

Article

Bayesian Prediction of Future Network Latency Times Under Two-Parameter Exponential Current Records

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Abstract

Modeling and predicting extreme observations in network latency data is an important task for monitoring the performance and reliability of communication systems. In this paper, we develop a comprehensive inferential and predictive framework for network latency times using upper and lower current record values from the two-parameter exponential distribution. First, explicit expressions for the probability density functions and cumulative distribution functions of the lower and upper current records are derived. Closed-form expressions for the r -th moments of these records are also obtained. Parameter estimation for the location and scale parameters is then investigated using both maximum likelihood estimation and Bayesian estimation. Because the Bayesian estimators do not admit closed-form solutions, a Markov Chain Monte Carlo approach based on the Metropolis–Hastings algorithm is employed to compute the posterior summaries. To forecast future latency behavior, classical predictive intervals and Bayesian predictive procedures are developed for future upper and lower current records as well as for future record ranges. The Bayesian framework additionally provides Bayesian predictive values and Bayesian predictive intervals that incorporate parameter uncertainty. The performance of the proposed procedures is evaluated through a simulation study. Finally, the methodology is applied to real network latency time data from Saudi Arabia. The results demonstrate that record-based inference combined with Bayesian prediction provides effective tools for modeling and forecasting extreme latency observations in modern communication networks.

Keywords: statistical computing; network latency times; two-parameter exponential current records; Bayesian estimation and prediction; Saudi network data



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1. Introduction

Statistical inference plays a central role in predicting computer science data, a field often referred to as statistical computing, which integrates advanced statistical theory with computational algorithms to extract meaningful patterns from complex, large-scale, and high-dimensional datasets. In modern computer science applications—such as machine learning, artificial intelligence, cybersecurity, network analysis, cloud systems, and big data analytics—massive volumes of structured and unstructured data are generated continuously, requiring robust inferential frameworks to model uncertainty, estimate parameters, test hypotheses, and generate reliable predictions. Statistical computing relies on probabilistic models, including regression models, Bayesian inference, Markov chain Monte

Carlo (MCMC) simulations, stochastic processes, and time-series analysis, to analyze algorithm performance, predict system failures, optimize resource allocation, detect anomalies, and enhance decision-making under uncertainty. For example, in predictive modeling for network traffic, inferential techniques allow researchers to estimate unknown distribution parameters and forecast future loads, while in cybersecurity, statistical anomaly detection methods help identify irregular patterns that may signal intrusions or attacks. The integration of computational tools such as numerical optimization, simulation techniques, and high-performance computing environments enables the practical implementation of complex inferential procedures that would otherwise be mathematically intractable. Moreover, statistical computing emphasizes reproducibility, scalability, and efficiency through programming languages and environments such as Mathematica, R, and Python, which provide extensive libraries for data manipulation, visualization, and predictive modeling. By combining theoretical inference with algorithmic implementation, statistical computing transforms raw computer science data into actionable insights, improves model accuracy, quantifies prediction uncertainty, and supports evidence-based technological innovation across domains ranging from software engineering to artificial intelligence systems.

Statistical computing has emerged as a central discipline integrating statistical inference with advanced computational methodologies to address complex, large-scale, and high-dimensional problems in computer science. The evolution and mutual development of statistics and computation are comprehensively discussed in *Statistics and Computing: The Genesis of Data Science* [1], which highlights how algorithmic advances have reshaped modern data-driven scientific inquiry. Foundational treatments of computational techniques for inference are provided in *Computational Statistics* [2], which systematically develops simulation-based methods and numerical algorithms for statistical analysis. More recently, *Statistical Modeling and Computation* [3] integrates inferential modeling with contemporary computational tools, emphasizing Bayesian analysis and Monte Carlo methods for predictive applications. In the context of large-scale and machine learning problems, Bayesian approaches linking statistical inference with modern learning theory have been explored in [4], while scalable procedures for distributed computing, subsampling, and minibatch optimization have been systematically reviewed in [5]. The fundamental interplay between statistical accuracy and computational complexity has been theoretically examined through convex relaxation frameworks in [6], providing insight into the limits of efficient large-data inference. Furthermore, probabilistic numerical methods that embed uncertainty quantification directly into computational algorithms have been developed in [7], strengthening the theoretical and practical foundations of statistical computing for predictive modeling and uncertainty assessment in modern computer science applications.

One important application of statistical inference in computer science arises in the prediction and monitoring of network latency times, where the stochastic behavior of delay measurements can be effectively modeled using continuous lifetime distributions. In many real-world communication systems, latency values exhibit right-skewed behavior and memoryless characteristics over short intervals, making the two-parameter exponential distribution a suitable probabilistic model for describing their variability. By focusing on upper and lower current records, it becomes possible to extract critical information about extreme delays that directly affect quality of service, congestion control, and system reliability. Statistical analysis of current records provides efficient estimates of the location and scale parameters of the two-parameter exponential model and enables predictive inference about future extreme latency events. Such record-based modeling is particularly attractive in high-frequency network environments because it reduces data storage requirements and emphasizes the most informative observations, especially those associated with peak delays.

Through likelihood-based and Bayesian inferential procedures, one can derive estimators, confidence intervals, and predictive distributions for upcoming record-breaking latency times, thereby enabling proactive network management and performance optimization under uncertainty.

Although communication network latency may exhibit heavy-tailed behavior under certain operating conditions, particularly during periods of severe congestion or highly bursty traffic, the present study considers the two-parameter exponential distribution as the underlying probabilistic model. This choice is motivated by two considerations. First, the exponential model provides a mathematically tractable framework that enables closed-form derivation of the distributions, moments, estimation procedures, and predictive intervals for current records. Second, for latency measurements collected over relatively short and stable observation periods, the exponential distribution provides a reasonable approximation of the observed variability and allows the proposed record-based methodology to be developed in a complete analytical form.

The novelty of our paper lies in developing a comprehensive inferential and predictive framework based specifically on upper and lower current record values from a two-parameter exponential distribution for modeling network latency times. Unlike traditional approaches that rely on complete samples or aggregate summaries, our methodology leverages record statistics to construct parameter estimation procedures and predictive models tailored to extreme latency behavior. The main purpose of the paper is therefore to derive the theoretical properties of the estimators, establish predictive distributions for future record latency times, and demonstrate how record-based inference can enhance reliability assessment and decision-making in modern computer network systems. The study is especially pertinent to the Kingdom of Saudi Arabia's Vision 2030 framework, which includes numerous national objectives.

Record statistics have attracted considerable attention in statistical theory because they arise naturally in situations where only extreme observations are observed over time. In particular, current record values provide a useful framework for studying sequential extremes without requiring the full data sequence. The mathematical properties of current records, including their distributional characteristics, moments, and recurrence relations, have been investigated under various probability models. These theoretical developments provide the foundation for statistical inference based on record data; see, for example, the contributions of Barakat et al. [8], Raqab [9,10], and Ahmadi and Balakrishnan [11]. Extensions of these results to more generalized record structures, such as (k) -record models, were later discussed by Chahkandi and Ahmadi [12].

Another important direction in the literature focuses on prediction and inference using record observations. Distribution-free and parametric prediction intervals for future record values have been studied extensively, particularly in the context of order statistics and sequential extremes. Several works have addressed the construction of prediction bounds and the statistical properties of estimators derived from record samples; see, for instance, Barakat et al. [13], Raqab [14], and Ahmadi and Balakrishnan [15]. These approaches highlight the usefulness of record-based inference when only partial information about the underlying sample is available.

More recently, record-based methodologies have been applied to a range of parametric distributions and real datasets, demonstrating their practical value in modeling extreme observations. Examples include studies involving Laplace, Kumaraswamy, and generalized exponential distributions, where current record statistics have been used to estimate model parameters and to construct prediction intervals for future record values [16–20]. A comprehensive overview of the theory and applications of current records can be found in the recent survey by Aldallal and Barakat [21].

For a series $\{X_\xi\}$, we use the following notation in this paper: At the ξ th record event, let $L_\xi^{c_r}$ and $U_\xi^{c_r}$ represent the lower and upper current records, respectively. By definition, $L_{\xi+1}^{c_r} = L_\xi^{c_r}$ for new upper records $U_{\xi+1}^{c_r} > U_\xi^{c_r}$, and $U_{\xi+1}^{c_r} = U_\xi^{c_r}$ for new lower records $L_{\xi+1}^{c_r} < L_\xi^{c_r}$. With initial values $U_0^{c_r} = L_0^{c_r} = X_1$, $L_\xi^{c_r}$ thus denotes the minimal observation up to the ξ th record point. For $\xi \geq 1$, the current record range is defined as $R_\xi^{c_r} = U_\xi^{c_r} - L_\xi^{c_r}$, and the interval $(L_\xi^{c_r}, U_\xi^{c_r})$ is referred to as the record coverage. Every time a new upper or lower current record is set, a new range is noted.

2. PDF and CDF of Two-Parameter Exponential Current Records

Understanding the probabilistic structure of network latency measurements is essential for developing reliable statistical models and prediction procedures. In particular, the probability density function (PDF) and cumulative distribution function (CDF) describe the distribution of latency times and provide the foundation for deriving statistical properties of record observations. These functions play a central role in parameter estimation, inference, and the prediction of future extreme latency events.

Since the present study focuses on inference based on current record values, it is necessary to derive the corresponding distributional forms associated with these records. When the underlying latency observations follow a two-parameter exponential distribution, the distributional behavior of the lower and upper current records can be obtained from the baseline model. Therefore, in this section, we derive the CDF and PDF of the lower and upper current record values under the assumption that the original data follow a two-parameter exponential distribution.

The probability density function and cumulative distribution function of the two-parameter exponential distribution with location parameter $\theta > 0$ and scale parameter $\beta > 0$ are given, respectively, by

$$f(x; \theta, \beta) = \frac{1}{\beta} \exp\left(-\frac{x - \theta}{\beta}\right), \quad x \geq \theta, \quad (1)$$

$$F(x; \theta, \beta) = 1 - \exp\left(-\frac{x - \theta}{\beta}\right), \quad x \geq \theta. \quad (2)$$

The parameter θ shifts the distribution along the horizontal axis. Statistically, it represents a threshold or minimum value below which observations cannot occur. Increasing θ shifts the entire distribution to the right without changing its shape.

Moreover, the parameter β controls the scale of the distribution:

- Larger values of β produce a more spread-out distribution with a heavier right tail.
- Smaller values of β concentrate the distribution more closely around θ .
- The mean of the distribution is $\theta + \beta$.
- The variance is β^2 .

The two-parameter exponential distribution is a special case of the three-parameter Weibull distribution when the shape parameter equals one. It is widely used for modeling waiting times, failure times, reliability data, and lifetimes under the assumption of a constant hazard rate.

Theorem 1. For $\xi > 1$, the CDF and PDF of the lower (upper) current records, when the original data follows a two-parameter exponential distribution with location parameter $\theta > 0$ and scale parameter $\beta > 0$, are respectively given by

$$F_{L_\xi^{c_r}}(x) = 1 + 2^\xi \left(1 - e^{-\frac{x - \theta}{\beta}}\right) P\left(\xi, -\log\left(1 - e^{-\frac{x - \theta}{\beta}}\right)\right) - P\left(\xi, -2\log\left(1 - e^{-\frac{x - \theta}{\beta}}\right)\right), \quad (3)$$

$$f_{L_{\xi}^{cr}}(x) = \frac{2^{\xi}}{\beta} e^{-\frac{x-\theta}{\beta}} P\left(\xi, -\log\left(1 - e^{-\frac{x-\theta}{\beta}}\right)\right), \quad (4)$$

$$F_{U_{\xi}^{cr}}(x) = P\left(\xi, 2\left(\frac{x-\theta}{\beta}\right)\right) - 2^{\xi} e^{-\frac{x-\theta}{\beta}} P\left(\xi, \frac{x-\theta}{\beta}\right), \quad (5)$$

$$f_{U_{\xi}^{cr}}(x) = \frac{2^{\xi}}{\beta} e^{-\frac{x-\theta}{\beta}} P\left(\xi, \frac{x-\theta}{\beta}\right), \quad (6)$$

where $x > \theta$, and $P(a, b) = \gamma(a, b) / \Gamma(a)$ denotes the regularized lower Gamma function.

Proof. Let X follow a two-parameter exponential distribution with CDF and PDF given by (1) and (2). To derive the CDF and PDF of the current records, we begin with the general formulas of Houchens:

$$F_{L_{\xi}^{cr}}(x) = 2^{\xi} F(x) \left\{ 1 - F(x) \sum_{\varphi=0}^{\xi-1} 2^{-(\varphi+1)} \sum_{\eta=0}^{\varphi} \frac{[-2 \log F(x)]^{\eta}}{\eta!} \right\}, \quad (7)$$

$$f_{L_{\xi}^{cr}}(x) = 2^{\xi} f(x) \left\{ 1 - F(x) \sum_{\varphi=0}^{\xi-1} \frac{[-\log F(x)]^{\varphi}}{\varphi!} \right\}, \quad (8)$$

$$F_{U_{\xi}^{cr}}(x) = 1 - 2^{\xi} \bar{F}(x) \left\{ 1 - \bar{F}(x) \sum_{\varphi=0}^{\xi-1} 2^{-(\varphi+1)} \sum_{\eta=0}^{\varphi} \frac{[-2 \log \bar{F}(x)]^{\eta}}{\eta!} \right\}, \quad (9)$$

$$f_{U_{\xi}^{cr}}(x) = 2^{\xi} f(x) \left\{ 1 - \bar{F}(x) \sum_{\varphi=0}^{\xi-1} \frac{[-\log \bar{F}(x)]^{\varphi}}{\varphi!} \right\}, \quad (10)$$

where $\bar{F}(x) = 1 - F(x)$.

Lower current records: Substituting $F(x) = 1 - e^{-z}$ into (7) and defining

$$z = \frac{x - \theta}{\beta},$$

we obtain

$$F_{L_{\xi}^{cr}}(x) = 2^{\xi} (1 - e^{-z}) \left[1 - (1 - e^{-z}) \sum_{\varphi=0}^{\xi-1} 2^{-(\varphi+1)} \sum_{\eta=0}^{\varphi} \frac{[-2 \log(1 - e^{-z})]^{\eta}}{\eta!} \right].$$

Let

$$A = \sum_{\varphi=0}^{\xi-1} 2^{-(\varphi+1)} \sum_{\eta=0}^{\varphi} \frac{[-2 \log(1 - e^{-z})]^{\eta}}{\eta!}.$$

Changing the order of summation yields

$$A = \sum_{\eta=0}^{\xi-1} \frac{[-\log(1 - e^{-z})]^{\eta}}{\eta!} - 2^{-\xi} \sum_{\eta=0}^{\xi-1} \frac{[-2 \log(1 - e^{-z})]^{\eta}}{\eta!}.$$

Using the identity

$$\sum_{r=0}^{\xi-1} \frac{w^r}{r!} = e^w (1 - P(\xi, w)),$$

with $w = -\log(1 - e^{-z})$ and $w = -2 \log(1 - e^{-z})$, yields after simplification

$$F_{L_{\xi}^{cr}}(x) = 1 + 2^{\xi} (1 - e^{-z}) P(\xi, -\log(1 - e^{-z})) - P(\xi, -2 \log(1 - e^{-z})).$$

Which proves (3), and differentiating with respect to x and using

$$\frac{d}{du}P(\xi, u) = \frac{u^{\xi-1}e^{-u}}{\Gamma(\xi)}, \quad u = -\log(1 - e^{-z}), \quad \frac{dz}{dx} = \frac{1}{\beta},$$

gives

$$f_{L_{\xi}^{cr}}(x) = \frac{2^{\xi}}{\beta} e^{-z} P(\xi, -\log(1 - e^{-z})),$$

which proves (4).

