{\partial \mathbb{A}}{\partial x_1} + \frac{\partial \mathbb{B}}{\partial x_2} = 0. \quad (18)$$

Let $x_1(t)$ and $x_2(t)$ represent a particular solution of the system. Then, along the corresponding phase path, the time derivative of the Hamiltonian function $\mathbb{H}$ is given by:

$$\begin{aligned} \frac{d\mathbb{H}}{dt} &= \frac{\partial \mathbb{H}}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial \mathbb{H}}{\partial x_2} \frac{dx_2}{dt}, \\ &= -\mathbb{B}\mathbb{A} + \mathbb{A}\mathbb{B}, \\ &= 0, \end{aligned} \quad (19)$$

in the second step, we have utilized Equation (16) and Equation (17). We have

$$\mathbb{H}(x_1, x_2) = \text{Constant} \quad (20)$$

which holds along any phase trajectory. According to Equation (20), the phase trajectories correspond to the level curves or contour lines, given by

$$\mathbb{H}(x_1, x_2) = C, \quad (21)$$

on the surface

$$z = \mathbb{H}(x_1, x_2), \quad (22)$$

in three-dimensional space. Based on the previously stated definition, Equation (10) is classified as a Hamiltonian dynamical system since it fulfills the state equation

$$\frac{\partial \mathbb{A}}{\partial x_1} + \frac{\partial \mathbb{B}}{\partial x_2} = 0. \quad (23)$$

and provides the Hamiltonian function as shown below:

$$\mathbb{H}(x_1, x_2) = \frac{1}{2}x_2^2 - \frac{\epsilon_1}{2}x_1^2 + \frac{\epsilon_2}{4}x_1^4. \quad (24)$$

2.4. Phase Portraits

A phase portrait is a graphical tool for investigating the qualitative dynamics of a dynamical system. It depicts how the system evolves under various initial conditions by plotting its state variables against one another. This can find important qualitative properties such as equilibrium points, stability, and oscillatory or periodic behavior, and gives a picture of the long-term evolution of the system without necessarily requiring explicit solutions. In the mKdV-ZK model, many different trajectory patterns can be seen in the phase portrait, each corresponding to a different type of system behavior depending on the nature and stability of the equilibrium points:

- **Periodic Trajectory (PT):** A trajectory that repeats itself after a fixed period, returning to the same state at the end of each cycle.
- **Homoclinic Trajectory (HT):** A trajectory that departs from an equilibrium point, follows a distinct path, and eventually returns to the same equilibrium point.
- **Quasi-Periodic Trajectory (QPT):** A trajectory governed by multiple incommensurate frequencies, leading to motion that is non-repeating yet structured. Such behavior is characteristic of systems with interacting periodic components.
- **Escaping Trajectory (ET):** A trajectory that diverges from a fixed point or attractor, moving toward infinity or beyond the system's limits, signifying unbounded or unstable dynamics.
- **Super Nonlinear Homoclinic Trajectory (SNHT):** They can occur due to very nonlinear dynamics close to saddle points and can demonstrate extremely complex, sensitive behavior—the hallmark of chaos and complex system behavior.

The appearance of these types of trajectories explains the structure, stability, and evolution of the mKdV-ZK system. Analyzing them provides a solid basis for deeper exploration into the equation's nonlinear dynamics.

For the dynamical system described by Equation (10), the nature and classification of equilibrium points vary depending on the specific values of the governing parameters. This classification is crucial for understanding the system's qualitative behavior and provides deeper insight into the model's structural dynamics.

Scenario 01: When $p > 0$, $q = 0$, $r = 0$, $w < 0$, and $\kappa < 0$, System (10) admits three equilibrium states: Here, $\mathbb{E}_0 = (0, 0)$ serves as a *center-type* equilibrium point and is *stable*, whereas $\mathbb{E}_1 = \left(\sqrt{\frac{\epsilon_1}{\epsilon_2}}, 0\right)$ and $\mathbb{E}_2 = \left(-\sqrt{\frac{\epsilon_1}{\epsilon_2}}, 0\right)$ are *saddle points* and thus *unstable*. As seen in Figure 1a, the phase portrait reveals kink and anti-kink structures that emerge from heteroclinic connections between the saddle points, indicative of nonlinear interactions.

Scenario 02: Under the configuration $p > 0$, $q > 0$, $r > 0$, $w > 0$, and $\kappa > 0$, System (10) also possesses three equilibrium points: $\mathbb{E}_0 = (0, 0)$, $\mathbb{E}_1 = \left(\sqrt{\frac{\epsilon_1}{\epsilon_2}}, 0\right)$, and $\mathbb{E}_2 = \left(-\sqrt{\frac{\epsilon_1}{\epsilon_2}}, 0\right)$. In this case, $\mathbb{E}_0$ becomes a *saddle point* and is hence *unstable*, while the other two, $\mathbb{E}_1$ and $\mathbb{E}_2$, behave as *centers* exhibiting *stable*, oscillatory motion. This is illustrated in Figure 1b. The corresponding trajectories form closed curves in phase space, reflecting a wide variety of dynamical solutions. These include hyperperiodic orbits (green), periodic loops (pink), and homoclinic trajectories (blue), each representing distinct aspects of the system's nonlinear response.

Scenario 03: For the parameter set $p > 0$, $q = 0$, $r = 0$, $w < 0$, and $\kappa = 0$, the dynamical behavior of System (10) reveals a single equilibrium point: $\mathbb{E}_0 = (0, 0)$. As illustrated in Figure 1c, this point acts as a *stable center*. The corresponding phase portrait

features a single family of periodic orbits, arising as solutions to System (10), highlighting the system's inherently regular and oscillatory nature in this parameter regime.

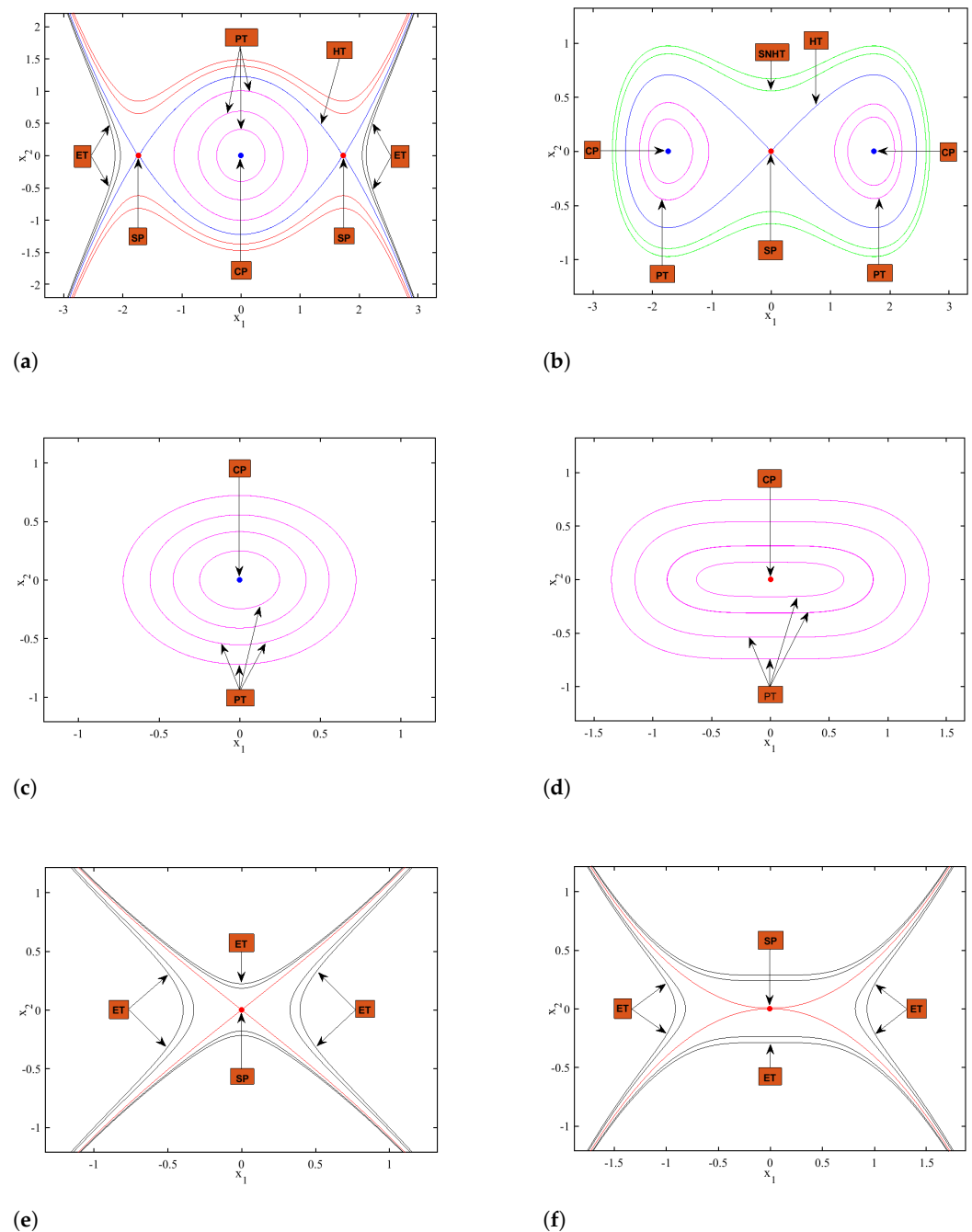


Figure 1. Phase portraits of System (10) for six different parameter regimes: (a) $p > 0, q = r = 0, w < 0, \kappa < 0$; (b) $p, q, r, w, \kappa > 0$; (c) $p > 0, q = r = 0, w < 0, \kappa > 0$; (d) $p > 0, q = r = 0, w = 0, \kappa > 0$; (e) $p, q, r, w, \kappa < 0$; (f) $p > 0, q = r = w = 0, \kappa < 0$.

Scenario 04: For the parameter set $p > 0, q = 0, r = 0, w = 0$, and $\kappa > 0$, System (10) yields a single equilibrium point at $\mathbb{E}_0 = (0, 0)$, which acts as a *stable center*, as shown in Figure 1d. The phase portrait shows smooth, nested periodic orbits around $\mathbb{E}_0$, with no *saddle points*. The entire phase space is filled with closed trajectories, indicating a purely oscillatory and stable system with uniform dynamics.

Scenario 05: For the parameter set $p < 0, q < 0, r < 0, w < 0$, and $\kappa < 0$, the dynamical system described by Equation (10) exhibits a single equilibrium point: $\mathbb{E}_0(x_1, x_2) = (0, 0)$, as illustrated in Figure 1e. In this configuration, the equilibrium point $\mathbb{E}_0$ behaves as a *saddle*

point, indicating that it is inherently *unstable*. Notably, the system does not produce any closed trajectories under these conditions, highlighting the absence of periodic or bounded cyclic behavior.

Scenario 06: For the parameter set $p > 0$, $q = 0$, $r = 0$, $w = 0$, and $\kappa < 0$, the system Equation (10) has a single equilibrium at $\mathbb{E}_0(x_1, x_2) = (0, 0)$, as illustrated in Figure 1f. This *saddle point* is inherently *unstable*. The system exhibits no periodic or bounded behavior, and perturbations grow over time, indicating a dissipative, structurally unstable regime.

The phase portraits corresponding to the aforementioned parameter scenarios, along with their associated trajectories, are illustrated in Figure 1. These diagrams provide a visual representation of the system's qualitative dynamics under varying conditions, highlighting the nature of the equilibrium points and the overall flow structure in the phase space.

Table 1 presents a comprehensive classification of the equilibrium points of System (10) across six distinct parameter configurations, detailing their existence, type, and stability properties in relation to the parameters p, q, r, w , and κ . This table effectively summarizes the key results discussed in earlier subsections.

Table 1. Classification of equilibrium points of System (10) under different parameter cases.