Upper current records: For the upper records,

$$\bar{F}(x) = e^{-z}, \quad -\log \bar{F}(x) = z.$$

Substituting into (9) and simplifying using the same summation identity gives

$$F_{U_{\xi}^{cr}}(x) = P(\xi, 2z) - 2^{\xi} e^{-z} P(\xi, z),$$

which proves (5). Finally, substituting into (10) gives

$$f_{U_{\xi}^{cr}}(x) = \frac{2^{\xi}}{\beta} e^{-z} \left[1 - e^{-z} \sum_{\varphi=0}^{\xi-1} \frac{z^{\varphi}}{\varphi!} \right].$$

Using

$$\sum_{\varphi=0}^{\xi-1} \frac{z^{\varphi}}{\varphi!} = e^z (1 - P(\xi, z)),$$

we obtain

$$f_{U_{\xi}^{cr}}(x) = \frac{2^{\xi}}{\beta} e^{-z} P(\xi, z),$$

which proves (6). $\square$

Figures 1–4 show the results of testing various values of θ , β , and ξ to illustrate the forms of the PDFs and CDFs of the upper and lower current records found in Theorem 1.

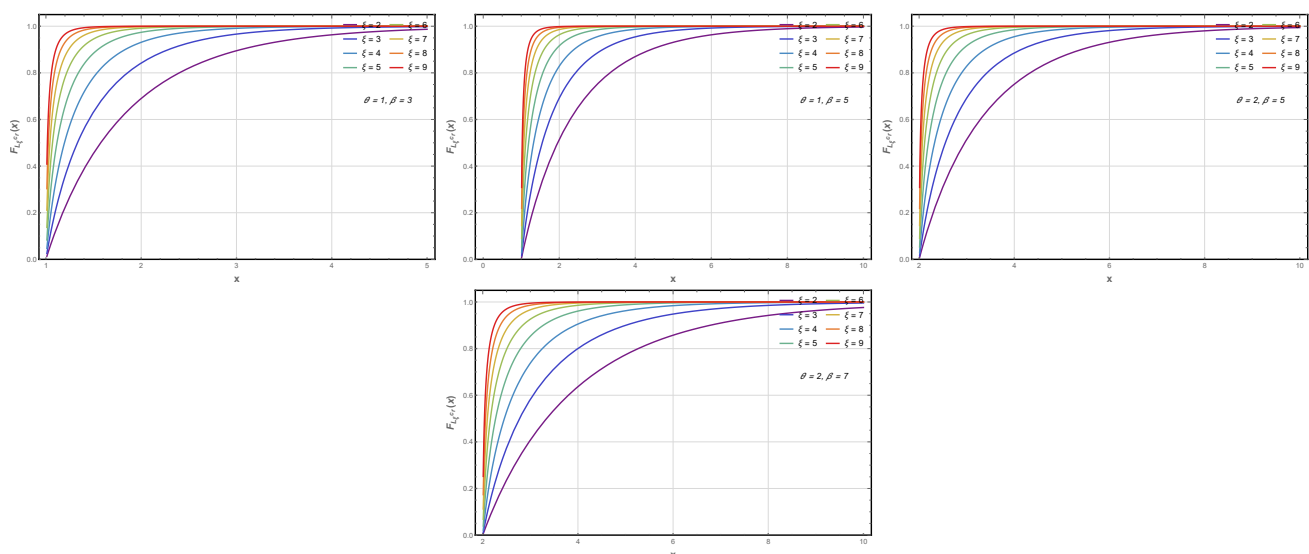


Figure 1. Plot of $F_{L_{\xi}^{cr}}(x; \theta, \beta)$ for different values of ξ , θ , and β .

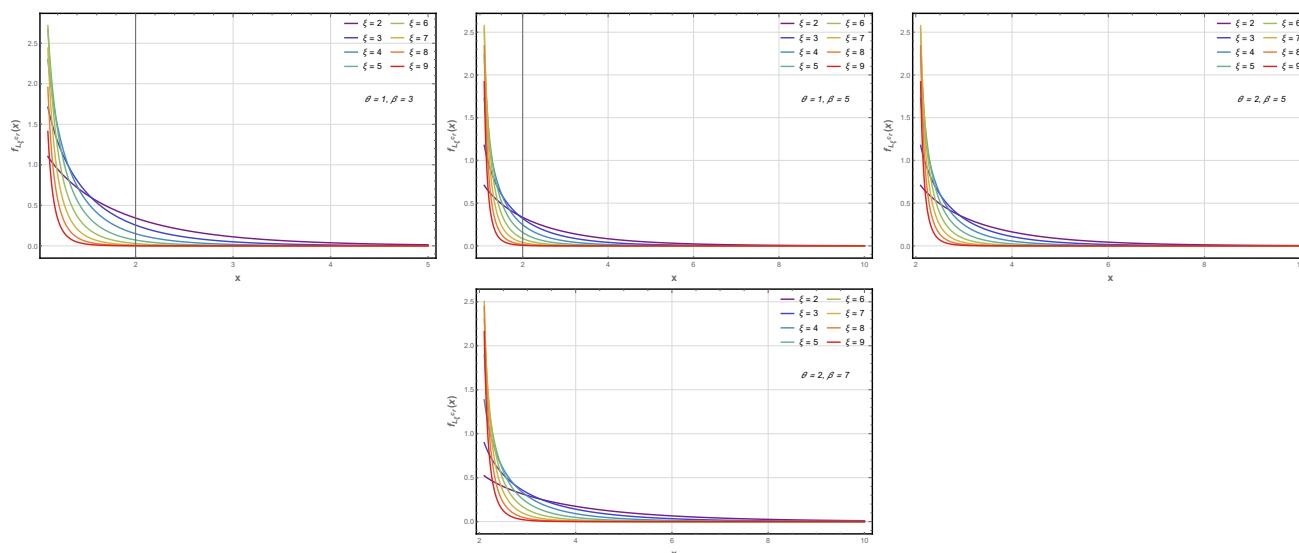


Figure 2. Plot of $f_{L_{\xi}^{cr}}(x; \theta, \beta)$ for different values of ξ , θ , and β .

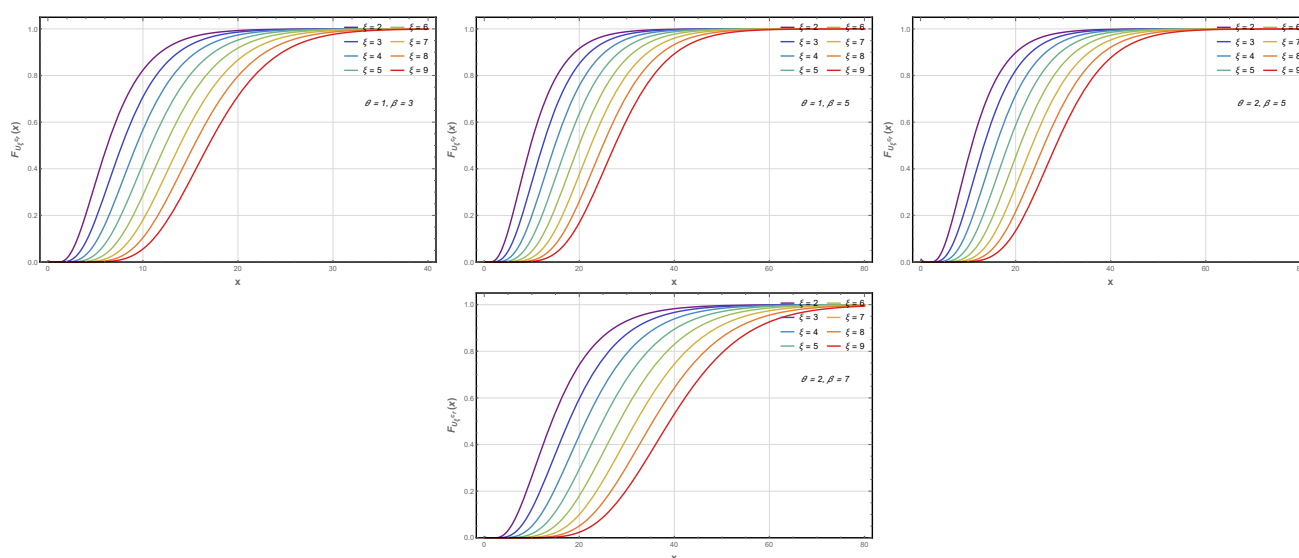


Figure 3. Plot of $F_{U_{\xi}^{cr}}(x; \theta, \beta)$ for different values of ξ , θ , and β .

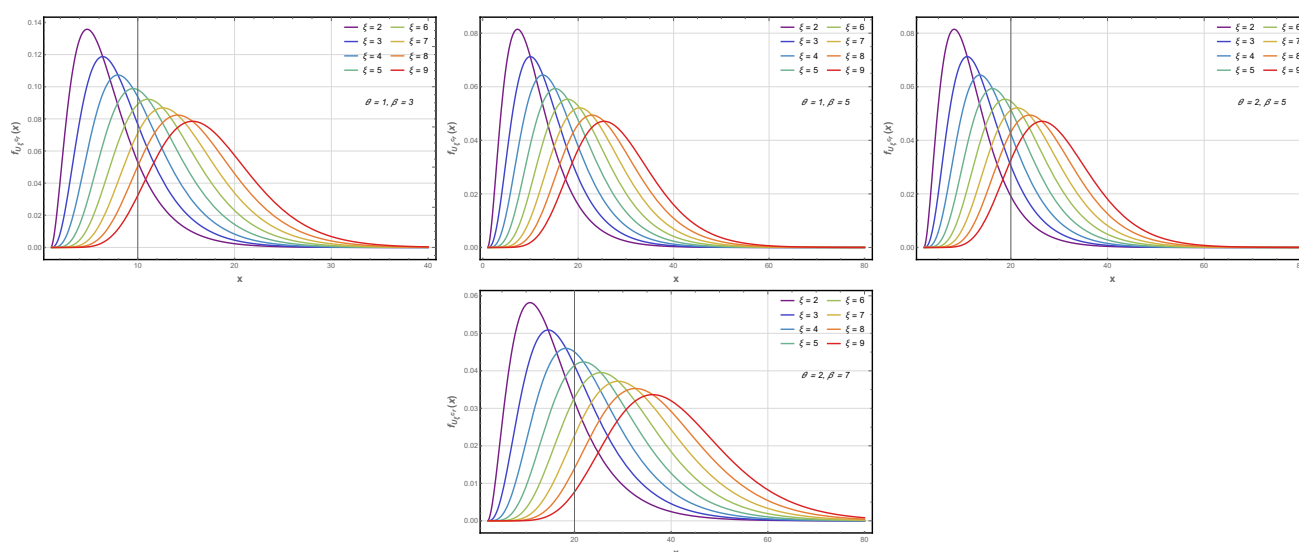


Figure 4. Plot of $f_{U_{\xi}^{cr}}(x; \theta, \beta)$ for different values of ξ , θ , and β .

Houchens [22] also obtained the joint PDF of two lower current records and two upper current records as

$$f_{L_{\xi}^{c_r}, L_{\tau}^{c_r}}(l_{\xi}, l_{\tau}) = \frac{f(l_{\xi})f(l_{\tau})}{F(l_{\xi})(\tau - \xi - 1)!} \left(H(l_{\xi}) - H(l_{\tau}) \right)^{\tau - \xi - 1}, \quad l_{\tau} < l_{\xi}, \quad 1 < \xi < \tau, \quad (11)$$

where $H(x) = \log(F(x))$.

Similarly,

$$f_{U_{\xi}^{c_r}, U_{\tau}^{c_r}}(u_{\xi}, u_{\tau}) = \frac{f(u_{\xi})f(u_{\tau})}{(1 - F(u_{\xi}))(\tau - \xi - 1)!} \left(R(u_{\tau}) - R(u_{\xi}) \right)^{\tau - \xi - 1}, \quad u_{\tau} > u_{\xi}, \quad 1 < \xi < \tau, \quad (12)$$

where $R(x) = -\log(1 - F(x))$. From (11) and (12), the conditional densities of $L_{\tau}^{c_r} \mid L_{\xi}^{c_r} = l_{\xi}$ and $U_{\tau}^{c_r} \mid U_{\xi}^{c_r} = u_{\xi}$ follow immediately as

$$f_{L_{\tau}^{c_r} \mid L_{\xi}^{c_r}}(l_{\tau} \mid l_{\xi}) = \frac{f(l_{\tau})}{F(l_{\xi})(\tau - \xi - 1)!} \left(H(l_{\xi}) - H(l_{\tau}) \right)^{\tau - \xi - 1}, \quad (13)$$

$$f_{U_{\tau}^{c_r} \mid U_{\xi}^{c_r}}(u_{\tau} \mid u_{\xi}) = \frac{f(u_{\tau})}{(1 - F(u_{\xi}))(\tau - \xi - 1)!} \left(R(u_{\tau}) - R(u_{\xi}) \right)^{\tau - \xi - 1}. \quad (14)$$

3. Moments of Two-Parameter Exponential Current Records

Moments provide fundamental measures that describe the probabilistic behavior of a distribution, including its location, variability, and higher-order structural characteristics. In the context of network latency time analysis, the moments of current record values are particularly informative because they capture the statistical behavior of extreme delays and unusually low response times observed over time. Such quantities are important for understanding the variability and expected magnitude of record-breaking latency events that may affect network performance and reliability. In this section, we derive analytical expressions for the r -th moments of the lower and upper current records when the underlying observations follow a two-parameter exponential distribution. These results provide theoretical insight into the distributional properties of record latency times and form a useful foundation for the estimation and predictive procedures developed in the subsequent sections.

Theorem 2. Assume that the random variable X follows a two-parameter exponential distribution with location parameter $\theta > 0$ and scale parameter $\beta > 0$. Let $L_{\xi}^{c_r}$ and $U_{\xi}^{c_r}$ denote the ξ -th lower and upper current record values, respectively, derived from the sequence of observations. Then the r -th moments associated with the lower and upper current records can be expressed as follows:

$$E\left[(L_{\xi}^{c_r})^r\right] = 2^{\xi} \sum_{j=0}^r \binom{r}{j} \theta^{r-j} \beta^j \left[\Gamma(j+1) - \sum_{k=0}^{\xi-1} \sum_{p=0}^{\infty} a_p(j) \frac{1}{(p+j+2)^{k+1}} \right], \quad (15)$$

$$E\left[(U_{\xi}^{c_r})^r\right] = 2^{\xi} \sum_{j=0}^r \binom{r}{j} \theta^{r-j} \beta^j \left[\Gamma(j+1) - \sum_{k=0}^{\xi-1} \frac{\Gamma(j+k+1)}{k! 2^{j+k+1}} \right]. \quad (16)$$

where $a_p(j)$ are the coefficients from

$$[-\ln(1-y)]^j = \sum_{p=0}^{\infty} a_p(j) y^{p+j}, \quad |y| < 1.$$

Proof. The lower and upper current records' r -th moment is, by definition,

$$E[(L_{\xi}^{c_r})^r] = \int_{\theta}^{\infty} x^r f_{L_{\xi}^{c_r}}(x; \theta, \beta) dx, \quad E[(U_{\xi}^{c_r})^r] = \int_{\theta}^{\infty} x^r f_{U_{\xi}^{c_r}}(x; \theta, \beta) dx.$$

Using the previously derived PDFs (4) and (6) and introducing the transformation

$$z = \frac{x - \theta}{\beta}, \quad x = \theta + \beta z, \quad dx = \beta dz,$$

we obtain

$$E[(L_{\xi}^{c_r})^r] = 2^{\xi} \int_0^{\infty} (\theta + \beta z)^r e^{-z} P(\xi, -\log(1 - e^{-z})) dz,$$

$$E[(U_{\xi}^{c_r})^r] = 2^{\xi} \int_0^{\infty} (\theta + \beta z)^r e^{-z} P(\xi, z) dz.$$

Expanding $(\theta + \beta z)^r$ using the binomial theorem gives

$$(\theta + \beta z)^r = \sum_{j=0}^r \binom{r}{j} \theta^{r-j} \beta^j z^j.$$

Substituting this expansion into the integrals yields

$$E[(L_{\xi}^{c_r})^r] = 2^{\xi} \sum_{j=0}^r \binom{r}{j} \theta^{r-j} \beta^j \int_0^{\infty} z^j e^{-z} P(\xi, -\log(1 - e^{-z})) dz,$$

$$E[(U_{\xi}^{c_r})^r] = 2^{\xi} \sum_{j=0}^r \binom{r}{j} \theta^{r-j} \beta^j \int_0^{\infty} z^j e^{-z} P(\xi, z) dz.$$

Lower current records: Let

$$y = 1 - e^{-z} \implies z = -\ln(1 - y), \quad dz = \frac{dy}{1 - y}.$$

Then

$$e^{-z} dz = dy, \quad z^j = [-\ln(1 - y)]^j.$$

Hence, the integral becomes

$$\int_0^{\infty} z^j e^{-z} P(\xi, -\log(1 - e^{-z})) dz = \int_0^1 [-\ln(1 - y)]^j P(\xi, -\ln y) dy.$$

Expand the regularized gamma function For integer ξ ,

$$P(\xi, -\ln y) = 1 - y \sum_{k=0}^{\xi-1} \frac{(-\ln y)^k}{k!}.$$

Thus, the integral splits as

$$\int_0^1 [-\ln(1 - y)]^j dy - \sum_{k=0}^{\xi-1} \frac{1}{k!} \int_0^1 [-\ln(1 - y)]^j y (-\ln y)^k dy.$$

Using the series expansion

$$[-\ln(1 - y)]^j = \sum_{p=0}^{\infty} a_p(j) y^{p+j}, \quad |y| < 1,$$

we have

$$\int_0^1 [-\ln(1-y)]^j y (-\ln y)^k dy = \sum_{p=0}^{\infty} a_p(j) \int_0^1 y^{p+j+1} (-\ln y)^k dy = \sum_{p=0}^{\infty} a_p(j) \frac{k!}{(p+j+2)^{k+1}}.$$

Substituting back, we obtain (15).

Upper current records: Expand the regularized gamma function; hence, the integral splits as

$$\int_0^{\infty} z^j e^{-z} dz - \sum_{k=0}^{\xi-1} \frac{1}{k!} \int_0^{\infty} z^j e^{-2z} z^k dz.$$

The first integral is

$$\int_0^{\infty} z^j e^{-z} dz = \Gamma(j+1).$$

While the second integral is

$$\int_0^{\infty} z^{j+k} e^{-2z} dz = \frac{\Gamma(j+k+1)}{2^{j+k+1}}.$$

substituting back gives the closed-form series (16). This completes the proof. $\square$

4. Parameter Estimation of Network Latency Times Data

In practical applications involving network latency measurements, the location parameter θ and the scale parameter β of the two-parameter exponential distribution are typically unknown and must be estimated from the observed data. Accurate estimation of these parameters is essential because they determine the baseline latency level, the variability of delay measurements, and the probabilistic behavior of extreme latency events that directly influence network performance and reliability.

When inference is based on current record values rather than complete samples, the estimation problem becomes more specialized, since record observations primarily capture extreme behavior in the data. To address this, we consider two widely used inferential approaches: maximum likelihood estimation (MLE) and Bayesian estimation (BE). The MLE method relies solely on observed data to obtain parameter estimates via likelihood maximization, whereas the Bayesian approach incorporates prior knowledge alongside the observed data to produce posterior estimates that account for parameter uncertainty. These estimation procedures provide the foundation for the predictive modeling of future latency records developed in the subsequent sections.

4.1. Maximum Likelihood Estimation (MLE)

Let $U_1^{cr}, U_2^{cr}, \dots, U_{\xi}^{cr}$ denote the first ξ upper current record values. The joint PDF definition for current records given by [13] is

$$f_{U_1^{cr}, U_2^{cr}, \dots, U_{\xi}^{cr}}(x_1, x_2, \dots, x_{\xi}) = \frac{\bar{F}^2(x_{\xi}; \theta, \beta) F(x_1; \theta, \beta)}{\bar{F}(x_1; \theta, \beta)} \left(\prod_{j=1}^{\xi} \frac{2f(x_j; \theta, \beta)}{\bar{F}(x_j; \theta, \beta)} \right), \quad x_1 < x_2 < \dots < x_{\xi}, \quad (17)$$

For observed two-parameter exponential upper current records, the likelihood function becomes

$$L(\theta, \beta \mid U^{cr}) = \frac{2^{\xi}}{\beta^{\xi}} \left(e^{(x_1 - \theta)/\beta} - 1 \right) e^{-2(x_{\xi} - \theta)/\beta}, \quad (18)$$

and the log-likelihood is

$$\ell(\theta, \beta \mid U^{cr}) = \xi \ln 2 - \xi \ln \beta + \ln \left(e^{(x_1 - \theta)/\beta} - 1 \right) - 2 \frac{x_{\xi} - \theta}{\beta}. \quad (19)$$

Derivative with respect to θ

$$\frac{\partial \ell}{\partial \theta} = -\frac{1}{\beta} \frac{e^{(x_1-\theta)/\beta}}{e^{(x_1-\theta)/\beta} - 1} + \frac{2}{\beta} = 0 \Rightarrow \frac{e^{(x_1-\theta)/\beta}}{e^{(x_1-\theta)/\beta} - 1} = 2 \Rightarrow \hat{\theta} = x_1 - \beta \ln 2.$$

Derivative with respect to β

$$\frac{\partial \ell}{\partial \beta} = -\frac{\xi}{\beta} - \frac{x_1 - \theta}{\beta^2} \frac{e^{(x_1-\theta)/\beta}}{e^{(x_1-\theta)/\beta} - 1} + \frac{2(x_\xi - \theta)}{\beta^2} = 0.$$

Substituting $\frac{e^{(x_1-\theta)/\beta}}{e^{(x_1-\theta)/\beta} - 1} = 2$ gives

$$-\xi\beta - 2(x_1 - \theta) + 2(x_\xi - \theta) = 0 \Rightarrow \hat{\beta} = \frac{2}{\xi}(x_\xi - x_1).$$

Finally, the MLEs of the parameters based on the upper current records are

$$\hat{\beta}_{U^{cr}}^{ML} = \frac{2}{\xi}(x_\xi - x_1), \quad \hat{\theta}_{U^{cr}}^{ML} = x_1 - \hat{\beta}_{U^{cr}}^{ML} \ln 2. \quad (20)$$

Now, let $L_1^{cr}, L_2^{cr}, \dots, L_\xi^{cr}$ denote the first ξ lower current record values. Using the joint PDF definition for lower current records from [13], which is given by

$$f_{L_1^{cr}, \dots, L_\xi^{cr}}(x_1, \dots, x_\xi) = \frac{\bar{F}^2(x_\xi; \theta, \beta) \bar{F}(x_1; \theta, \beta)}{F(x_1; \theta, \beta)} \prod_{j=1}^{\xi} \frac{2f(x_j; \theta, \beta)}{F(x_j; \theta, \beta)}, \quad x_1 > \dots > x_\xi, \quad (21)$$

and (1) and (2) gives the likelihood function

$$L(\theta, \beta \mid L^{cr}) = \frac{2^\xi e^{-2(x_1-\theta)/\beta} e^{-(x_\xi-\theta)/\beta}}{\beta^\xi (1 - e^{-(x_1-\theta)/\beta})} \prod_{j=1}^{\xi} \frac{e^{-(x_j-\theta)/\beta}}{\beta(1 - e^{-(x_j-\theta)/\beta})}. \quad (22)$$

The log-likelihood function will be

$$\begin{aligned} \ell(\theta, \beta \mid L^{cr}) &= \xi \ln 2 - 2\xi \ln \beta - 2 \frac{x_1 - \theta}{\beta} - \frac{x_\xi - \theta}{\beta} - \sum_{j=1}^{\xi} \frac{x_j - \theta}{\beta} - \ln(1 - e^{-(x_1-\theta)/\beta}) \\ &\quad - \sum_{j=1}^{\xi} \ln(1 - e^{-(x_j-\theta)/\beta}). \end{aligned} \quad (23)$$

Derivative with respect to θ

$$\frac{\partial \ell}{\partial \theta} = \frac{2}{\beta} + \frac{1}{\beta} + \sum_{j=1}^{\xi} \frac{1}{\beta} - \frac{(1/\beta)e^{-(x_1-\theta)/\beta}}{1 - e^{-(x_1-\theta)/\beta}} - \sum_{j=1}^{\xi} \frac{(1/\beta)e^{-(x_j-\theta)/\beta}}{1 - e^{-(x_j-\theta)/\beta}} = 0.$$

Multiply through by β to simplify

$$3 + \xi - \frac{e^{-(x_1-\theta)/\beta}}{1 - e^{-(x_1-\theta)/\beta}} - \sum_{j=1}^{\xi} \frac{e^{-(x_j-\theta)/\beta}}{1 - e^{-(x_j-\theta)/\beta}} = 0. \quad (24)$$

This equation can be solved numerically to find $\hat{\theta}_{L^{cr}}^{ML}$. Now derivative with respect to β

$$\frac{\partial \ell}{\partial \beta} = -\frac{2\xi}{\beta} + \frac{2(x_1 - \theta)}{\beta^2} + \frac{x_\xi - \theta}{\beta^2} + \sum_{j=1}^{\xi} \frac{x_j - \theta}{\beta^2} - \frac{(x_1 - \theta)e^{-(x_1-\theta)/\beta}}{\beta^2(1 - e^{-(x_1-\theta)/\beta})} - \sum_{j=1}^{\xi} \frac{(x_j - \theta)e^{-(x_j-\theta)/\beta}}{\beta^2(1 - e^{-(x_j-\theta)/\beta})} = 0.$$

Multiply through by β^2 and some simplifications we get

$$-2\xi\beta + 2x_1 + x_\xi - 3\theta + \sum_{j=1}^{\xi} (x_j - \theta) \frac{1 - 2e^{-(x_j - \theta)/\beta}}{1 - e^{-(x_j - \theta)/\beta}} - \frac{(x_1 - \theta)e^{-(x_1 - \theta)/\beta}}{1 - e^{-(x_1 - \theta)/\beta}} = 0 \quad (25)$$

Unlike the upper current record case, closed-form solutions are not generally available for lower current records; the MLEs $\hat{\theta}_{L_{cr}}^{ML}$ and $\hat{\beta}_{L_{cr}}^{ML}$ must be computed numerically by solving the above nonlinear equations using methods such as Newton–Raphson or Fisher scoring.

4.2. Bayesian Estimation (BE)

BE provides a flexible inferential framework for estimating the unknown parameters of the two-parameter exponential distribution when inference is based on current record observations. Unlike classical approaches that rely solely on the observed sample, the Bayesian methodology incorporates prior knowledge about the parameters and combines it with the information in the observed data using Bayes' theorem. The resulting posterior distribution reflects both sources of information and provides a complete probabilistic description of parameter uncertainty.

In the context of network latency time analysis, Bayesian estimation is particularly attractive because record values often represent only a small subset of the available observations while still capturing the most extreme and informative latency events. By accounting for parameter uncertainty, the Bayesian framework produces more stable and interpretable estimates of the location parameter θ , which represents the baseline latency level, and the scale parameter β , which controls the variability of delay measurements. These posterior estimates naturally support predictive inference and form the basis for the Bayesian predictive procedures for future latency records developed in the subsequent sections.

4.2.1. BE for Upper Current Records

We assume independent priors for the estimators of the BE of the upper current record distribution:

$$\pi(\theta) \propto \theta^{a_1-1} e^{-b_1\theta}, \quad \pi(\beta) \propto \beta^{a_2-1} e^{-b_2\beta},$$

and the posterior joint is then proportional to the probability function provided by (18).

$$\pi(\theta, \beta \mid \underline{U}^{cr}) \propto 2^\xi \theta^{a_1-1} \beta^{a_2-\xi-1} \left(e^{(x_1-\theta)/\beta} - 1 \right) \exp \left[-\frac{2(x_\xi - \theta)}{\beta} - b_1\theta - b_2\beta \right]. \quad (26)$$

Under the Square Error Loss (SEL) function, the BE of parameter θ , represented as $\hat{\theta}_{U_{cr}}^{BE}$, is obtained as the posterior expectation of θ conditional on the observed upper current record values. Let $\underline{U}^{cr} = (U_1^{cr}, U_2^{cr}, \dots, U_\xi^{cr})$, which will be provided by

$$E(\theta \mid \underline{U}^{cr}) = \int_0^\infty \int_0^\infty \theta \pi(\theta, \beta \mid \underline{U}^{cr}) d\theta d\beta, \quad (27)$$

whereas the BE of β under the same set $\underline{U}^{cr}$, $\hat{\beta}_{U_{cr}}^{BE}$, may be computed from

$$E(\beta \mid \underline{U}^{cr}) = \int_0^\infty \int_0^\infty \beta \pi(\theta, \beta \mid \underline{U}^{cr}) d\theta d\beta. \quad (28)$$

4.2.2. BE for Lower Current Records

Again, we assume independent priors for the estimators of the BE of the lower current record distribution:

$$\pi(\theta) \propto \theta^{c_1-1} e^{-d_1\theta}, \quad \pi(\beta) \propto \beta^{c_2-1} e^{-d_2\beta}.$$

Now, we obtain the following posterior proportional function using the likelihood function shown in (22).

$$\pi(\theta, \beta \mid \underline{L}^{cr}) \propto 2^{\xi} \theta^{c_1-1} \beta^{c_2-1-2\xi} \frac{\exp\left[-\frac{2(x_1-\theta) + (x_{\xi}-\theta) + \sum_{j=1}^{\xi}(x_j-\theta)}{\beta}\right]}{(1 - e^{-(x_1-\theta)/\beta}) \prod_{j=1}^{\xi}(1 - e^{-(x_j-\theta)/\beta})} \exp(-d_1\theta - d_2\beta). \quad (29)$$

To obtain the Bayes estimators of the parameters θ and β , denoted by $\hat{\theta}_{\underline{L}^{cr}}^{BE}$ and $\hat{\beta}_{\underline{L}^{cr}}^{BE}$ under the squared error loss (SEL) function based on the observed lower current records $\underline{L}^{cr} = (L_1^{cr}, L_2^{cr}, \dots, L_{\xi}^{cr})$, we use the posterior means given by

$$E(\theta \mid \underline{L}^{cr}) = \int_0^{\infty} \int_0^{\infty} \theta \pi(\theta, \beta \mid \underline{L}^{cr}) d\theta d\beta, \quad (30)$$

$$E(\beta \mid \underline{L}^{cr}) = \int_0^{\infty} \int_0^{\infty} \beta \pi(\theta, \beta \mid \underline{L}^{cr}) d\theta d\beta. \quad (31)$$

The posterior expressions in (27), (28), (30) and (31) involve the complicated forms given in (26) and (29), which makes obtaining analytical solutions for the corresponding integrals infeasible. Consequently, closed-form Bayesian estimators cannot be derived directly. To approximate these quantities, numerical integration techniques are required. In this study, a simulation-based approach using Markov Chain Monte Carlo (MCMC) methods is adopted. Specifically, posterior samples are generated through the Metropolis–Hastings (MH) algorithm, allowing the Bayesian estimates to be approximated using the generated posterior samples.

5. Prediction of Future Network Latency Time Data

In this section, we investigate the prediction of future network latency time under current records, which are critical performance indicators in modern communication systems. Network latency times reflect transmission delay, response time, packet propagation delay, queuing delay, and overall system throughput efficiency. Accurate prediction of extreme or record-breaking latency observations is essential for performance optimization, congestion control, quality-of-service (QoS) assessment, and reliability monitoring in real-time network environments.