Scenario	Parameter Set	Equilibrium Points	Type	Stability
01	$p > 0, q = r = 0, w < 0, \kappa < 0$	$\mathbb{E}_0 = (0, 0)$ $\mathbb{E}_{1,2} = \pm \sqrt{\frac{\epsilon_1}{\epsilon_2}}$	Center point Saddle points	Stable Unstable
02	$p, q, r, w, \kappa > 0$	$\mathbb{E}_0 = (0, 0)$ $\mathbb{E}_{1,2} = \pm \sqrt{\frac{\epsilon_1}{\epsilon_2}}$	Saddle point Center points	Unstable Stable
03	$p > 0, q = r = 0, w < 0, \kappa = 0$	$\mathbb{E}_0 = (0, 0)$	Center point	Stable
04	$p > 0, q = r = w = 0, \kappa > 0$	$\mathbb{E}_0 = (0, 0)$	Center point	Stable
05	$p, q, r, w, \kappa < 0$	$\mathbb{E}_0 = (0, 0)$	Saddle point	Unstable
06	$p > 0, q = r = w = 0, \kappa < 0$	$\mathbb{E}_0 = (0, 0)$	Saddle point	Unstable

2.5. Nonlinear Wave Solutions

The nonlinear wave behavior of System (10) can be investigated through phase plane analysis by varying key physical parameters such as p, q, r, w , and κ . The phase trajectories derived from System (10) reveal a rich diversity of wave patterns depending on the nature and stability of the fixed points. Each trajectory within the phase plane corresponds uniquely to a specific wave solution, establishing a one-to-one correspondence between trajectories and nonlinear waveforms. Therefore, to uncover the full set of possible nonlinear wave solutions, it is necessary to systematically explore all relevant phase trajectories by adjusting the system parameters.

By selecting appropriate parameter values, both periodic and quasiperiodic wave behaviors can be observed. For instance, when $p = 1$, $q = 0$, $r = 0$, $w = -1$, and $\kappa = -1$, the system exhibits *periodic wave solutions*, as illustrated in Figure 2. The temporal evolution of the wave in the x_1 - and x_2 -directions is shown in Figure 2a and Figure 2b, respectively, while a helical 3D trajectory representing the full periodic structure is presented in Figure 2c.

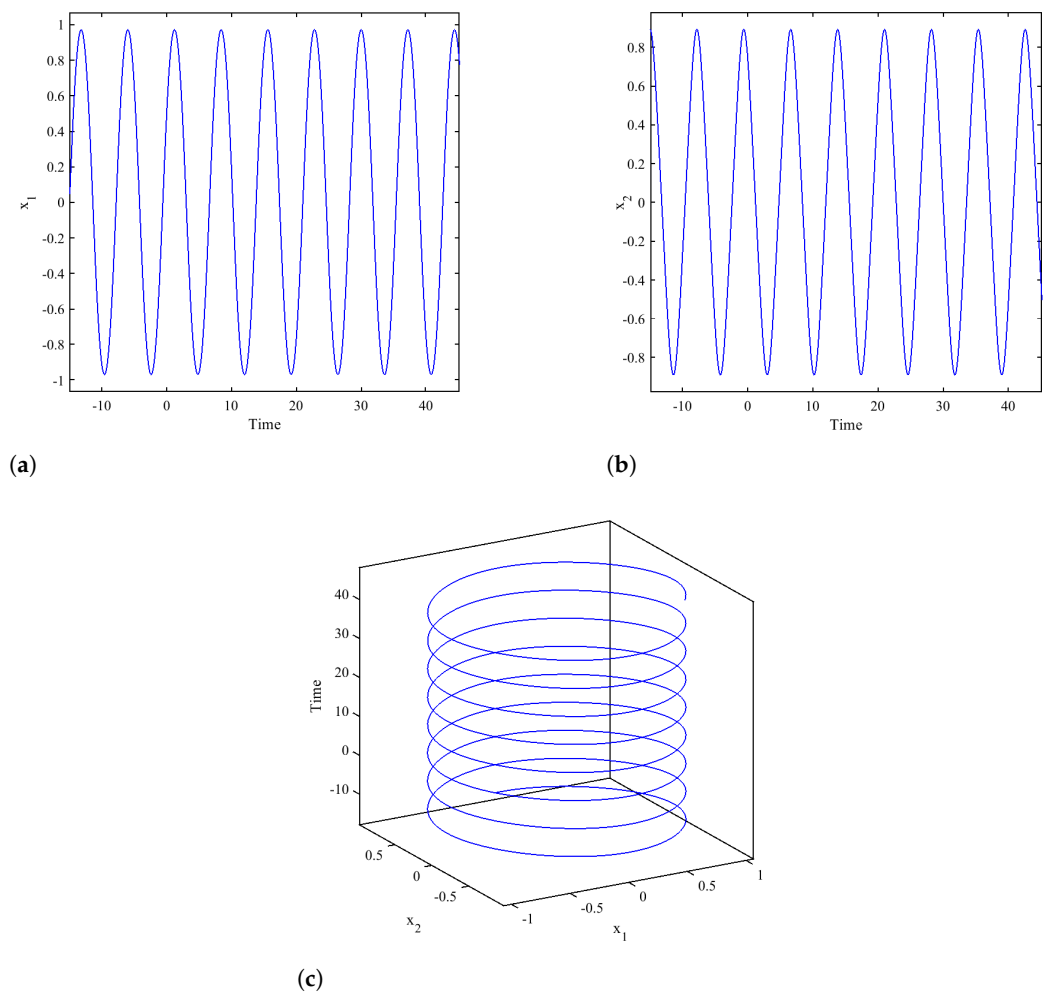


Figure 2. Periodic wave solutions of System (10) in 2D and 3D views: (a) Periodic wave profile along the x_1 -direction. (b) Periodic wave profile along the x_2 -direction. (c) 3D helical representation of the periodic wave solution.

On the other hand, *quasiperiodic wave patterns* emerge when $p = 01$, $q = 01$, $r = 01$, $w = 01$, and $\kappa = 01$, as depicted in Figure 3. These solutions feature more complex temporal structures compared to periodic ones, with Figure 3a and Figure 3b showing the evolution in the x_1 - and x_2 -directions, respectively. The corresponding 3D quasiperiodic structure is visualized in Figure 3c, highlighting the intricate interplay of nonlinear dynamics.

2.6. Implications of the Bifurcation Analysis

The bifurcation analysis provides valuable insight into the core processes that govern the nonlinear behavior of the mKdV-ZK equation when reduced to a two-dimensional system. It shows that the system can exhibit a variety of equilibrium states, some stable (center points) and others unstable (saddles), depending on the parameter choices. This parameter sensitivity leads to a wide spectrum of dynamic behaviors, such as regular oscillations, transitions between states (heteroclinic and homoclinic trajectories), and unbounded (escaping) motion. Although this reduced system does not exhibit true chaotic dynamics, it still captures essential nonlinear characteristics, particularly the emergence and stability shifts of equilibrium points, a defining feature of bifurcation. These bifurcations help explain the appearance of different wave structures in the full mKdV-ZK model. For example, kink and anti-kink solitons, as well as periodic and quasi-periodic wave patterns, are tied directly to the types of trajectories observed in the phase plane. Understanding which

parameter ranges lead to stability versus instability is crucial. It shows the precise balance between nonlinear effects and dispersive forces in wave propagation. In practical terms, the analysis might tell us how to regulate or maintain particular patterns of wave behavior in real situations, whether by suppressing variations or preventing instabilities. In short, the simplified system does not exhibit a chaotic motion, but the dynamics are complex and sensitive. This discussion provides the basis for further research on more sophisticated, higher-dimensional behavior in the mKdV-ZK framework.

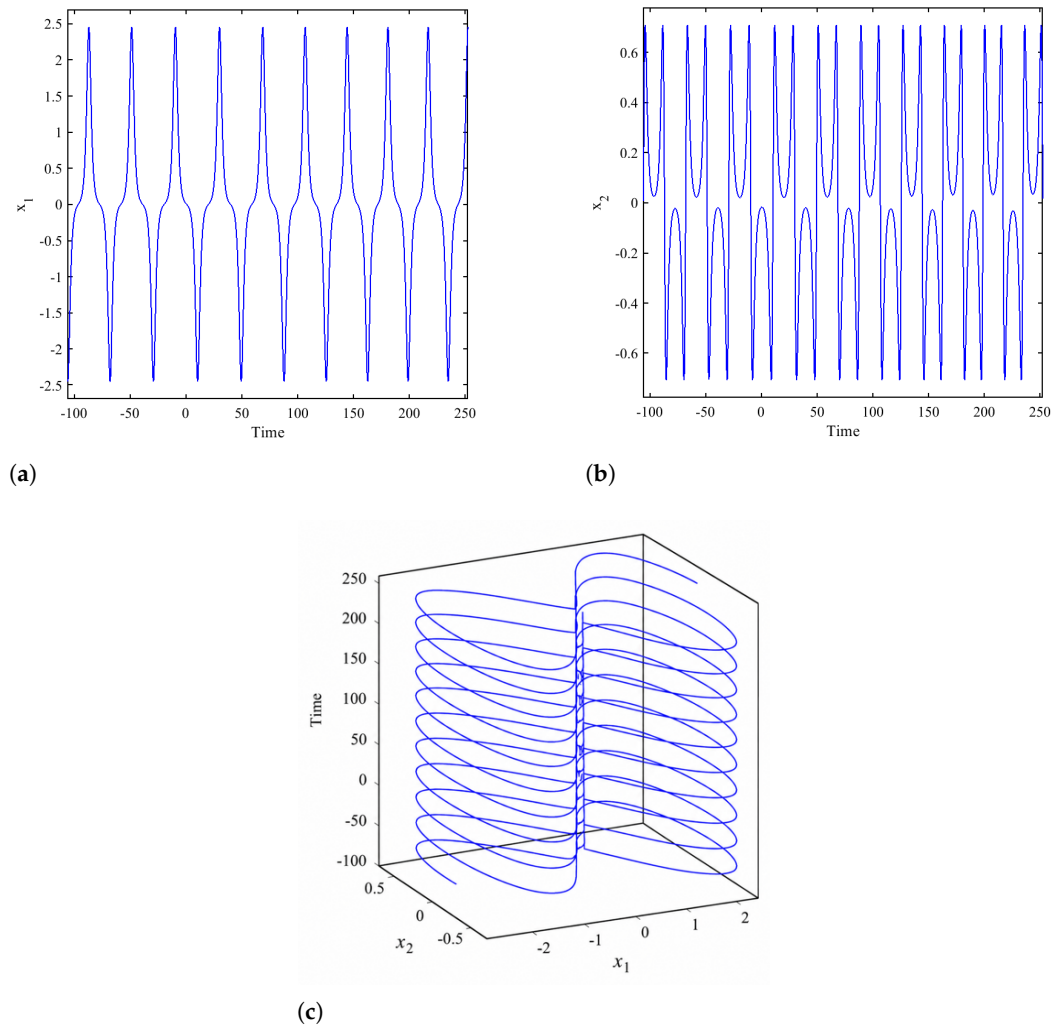


Figure 3. Quasiperiodic wave solutions of System (10) in 2D and 3D views: (a) Quasiperiodic wave pattern along the x_1 -direction. (b) Quasiperiodic wave pattern along the x_2 -direction. (c) 3D projection illustrating the complex structure of the quasiperiodic wave.

Remark 1. In Section 2 of this research, it was shown that the mKdV-ZK equation can be transformed, using a traveling wave and a Galilean transformation, into a two-dimensional dynamical system, allowing a qualitative analysis of its nonlinear dynamics. The equilibria were determined as stable center points and unstable saddle points based on the parameters, and their stability was assessed using linear stability analysis. The Hamiltonian structure and the phase portraits of the system exhibited a variety of dynamics, including periodic, quasi-periodic, homoclinic, and heteroclinic orbits, which are signatures of complex nonlinear dynamics. The bifurcation analysis revealed the dependence of parameter variations as the transition between stable and unstable regimes, where nonlinearity and dispersion are revealed. This discussion gives a solid basis for studying more complex interactions in higher dimensions.

The previously obtained planar dynamical system has a Hamiltonian structure, which rules out the occurrence of chaotic dynamics. To explore the potential for more complicated, including chaotic, behaviour, the associated third-order differential equation is recast into a three-dimensional autonomous dynamical system.