By incorporating specialized prediction intervals tailored to the current record structure and leveraging refined Bayesian and likelihood-based estimators, more precise and informative forecasts of future latency spikes can be obtained. These enhanced predictive procedures allow for better anomaly detection, improved traffic management strategies, stronger statistical inference, and more robust predictive modeling for high-speed and large-scale network infrastructures.

5.1. Classical Prediction Intervals (CPI)

Predicting future lower and upper current records, as well as future record ranges, was first systematically addressed by Barakat et al. [13], who developed prediction intervals tailored to different types of current records. Their construction relies on the most recently observed current record and an adjustment quantity that depends on whether the prediction concerns a lower record, an upper record, or a record range. A key component of these intervals is the tabulated constant $Y_{\tau, \alpha}$, where τ denotes the order of the future current record to be predicted (with $\zeta = 1$ representing the first future record, $\zeta = 2$ the second, and so forth), and α is the significance level.

In the present framework, which focuses on network latency time current records, we refine the interval construction procedure by incorporating estimators that are specifically

derived from the relevant record type. Rather than using parameter estimates obtained from the complete dataset, we employ record-based estimators to better capture the stochastic behavior of extreme latency observations.

For predictive intervals of upper latency time current records, we utilize $\hat{\theta}_{U_{cr}}^{ML}$ or $\hat{\theta}_{U_{cr}}^{BE}$ and $\hat{\beta}_{U_{cr}}^{ML}$ or $\hat{\beta}_{U_{cr}}^{BE}$. Similarly, for lower latency time current records, we use $\hat{\theta}_{L_{cr}}^{ML}$ or $\hat{\theta}_{L_{cr}}^{BE}$ and $\hat{\beta}_{L_{cr}}^{ML}$ or $\hat{\beta}_{L_{cr}}^{BE}$. This targeted estimation strategy improves the precision of predictive intervals for future latency spikes and unusually low delay events, thereby enhancing reliability assessment and performance forecasting in network systems.

The prediction procedure can be summarized in the following algorithm:

First step: For a given latency time dataset, identify the probability distribution F that provides the best fit to the observed delay, response time, and transmission latency measurements.

Second step: Identify the sequences of lower and upper current latency record values from the observed data and compute the associated record ranges. Let ζ denote the total number of observed lower and upper current records.

Third step: Estimate the parameters using the appropriate record-based methods. Specifically, compute $\hat{\theta}_{L_{cr}}^{ML}$ and $\hat{\beta}_{L_{cr}}^{ML}$ or $\hat{\theta}_{L_{cr}}^{BE}$ and $\hat{\beta}_{L_{cr}}^{BE}$ for lower records, and $\hat{\theta}_{U_{cr}}^{ML}$ and $\hat{\beta}_{U_{cr}}^{ML}$ or $\hat{\theta}_{U_{cr}}^{BE}$ and $\hat{\beta}_{U_{cr}}^{BE}$ for upper records.

Fourth step: Select the prediction order τ corresponding to the future latency current record of interest and specify the significance level α . The associated critical value $Y_{\tau,\alpha}$ can then be retrieved from the relevant statistical tables.

Fifth step: Using the estimates obtained in Step 3, construct the predictive confidence intervals as follows:

- a- $\left(U_{\zeta}^{cr}, F^{-1}(1 - \bar{F}^{1+Y_{\tau,\alpha}}(U_{\zeta}^{cr})) \right)$ is a $(1 - \alpha)\%$ predictive interval for $U_{\zeta+\tau}^{cr}$.
- b- $\left(F^{-1}(F^{1+Y_{\tau,\alpha}}(L_{\zeta}^{cr})), L_{\zeta}^{cr} \right)$ is a $(1 - \alpha)\%$ predictive interval for $L_{\zeta+\tau}^{cr}$.
- c- $\left(U_{\zeta}^{cr} - L_{\zeta}^{cr}, F^{-1}(1 - \bar{F}^{1+Y_{\tau,\alpha}}(U_{\zeta}^{cr})) - F^{-1}(F^{1+Y_{\tau,\alpha}}(L_{\zeta}^{cr})) \right)$ is a $\gamma\%$ predictive interval for $R_{\zeta+\tau}^{cr}$, where $\gamma \geq \max(1 - 2\alpha, 0)$.

5.2. Bayesian Predictive Interval (BPI) and Bayesian Prediction Value (BPV)

BPV provides a coherent probabilistic framework for forecasting future network latency time current records by combining prior information about the unknown parameters with the evidence contained in the observed record data. Within this framework, inference is based on the posterior distribution of the parameters derived from the lower and upper latency current records. Rather than relying on single-point estimates, Bayesian prediction accounts explicitly for parameter uncertainty by integrating over the joint posterior distribution.

This integration yields the posterior predictive distribution of a future latency time current record, which incorporates both sampling variability and uncertainty in the model parameters. From this predictive distribution, BPIs can be constructed to provide a range of plausible values for future latency observations, such as potential delay spikes or unusually low response times with a specified probability of coverage.

As additional latency measurements or new current records become available, the posterior distribution and the associated predictive intervals are updated in a coherent and consistent manner. This adaptive feature makes Bayesian prediction particularly attractive in network performance analysis, especially in environments with limited observations, rapidly changing traffic conditions, or when prior knowledge about latency behavior is available.

5.2.1. BPV and BPI for Upper Current Records

Let U_{τ}^{cr} denote the future upper current record to be predicted, given the observed sequence of upper current records $\underline{U}^{cr}$. The posterior predictive density of U_{τ}^{cr} , conditional on the observed records, is defined by integrating the conditional density (14) with respect to the posterior of the parameters θ and β given in (26):

$$f_{U_{\tau}^{cr}|\underline{U}^{cr}}^p(u_{\tau}) = \int_0^{\infty} \int_0^{\infty} f_{U_{\tau}^{cr}|\underline{U}^{cr}}(u_{\tau} | u_{\xi}; \theta, \beta) \pi(\theta, \beta | \underline{U}^{cr}) d\theta d\beta, \quad (32)$$

where $f_{U_{\tau}^{cr}|\underline{U}^{cr}}(u_{\tau} | u_{\xi}; \theta, \beta)$ is the conditional density of the next upper current record from (14). With some routine calculations, we reach

$$f_{U_{\tau}^{cr}|\underline{U}^{cr}}^p(u_{\tau}) \propto \int_0^{\infty} \int_0^{\infty} \frac{2^{\xi}}{(\tau - \xi - 1)!} \theta^{a_1-1} \beta^{a_2-\tau-1} (u_{\tau} - u_{\xi})^{\tau-\xi-1} \left(e^{(x_1-\theta)/\beta} - 1 \right) \times \exp \left[-\frac{u_{\tau} + 2x_{\xi} - u_{\xi} - 2\theta}{\beta} - b_1\theta - b_2\beta \right] d\theta d\beta, \quad u_{\tau} \geq u_{\xi}. \quad (33)$$

Under the squared-error loss function, the Bayesian point predictor of U_{τ}^{cr} is the posterior predictive mean:

$$\widehat{U_{\tau}^{cr}}^{\text{BPV}} = E(U_{\tau}^{cr} | \underline{U}^{cr}) = \int_{u_{\xi}}^{\infty} u_{\tau} f_{U_{\tau}^{cr}|\underline{U}^{cr}}^p(u_{\tau}) du_{\tau}. \quad (34)$$

Since (33) does not admit a closed-form solution, the posterior predictive mean is approximated via simulation:

$$\widehat{U_{\tau}^{cr}}^{\text{BPV}} \approx \frac{1}{T} \sum_{t=1}^T U_{\tau}^{cr(t)},$$

where $\{U_{\tau}^{cr(t)}\}_{t=1}^T$ are draws from the posterior predictive distribution.

For Bayesian predictive intervals (BPIs), a $100(1 - \alpha)\%$ interval for U_{τ}^{cr} is defined as

$$P(L_{\alpha} \leq U_{\tau}^{cr} \leq U_{\alpha} | \underline{U}^{cr}) = 1 - \alpha,$$

with bounds determined from the posterior predictive distribution:

$$\int_{u_{\xi}}^{L_{\alpha}} f_{U_{\tau}^{cr}|\underline{U}^{cr}}^p(u_{\tau}) du_{\tau} = \frac{\alpha}{2}, \quad (35)$$

$$\int_{u_{\xi}}^{U_{\alpha}} f_{U_{\tau}^{cr}|\underline{U}^{cr}}^p(u_{\tau}) du_{\tau} = 1 - \frac{\alpha}{2}. \quad (36)$$

In practice, the BPI can be obtained from the empirical quantiles of the simulated posterior predictive sample $\{U_{\tau}^{cr(t)}\}_{t=1}^T$:

$$(L_{\alpha}, U_{\alpha}) = (Q_{\alpha/2}, Q_{1-\alpha/2}),$$

where Q_p denotes the p th quantile of the posterior predictive sample.

5.2.2. BPV and BPI for Lower Current Records

The prediction procedure for lower current records follows a framework similar to that used for upper records, with appropriate consideration of the decreasing pattern inherent in lower record sequences. Let L_{τ}^{cr} denote the τ -th future lower current record value, where $\tau > \xi$ and ξ denotes the number of observed lower current records.

The posterior predictive density of $L_\tau^{c_r}$ given the observed sequence $\underline{L}^{c_r}$ is

$$f_{L_\tau^{c_r}|\underline{L}^{c_r}}^p(l_\tau) = \int_0^\infty \int_0^\infty f_{L_\tau^{c_r}|\underline{L}_\xi^{c_r}}(l_\tau | l_\xi; \theta, \beta) \pi(\theta, \beta | \underline{L}^{c_r}) d\theta d\beta, \quad (37)$$

where the conditional density $f_{L_\tau^{c_r}|\underline{L}_\xi^{c_r}}$ is given by (13). And the posterior distribution is given by (29).

$$f_{L_\tau^{c_r}|\underline{L}^{c_r}}^p(l_\tau) \propto \int_0^\infty \int_0^\infty \frac{2^\xi \theta^{c_1-1}}{(\tau - \xi - 1)!} \beta^{c_2-2-2\xi} \frac{\left[\ln(1 - e^{-(l_\xi-\theta)/\beta}) - \ln(1 - e^{-(l_\tau-\theta)/\beta}) \right]^{\tau-\xi-1}}{1 - e^{-(l_\xi-\theta)/\beta}} \\ \times \frac{\exp \left[-\frac{l_\tau + 2x_1 + x_\xi + \sum_{j=1}^\xi x_j - (\xi+4)\theta}{\beta} - d_1\theta - d_2\beta \right]}{(1 - e^{-(x_1-\theta)/\beta}) \prod_{j=1}^\xi (1 - e^{-(x_j-\theta)/\beta})} d\theta d\beta, \quad l_\tau \leq l_\xi. \quad (38)$$

Under the squared-error loss, the Bayesian point predictor of $L_\tau^{c_r}$ is the posterior predictive mean:

$$\widehat{L}_\tau^{c_r \text{BP}} = E(L_\tau^{c_r} | \underline{L}^{c_r}) = \int_0^{l_\xi} l_\tau f_{L_\tau^{c_r}|\underline{L}^{c_r}}^p(l_\tau) dl_\tau. \quad (39)$$

Because this integral has no closed form, it is approximated via Monte Carlo:

$$\widehat{L}_\tau^{c_r \text{BP}} \approx \frac{1}{T} \sum_{t=1}^T L_\tau^{c_r(t)}, \quad (40)$$

where $\{L_\tau^{c_r(t)}\}_{t=1}^T$ are draws from the posterior predictive distribution.

Now for BPI, a $100(1 - \gamma)\%$ Bayesian predictive interval for $L_\tau^{c_r}$ is defined as:

$$P(L_\gamma \leq L_\tau^{c_r} \leq U_\gamma | \underline{L}^{c_r}) = 1 - \gamma,$$

with the bounds determined by the posterior predictive distribution:

$$\int_0^{L_\gamma} f_{L_\tau^{c_r}|\underline{L}^{c_r}}^p(l_\tau) dl_\tau = \frac{\gamma}{2}, \quad \int_0^{U_\gamma} f_{L_\tau^{c_r}|\underline{L}^{c_r}}^p(l_\tau) dl_\tau = 1 - \frac{\gamma}{2}.$$

In practice, these are approximated using the empirical quantiles of the simulated posterior predictive sample:

$$(L_\gamma, U_\gamma) = (q_{\gamma/2}, q_{1-\gamma/2}),$$

where q_p denotes the p th quantile of $\{L_\tau^{c_r(t)}\}_{t=1}^T$.

Note that, because lower current records are decreasing, $L_\tau^{c_r} \leq L_\xi^{c_r}$ for $\tau > \xi$, so any credible interval must respect this upper bound in simulations and interpretation.

For lower current records, the posterior distribution may become highly concentrated near the last observed lower current record when only a limited number of records are available. This behavior is a consequence of the record structure itself, since future lower current records cannot exceed the current minimum record and therefore contain relatively little additional information. Consequently, the Bayesian estimates may exhibit an apparent degeneracy by concentrating around the boundary of the parameter space. This phenomenon reflects the limited information contained in lower current records rather than numerical instability of the Metropolis–Hastings algorithm. The corresponding MCMC chains remained stable and converged satisfactorily, indicating that the observed concentration is an inherent characteristic of record-based inference.

6. Numerical Illustration

6.1. Simulation Study

This section presents a comprehensive simulation study to evaluate the performance of the proposed estimation procedures for current-record data arising from the two-parameter exponential distribution. Simulation studies play an essential role in statistical research because they allow the behavior of estimation methods to be examined under controlled and reproducible conditions. In particular, simulations enable the investigation of the finite-sample properties of estimators, including their bias, variability, and overall stability, and provide a practical framework for comparing different inferential approaches when theoretical analysis alone is insufficient.

In the context of record-based inference, where only upper and lower current records are observed instead of the complete sample, simulation experiments are especially important. Record values represent extreme observations and therefore contain limited but highly informative information about the underlying distribution. Evaluating estimation methods under such conditions helps assess their reliability when applied to real-world problems involving extreme events, such as unusually large or small latency times in communication networks. Consequently, the simulation analysis provides insight into how well the proposed procedures are expected to perform when modeling and predicting extreme latency behavior.

To conduct the simulation study, 1000 random samples of size 100 were generated from the two-parameter exponential distribution. For each generated sample, the corresponding upper and lower current records were extracted and used to estimate the model parameters using both MLE and BE. For each observed record sample size ζ , the mean of the parameter estimates and their standard deviations (SD) were computed to evaluate the variability of the estimators. The number of samples corresponding to each record size ζ was also recorded and denoted by n .

In addition, the accuracy of the estimators was assessed using two widely used performance measures: the mean absolute deviation (MAD) and the root mean square error (RMSE). Furthermore, bootstrap confidence intervals (BCIs) were constructed for each model parameter to assess the uncertainty around the estimates. The complete simulation results are presented in Tables 1–4, while Figure 5 provide graphical summaries of the principal performance measures. These figures facilitate comparison between the maximum likelihood and Bayesian estimators and illustrate how the estimation accuracy changes as the number of current records increases.