3. Higher-Order Dynamical Analysis

To explore the richer nonlinear dynamics of the mKdV-ZK system, the third-order equation as shown in Equation (5) is transformed into an equivalent first-order three-dimensional dynamical system. This reformulation facilitates a more detailed investigation of both local and global behavior, including the analysis of equilibrium points and their stability properties. It also provides a convenient framework for studying how the system evolves as parameters vary, leading to transitions between steady-state, periodic, and chaotic regimes. Particular attention is given to the onset of fold, fold–Hopf, and triplet bifurcations. These critical transitions are identified through changes in the eigenvalue spectrum of the Jacobian matrix, which signal qualitative shifts in the system’s dynamics.

To facilitate this higher-order analysis, we introduce a Galilean-type transformation that reduces Equation (5) into a three-dimensional first-order system of ODEs. Letting

$$x_1 = \mathbb{V}, \quad x_2 = \mathbb{V}', \quad x_3 = \mathbb{V}'',$$

we obtain the following system:

$$\begin{cases} \frac{dx_1}{d\xi} = x_2, \\ \frac{dx_2}{d\xi} = x_3, \\ \frac{dx_3}{d\xi} = \frac{\omega x_2 - p\kappa x_1^2 x_2}{p^3 + pq^2 + pr^2}, \end{cases} \quad (25)$$

where the prime notation $(') = \frac{d}{d\xi}$ denotes differentiation with respect to the wave variable ξ .

To determine the equilibrium points of the dynamical System (25), we set the derivatives equal to zero. This yields the algebraic system:

$$\begin{cases} x_2 = 0, \\ x_3 = 0, \\ \frac{\omega x_2 - p\kappa x_1^2 x_2}{p^3 + pq^2 + pr^2} = 0. \end{cases} \quad (26)$$

Substituting $x_2 = 0$ into the third equation confirms that the entire expression is identically satisfied, regardless of the value of x_1 . Therefore, the system admits a family of equilibrium points given by $\mathbb{E}(x_1, x_2, x_3) = (\eta, 0, 0)$, where $\eta \in \mathbb{R}$. This continuum of equilibrium states indicates that the system possesses a line of critical points in the phase space, whose stability characteristics are explored in the following sections via Jacobian and bifurcation analysis.

The Jacobian matrix $\mathbb{J}$ associated with the dynamical System (25) is given by:

$$\mathbb{J} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-2p\kappa x_1 x_2}{p^3 + pq^2 + pr^2} & \frac{\omega - p\kappa x_1^2}{p^3 + pq^2 + pr^2} & 0 \end{bmatrix}. \quad (27)$$

Evaluating the Jacobian at the equilibrium points $\mathbb{E}(x_1, x_2, x_3) = (\eta, 0, 0)$ yields:

$$\mathbb{J}|_{\mathbb{E}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{\omega - p\kappa\eta^2}{p^3 + pq^2 + pr^2} & 0 \end{bmatrix}. \quad (28)$$

The characteristic equation of (28) is derived by calculating $\det(\mathbb{J}|_{\mathbb{E}} - \lambda\mathbb{I}) = 0$, where $\mathbb{I}$ is the identity matrix and λ represents the eigenvalues. This yields the characteristic equation as:

$$\lambda^3 - \frac{\omega - p\kappa\eta^2}{p^3 + pq^2 + pr^2}\lambda = 0. \quad (29)$$

Introducing the parameter

$$\delta = \frac{\omega - p\kappa\eta^2}{p^3 + pq^2 + pr^2},$$

the characteristic equation simplifies to:

$$\lambda(\lambda^2 - \delta) = 0. \quad (30)$$

Solving Equation (30) yields the eigenvalues of the system:

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm\sqrt{\delta}. \quad (31)$$

- **Bifurcation Analysis of the System**

The dynamical behavior of System (25) is governed by the eigenvalue spectrum of the Jacobian matrix evaluated at the equilibrium points $\mathbb{E}(\eta, 0, 0)$. The eigenvalues are given by

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm\sqrt{\delta}, \quad \text{where} \quad \delta = \frac{\omega - p\kappa\eta^2}{p^3 + pq^2 + pr^2}.$$

Depending on the sign of δ , the system exhibits distinct local bifurcation behaviors:

- **Fold Bifurcation (Saddle-Node Type):**

For $\delta > 0$, the eigenvalues consist of one zero and two real values of opposite signs:

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm\sqrt{\delta} \in \mathbb{R}.$$

This corresponds to a fold (saddle-node) bifurcation, where two equilibria merge and annihilate. The presence of a neutral direction and a saddle structure indicates a local topological change in the phase space.

- **Fold–Hopf Bifurcation (Zero-Hopf Type):**

When $\delta < 0$, the eigenvalues become:

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm i\sqrt{|\delta|}.$$

This configuration of eigenvalues confirms that $\mathbb{E}$ is a Fold–Hopf equilibrium point. The interaction of a zero eigenvalue with purely imaginary conjugates gives rise to complex dynamics, including the possibility of invariant tori, quasi-periodicity, or transition to chaos through secondary bifurcations.

- **Triplet Bifurcation (Zero Bifurcation):**

For $\delta = 0$, all three eigenvalues vanish:

$$\lambda_1 = \lambda_2 = \lambda_3 = 0.$$

This degenerate case corresponds to a triplet or zero bifurcation, where the system loses all linear restoring forces near the equilibrium. Such a scenario typically indicates a highly non-hyperbolic equilibrium, and the resulting dynamics may involve unfolding into multiple bifurcation branches under small perturbations.

3.1. Non-Integrability of System

Before applying chaos diagnostics to System (25), we establish that the system is non-integrable for the chosen parameter values used in our simulations. This is necessary because chaotic motion in an autonomous continuous system cannot coexist with a sufficient set of smooth first integrals; therefore, establishing non-integrability provides the analytical foundation on which the following chaos diagnostics are built.

For a three-dimensional continuous dynamical system

$$\frac{dx}{d\zeta} = f(x), \quad x \in \mathbb{R}^3,$$

Liouville integrability requires the existence of two independent first integrals $I_1(x)$ and $I_2(x)$ in involution, that is, satisfying the Poisson bracket $\{I_1, I_2\} = 0$. Integrable systems confine trajectories to invariant tori or surfaces and satisfy $\lambda_{max} = 0$, whereas non-integrable systems may admit positive Lyapunov exponents and chaotic behavior. Thus, any claim of chaos must be supported by establishing, or at least strongly justifying, the non-integrability of the governing system.

After using the parameter values and initial condition for chaotic simulation as presented in Section 3.2.1, the denominator term of System (25) will become

$$p^3 + pq^2 + pr^2 = p(p^2 + q^2 + r^2) = 0.000564,$$

Therefore, the third part of System (25) reduces to

$$\dot{x}_3 = \frac{0.016x_2 - (0.04)(0.1)x_1^2x_2}{0.000564} = \frac{x_2(0.016 - 0.004x_1^2)}{0.000564}.$$

Introducing $k = 1/0.000564 \approx 1773.0496$ gives

$$\dot{x}_3 = kx_2(0.016 - 0.004x_1^2).$$

Under the chosen parameter values, the governing system has a non-Hamiltonian structure in the three-dimensional phase space, is volume-preserving (i.e., its divergence is zero), and carries the cubic nonlinearity $-0.004kx_1^2x_2$, which is responsible for the breakdown of regular motion.

To rule out low-degree polynomial first integrals, we first consider a linear first integral of the form

$$I = ax_1 + bx_2 + cx_3.$$

differentiation along the flow yields:

$$\dot{I} = ax_2 + bx_3 + c \frac{\omega x_2 - p\kappa x_1^2 x_2}{p^3 + pq^2 + pr^2}.$$

For $\dot{I}$ to vanish identically, the coefficient of the cubic monomial $x_1^2x_2$ must be zero, which yields $c(-pk/p^3 + pq^2 + pr^2) = 0$. But as $pk/(p^3 + pq^2 + pr^2) \neq 0$, we obtain $c = 0$, and the residue $\dot{I} = ax_2 + bx_3$ then forces $a = b = 0$. Thus, no non-trivial linear first integral exists.

Next, we consider a quadratic first integral

$$I = \sum_{1 \leq i \leq j \leq 3} a_{ij} x_i x_j.$$

Now, by calculating

$$\dot{I} = \sum_{i=1}^3 \frac{\partial I}{\partial x_i} \dot{x}_i$$

with

$$\frac{\partial I}{\partial x_1} = 2a_{11}x_1 + a_{12}x_2 + a_{13}x_3,$$

$$\frac{\partial I}{\partial x_2} = a_{12}x_1 + 2a_{22}x_2 + a_{23}x_3,$$

$$\frac{\partial I}{\partial x_3} = a_{13}x_1 + a_{23}x_2 + 2a_{33}x_3,$$

the nonlinear contribution from $\dot{x}_3$ produces, as its highest-degree monomial,

$$-\frac{2pka_{33}}{p^3 + pq^2 + pr^2} x_1^2 x_2 x_3.$$

For $\dot{I} = 0$ to hold identically, the coefficient of $x_1^2 x_2 x_3$ must vanish. Therefore, vanishing of this term forces $a_{33} = 0$. Propagating the coefficient-matching procedure through the remaining independent monomials yields $a_{ij} = 0$ for all i, j . Hence, no non-trivial linear or quadratic first integral exists.

For higher-degree polynomial assumptions, the cubic nonlinearity $-pkx_1^2 x_2 / p^3 + pq^2 + pr^2$ couples monomials of different degrees and produces a hierarchy of constraints that fails to close, as confirmed by a Darboux-type argument. Furthermore, although System (25) is volume-preserving, this property fixes only the phase-space measure and does not by itself produce a scalar invariant. The system possesses neither an evident canonical Hamiltonian structure nor an obvious continuous symmetry that would generate a conserved quantity. Consequently, no global analytic first integral is known or expected in the parameter regime under consideration.

This analytical conclusion is reinforced by numerical evidence. A positive maximal Lyapunov exponent is incompatible with motion constrained by a complete set of smooth first integrals and thus signals non-integrability. As shown in Figure 6a–c, the largest Lyapunov exponent of System (25) under the stated parameter values remains positive across the relevant ranges of p, q, r . Such behavior is inconsistent with integrability and provides strong numerical evidence that the system is non-integrable.

Additional support comes from the Painlevé test [34], which gives a necessary condition for integrability; an integrable equation admits, in the complex plane, only poles as movable singularities. Failure of this property for general parameter values is widely regarded as analytic evidence of non-integrability. Reducing System (25) to a third-order differential equation for $x_1(\xi)$ via $x_2 = x_1'$ and $x_3 = x_1''$ yields

$$x_1''' = \frac{\omega x_1' - pkx_1^2 x_1'}{p^3 + pq^2 + pr^2},$$

or, where $x_1' \neq 0$,

$$\frac{x_1'''}{x_1'} = \frac{\omega - pkx_1^2}{p^3 + pq^2 + pr^2},$$

and differentiating with respect to ζ yields

$$\frac{d}{d\zeta} \left(\frac{x_1'''}{x_1'} \right) = \frac{-2pk}{p^3 + pq^2 + pr^2} x_1 x_1'.$$

Substituting the leading-order form

$$x_1 \sim a(\zeta - \zeta_0)^\alpha$$

into the associated third-order equation yields non-integer dominant exponents for general parameter values, showing the presence of movable algebraic branch points. According to the Ablowitz et al. [35] criterion, this failure of the Painlevé property is analytic evidence of non-integrability.

Overall, these three independent arguments, the absence of non-trivial polynomial first integrals, the positivity of the largest Lyapunov exponent, and the failure of the Painlevé property, establish that System (25) is non-integrable for the chosen parameter set.

3.2. Tools for Chaos Detection

To thoroughly explore the chaotic nature of System (25), a comprehensive suite of numerical techniques is employed. Each method offers a distinct lens through which the system's intricate dynamics can be examined, contributing to a deeper understanding of its potential for chaos [36–38]. We start with numerical simulations, which help visualize how the system evolves by plotting its trajectories in phase space. These visuals provide a first glimpse of whether the system behaves consistently or shows signs of chaos. Time series analysis then helps identify the type of motion, whether it is repeating (periodic), nearly repeating (quasi-periodic), or completely irregular (chaotic). To quantify how the system responds to small changes, the largest Lyapunov exponent (LLE) is calculated. A positive LLE indicates that small differences in initial conditions lead to exponentially diverging outcomes, a hallmark of chaotic systems. Bifurcation diagrams are then used to examine how the system's behavior shifts as certain parameters are varied. These diagrams help identify transitions from stability to more complex, possibly chaotic regimes. In addition, sensitivity analysis is conducted by slightly altering the initial conditions and observing how the trajectories evolve. A high degree of divergence between similar starting points further confirms chaotic dynamics. Finally, a Poincaré section is employed. This technique involves slicing through the system's trajectory in phase space at regular intervals and recording the intersection points. In systems with chaotic behavior, these points tend to appear scattered and unstructured, providing a clear visual indicator of underlying complexity.

3.2.1. Numerical Simulations

To examine the dynamical behavior of the autonomous system given in Equation (25), numerical simulations were performed over the time interval $[0, 500]$ with a fixed step size of 0.1. The simulations were implemented in MATLAB R2018a using the ode45 solver, which is widely used for the numerical integration of nonlinear dynamical systems. The initial conditions were chosen as $x_1(0) = 2$, $x_2(0) = 2$, and $x_3(0) = 2$, while the system parameters were fixed at $p = 0.04$, $q = 0.05$, $r = 0.1$, $w = 0.016$, and $\kappa = 0.1$. The outcomes of these simulations clearly indicate chaotic dynamics. The system displays non-repeating (aperiodic) behavior over long periods and is highly sensitive to its starting conditions and parameter values. This means that even very small changes can produce vastly different outcomes, a key feature of chaotic systems and a sign of their unpredictable nature.

Figure 4 illustrates the chaotic attractors observed in the system. Three-dimensional plots are shown in Figure 4a,c,e, each representing different combinations of the system's variables. Their corresponding two-dimensional projections appear in Figure 4b,d,f, helping to highlight the structure and complexity of the phase space. These visualizations confirm the presence of chaos in the system and provide valuable insight into its dynamic and nonlinear behavior.

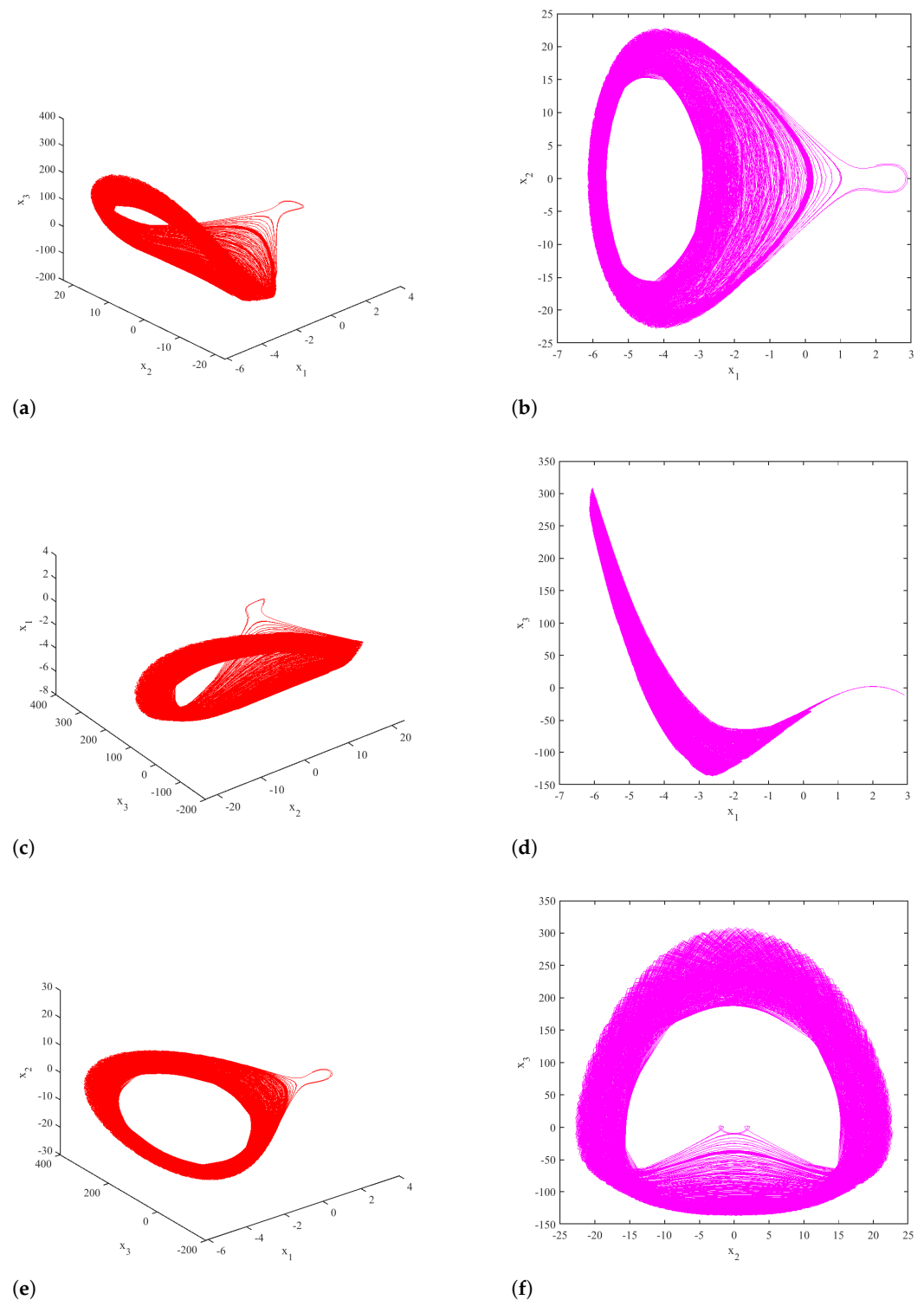


Figure 4. Phase plots of the chaotic attractor of Equation (25): (a) 3D attractor in (x_1, x_2, x_3) space. (b) 2D projection on the (x_1, x_2) plane. (c) 3D attractor in (x_2, x_3, x_1) space. (d) 2D projection on the (x_1, x_3) plane. (e) 3D attractor in (x_1, x_3, x_2) space. (f) 2D projection on the (x_2, x_3) plane.

3.2.2. Time Series Analysis

Time series analysis is a key method for understanding the behavior of nonlinear dynamical systems over time [39]. By examining how the system's state variables evolve, we can identify distinct behaviors such as steady motion, regular oscillations, quasi-periodicity, or chaotic fluctuations. For the system described by Equation (25), the time series for the variables x_1 , x_2 , and x_3 reveal strong indications of chaotic behavior, as shown in Figure 5.

In Figure 5a, the trajectory of x_1 is characterized by irregular, high-amplitude oscillations, generally fluctuating within the range $[-7, 3]$. The absence of a periodic pattern and the presence of erratic fluctuations point to the system's underlying instability, characteristic of chaotic regimes. Similarly, Figure 5b illustrates the behavior of x_2 , which oscillates unpredictably between $[-25, 25]$. The signal is marked by sharp transitions and a lack of convergence, reflecting sensitivity to initial conditions—one of the hallmarks of chaos. In Figure 5c, the trajectory of x_3 displays asymmetric oscillations ranging from $[-150, 350]$. Although the range is broader, the waveform remains irregular and non-repeating, further reinforcing the conclusion that the behavior is aperiodic and chaotic. These time series confirm the presence of chaos in the system governed by Equation (25). The bounded yet unpredictable, non-repetitive nature of the trajectories across all three state variables demonstrates sensitive dependence on initial conditions and a lack of long-term regularity, which are defining features of chaotic systems.

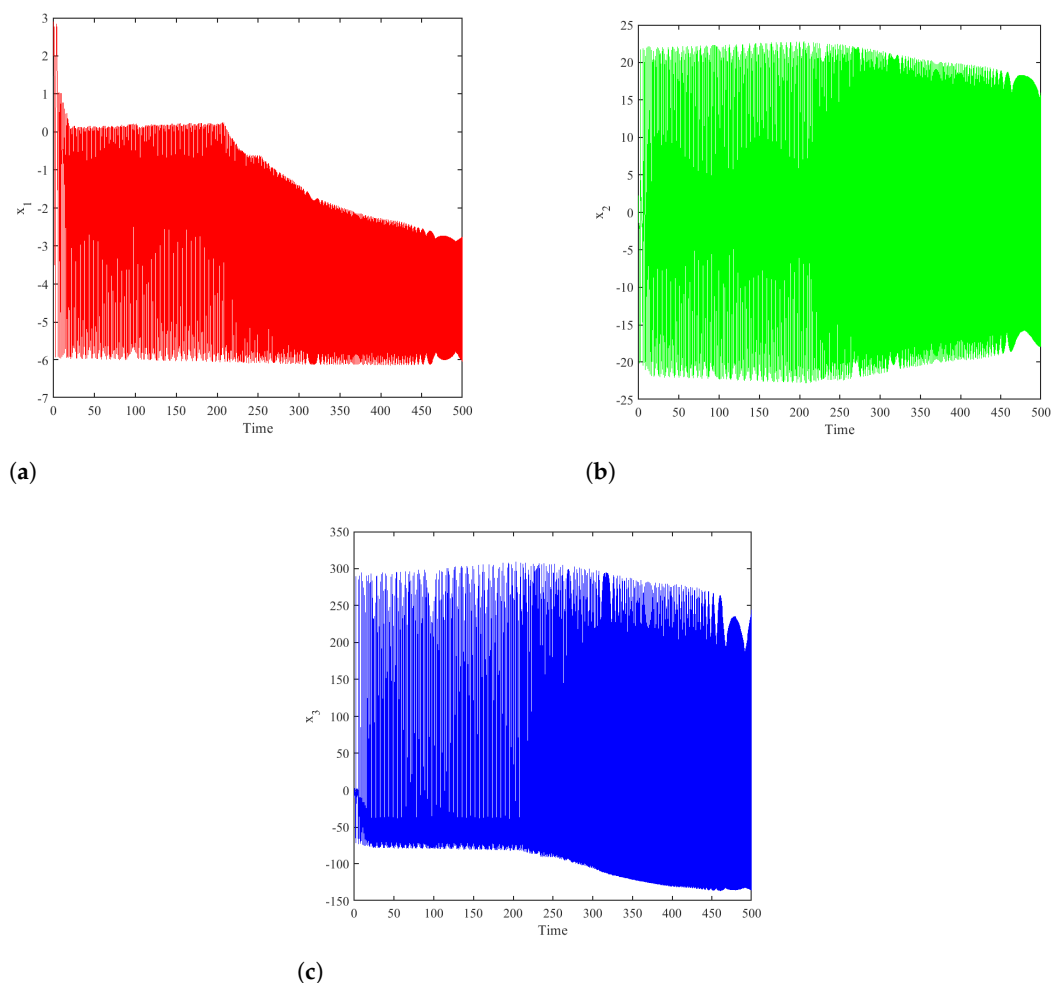


Figure 5. Time series analysis of state variables of System (25): (a) Time series plot for x_1 . (b) Time series plot for x_2 . (c) Time series plot for x_3 .

3.2.3. Lyapunov Exponents

Lyapunov exponents measure the average exponential rate at which nearby trajectories in phase space diverge or converge, and they provide the standard quantitative test of chaos in a dynamical system. For an n -dimensional flow, the spectrum $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ classifies the dynamics. A positive largest exponent ($\lambda_{\max} > 0$) signals exponential sensitivity to initial conditions and is the defining fingerprint of chaotic motion. In contrast, ($\lambda_{\max} < 0$) indicates contraction onto a stable attractor, and ($\lambda_{\max} = 0$) corresponds to neutrally stable, typically quasi-periodic, behaviour [40,41].

The largest Lyapunov exponent for System (25) is estimated from the simulated trajectory through the nearest neighbor divergence formula

$$\lambda_{\max} = \frac{1}{(E_{\max} - E_{\min} + 1) \Delta t} \sum_{E=E_{\min}}^{E_{\max}} \frac{1}{E} \ln \frac{\|X_{i+E} - X_{i^*+E}\|}{\|X_i - X_{i^*}\|},$$

where X_i is the reference state at step i , X_{i^*} is its nearest neighbour in phase space, E is the evolution time over which their separation is tracked, $[E_{\min}, E_{\max}]$ is the averaging window, and Δt is the integration step.