Overall, the graphical summaries presented in Figure 5 clearly support the numerical findings reported in Tables 1–4. In particular, both the RMSE and MAD generally decrease as the number of current records increases, indicating improved estimation accuracy with larger record samples. Moreover, the Bayesian estimator exhibits smaller variability than the maximum likelihood estimator in most simulation scenarios involving small and moderate record sizes. The quantity n reported in the simulation tables denotes the number of Monte Carlo samples producing the corresponding record size rather than the original simulation sample size. Consequently, some record sizes occur infrequently, resulting in small effective sample sizes. The associated bootstrap confidence intervals should therefore be interpreted with appropriate caution. Also, The relatively large standard deviations observed in a few scenarios correspond to rare record configurations that occurred infrequently during the Monte Carlo simulations. Because current records represent only a sparse subset of the original observations, these configurations naturally exhibit greater sampling variability. The simulation results demonstrate that predicting network latency using current-record data is feasible, but the accuracy of parameter estimates depends on record type, sample size, and estimation method. For upper current records (U^{Cr}), MLE es-

estimates of the location parameter θ tend to overestimate the true value when the record size is small ($\xi = 4, 5$) due to the limited information content of early records. As the number of observed upper records increases, both MLE and Bayesian estimation (BE) converge toward the true value, and the associated variability decreases. Similarly, MLE estimates of the scale parameter β exhibit large standard deviations and occasional overestimation for small record sizes, whereas BE provides more stable and reliable estimates with lower RMSE, making it particularly suitable for forecasting extreme high-latency events. For lower current records (L^c), both MLE and BE produce estimates closer to the true values even with small ξ , but MLE is more variable, especially for the scale parameter. BE, in contrast, yields remarkably stable estimates with tight credible intervals, reflecting high reliability in predicting minimal latency events. Overall, increasing the number of observed records improves estimation accuracy and reduces variability for both upper and lower records, indicating that longer sequences of observed latency extremes yield more confident predictions of future record values. Comparisons between MLE and BE highlight that Bayesian methods generally reduce bias, produce smaller SDs and RMSE/MAD values, and provide narrower, more reliable confidence intervals, particularly in small-sample scenarios. These findings suggest that BE is advantageous for network latency prediction, offering more stable and realistic forecasts for both extremely high and low latency events, which is critical in latency-sensitive applications such as real-time streaming or financial networks.

The simulation results also demonstrate the practical value of the proposed methodology for communication network management. In particular, lower RMSE and MAD values indicate more accurate prediction of future extreme latency observations, reducing the uncertainty associated with forecasting latency spikes. More reliable predictions enable network operators to anticipate abnormal latency conditions earlier, allowing timely resource allocation and traffic management decisions that improve network capacity planning and maintain the desired quality of service (QoS). Therefore, the statistical performance measures reported in Tables 1–4 have direct practical significance beyond their numerical values.

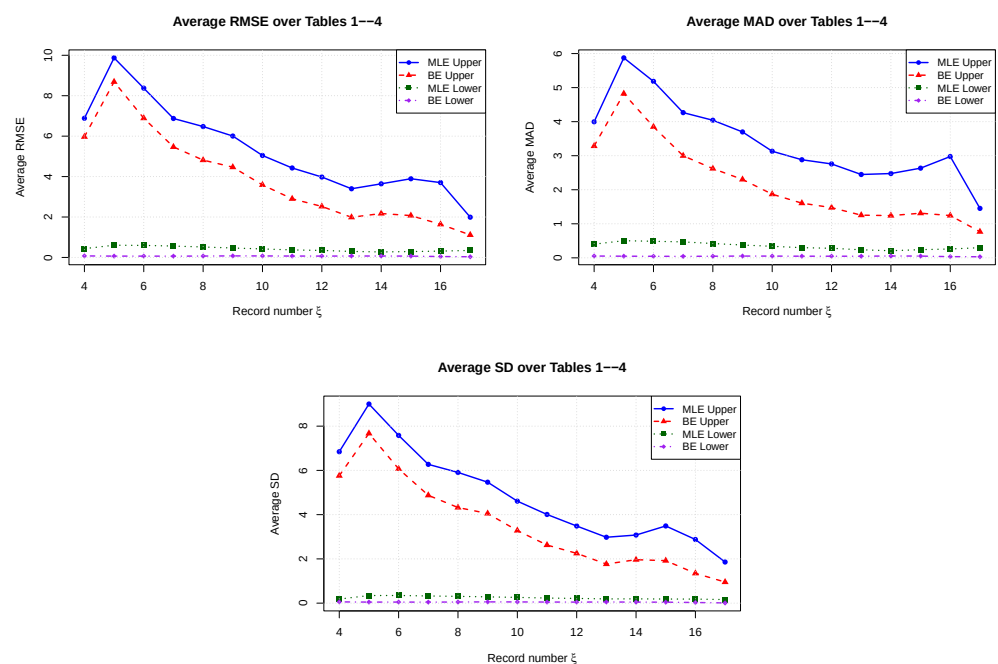


Figure 5. Average RMSE, MAD and SD of the MLE and BEs for upper and lower current records across Tables 1–4.

Table 1. Simulation results when $\theta = 1$ and $\beta = 3$.

ξ	Parameter	MLE					BE				
		Mean	SD	RMSE	MAD	95% BCI	Mean	SD	RMSE	MAD	95% BCI
4 ($n = 13$)	$\theta_{U^{\epsilon r}}$	3.9615	7.3278	7.6153	4.2948	[0.5044, 7.9408]	3.8289	6.0717	6.4650	3.3866	[0.9853, 7.5845]
	$\beta_{U^{\epsilon r}}$	5.8784	3.1683	4.1818	3.8451	[3.8438, 7.4302]	5.1709	1.4485	2.5761	2.4216	[4.2651, 5.8814]
	$\theta_{L^{\epsilon r}}$	0.5915	0.1851	0.4453	0.4093	[0.4888, 0.6980]	1.0368	0.0464	0.0577	0.0405	[1.0128, 1.0633]
	$\beta_{L^{\epsilon r}}$	1.8190	1.8791	2.1521	2.0617	[0.8606, 2.9847]	5.4991	7.9799	8.0385	5.7387	[1.4766, 10.5652]
5 ($n = 27$)	$\theta_{U^{\epsilon r}}$	2.7301	3.7802	4.0931	2.6515	[1.3940, 4.3136]	2.5093	2.7113	3.0589	1.7967	[1.6216, 3.5964]
	$\beta_{U^{\epsilon r}}$	4.4952	1.8274	2.3348	1.9792	[3.8297, 5.1346]	5.2517	0.9405	2.4335	2.3067	[4.9153, 5.5964]
	$\theta_{L^{\epsilon r}}$	0.5643	0.2619	0.5058	0.4395	[0.4706, 0.6558]	1.0397	0.0277	0.0481	0.0397	[1.0300, 1.0505]
	$\beta_{L^{\epsilon r}}$	1.3283	0.9249	1.9022	1.7393	[1.0029, 1.6847]	1.8779	1.4791	1.8347	1.6399	[1.3499, 2.4284]
6 ($n = 66$)	$\theta_{U^{\epsilon r}}$	1.9421	3.2467	3.3569	2.0749	[1.2642, 2.6970]	1.7837	2.0853	2.2129	1.1958	[1.3643, 2.3460]
	$\beta_{U^{\epsilon r}}$	4.2515	1.6110	2.0303	1.6293	[3.9146, 4.6509]	5.5305	0.8806	2.6771	2.5574	[5.3138, 5.7410]
	$\theta_{L^{\epsilon r}}$	0.6178	0.2282	0.4443	0.3837	[0.5673, 0.6702]	1.0248	0.0214	0.0327	0.0248	[1.0200, 1.0302]
	$\beta_{L^{\epsilon r}}$	1.0221	0.6654	2.0852	1.9918	[0.8698, 1.2041]	1.7709	2.2880	2.5820	2.1382	[1.3030, 2.3519]
7 ($n = 110$)	$\theta_{U^{\epsilon r}}$	2.7620	4.3350	4.6611	2.7682	[2.0288, 3.6145]	2.4506	3.1983	3.4986	1.7396	[1.8819, 3.0908]
	$\beta_{U^{\epsilon r}}$	3.6891	1.6873	1.8155	1.4441	[3.3538, 3.9835]	5.5300	1.1191	2.7644	2.5802	[5.3109, 5.7255]
	$\theta_{L^{\epsilon r}}$	0.6079	0.2115	0.4450	0.3980	[0.5671, 0.6465]	1.0318	0.0345	0.0468	0.0321	[1.0259, 1.0384]
	$\beta_{L^{\epsilon r}}$	1.1619	0.9944	2.0877	1.9842	[0.9995, 1.3692]	1.6123	1.6880	2.1792	1.8723	[1.2993, 1.8944]
8 ($n = 145$)	$\theta_{U^{\epsilon r}}$	2.5832	3.3036	3.6530	2.4790	[2.0641, 3.0858]	1.9898	1.9215	2.1556	1.3137	[1.6992, 2.3202]
	$\beta_{U^{\epsilon r}}$	3.0072	1.2271	1.2229	0.9241	[2.8061, 3.1945]	5.3954	0.8790	2.5505	2.3954	[5.2597, 5.5509]
	$\theta_{L^{\epsilon r}}$	0.6455	0.2174	0.4155	0.3575	[0.6080, 0.6813]	1.0288	0.0310	0.0423	0.0289	[1.0241, 1.0341]
	$\beta_{L^{\epsilon r}}$	0.9135	0.5416	2.1552	2.0865	[0.8293, 1.0018]	1.3174	1.1106	2.0140	1.8320	[1.1435, 1.4990]
9 ($n = 167$)	$\theta_{U^{\epsilon r}}$	2.8080	3.4128	3.8531	2.4565	[2.3018, 3.2978]	2.0886	2.2575	2.5002	1.3408	[1.7722, 2.4669]
	$\beta_{U^{\epsilon r}}$	2.6721	1.0771	1.1228	0.8463	[2.5032, 2.8314]	5.4062	0.9586	2.5890	2.4570	[5.2670, 5.5486]
	$\theta_{L^{\epsilon r}}$	0.6862	0.1888	0.3659	0.3161	[0.6558, 0.7170]	1.0303	0.0333	0.0449	0.0305	[1.0255, 1.0354]
	$\beta_{L^{\epsilon r}}$	0.8694	0.4749	2.1826	2.1308	[0.8000, 0.9412]	1.3402	1.1543	2.0198	1.8867	[1.1699, 1.5213]
10 ($n = 165$)	$\theta_{U^{\epsilon r}}$	1.7870	2.3055	2.4295	1.5976	[1.4460, 2.1367]	1.4585	1.2806	1.3565	0.7725	[1.2466, 1.6624]
	$\beta_{U^{\epsilon r}}$	2.6329	0.9004	0.9698	0.7802	[2.4967, 2.7710]	5.5932	0.7909	2.7104	2.5942	[5.4723, 5.7169]
	$\theta_{L^{\epsilon r}}$	0.6957	0.1917	0.3594	0.3124	[0.6666, 0.7249]	1.0306	0.0315	0.0438	0.0308	[1.0258, 1.0357]
	$\beta_{L^{\epsilon r}}$	0.7344	0.3093	2.2865	2.2656	[0.6878, 0.7802]	0.9244	0.6149	2.1642	2.0770	[0.8320, 1.0241]
11 ($n = 124$)	$\theta_{U^{\epsilon r}}$	2.3409	2.8259	3.1176	1.9851	[1.8534, 2.8543]	1.7838	1.6925	1.8590	1.0053	[1.5129, 2.1013]
	$\beta_{U^{\epsilon r}}$	2.4539	0.9190	1.0658	0.8529	[2.3008, 2.6290]	5.6656	0.8342	2.7921	2.6656	[5.5204, 5.8177]
	$\theta_{L^{\epsilon r}}$	0.7521	0.1988	0.3173	0.2530	[0.7161, 0.7875]	1.0316	0.0305	0.0439	0.0316	[1.0267, 1.0370]
	$\beta_{L^{\epsilon r}}$	0.6892	0.3510	2.3371	2.3108	[0.6299, 0.7477]	0.9905	0.7789	2.1541	2.0543	[0.8515, 1.1282]
12 ($n = 71$)	$\theta_{U^{\epsilon r}}$	1.7880	1.6504	1.8184	1.3297	[1.4199, 2.1756]	1.3180	0.7573	0.8164	0.5840	[1.1393, 1.5027]
	$\beta_{U^{\epsilon r}}$	2.2848	0.6763	0.9810	0.8550	[2.1350, 2.4400]	5.6961	0.7080	2.7862	2.6961	[5.5397, 5.8667]
	$\theta_{L^{\epsilon r}}$	0.7128	0.1911	0.3442	0.2893	[0.6722, 0.7555]	1.0243	0.0284	0.0373	0.0244	[1.0182, 1.0316]
	$\beta_{L^{\epsilon r}}$	0.6750	0.2395	2.3372	2.3250	[0.6205, 0.7278]	0.8126	0.4258	2.2279	2.1874	[0.7132, 0.9122]
13 ($n = 54$)	$\theta_{U^{\epsilon r}}$	1.8042	1.8057	1.9613	1.3612	[1.3663, 2.2870]	1.2492	0.7196	0.7552	0.5379	[1.0795, 1.4485]
	$\beta_{U^{\epsilon r}}$	2.0949	0.6954	1.1374	0.9901	[1.9119, 2.2835]	5.6643	0.7636	2.7697	2.6643	[5.4533, 5.8618]
	$\theta_{L^{\epsilon r}}$	0.7500	0.1925	0.3144	0.2515	[0.6991, 0.8008]	1.0229	0.0208	0.0308	0.0229	[1.0177, 1.0285]
	$\beta_{L^{\epsilon r}}$	0.6258	0.2553	2.3876	2.3742	[0.5580, 0.6918]	0.7862	0.4609	2.2603	2.2138	[0.6696, 0.9115]
14 ($n = 33$)	$\theta_{U^{\epsilon r}}$	3.0621	3.0096	3.6105	2.4756	[2.0792, 4.0799]	2.0245	1.7087	1.9700	1.1954	[1.4584, 2.6311]
	$\beta_{U^{\epsilon r}}$	1.9824	1.0958	1.4832	1.2983	[1.6206, 2.3744]	5.6888	1.2657	2.9636	2.6888	[5.2742, 6.1270]
	$\theta_{L^{\epsilon r}}$	0.7956	0.1846	0.2736	0.2115	[0.7329, 0.8571]	1.0306	0.0280	0.0412	0.0309	[1.0217, 1.0409]
	$\beta_{L^{\epsilon r}}$	0.6757	0.3506	2.3498	2.3243	[0.5656, 0.7894]	0.9792	0.6837	2.1300	2.0272	[0.7559, 1.2065]
15 ($n = 14$)	$\theta_{U^{\epsilon r}}$	2.2166	2.2088	2.4517	1.6527	[1.1966, 3.4528]	1.4131	1.0170	1.0636	0.7180	[0.9640, 1.9908]
	$\beta_{U^{\epsilon r}}$	1.6594	0.5211	1.4316	1.3406	[1.4051, 1.9130]	5.4774	0.7198	2.5727	2.4774	[5.0828, 5.8368]
	$\theta_{L^{\epsilon r}}$	0.6987	0.1901	0.3526	0.3013	[0.6052, 0.8045]	1.0273	0.0243	0.0359	0.0273	[1.0170, 1.0423]
	$\beta_{L^{\epsilon r}}$	0.7428	0.2528	2.2703	2.2572	[0.6165, 0.8688]	0.9305	0.5677	2.1406	2.0695	[0.6711, 1.2297]
16 ($n = 7$)	$\theta_{U^{\epsilon r}}$	1.8022	1.3364	1.4746	1.1827	[0.9255, 2.7750]	1.1902	0.7529	0.7226	0.4939	[0.7422, 1.8136]
	$\beta_{U^{\epsilon r}}$	1.4557	0.4898	1.6095	1.5443	[1.1585, 1.8315]	5.3004	0.6748	2.3837	2.3004	[4.8952, 5.8144]
	$\theta_{L^{\epsilon r}}$	0.6815	0.2128	0.3745	0.3185	[0.5554, 0.8329]	1.0178	0.0133	0.0216	0.0178	[1.0095, 1.0278]
	$\beta_{L^{\epsilon r}}$	0.6899	0.3049	2.3273	2.3101	[0.4652, 0.9030]	0.6715	0.2150	2.3370	2.3285	[0.5451, 0.8411]
17 ($n = 4$)	$\theta_{U^{\epsilon r}}$	1.4466	0.2097	0.4705	0.4466	[1.2983, 1.5948]	1.0615	0.0070	0.0617	0.0615	[1.0565, 1.0665]
	$\beta_{U^{\epsilon r}}$	1.8451	0.1833	1.1622	1.1549	[1.7155, 1.9747]	6.1225	0.1769	3.1250	3.1225	[5.9974, 6.2476]
	$\theta_{L^{\epsilon r}}$	0.5065	0.0033	0.4935	0.4935	[0.5042, 0.5088]	1.0060	0.0027	0.0063	0.0060	[1.0041, 1.0079]
	$\beta_{L^{\epsilon r}}$	0.9004	0.0642	2.1001	2.0996	[0.8550, 0.9458]	0.5795	0.0850	2.4212	2.4205	[0.5194, 0.6396]

Table 2. Simulation results when $\theta = 1$ and $\beta = 5$.

ξ	Parameter	MLE					BE				
		Mean	SD	RMSE	MAD	95% BCI	Mean	SD	RMSE	MAD	95% BCI
4 ($n = 13$)	$\theta_{U^{cr}}$	3.0959	5.2996	5.4898	3.4292	[0.4513, 6.2030]	3.3343	4.2066	4.6551	2.8394	[1.2879, 5.8739]
	$\beta_{U^{cr}}$	9.8026	4.4754	6.4362	5.6107	[7.2842, 12.004]	6.5734	1.4774	2.1157	1.9421	[5.6632, 7.2902]
	$\theta_{L^{cr}}$	0.4826	0.2184	0.5581	0.5174	[0.3594, 0.5881]	1.0465	0.0469	0.0647	0.0465	[1.0246, 1.0745]
	$\beta_{L^{cr}}$	1.7942	1.5917	3.5495	3.2058	[0.9478, 2.6669]	2.7823	3.0686	3.6810	3.3470	[1.1810, 4.5650]
5 ($n = 29$)	$\theta_{U^{cr}}$	5.0469	10.805	11.362	5.3572	[1.6978, 9.2487]	5.1432	9.3167	10.0486	4.6475	[2.2053, 9.1305]
	$\beta_{U^{cr}}$	7.6167	3.8999	4.6402	4.0574	[6.0806, 9.0002]	6.3587	1.8716	2.2865	2.1283	[5.6818, 6.9855]
	$\theta_{L^{cr}}$	0.6095	0.1732	0.4260	0.4007	[0.5508, 0.6761]	1.0423	0.0530	0.0671	0.0426	[1.0263, 1.0632]
	$\beta_{L^{cr}}$	1.8575	2.2928	3.8666	3.6875	[1.1295, 2.7658]	5.8719	14.563	14.336	6.6310	[2.0073, 12.191]
6 ($n = 64$)	$\theta_{U^{cr}}$	4.0680	6.1098	6.7941	4.2423	[2.5908, 5.7083]	3.8286	4.5373	5.3166	3.0738	[2.8304, 4.9044]
	$\beta_{U^{cr}}$	6.6575	2.6155	3.0791	2.6152	[6.0002, 7.2960]	6.7629	1.2752	2.1699	1.9425	[6.4221, 7.0644]
	$\theta_{L^{cr}}$	0.6184	0.1949	0.4278	0.3836	[0.5735, 0.6655]	1.0421	0.0443	0.0609	0.0437	[1.0311, 1.0538]
	$\beta_{L^{cr}}$	1.7133	1.3492	3.5488	3.3402	[1.4107, 2.0491]	2.8952	2.9657	3.6178	3.2028	[2.2362, 3.6549]
7 ($n = 121$)	$\theta_{U^{cr}}$	4.4078	6.4853	7.3023	4.3306	[3.3135, 5.5558]	3.8954	5.0485	5.8017	3.1547	[3.0280, 4.7673]
	$\beta_{U^{cr}}$	5.6697	2.4325	2.5133	2.0512	[5.2168, 6.1068]	6.7531	1.3649	2.2183	2.0199	[6.5206, 6.9850]
	$\theta_{L^{cr}}$	0.6123	0.2277	0.4491	0.3950	[0.5716, 0.6522]	1.0439	0.0475	0.0646	0.0444	[1.0360, 1.0530]
	$\beta_{L^{cr}}$	1.6259	1.3714	3.6400	3.4519	[1.3925, 1.8677]	2.8664	3.1344	3.7809	3.2479	[2.3727, 3.4250]
8 ($n = 143$)	$\theta_{U^{cr}}$	3.7363	6.1111	6.6762	3.8298	[2.7825, 4.7438]	3.3538	4.5533	5.1115	2.6040	[2.6574, 4.1723]
	$\beta_{U^{cr}}$	5.3014	2.2280	2.2406	1.7532	[4.9270, 5.6794]	6.9344	1.3440	2.3528	2.1044	[6.7066, 7.1449]
	$\theta_{L^{cr}}$	0.6513	0.2239	0.4140	0.3519	[0.6140, 0.6842]	1.0433	0.0449	0.0623	0.0436	[1.0367, 1.0511]
	$\beta_{L^{cr}}$	1.3108	1.0840	3.8441	3.7071	[1.1359, 1.4850]	2.1824	2.2413	3.5954	3.3702	[1.8338, 2.5671]
9 ($n = 162$)	$\theta_{U^{cr}}$	2.5621	3.8724	4.1645	2.6058	[1.9964, 3.1308]	2.4412	2.7928	3.1350	1.7189	[2.0471, 2.8690]
	$\beta_{U^{cr}}$	4.8748	1.7604	1.7595	1.3121	[4.5885, 5.1321]	7.0450	1.1307	2.3351	2.1217	[6.8747, 7.2186]
	$\theta_{L^{cr}}$	0.6604	0.2242	0.4065	0.3481	[0.6281, 0.6947]	1.0516	0.0568	0.0766	0.0517	[1.0436, 1.0613]
	$\beta_{L^{cr}}$	1.0132	0.6103	4.0329	3.9868	[0.9211, 1.1146]	1.5509	1.5614	3.7841	3.5949	[1.3266, 1.7907]
10 ($n = 173$)	$\theta_{U^{cr}}$	3.2076	4.3727	4.8871	3.0211	[2.6124, 3.8509]	2.6947	3.1248	3.5468	1.8609	[2.2343, 3.2187]
	$\beta_{U^{cr}}$	4.3335	1.5361	1.6704	1.3369	[4.1015, 4.5541]	7.0531	1.1345	2.3441	2.1433	[6.8654, 7.2196]
	$\theta_{L^{cr}}$	0.7003	0.2160	0.3691	0.3050	[0.6693, 0.7321]	1.0459	0.0516	0.0690	0.0461	[1.0382, 1.0537]
	$\beta_{L^{cr}}$	1.0552	0.7421	4.0136	3.9631	[0.9560, 1.1681]	1.6764	1.4971	3.6434	3.4798	[1.4534, 1.9263]
11 ($n = 115$)	$\theta_{U^{cr}}$	2.7499	3.5493	3.9433	2.6052	[2.1567, 3.4422]	2.2322	2.1327	2.4550	1.4373	[1.8542, 2.6547]
	$\beta_{U^{cr}}$	3.9443	1.1994	1.5939	1.3563	[3.7259, 4.1624]	7.0718	0.9025	2.2583	2.0718	[6.9245, 7.2330]
	$\theta_{L^{cr}}$	0.7176	0.2174	0.3558	0.2874	[0.6757, 0.7536]	1.0483	0.0449	0.0658	0.0483	[1.0404, 1.0576]
	$\beta_{L^{cr}}$	0.9382	0.5193	4.0946	4.0618	[0.8506, 1.0360]	1.3527	1.0134	3.7843	3.6494	[1.1802, 1.5394]
12 ($n = 89$)	$\theta_{U^{cr}}$	2.9937	3.3541	3.8856	2.7076	[2.2892, 3.7286]	2.3048	1.9950	2.3743	1.4653	[1.9233, 2.6912]
	$\beta_{U^{cr}}$	3.6023	1.1695	1.8182	1.6009	[3.3558, 3.8520]	7.0541	0.9734	2.2708	2.0654	[6.8607, 7.2496]
	$\theta_{L^{cr}}$	0.7400	0.2064	0.3312	0.2659	[0.6957, 0.7818]	1.0445	0.0382	0.0585	0.0446	[1.0369, 1.0531]
	$\beta_{L^{cr}}$	0.9359	0.4743	4.0914	4.0641	[0.8406, 1.0326]	1.3652	0.9122	3.7463	3.6348	[1.1810, 1.5481]
13 ($n = 48$)	$\theta_{U^{cr}}$	2.8120	3.1016	3.5641	2.4878	[1.9859, 3.6895]	2.2768	1.9124	2.2828	1.4328	[1.7450, 2.8168]
	$\beta_{U^{cr}}$	3.3484	0.8687	1.8619	1.6849	[3.1157, 3.5817]	7.1176	0.7785	2.2534	2.1176	[6.8985, 7.3471]
	$\theta_{L^{cr}}$	0.7572	0.1807	0.3016	0.2523	[0.7071, 0.8086]	1.0461	0.0480	0.0662	0.0462	[1.0342, 1.0605]
	$\beta_{L^{cr}}$	0.8339	0.3690	4.1820	4.1661	[0.7282, 0.9390]	1.2390	0.8101	3.8455	3.7610	[1.0197, 1.4583]
14 ($n = 22$)	$\theta_{U^{cr}}$	2.9236	2.3439	2.9907	2.0438	[2.0014, 3.8625]	1.9767	1.2942	1.5977	1.0323	[1.4817, 2.5083]
	$\beta_{U^{cr}}$	2.8341	0.8044	2.3041	2.1659	[2.5167, 3.1751]	6.8068	0.8209	1.9768	1.8068	[6.5024, 7.1431]
	$\theta_{L^{cr}}$	0.7983	0.1638	0.2575	0.2037	[0.7271, 0.8617]	1.0438	0.0374	0.0571	0.0438	[1.0294, 1.0598]
	$\beta_{L^{cr}}$	0.7744	0.2551	4.2329	4.2256	[0.6776, 0.8845]	1.1853	0.6369	3.8651	3.8147	[0.9459, 1.4740]
15 ($n = 12$)	$\theta_{U^{cr}}$	2.1645	2.4530	2.6214	1.8626	[0.9801, 3.5012]	1.8892	1.3424	1.5629	1.0521	[1.2363, 2.6900]
	$\beta_{U^{cr}}$	3.2205	1.2218	2.1295	1.8840	[2.5655, 3.9430]	7.2915	1.3049	2.6100	2.2915	[6.6005, 7.9863]
	$\theta_{L^{cr}}$	0.7614	0.1766	0.2925	0.2412	[0.6622, 0.8558]	1.0381	0.0333	0.0497	0.0381	[1.0219, 1.0575]
	$\beta_{L^{cr}}$	0.7666	0.2922	4.2427	4.2334	[0.6012, 0.9136]	0.9450	0.5384	4.0876	4.0550	[0.6800, 1.2539]
16 ($n = 5$)	$\theta_{U^{cr}}$	3.1513	1.7916	2.6825	2.3685	[1.7251, 4.4499]	2.0168	0.7232	1.2051	1.0168	[1.4273, 2.6124]
	$\beta_{U^{cr}}$	2.5505	0.6485	2.5172	2.4495	[2.0617, 3.0382]	6.9872	0.6250	2.0643	1.9872	[6.4814, 7.4693]
	$\theta_{L^{cr}}$	0.8683	0.1969	0.2200	0.1384	[0.6914, 0.9857]	1.0363	0.0213	0.0410	0.0363	[1.0230, 1.0547]
	$\beta_{L^{cr}}$	0.6414	0.2541	4.3645	4.3586	[0.5103, 0.8673]	0.9987	0.1286	4.0030	4.0013	[0.8850, 1.0868]
17 ($n = 4$)	$\theta_{U^{cr}}$	3.8381	4.5578	4.8615	3.3381	[0.6717, 8.0376]	2.8578	2.3594	2.7616	1.8578	[1.4196, 5.2067]
	$\beta_{U^{cr}}$	4.2691	1.7391	1.6741	1.6269	[3.0457, 5.9199]	9.0064	1.4219	4.1914	4.0064	[7.9398, 10.3397]
	$\theta_{L^{cr}}$	0.8403	0.1739	0.2195	0.1597	[0.6577, 0.9724]	1.0490	0.0231	0.0529	0.0490	[1.0341, 1.0724]
	$\beta_{L^{cr}}$	0.9565	0.5581	4.0723	4.0435	[0.6348, 1.5226]	1.6947	1.1346	3.4483	3.3053	[0.9738, 2.8049]

Table 3. Simulation results when $\theta = 2$ and $\beta = 5$.