We first examine how $\lambda_{\max}$ depends on each of the three governing parameters p , q , and r . The results are presented in Figure 6a–c. In Figure 6a, $\lambda_{\max}$ remains positive over $0.01 \leq p \leq 0.055$, indicating sustained chaotic behavior throughout this range; in particular, for $p > 0.055$, the exponent drops into the negative region, signaling a transition toward regular dynamics. Figure 6b shows that for $-0.2 \leq q \leq 0.3$, the exponent stays predominantly positive with visible fluctuations, confirming persistent sensitivity to initial conditions. The same pattern holds along the r -axis in Figure 6c, where $\lambda_{\max}$ remains positive and oscillatory across $-0.3 \leq r \leq 0.5$. Together, these three panels demonstrate that chaos is not a fragile, parameter-tuned artifact but a robust feature of the system over an extended region of parameter space.

To further support this single-exponent analysis, the full Lyapunov spectrum $\{\lambda_1, \lambda_2, \lambda_3\}$ was then computed along the parameter q using the Wolf algorithm [42], which tracks the simultaneous evolution of an orthonormal set of perturbation vectors. The results are shown in Figure 6d, with extremal values listed in Table 2. The largest exponent λ_1 remains positive throughout the range and attains its maximum value $\lambda_{\max} = 0.0426$ at $q \approx 0.152$, identifying the parameter range of strongest sensitivity. The second exponent λ_2 stays near zero, consistent with the neutral direction along the flow, while λ_3 is negative, ensuring net volume contraction. This $(+, 0, -)$ signature is the canonical spectral fingerprint of a low-dimensional chaotic attractor and completely agrees with the non-integrability argument as discussed in the earlier section, offering independent confirmation that System (25) supports genuine chaotic dynamics.

Table 2. Maximum values of the Lyapunov spectrum with respect to the parameter q .

Exponent	Maximum Value	Corresponding q
λ_1	0.0426183903	0.1524848485
λ_2	0.0039977302	5.0000000000
λ_3	−0.0046017083	1.2633737374
$\lambda_{\max}$	0.0426183903	0.1524848485

3.2.4. Bifurcation Values Diagram

To analyze how the system's dynamics evolve with changes in parameters, bifurcation value diagrams are used [43]. These diagrams plot the final values of a state variable as a control parameter varies, offering insight into the system's qualitative behavior. Stable

states typically appear as smooth curves, periodic motions as discrete bands, and chaotic behavior as dense, scattered points [44]. Figure 7 presents three such diagrams based on Equation (25), each corresponding to a different control parameter. In Figure 7a, as parameter q varies, x_1 exhibits a dense scatter throughout, indicating persistent chaos. Figure 7b shows x_2 with respect to r , displaying a symmetric bell-shaped spread, suggesting a transition from regular to chaotic dynamics. Figure 7c illustrates how x_3 changes with p , revealing two diverging branches that suggest multistability and complex attractor structures. These diagrams collectively highlight the system's sensitivity to parameter variations and its rich, nonlinear, chaotic behavior across different state variables.

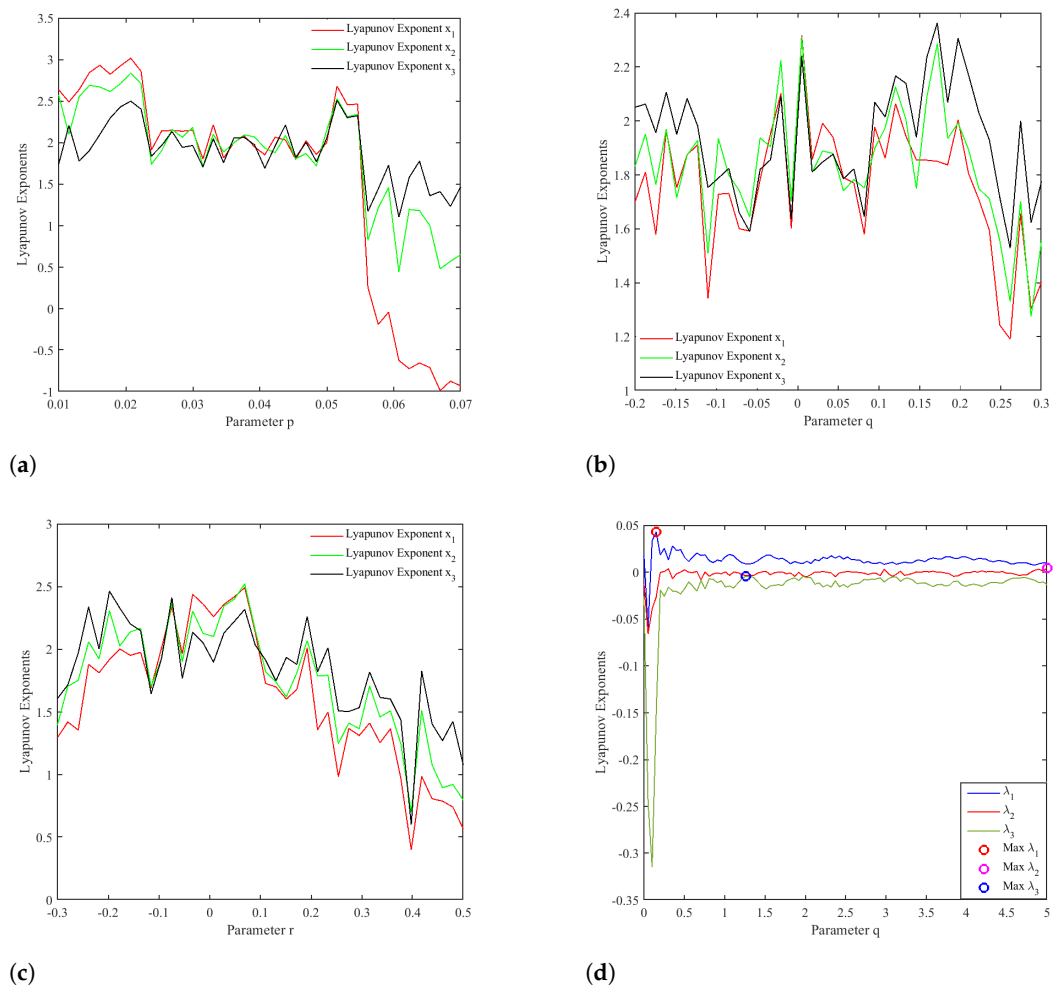


Figure 6. Lyapunov exponent plots of System (25): (a) Lyapunov exponents along p . (b) Lyapunov exponents along q . (c) Lyapunov exponents along r . (d) Dynamics of the full Lyapunov spectrum using the Wolf algorithm along parameter q .

3.2.5. Sensitivity Analysis

Sensitivity analysis evaluates how small changes in initial conditions influence the evolution of a dynamical system [13,45]. In chaotic systems, even minimal perturbations can cause drastically different trajectories, complicating long-term predictions. To examine the sensitivity of the system defined by Equation (25), perturbations were applied individually to each of the initial state variables while keeping the others constant. The base condition $(x_1, x_2, x_3) = (2, 2, 2)$ was modified to $(2.2, 2, 2)$ for x_1 , $(2, 1.85, 2)$ for x_2 , and $(2, 2, 3)$ for x_3 . The resulting trajectories are shown in Figure 8. Figure 8a reveals a rapid and continuous divergence due to a small change in x_1 , highlighting strong sensitivity. Figure 8b shows similar divergence from a slight perturbation in x_2 , further confirming the

system's instability. In Figure 8c, altering x_3 by just one unit also produces significant separation in trajectories. These results confirm the system's chaotic nature: even infinitesimal deviations in initial conditions yield markedly different outcomes, a hallmark of nonlinear dynamical chaos.

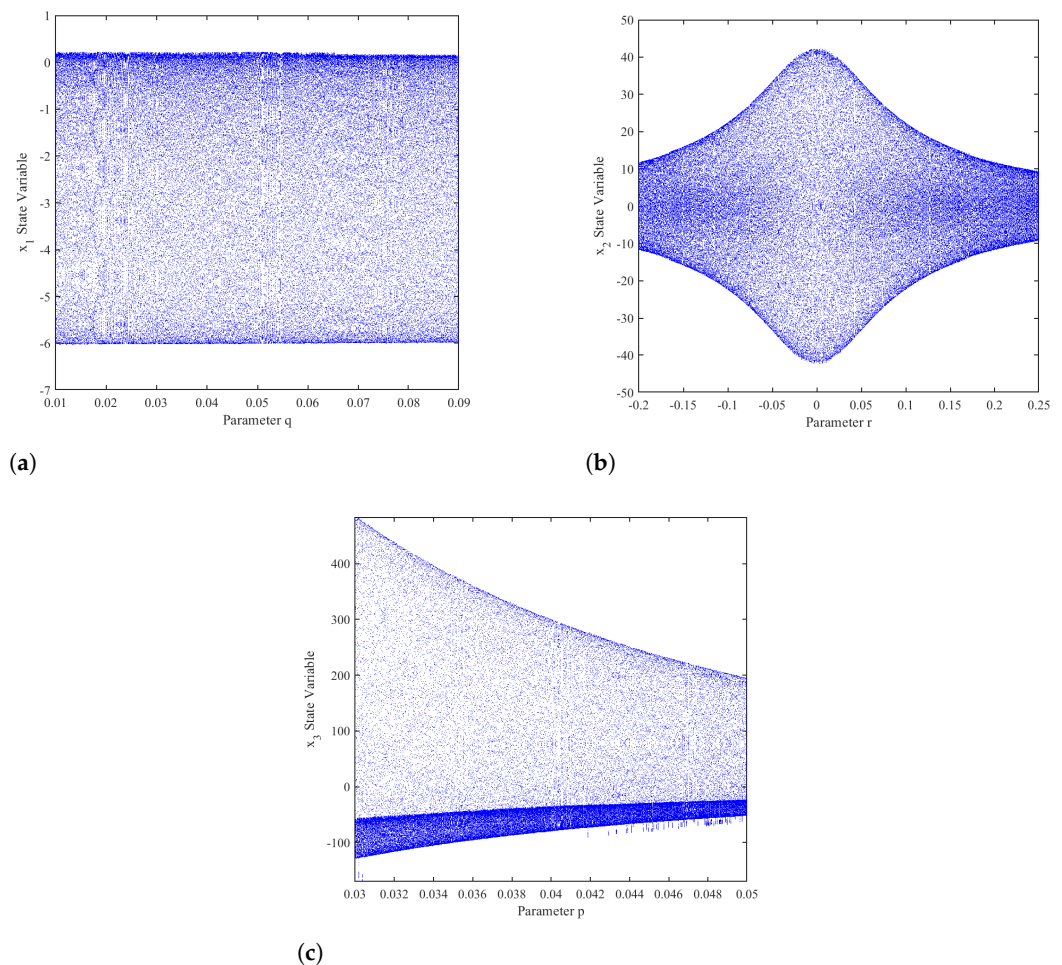


Figure 7. Bifurcation diagrams for system parameters: (a) Bifurcation diagram of x_1 with q . (b) Bifurcation diagram of x_2 with r . (c) Bifurcation diagram of x_3 with p .

3.2.6. Poincaré Section Analysis

The Poincaré section is a classical technique introduced by Henri Poincaré for analyzing the qualitative behavior of dynamical systems by examining their trajectories in a reduced-dimensional phase space [46]. It is constructed by recording selected points from the trajectory of a continuous dynamical system, thereby converting the continuous flow into a discrete map. The distribution of these points provides useful information about the nature of the system motion. A finite number of isolated points usually indicates periodic behavior, and a closed curve corresponds to quasi-periodic motion, typically associated with toroidal trajectories. In contrast, a scattered and irregular distribution of points is characteristic of chaotic dynamics. In this scenario, the Poincaré-type sections were obtained directly from the simulation of the targeted system as described by Equation (25). Firstly, the system was solved, and to remove transient effects, the first 20% of the trajectory was discarded. The remaining trajectory was then sampled at a fixed interval to obtain the Poincaré points. Thus, the plotted points are obtained from the transformed trajectory of the target ODE system.