ζ	Parameter	MLE					BE				
		Mean	SD	RMSE	MAD	95% BCI	Mean	SD	RMSE	MAD	95% BCI
4 ($n = 10$)	θ_{Ucr}	1.5290	0.0081	0.4710	0.4710	[1.5233, 1.5348]	1.1561	0.8088	1.1283	1.1131	[0.7694, 1.7766]
	β_{Ucr}	0.6615	0.7941	4.3747	4.3385	[0.1000, 1.2230]	2.6268	1.1482	2.5082	2.3732	[1.8150, 3.4387]
	θ_{Lcr}	1.7398	0.2249	0.3333	0.2883	[1.5925, 1.8943]	2.0657	0.0344	0.0730	0.0657	[2.0434, 2.0882]
	β_{Lcr}	0.7932	0.7224	4.2596	4.2068	[0.3286, 1.3289]	1.8528	3.3075	4.3911	4.3425	[0.2612, 4.4226]
5 ($n = 35$)	θ_{Ucr}	4.2118	5.9337	6.2526	4.2919	[2.4571, 6.2636]	4.3118	4.7562	5.2268	3.1068	[2.8607, 5.9832]
	β_{Ucr}	8.0774	3.3259	4.4962	3.6496	[7.0030, 9.1367]	6.8508	1.2690	2.2338	2.0139	[6.4305, 7.2357]
	θ_{Lcr}	1.3850	0.5098	0.7941	0.6170	[1.2002, 1.5462]	2.0421	0.0389	0.0569	0.0422	[2.0301, 2.0549]
	β_{Lcr}	1.8882	1.5729	3.4766	3.1654	[1.4061, 2.4272]	3.3235	2.9146	3.3261	2.9966	[2.4238, 4.2038]
6 ($n = 82$)	θ_{Ucr}	5.4046	7.4172	8.1201	5.4245	[3.9088, 7.1195]	4.8534	5.5334	6.1957	3.6686	[3.7115, 5.9919]
	β_{Ucr}	6.7931	3.2082	3.6581	2.9228	[6.1182, 7.4847]	6.8529	1.5795	2.4285	2.1890	[6.5013, 7.1647]
	θ_{Lcr}	1.4344	0.4495	0.7208	0.5719	[1.3306, 1.5296]	2.0563	0.0665	0.0868	0.0566	[2.0428, 2.0715]
	β_{Lcr}	1.8360	1.4997	3.4976	3.2056	[1.5377, 2.1780]	3.5522	4.2292	4.4457	3.6339	[2.6836, 4.4653]
7 ($n = 101$)	θ_{Ucr}	3.8302	5.3584	5.6372	3.9305	[2.8771, 4.8504]	3.4767	3.7763	4.0373	2.3286	[2.7815, 4.2654]
	β_{Ucr}	6.0197	2.5301	2.7162	1.9733	[5.5189, 6.4929]	6.9768	1.3092	2.3675	2.1088	[6.7328, 7.2311]
	θ_{Lcr}	1.5374	0.3891	0.6032	0.4723	[1.4559, 1.6079]	2.0416	0.0426	0.0594	0.0422	[2.0330, 2.0496]
	β_{Lcr}	1.3308	1.0560	3.8167	3.6692	[1.1327, 1.5271]	2.4228	2.5110	3.5895	3.3030	[1.9522, 2.9391]
8 ($n = 158$)	θ_{Ucr}	4.7457	6.2634	6.8206	4.4439	[3.9156, 5.7467]	4.0096	4.4712	4.8892	2.7484	[3.3210, 4.7256]
	β_{Ucr}	4.9297	2.2424	2.2364	1.6624	[4.5910, 5.3055]	6.7828	1.4066	2.2682	2.0119	[6.5385, 7.0128]
	θ_{Lcr}	1.5326	0.3533	0.5852	0.4721	[1.4788, 1.5871]	2.0436	0.0459	0.0632	0.0439	[2.0364, 2.0509]
	β_{Lcr}	1.3911	1.0483	3.7571	3.6266	[1.2278, 1.5647]	2.3259	2.2809	3.5101	3.1626	[1.9970, 2.7236]
9 ($n = 167$)	θ_{Ucr}	5.0363	6.0476	6.7508	4.4976	[4.1121, 5.9260]	3.9724	4.3912	4.8019	2.5592	[3.3127, 4.6455]
	β_{Ucr}	4.4973	1.7180	1.7850	1.4047	[4.2461, 4.7681]	6.9081	1.1358	2.2188	2.0020	[6.7445, 7.0674]
	θ_{Lcr}	1.6001	0.3195	0.5113	0.4038	[1.5466, 1.6453]	2.0565	0.0549	0.0787	0.0566	[2.0483, 2.0647]
	β_{Lcr}	1.2852	0.9772	3.8404	3.7339	[1.1446, 1.4232]	2.2266	2.2227	3.5500	3.2679	[1.8987, 2.5731]
10 ($n = 138$)	θ_{Ucr}	3.5472	4.2059	4.4671	3.0688	[2.8915, 4.2720]	2.8876	2.5809	2.7204	1.6309	[2.4726, 3.3449]
	β_{Ucr}	4.3065	1.4903	1.6389	1.3273	[4.0757, 4.5697]	7.0955	1.0378	2.3368	2.1327	[6.9258, 7.2765]
	θ_{Lcr}	1.6757	0.2419	0.4041	0.3329	[1.6348, 1.7138]	2.0575	0.0578	0.0814	0.0576	[2.0483, 2.0669]
	β_{Lcr}	1.0258	0.6146	4.0211	3.9742	[0.9271, 1.1326]	1.5556	1.2730	3.6705	3.4751	[1.3591, 1.7746]
11 ($n = 125$)	θ_{Ucr}	3.9551	3.7322	4.2001	3.0299	[3.3132, 4.6510]	2.8964	2.3063	2.4658	1.4007	[2.5548, 3.3066]
	β_{Ucr}	3.8386	1.3083	1.7455	1.5163	[3.6142, 4.0551]	7.0556	0.9912	2.2804	2.0695	[6.8975, 7.2252]
	θ_{Lcr}	1.6959	0.1966	0.3617	0.3090	[1.6608, 1.7283]	2.0495	0.0535	0.0727	0.0495	[2.0405, 2.0590]
	β_{Lcr}	0.9892	0.4730	4.0384	4.0108	[0.9083, 1.0801]	1.4923	1.0582	3.6626	3.5322	[1.3120, 1.6775]
12 ($n = 77$)	θ_{Ucr}	3.9488	3.7480	4.2027	2.7918	[3.1735, 4.8679]	2.9792	2.5533	2.7191	1.4164	[2.4922, 3.5783]
	β_{Ucr}	3.7924	1.4727	1.8971	1.6485	[3.4793, 4.1368]	7.2613	1.2691	2.5891	2.3091	[6.9777, 7.5449]
	θ_{Lcr}	1.7377	0.1895	0.3229	0.2623	[1.6926, 1.7757]	2.0414	0.0427	0.0592	0.0414	[2.0323, 2.0513]
	β_{Lcr}	0.9567	0.4932	4.0729	4.0433	[0.8578, 1.0689]	1.4802	1.0567	3.6730	3.5651	[1.2659, 1.7137]
13 ($n = 55$)	θ_{Ucr}	3.8050	3.0026	3.4799	2.6439	[2.9738, 4.5863]	2.6955	1.8379	1.9494	1.2473	[2.2389, 3.1853]
	β_{Ucr}	3.5198	1.2627	1.9381	1.7463	[3.1833, 3.8544]	7.2819	1.0565	2.5106	2.2849	[7.0185, 7.5791]
	θ_{Lcr}	1.8155	0.1760	0.2539	0.2015	[1.7686, 1.8631]	2.0576	0.0624	0.0845	0.0576	[2.0416, 2.0750]
	β_{Lcr}	0.7721	0.3618	4.2431	4.2279	[0.6842, 0.8734]	1.2535	0.8261	3.8348	3.7465	[1.0378, 1.4655]
14 ($n = 29$)	θ_{Ucr}	3.9969	2.8537	3.4425	2.3287	[3.0125, 5.0715]	2.6122	1.7300	1.8068	1.1034	[2.0455, 3.3581]
	β_{Ucr}	2.7929	0.6677	2.3026	2.2071	[2.5533, 3.0244]	6.8375	0.6940	1.9600	1.8375	[6.5726, 7.0897]
	θ_{Lcr}	1.8323	0.1777	0.2421	0.1846	[1.7728, 1.8949]	2.0517	0.0333	0.0612	0.0519	[2.0399, 2.0642]
	β_{Lcr}	0.7728	0.3620	4.2422	4.2272	[0.6463, 0.8984]	1.2670	0.7714	3.8092	3.7330	[1.0094, 1.5563]
15 ($n = 12$)	θ_{Ucr}	4.6863	4.6307	5.1838	3.6863	[2.2542, 7.3950]	2.9589	2.3092	2.4099	1.7402	[1.7963, 4.2608]
	β_{Ucr}	2.7129	1.2301	2.5726	2.3292	[2.0701, 3.3871]	6.8780	1.2900	2.2478	1.9267	[6.1400, 7.5695]
	θ_{Lcr}	1.7487	0.1701	0.2995	0.2513	[1.6581, 1.8382]	2.0299	0.0245	0.0380	0.0299	[2.0178, 2.0450]
	β_{Lcr}	0.8459	0.4163	4.1732	4.1541	[0.6233, 1.0652]	1.0916	0.8376	3.9899	3.9084	[0.6830, 1.5590]
16 ($n = 8$)	θ_{Ucr}	5.2211	2.5462	4.0061	3.5206	[3.5701, 6.9010]	2.7984	0.9575	1.1998	0.9764	[2.1985, 3.3854]
	β_{Ucr}	2.5644	1.2201	2.6898	2.4356	[1.8784, 3.3773]	7.0580	1.3189	2.3994	2.0580	[6.2892, 7.9777]
	θ_{Lcr}	1.8271	0.1074	0.2000	0.1748	[1.7589, 1.8934]	2.0465	0.0421	0.0609	0.0467	[2.0210, 2.0730]
	β_{Lcr}	0.8916	0.2428	4.1147	4.1084	[0.7319, 1.0522]	1.3662	0.4712	3.6604	3.6338	[1.0511, 1.6589]
17 ($n = 3$)	θ_{Ucr}	2.2844	0.8104	0.6397	0.5730	[1.7114, 2.8574]	1.6180	0.4937	0.5175	0.3820	[1.2689, 1.9671]
	β_{Ucr}	2.3395	0.2042	2.6644	2.6605	[2.1951, 2.4839]	6.7283	0.3242	1.7435	1.7283	[6.4991, 6.9576]
	θ_{Lcr}	1.7583	0.3104	0.3265	0.2417	[1.5388, 1.9778]	2.0382	0.0008	0.0382	0.0382	[2.0377, 2.0388]
	β_{Lcr}	0.8240	0.2887	4.1810	4.1760	[0.6198, 1.0281]	1.0845	0.0200	3.9155	3.9155	[1.0703, 1.0986]

Table 4. Simulation results when $\theta = 2$ and $\beta = 7$.

ζ	Parameter	MLE					BE				
		Mean	SD	RMSE	MAD	95% BCI	Mean	SD	RMSE	MAD	95% BCI
4 ($n = 6$)	θ_{Ucr}	6.5915	14.739	13.96	7.7915	[0, 19.7745]	6.5808	11.949	11.627	5.7981	[0.941, 17.247]
	β_{Ucr}	14.637	8.1872	10.581	9.6732	[8.152, 20.352]	7.8478	2.1602	2.11	1.9201	[6.0451, 9.288]
	θ_{Lcr}	1.6078	0.1062	0.4035	0.3922	[1.5245, 1.6908]	2.0567	0.1004	0.1063	0.0605	[2.0023, 2.1506]
	β_{Lcr}	2.5133	3.4119	5.4262	5.1207	[0.7970, 5.6743]	6.1107	9.7821	8.7944	7.4667	[0.8915, 14.901]
5 ($n = 30$)	θ_{Ucr}	11.195	15.478	17.781	11.195	[5.9750, 17.453]	11.089	13.901	16.414	9.7435	[6.7515, 16.504]
	β_{Ucr}	9.5257	5.0065	5.5325	4.9032	[7.5899, 11.237]	7.2125	2.2408	2.2134	1.7583	[6.3294, 7.9256]
	θ_{Lcr}	1.4517	0.4310	0.6930	0.5576	[1.3061, 1.5907]	2.0609	0.0683	0.0906	0.0689	[2.0377, 2.0837]
	β_{Lcr}	4.2108	4.5722	5.2903	4.5701	[2.6799, 5.9147]	9.5697	12.565	12.618	8.6385	[5.5318, 14.299]
6 ($n = 73$)	θ_{Ucr}	9.1110	13.552	15.222	8.9926	[6.1692, 12.462]	8.8064	12.123	13.83	7.4577	[6.1686, 11.902]
	β_{Ucr}	8.6651	4.7379	4.9913	4.0090	[7.6591, 9.7593]	7.5082	2.2510	2.2925	1.8171	[6.9899, 8.0048]
	θ_{Lcr}	1.3938	0.5116	0.7909	0.6073	[1.2681, 1.5057]	2.0566	0.0580	0.0808	0.0579	[2.0439, 2.0694]
	β_{Lcr}	2.8442	2.7753	4.9867	4.6180	[2.2418, 3.4657]	6.6409	8.6446	8.5927	6.2682	[4.7728, 8.7699]
7 ($n = 127$)	θ_{Ucr}	6.3671	8.9123	9.8932	6.0336	[4.8678, 8.0186]	6.1623	7.4733	8.5284	4.7695	[4.9543, 7.5329]
	β_{Ucr}	7.9787	3.5521	3.6710	2.6175	[7.3674, 8.5800]	7.9362	1.6702	1.9090	1.4751	[7.6487, 8.2138]
	θ_{Lcr}	1.4054	0.4679	0.7555	0.6074	[1.3208, 1.4867]	2.0570	0.0577	0.0810	0.0573	[2.0476, 2.0669]
	β_{Lcr}	2.0692	1.7090	5.2163	4.9807	[1.7806, 2.4077]	4.0508	4.4629	5.3346	4.5105	[3.3097, 4.8157]
8 ($n = 123$)	θ_{Ucr}	5.7171	7.9551	8.7513	5.4133	[4.3368, 7.1816]	5.2503	6.3409	7.1024	3.8220	[4.1957, 6.3948]
	β_{Ucr}	7.0327	2.8136	2.8023	2.1905	[6.5396, 7.5151]	8.0002	1.5157	1.8108	1.4774	[7.7421, 8.2671]
	θ_{Lcr}	1.5131	0.4355	0.6521	0.5032	[1.4337, 1.5848]	2.0731	0.0806	0.1086	0.0733	[2.0601, 2.0871]
	β_{Lcr}	1.7722	1.5690	5.4564	5.2853	[1.5152, 2.0552]	3.1662	3.0438	4.8875	4.4575	[2.6467, 3.7046]
9 ($n = 168$)	θ_{Ucr}	5.6312	8.5340	9.2510	5.2319	[4.3979, 7.0395]	5.1178	6.7636	7.4293	3.5904	[4.1399, 6.3266]
	β_{Ucr}	6.4148	2.6018	2.6592	1.9551	[6.0351, 6.8133]	8.0765	1.5624	1.8935	1.5673	[7.8411, 8.2949]
	θ_{Lcr}	1.5759	0.4009	0.5828	0.4378	[1.5138, 1.6351]	2.0702	0.0802	0.1064	0.0704	[2.0588, 2.0835]
	β_{Lcr}	1.5721	1.3707	5.5973	5.4650	[1.3845, 1.7939]	2.8022	2.8068	5.0451	4.7201	[2.3878, 3.2516]
10 ($n = 180$)	θ_{Ucr}	5.6996	7.5527	8.3912	4.8362	[4.6530, 6.8907]	4.8025	6.1366	6.7308	3.2139	[3.9657, 5.7502]
	β_{Ucr}	5.5406	1.9453	2.4275	1.9455	[5.2579, 5.8126]	8.0034	1.2964	1.6365	1.3262	[7.7941, 8.1865]
	θ_{Lcr}	1.6108	0.3901	0.5503	0.4050	[1.5531, 1.6632]	2.0703	0.0756	0.1030	0.0705	[2.0601, 2.0817]
	β_{Lcr}	1.4539	1.0951	5.6525	5.5461	[1.3029, 1.6089]	2.6334	2.6244	5.0909	4.7981	[2.2656, 3.0182]
11 ($n = 110$)	θ_{Ucr}	4.5896	5.9249	6.4414	3.9053	[3.4933, 5.8119]	4.0754	4.3718	4.8214	2.5811	[3.3379, 4.9725]
	β_{Ucr}	5.6790	2.0100	2.3976	1.9906	[5.3033, 6.0639]	8.4088	1.3198	1.9264	1.5984	[8.1678, 8.6461]
	θ_{Lcr}	1.6654	0.3061	0.4526	0.3482	[1.6067, 1.7222]	2.0635	0.0669	0.0920	0.0635	[2.0515, 2.0758]
	β_{Lcr}	1.2484	0.8472	5.8131	5.7516	[1.0940, 1.4204]	2.0826	1.8827	5.2625	5.0531	[1.7705, 2.4260]
12 ($n = 78$)	θ_{Ucr}	5.0755	5.1897	6.0039	4.2045	[3.9645, 6.2447]	4.0039	3.6963	4.1837	2.4355	[3.2349, 4.8506]
	β_{Ucr}	5.0017	1.7036	2.6188	2.2436	[4.6143, 5.3498]	8.3004	1.2357	1.7884	1.4891	[8.0181, 8.5747]
	θ_{Lcr}	1.7151	0.2694	0.3909	0.3045	[1.6514, 1.7719]	2.0821	0.0782	0.1131	0.0821	[2.0660, 2.1010]
	β_{Lcr}	1.1984	0.6571	5.8382	5.8016	[1.0538, 1.3419]	1.9884	1.5081	5.2308	5.0333	[1.6586, 2.3150]
13 ($n = 55$)	θ_{Ucr}	4.3017	4.0022	4.5852	3.3005	[3.2969, 5.3843]	3.4601	2.5976	2.9592	1.7961	[2.8600, 4.1724]
	β_{Ucr}	4.8890	1.2211	2.4332	2.1843	[4.5753, 5.1955]	8.5358	0.9456	1.7990	1.5555	[8.2895, 8.7887]
	θ_{Lcr}	1.7630	0.2131	0.3174	0.2422	[1.7027, 1.8157]	2.0625	0.0765	0.0983	0.0626	[2.0443, 2.0842]
	β_{Lcr}	1.0218	0.5341	6.0015	5.9782	[0.8875, 1.1675]	1.6493	1.2813	5.4992	5.3507	[1.3325, 1.9913]
14 ($n = 30$)	θ_{Ucr}	4.0398	4.1052	4.5224	3.0560	[2.6682, 5.5350]	3.2496	3.1247	3.3166	1.6323	[2.3206, 4.3890]
	β_{Ucr}	4.4539	1.2173	2.8134	2.5461	[4.0318, 4.9218]	8.3995	1.0144	1.7185	1.4415	[8.0261, 8.7622]
	θ_{Lcr}	1.8091	0.2495	0.3108	0.2346	[1.7134, 1.8988]	2.0928	0.0863	0.1257	0.0928	[2.0645, 2.1248]
	β_{Lcr}	0.9742	0.5433	6.0495	6.0258	[0.7962, 1.1735]	1.4924	1.3400	5.6630	5.5076	[1.0669, 1.9776]
15 ($n = 9$)	θ_{Ucr}	4.9904	4.6601	5.3148	3.3347	[2.5041, 7.9878]	3.5651	3.0164	3.2461	1.7337	[2.1107, 5.6468]
	β_{Ucr}	3.6588	1.0655	3.4890	3.3412	[2.9914, 4.3329]	7.8794	1.0167	1.3008	1.0103	[7.2764, 8.5256]
	θ_{Lcr}	1.9013	0.2073	0.2190	0.1680	[1.7717, 2.0217]	2.1115	0.0891	0.1396	0.1115	[2.0594, 2.1699]
	β_{Lcr}	0.8887	0.6309	6.1402	6.1113	[0.5463, 1.2911]	1.3463	0.9031	5.7174	5.6537	[0.8772, 1.9685]
16 ($n = 11$)	θ_{Ucr}	5.7895	5.8345	6.6443	4.8326	[2.3503, 9.4935]	4.0233	2.9587	3.4283	2.4839	[2.2263, 6.1122]
	β_{Ucr}	3.9011	1.4483	3.3821	3.0989	[2.9537, 4.8048]	8.4181	1.3552	1.9021	1.6183	[7.5522, 9.2243]
	θ_{Lcr}	1.5899	0.2179	0.4580	0.4101	[1.4471, 1.7259]	2.0372	0.0398	0.0526	0.0372	[2.0146, 2.0637]
	β_{Lcr}	1.2635	0.4586	5.7525	5.7365	[1.0182, 1.6112]	1.8363	1.2123	5.2867	5.1637	[1.1009, 2.6941]