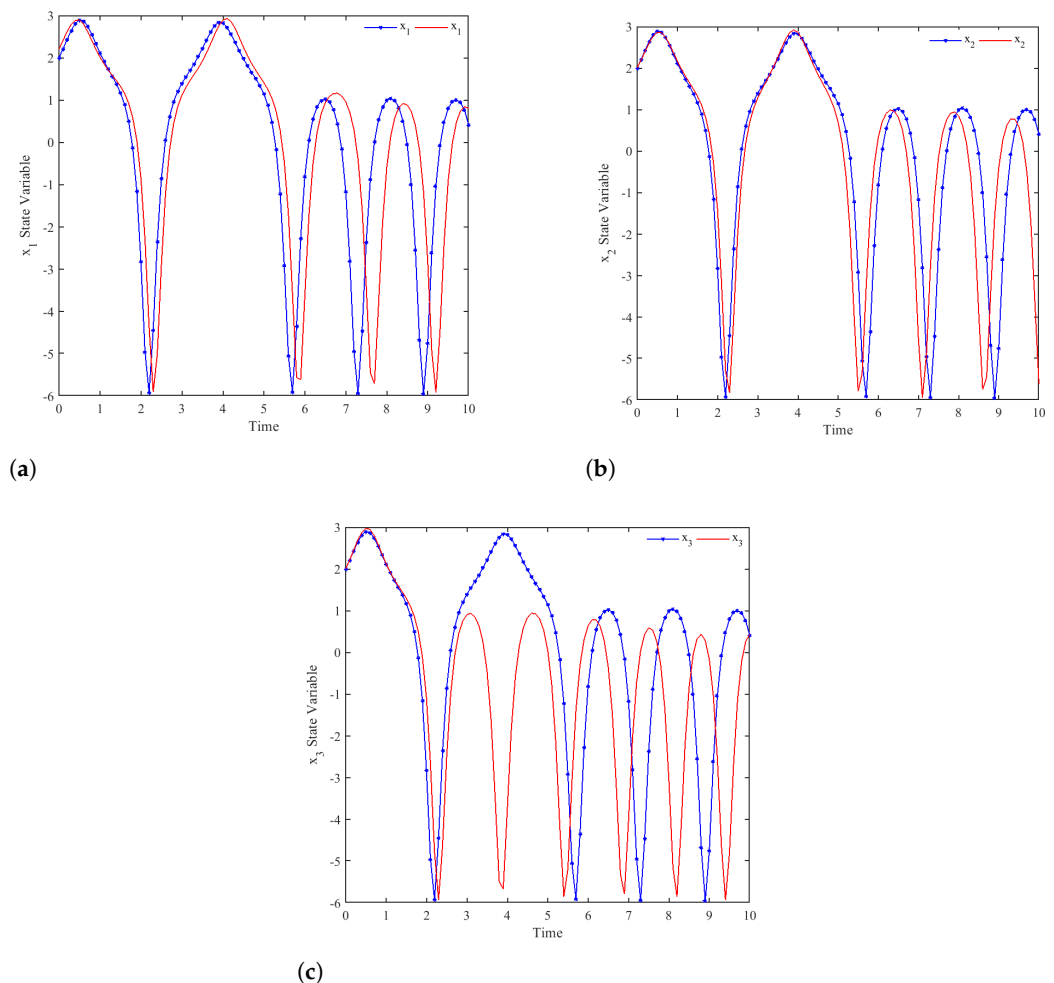


Figure 8. Sensitivity analysis of the state variables of System (25): (a) Sensitivity plot of state variable x_1 . (b) Sensitivity plot of state variable x_2 . (c) Sensitivity plot of state variable x_3 .

Furthermore, to examine the structure of these Poincaré points from different angles, the sampled points were projected onto two coordinate planes: x_1 – x_2 and x_2 – x_3 . These different projections for the system given in Equation (25) are shown in Figure 9. Specifically, Figure 9a highlights the projection in the x_1 – x_2 plane, which represents a widely scattered set of points without a regular pattern. A similar pattern is evident in Figure 9b, where the x_2 – x_3 plane also exhibits a dispersed and non-repeating structure. The scattered nature of the points in these projections indicates that the system shows chaotic motion. Therefore, the comparison of Poincaré sections at different projection angles further supports the presence of chaotic dynamics in the system.

3.3. Implications of Chaos in the mKdV-ZK Model

The chaotic dynamics uncovered in the mKdV-ZK system have important implications for understanding and predicting nonlinear wave behavior in multidimensional dispersive media. The observed chaos, marked by extreme sensitivity to initial conditions, implies that small perturbations can lead to vastly different wave evolutions, challenging the long-term stability of solitary and periodic structures. This highlights the intricate interplay between nonlinearity and dispersion, and reveals the system's ability to access a rich spectrum of states, including quasi-periodic and aperiodic oscillations. In practical contexts such as plasma physics, shallow water dynamics, and nonlinear optics, these chaotic regimes manifest as energy localization, wave breaking, and complex spatiotemporal patterns. Identifying and characterizing chaos in the mKdV-ZK model thus enhances its

descriptive power and provides valuable insights for controlling or exploiting nonlinear wave phenomena in real-world applications.

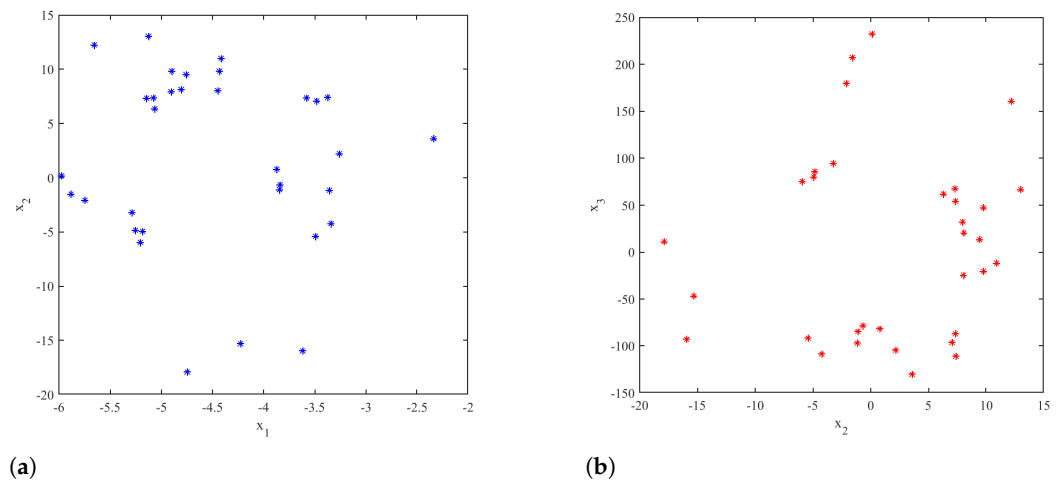


Figure 9. Poincaré sections of System (25) from different projection angles: (a) Projection on the x_1 - x_2 plane. (b) Projection on the x_2 - x_3 plane.

4. Artificial Neural Networks

ANNs have emerged as powerful computational tools for modeling highly nonlinear and chaotic systems [27,47–50]. Inspired by the structure and function of the human brain [51–53], ANNs consist of interconnected layers of neurons that adaptively learn from data, enabling them to approximate complex relationships where traditional mathematical methods may fail. Their ability to learn from historical data makes them particularly suitable for capturing the intricate dynamics of chaotic systems, which are characterized by high sensitivity to initial conditions and nonlinearity. Among various ANN architectures, the NARXNN stands out for its effectiveness in modeling time-dependent dynamical systems [54,55]. The NARX architecture utilizes both past values of the output (auto-regressive component) and external inputs (exogenous variables) to predict future states [56]. This recursive, feedback-driven framework aligns well with the dynamics of chaotic systems, making it a robust tool for their analysis and prediction [57,58].

The general architecture of an NARXNN is illustrated in Figure 10. It incorporates delayed input and output connections, forming feedback loops that give the network temporal memory. This memory capability significantly enhances the network's ability to model and predict complex, time-evolving behaviors accurately.

4.1. NARXNN for Modeling Nonlinear Chaotic System

In this study, the NARXNN was used to model the chaotic dynamics of the system described by Equation (25). It is particularly well-suited to time-series modeling [59,60] because it incorporates both delayed external inputs and delayed outputs to predict the system's future response. Due to its feedback structure, the network can learn the complex temporal dependencies and nonlinear behavior of chaotic trajectories [61].

Mathematically, the general NARXNN model is expressed as

$$\hat{y}(k+1) = \mathbb{F}\left(u(k), u(k-1), \dots, u(k-d_u), y(k), y(k-1), \dots, y(k-d_y)\right). \quad (32)$$

where $u(k)$ denotes the external input, which is time ($\mathbb{T}_{in}$), and $y(k)$ represents the network output response, corresponding to x_1 , x_2 , and x_3 , while d_u and d_y are the input and feedback delays, respectively, and $\mathbb{F}(\cdot)$ are nonlinear mappings learned by the neural network. The provided architecture shows that the delayed input sequence $u(k), u(k-1), \dots, u(k-d_u)$

and the delayed feedback sequence $y(k), y(k-1), \dots, y(k-d_y)$ are passed to the hidden layer. Each hidden neuron computes a weighted summation of these delayed signals together with a bias term, followed by a nonlinear activation function f_h . The outputs of the hidden neurons are then linearly combined and passed through the output activation function f_o to produce the predicted output $y(k+1)$. Therefore, the NARXNN structure can be written as

$$y(k+1) = f_o \left[\sum_{h=1}^{N_h} w_{ho} f_h \left(\sum_{i=0}^{d_u} w_{ih} u(k-i) + \sum_{j=0}^{d_y} w_{jh} y(k-j) + b_h \right) + b_o \right]. \quad (33)$$

where N_h denotes the total number of hidden neurons, w_{ih} shows the weights that connect the delayed input signals to the hidden layer, w_{jh} represents the weights that connect the delayed feedback signals to the hidden layer, w_{ho} are the weights that connect the hidden layer to the output layer, and b_h and b_o are the bias elements of the hidden and the output layer, respectively. Also, $f_h(\cdot)$ and $f_o(\cdot)$ represent the activation functions for the hidden layer and output layer, and z^{-1} indicates the unit delay operator.

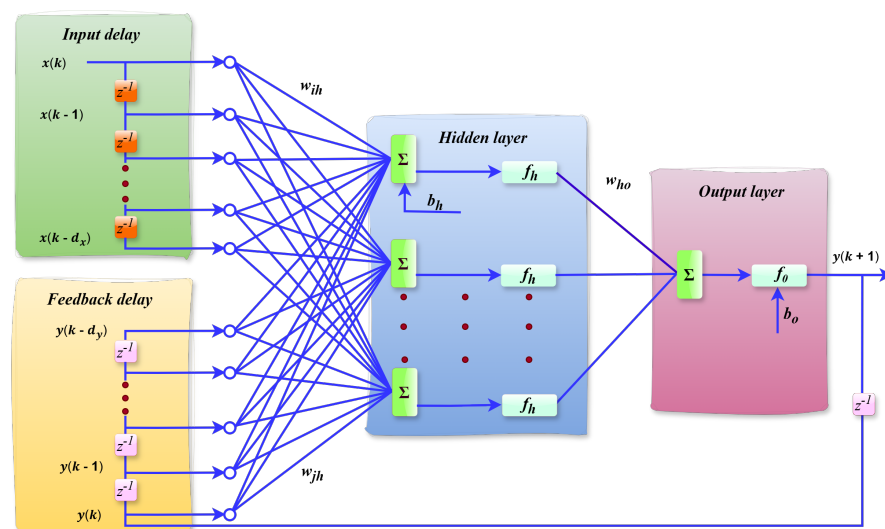


Figure 10. Architecture of the NARXNN.

In this scenario, the chaotic data are generated numerically from Equation (25) over a time interval of $[0, 500]$, with a time step of 0.1, yielding 5000 data points. In the utilized NARXNN model, the time sequence is taken as the external input, while the corresponding chaotic state responses are used as target outputs. The dataset is divided into three subsets: 70% for training, 15% for validation, and 15% for testing. The training data enabled the network to learn the underlying system dynamics. The validation set was used to tune the model and prevent overfitting by monitoring performance during training. Finally, the testing data provided an unbiased assessment of the model's predictive accuracy. Moreover, the number of hidden neurons is selected as $N_h = 10$, while the numbers of input and feedback delays are chosen as $d_u = d_y = 5$. This network configuration enables the NARXNN to effectively capture and reproduce the nonlinear and chaotic temporal behavior of the system. The architecture of the NARXNN used in this study is illustrated in Figure 11.

4.2. NARXNN Training and Performance Evaluation

To train the NARX neural network, the Levenberg–Marquardt optimization algorithm was used across 1000 training epochs. During this process, the network's weights were updated to minimize the mean squared error (MSE) loss function. Over time, this led to steady improvements in performance, culminating in the best validation result at epoch

1000, where the MSE reached as low as 10^{-4} . This low error indicates the network's strong ability to capture the chaotic behavior of the system with high accuracy.

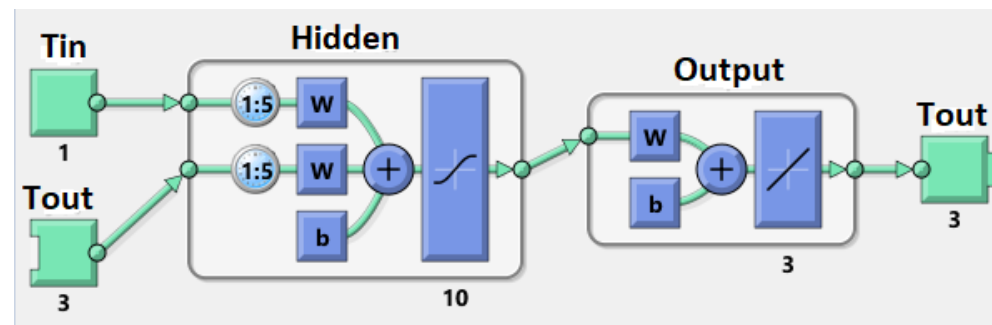


Figure 11. Training view of the NARXNN model.

Throughout training, the network showed consistent convergence across the training, validation, and testing datasets. This smooth learning curve suggests that the NARXNN successfully internalized the system's underlying dynamics without falling into overfitting.

To comprehensively evaluate the model's performance, we employed several key metrics and graphical analyses:

- **Training Performance:** As shown in Figure 12a, the MSE consistently declined over time, reaching its lowest point at epoch 1000. This gradual error reduction confirms that the network was learning efficiently and effectively throughout training.
- **Gradient and Hyperparameters:** In Figure 12b, the graphs provide a deeper look into the optimization process. The gradient shows the rate of change during training, with a value of 0.53956 at epoch 1000, indicating a decrease in gradient as the network nears convergence. The mu value of 0.0001 indicates that the training process was adjusting the learning rate to ensure stability. Finally, validation check = 0 indicates that the model achieved an optimal solution without further need for hyperparameter adjustments. This visualization demonstrates the efficient fine-tuning of the network's parameters during training.
- **Regression Plots:** Figure 13 displays the regression plots for training, validation, test, and overall data. All plots show a correlation coefficient (R) of 1, confirming that the NARXNN closely tracks the target values, with minimal deviation from the ideal $\hat{Y} = T$ line. These plots illustrate the network's capability to model chaotic dynamics with exceptional precision.
- **Error Histogram:** The error distribution, shown in Figure 14a, reveals a tight clustering around zero, indicating that prediction errors are minimal and centered, with few large errors. This confirms the model's robustness in predicting the system's future states.
- **Autocorrelation of Errors:** In Figure 14b, the autocorrelation analysis of the residuals demonstrates that the errors are randomly distributed without any significant patterns or biases. This random distribution further affirms the NARXNN's generalization capability and robustness, suggesting the absence of systemic errors.

So the NARXNN demonstrated exceptional performance in capturing the system's chaotic dynamics. It achieved smooth, consistent convergence during training and maintained high precision across all data subsets: training, validation, and testing. The model's ability to reliably predict the dynamics with minimal error highlights its effectiveness and robustness in modeling complex, nonlinear chaotic systems.

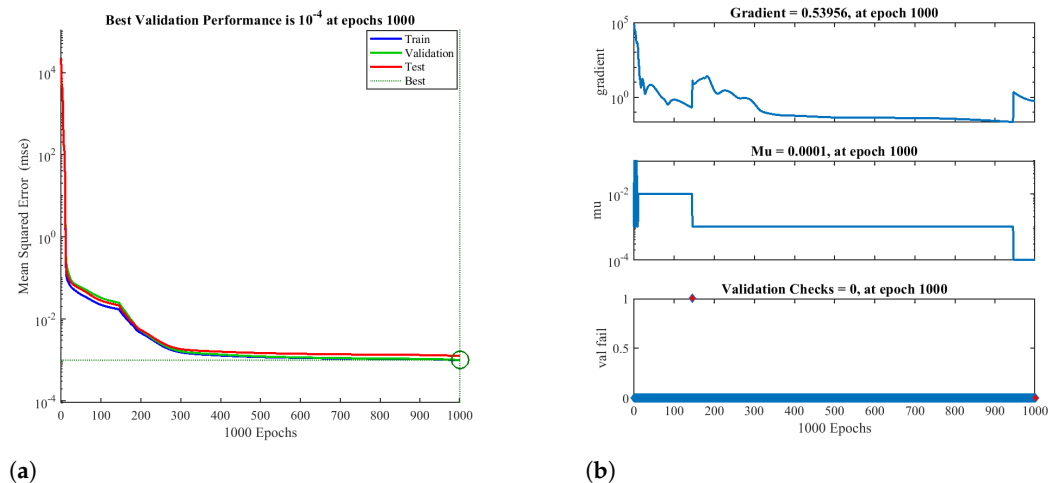


Figure 12. NARX model training and performance metrics: (a) Performance plot of the NARX training process. (b) NARX performance after the end of training.

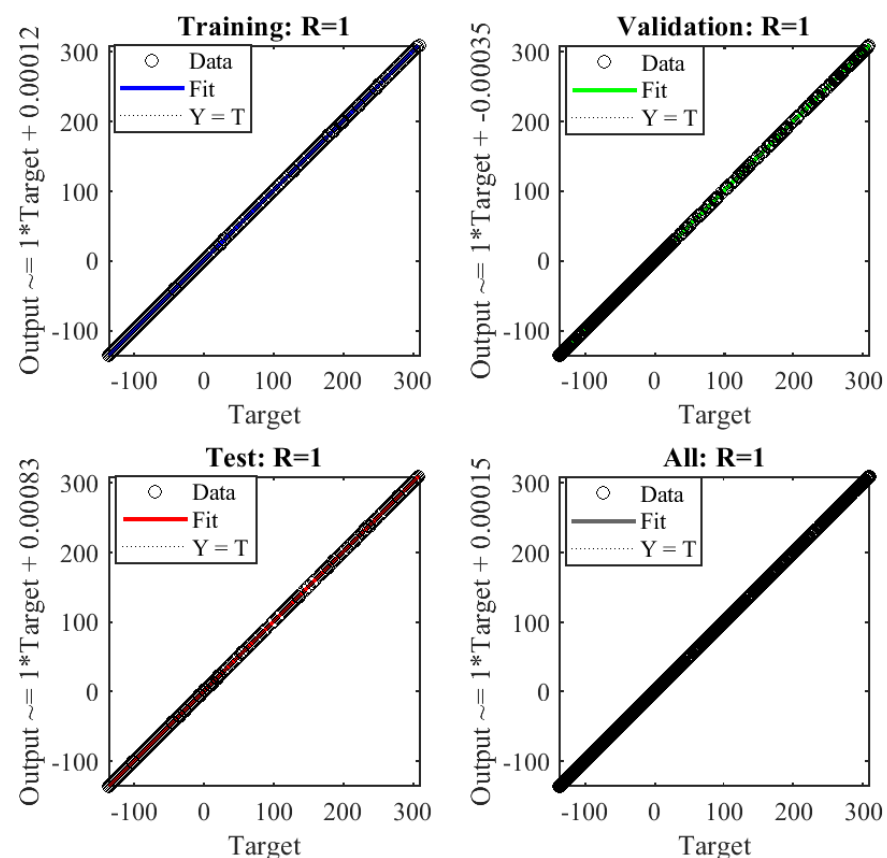


Figure 13. Regression plots between target output and model output.

4.3. Time Series Comparison of the Model

To evaluate the predictive performance of the NARXNN in modeling the chaotic dynamics of the system described by Equation (25), we conducted a time series comparison between the original system outputs and the network's predicted values. This qualitative analysis helps determine how effectively the NARXNN captures the nonlinear temporal evolution of each variable.

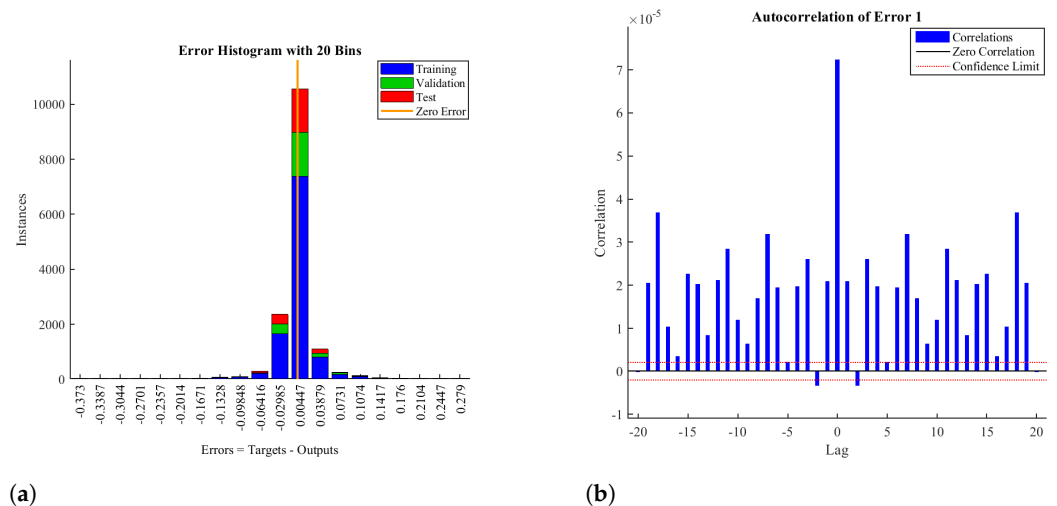


Figure 14. Error distribution and autocorrelation of NARX model errors: (a) Error histogram. (b) Autocorrelation of errors.

Figure 15a shows the time series of variable x_1 , characterized by irregular, high-amplitude oscillations within the range of $[-6, 3]$. These erratic, non-repeating fluctuations are indicative of chaotic behavior. The NARXNN successfully replicates this behavior, closely following the complex waveform and capturing the unpredictable transitions with high fidelity.

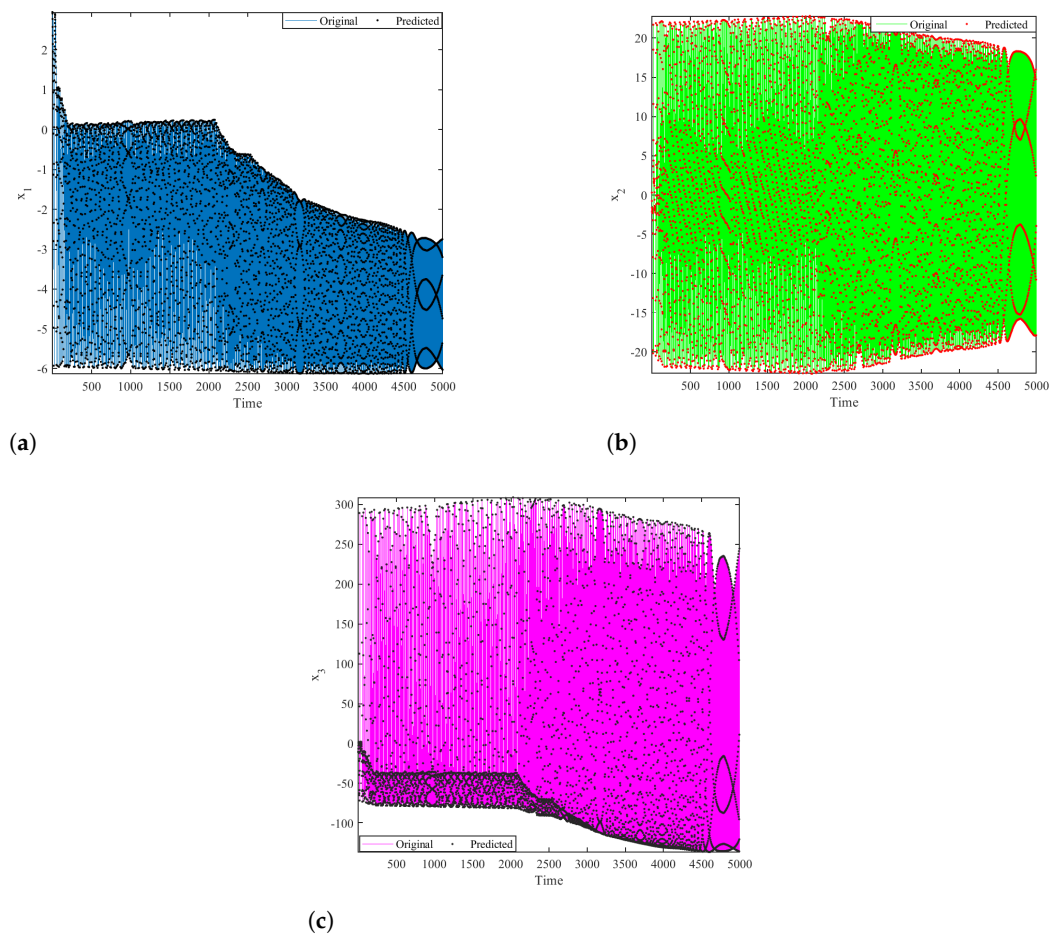


Figure 15. Comparisons of original and predicted values of the state variables: (a) Comparison of time versus x_1 . (b) Comparison of time versus x_2 . (c) Comparison of time versus x_3 .

Similarly, Figure 15b presents the temporal dynamics of x_2 , which exhibits rapid and unpredictable oscillations within the range $[-25, 25]$. The absence of periodicity and the pronounced sensitivity to initial conditions further affirm the chaotic nature of the system. Once again, the NARXNN effectively tracks the true trajectory, providing highly accurate predictions of the system's nonlinear and time-dependent dynamics.

In Figure 15c, the time series of x_3 reveals asymmetric and large-amplitude oscillations ranging between $[-150, 300]$. Despite the wider dynamic range, the predicted values remain well-aligned with the true trajectory, successfully reproducing the system's aperiodic and non-symmetric fluctuations.

These comparisons clearly demonstrate that the NARXNN effectively predicts the chaotic dynamics of the system, accurately capturing the nonlinear interactions and reproducing the system's erratic, sensitive behavior.

4.4. Phase Space Comparison of the Model

To further validate the NARXNN's ability to replicate the chaotic dynamics of the model described by Equation (25), we performed a detailed comparison between the original and predicted trajectories in both 3D and 2D phase spaces. These visual comparisons help assess how accurately the model reconstructs the system's nonlinear evolution and inter-variable relationships in state space.

Figure 16a illustrates the 3D phase trajectory in the (x_1, x_2, x_3) space, where the original trajectory is shown in green and the predicted path by NARXNN in red. The tight overlap between these trajectories highlights the network's ability to reconstruct the complex, irregular geometry characteristic of chaotic motion. A similar agreement is evident in Figure 16c, which shows the projection in the (x_2, x_3, x_1) space, further confirming the model's generalization across different coordinate frames.

A third 3D comparison in Figure 16e, based on the (x_1, x_3, x_2) arrangement, reaffirms this trend, with the predicted trajectory closely following the true system behavior, even in the presence of nonlinear interactions and sensitivity to initial conditions.

Complementing the 3D analysis, 2D trajectory plots further validate the NARXNN's effectiveness. In Figure 16b, the projection onto the (x_1, x_2) plane reveals an excellent match between the predicted and original trajectories, with the black dots (predicted) tightly tracing the magenta curve (original). A similar agreement is observed in Figure 16d, where the (x_1, x_3) comparison shows nearly perfect alignment between the red and green curves. Lastly, Figure 16f illustrates the phase portrait in the (x_2, x_3) plane, where the predicted dynamics again replicate the original system with minimal deviation.

The consistency between predicted and actual trajectories across multiple projections confirms that the NARXNN successfully captures the essential qualitative behavior and intricate nonlinear dynamics of the system. This agreement across high-dimensional phase space representations emphasizes the model's robustness and its strong potential for generalizing chaotic system behavior.

To summarize the configuration and performance of the NARXNN, Table 3 outlines the key parameters used during training. This includes the simulation setup, dataset division, optimization method, and performance metrics. The network was trained using the Levenberg–Marquardt algorithm for 1000 epochs, with the MSE as the loss function. The best validation performance was achieved with an MSE of 10^{-4} , alongside a correlation coefficient $R = 1$, and a minimal, centered error distribution, highlighting the model's precision and stability in modeling chaotic systems.

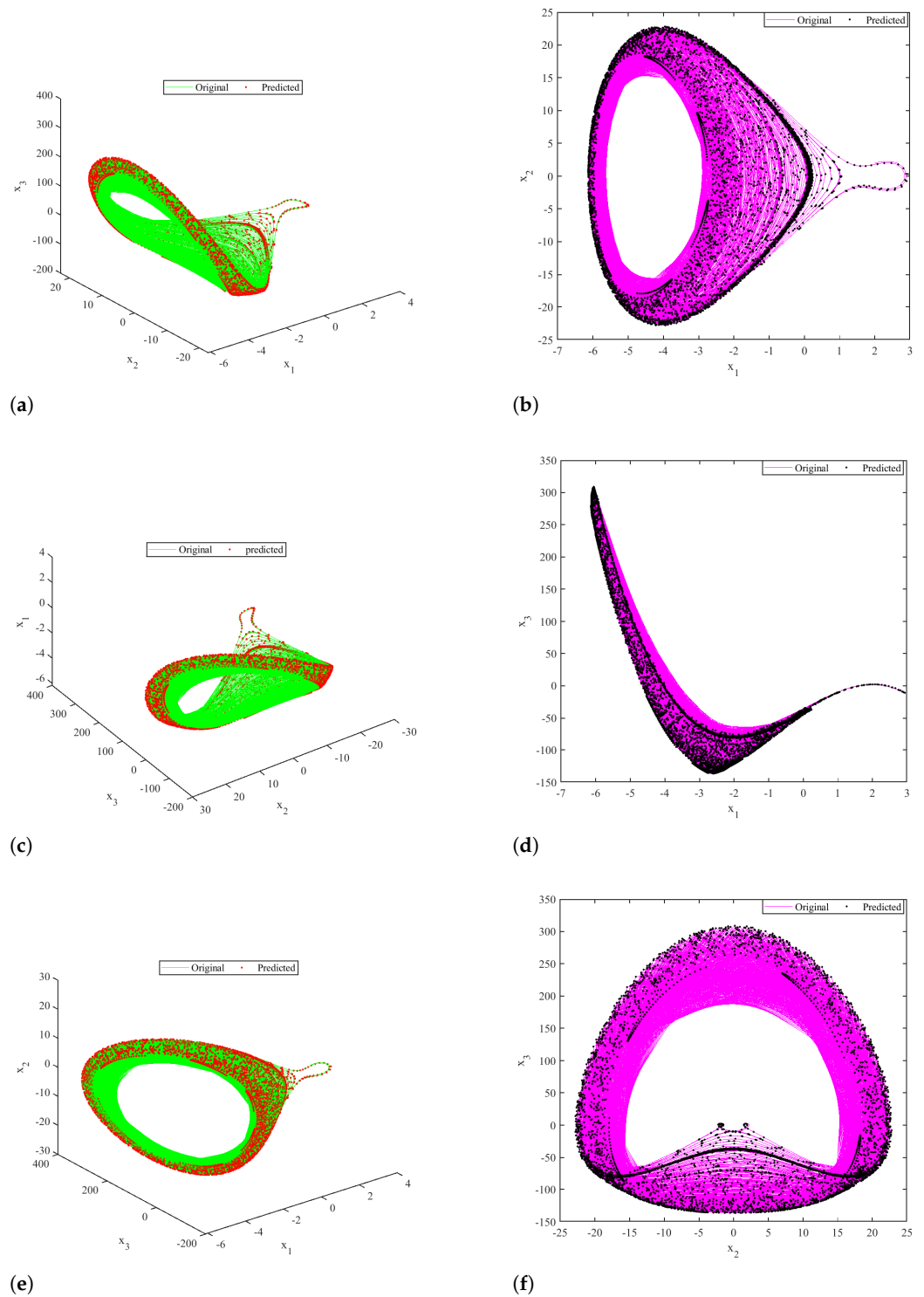


Figure 16. State comparisons of original and predicted values: (a) 3D comparison of states x_1 , x_2 , and x_3 . (b) 2D comparison of states x_1 and x_2 . (c) 3D comparison of states x_2 , x_3 , and x_1 . (d) 2D comparison of states x_1 and x_3 . (e) 3D comparison of states x_1 , x_3 , and x_2 . (f) 2D comparison of states x_2 and x_3 .

Table 3. Model Parameters for NARXNN Training.

Parameter	Description	Value
Time Interval	Time range for system simulation	[0, 500]
Time Step	Time step for the numerical simulation	0.1
Dataset Size	Total number of data points for each state variable	5000
Training Data	Percentage of the dataset used for training	70%
Validation Data	Percentage of the dataset used for validation	15%
Testing Data	Percentage of the dataset used for testing	15%
Epochs	Number of training epochs	1000
Loss Function	Loss function used during training	Mean Squared Error (MSE)
Training Algorithm	Optimization algorithm for training the model	Levenberg–Marquardt
Learning Rate	Learning rate during training (adaptive)	0.0001
Gradient at Epoch 1000	Gradient value indicating convergence	0.53956
Best Validation Performance	Validation performance at epoch 1000	MSE = 10^{-4}
Correlation Coefficient (R)	Regression correlation between predicted and target values	$R = 1$
Error Distribution	Distribution of prediction errors	Minimal errors, centered around zero
Autocorrelation of Errors	Autocorrelation of residuals	Randomly distributed, no significant bias

5. Conclusions

This study presents a detailed investigation into the nonlinear dynamics of the modified Korteweg–de Vries–Zakharov–Kuznetsov equation, combining classical analytical techniques with modern computational intelligence to form a robust framework for analyzing and predicting chaotic systems. By integrating the qualitative insights of dynamical systems theory with the predictive power of artificial neural networks, this work offers a powerful hybrid methodology for exploring nonlinear partial differential equations.

To begin, the mKdV-ZK equation was systematically reduced to an ordinary differential equation using a traveling wave transformation, enabling tractable analysis of solitary wave solutions. Reformulated as a dynamical system, the ODE was analyzed in detail to identify equilibrium points, determine their stability through eigenvalue analysis, and characterize the system’s phase portraits. This revealed a wide variety of nonlinear wave patterns, including periodic, quasi-periodic, homoclinic, heteroclinic, and unbounded trajectories, that shed light on the system’s long-term behavior and stability. To rigorously assess the presence of chaos, the study employed several numerical diagnostics, including time series plots, Lyapunov exponents, bifurcation diagrams, sensitivity to initial conditions, and Poincaré sections. The emergence of positive Lyapunov exponents and significant trajectory divergence confirmed the system’s chaotic nature across multiple parameter regimes. Moreover, the analysis revealed key bifurcation phenomena, such as fold bifurcations, fold–Hopf interactions, and triplet bifurcations, which play a crucial role in initiating and transitioning between dynamic regimes.

To model and predict the system’s chaotic dynamics, a Nonlinear Autoregressive Exogenous Neural Network was developed and trained. The Nonlinear Autoregressive Exogenous effectively learned the system’s intricate dynamics from time series data, showing strong generalization and minimal prediction error. Its architecture allowed it to retain memory of past states while incorporating external inputs, making it particularly well-suited for capturing chaotic, time-dependent behavior. The close agreement between the predicted and original trajectories in various 2D and 3D phase spaces further confirmed the network’s predictive power.

In summary, this study provides significant insights into the dynamics and chaotic behavior of the modified Korteweg–de Vries–Zakharov–Kuznetsov equation. It showcases the potential of a hybrid analytical–computational framework for studying complex nonlinear systems. The approach developed here opens promising avenues for future investigations of chaotic and nonlinear phenomena in a wide range of scientific and engineering applications.

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