6.2. Sensitivity Analysis of Prior Distributions

Since the proposed estimation and prediction procedures are Bayesian, it is important to examine the effect of prior selection on the posterior inference. To evaluate the robustness of the proposed methodology, a sensitivity analysis was performed using three

alternative independent Gamma prior specifications for the location parameter θ and the scale parameter β :

$$\begin{aligned}\text{Prior I: } & \theta \sim \Gamma(2, 1), & \beta & \sim \Gamma(2, 1), \\ \text{Prior II: } & \theta \sim \Gamma(1, 1), & \beta & \sim \Gamma(1, 1), \\ \text{Prior III: } & \theta \sim \Gamma(5, 2), & \beta & \sim \Gamma(5, 2).\end{aligned}$$

For each prior specification, posterior samples were generated using the Metropolis–Hastings algorithm described in Section 4.2, and the Bayesian estimates, Bayesian predictive values (BPVs), and Bayesian predictive intervals (BPIs) were recomputed under identical simulation settings.

The sensitivity analysis was performed using the observed upper and lower current records of size 10 obtained from a 100 random generated data of Two-Parameter Exponential $E(70, 4)$:

$$U^{cr} = \{73.5362, 78.8988, 81.1220, 81.1220, 81.5532, 82.0116, 83.4467, 83.4467, 83.8048, 83.8048\},$$

$$L^{cr} = \{73.5362, 73.5362, 73.5362, 71.5224, 71.5224, 71.5224, 71.5224, 69.6355, 69.6355, 68.2002\}.$$

Tables 5 and 6 present the Bayesian estimates and predictive results obtained under the three prior specifications.

Table 5. Sensitivity analysis of Bayesian estimates and predictions for upper current records under different prior specifications.

Prior	Prior for θ	Prior for β	$\hat{\theta}_{U^{cr}}^{BE}$	$\hat{\beta}_{U^{cr}}^{BE}$	$\hat{U}_{11}^{cr}$	95% of $\hat{U}_{11}^{cr}$
Prior I	$\Gamma(2, 1)$	$\Gamma(2, 1)$	2.4051	6.8242	90.6787	(83.9687, 109.9568)
Prior II	$\Gamma(1, 1)$	$\Gamma(1, 1)$	1.2663	6.2144	90.0728	(83.9545, 107.8964)
Prior III	$\Gamma(5, 2)$	$\Gamma(5, 2)$	2.7091	5.8439	89.6673	(83.9489, 105.9273)

Table 6. Sensitivity analysis of Bayesian estimates and predictions for lower current records under different prior specifications.

Prior	Prior for θ	Prior for β	$\hat{\theta}_{L^{cr}}^{BE}$	$\hat{\beta}_{L^{cr}}^{BE}$	$\hat{L}_{11}^{cr}$	95% of $\hat{L}_{11}^{cr}$
Prior I	$\Gamma(2, 1)$	$\Gamma(2, 1)$	68.1798	3.2313	68.1899	(68.1033, 68.2002)
Prior II	$\Gamma(1, 1)$	$\Gamma(1, 1)$	68.1871	2.2206	68.1936	(68.1359, 68.2002)
Prior III	$\Gamma(5, 2)$	$\Gamma(5, 2)$	68.1730	2.7758	68.1864	(68.0781, 68.2002)

The results show only minor changes in the posterior estimates, BPVs, and BPIs across the three prior specifications. In particular, the predicted upper and lower current records and their corresponding predictive intervals remain nearly unchanged, indicating that the observed current-record data dominate the posterior inference. Therefore, the proposed Bayesian estimation and prediction procedures are robust to reasonable choices of prior distributions, demonstrating that the reported predictive performance is driven primarily by the observed data rather than by the selected priors.

6.3. Saudi Arabian Network Latency Data: Real Example

The real dataset consists of communication network latency measurements collected for Saudi Arabia over successive observation periods. Network latency represents the time required for a data packet to travel between a source and its destination and is one of the primary indicators used to evaluate network performance. Although the average latency reflects the overall quality of the network, unusually large latency values are of greater practical importance because they may indicate temporary congestion, routing problems,

or abnormal traffic conditions. Consequently, monitoring extreme latency observations is essential for maintaining reliable network performance and ensuring compliance with quality-of-service requirements. As a demonstration of the effective work presented in this paper, the following real data on median latency times for the Saudi Arabia network from 16 Decmber 2025 to 14 January 2026 were used to achieve that. The data are in milliseconds (ms), and you can find it on this link: <https://radar.cloudflare.com/explorer?dataSet=quality.iqi&metric=latency&loc=sa&dt=12w> (accessed on 8 June 2026).

85.901145, 85.627304, 86.206969, 86.935237, 85.119027, 84.56502, 83.261236, 82.94845, 82.909258, 82.337733, 83.734564, 83.985641, 83.613285, 82.921656, 82.888806, 82.577017, 82.805566, 84.459886, 84.375657, 82.643467, 81.743459, 82.402109, 82.496672, 86.332507, 88.352858, 89.458687, 91.746836, 90.412028, 87.350245, 86.722171.

The two-parameter Exponential distribution $E(\theta, \beta)$, with an estimated location parameter $\hat{\theta} = 81.7435$ and an estimated scale parameter $\hat{\beta} = 3.15102$, is the best-suited distribution to these data, according to our test after comparing with the Weibull, Lognormal, and Pareto distributions. The Kolmogorov–Smirnov test (p -value = 0.565214) and the Cramér-von Mises test (p -value = 0.721624) are the goodness-of-fit tests utilized to confirm this with higher values than all other distributions. The Histogram of the data with the proposed fitted distribution, the Q-Q plot, and the probability Plot are displayed in Figure 6.

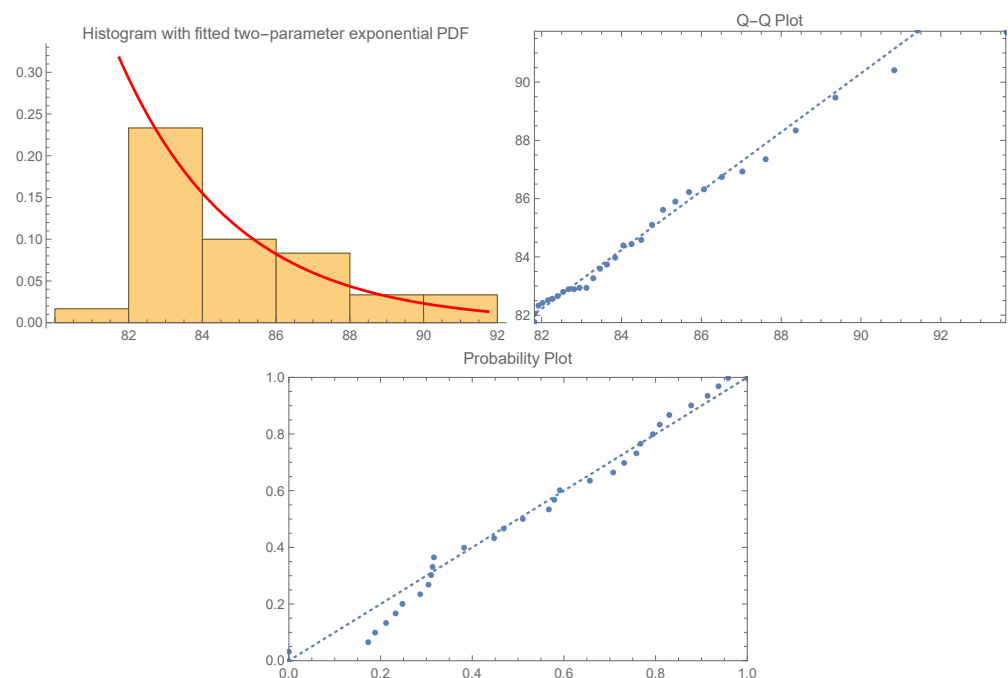


Figure 6. $E(81.7435, 3.15102)$ against the real data.

We now employ the two suggested methods, CPI and BPV with BPI, to forecast future upper and lower currents as well as the record range.

(1) **CPI:**

We use the suggested algorithm in Section 5.1 in the manner described below to do this:

First step: As mentioned, the data does fit the two-parameter Exponential distribution with $\hat{\theta} = 81.7435$ and $\hat{\beta} = 3.15102$.

Second step:

Third step: The BE equations from Section 4.2 and values from Table 7 can be used to determine $\hat{\theta}_{U_{cr}}^{BE} = 83.2511$ with 95% credible intervals (CI) (78.293, 85.5908), $\hat{\beta}_{U_{cr}}^{BE} = 2.49462$ with CI (1.38486, 4.06434), $\hat{\theta}_{L_{cr}}^{BE} = 81.7435$ with CI (81.7434, 81.7435), and $\hat{\beta}_{L_{cr}}^{BE} = 1.58817$ with CI (1.21452, 1.87945).

While the MLEs can be determined from the equations in Section 4.1 to equal $\hat{\theta}_{U_{cr}}^{ML} = 85.3223$, $\hat{\beta}_{U_{cr}}^{ML} = 0.8351$, $\hat{\theta}_{L_{cr}}^{ML} = 81.4697$, and $\hat{\beta}_{L_{cr}}^{ML} = 1.14669$.

Table 7. Current records U_{ξ}^{cr} , L_{ξ}^{cr} and their differences R_{ξ}^{cr} .

ξ	1	2	3	4	5	6	7
U_{ξ}^{cr}	85.9011	85.9011	86.207	86.9352	86.9352	86.9352	86.9352
L_{ξ}^{cr}	85.9011	85.6273	85.6273	85.6273	85.119	84.565	83.2612
R_{ξ}^{cr}	0.0000	0.2738	0.5797	1.3080	1.8162	2.3702	3.6740
ξ	8	9	10	11	12	13	14
U_{ξ}^{cr}	86.9352	86.9352	86.9352	86.9352	88.3529	89.4587	91.7468
L_{ξ}^{cr}	82.9485	82.9093	82.3377	81.7435	81.7435	81.7435	81.7435
R_{ξ}^{cr}	3.9867	4.0260	4.5975	5.1917	6.6094	7.7152	10.0033

Fourth step: The values of $Y_{\tau,\alpha}$ at $\alpha = 0.05$ and $\xi = 14$ will be $Y_{1,0.05} = 0.208182$, $Y_{2,0.05} = 0.338022$, and $Y_{3,0.05} = 0.45655$ to predict the next three upper and lower currents. Tables 8 and 9 display the CPI for the upcoming three current records in both cases of estimation techniques.

Table 8. CPI for the future current records under BE.

$\xi = 14$	$U_{\xi+\tau}^{cr}$	$L_{\xi+\tau}^{cr}$	$R_{\xi+\tau}^{cr}$
$\tau = 1$	(91.7468, 93.5155)	(81.7435, 81.7435)	(10.0033, 11.7720)
$\tau = 2$	(91.7468, 94.6185)	(81.7435, 81.7435)	(10.0033, 12.8750)
$\tau = 3$	(91.7468, 95.6255)	(81.7435, 81.7435)	(10.0033, 13.8820)

Table 9. CPI for the future current records under MLE.

$\xi = 14$	$U_{\xi+\tau}^{cr}$	$L_{\xi+\tau}^{cr}$	$R_{\xi+\tau}^{cr}$
$\tau = 1$	(91.7468, 93.0843)	(81.6613, 81.7435)	(10.0033, 11.4230)
$\tau = 2$	(91.7468, 93.9184)	(81.6239, 81.7435)	(10.0033, 12.2945)
$\tau = 3$	(91.7468, 94.6799)	(81.5965, 81.7435)	(10.0033, 13.0834)

Fifth step: In the present analysis, the predictive intervals for the future lower current records do not vary with the prediction order τ . This occurs because the last observed lower current record coincides with the estimated location parameter of the two-parameter exponential distribution. Specifically, the BE obtained from the lower current records yields $\hat{\theta}_{L_{cr}}^{BE} = 81.7435$, which is identical to the most recent lower record $L_{\xi}^{cr} = 81.7435$. For a two-parameter exponential distribution, the cumulative distribution function is given by $F(x) = 1 - \exp\left(-\frac{x-\theta}{\beta}\right)$ for $x > \theta$, and therefore $F(\theta) = 0$. Since $L_{\xi}^{cr} = \hat{\theta}_{L_{cr}}^{BE}$, it follows that $F(L_{\xi}^{cr}) = 0$, which implies that $F^{1+Y_{\tau,\alpha}}(L_{\xi}^{cr}) = 0$ for all τ . Consequently, the inverse transformation yields $F^{-1}(0) = \theta$, and the predictive interval $\left(F^{-1}\left(F^{1+Y_{\tau,\alpha}}(L_{\xi}^{cr})\right), L_{\xi}^{cr}\right)$ reduces to (θ, θ) . As a result, the lower predictive interval collapses to the single point (81.7435, 81.7435) for all predicted orders τ . To avoid that, we will use the MLE estimates instead, and the results are presented in Table 9.

(2) **BPV with BPI:**

Equations (34)–(36) are applied to the data in Table 4 in order to forecast future upper current records using BP with BPI. Equations (39) and (40) can be used to achieve analogous work to lower current records. Table 10 displays our predictions for the next three current records, along with their BPIs for $\xi = 1, 2$, and 3. The same thing occurs for the lower current record when we use the BE, for the same reasons previously mentioned. That's why, in Table 11, we used the MLE for the parameters. This results in narrower intervals and more accurate results.

Table 10. BPV and BPI for the future current records under BE.

$\xi = 14$	$\hat{U}_{\xi+\tau}^{cr}$	$\hat{L}_{\xi+\tau}^{cr}$	$\hat{R}_{\xi+\tau}^{cr}$
$\tau = 1$	94.2559 (91.8018,101.8265)	81.7435 (81.7434,81.7437)	12.5124 (10.0584,20.0830)
$\tau = 2$	96.7610 (92.2900,106.8942)	81.7435 (81.7433,81.7437)	15.0175 (10.5466,25.1507)
$\tau = 3$	99.3022 (93.0518,111.6608)	81.7435 (81.7433,81.7437)	17.5587 (11.3084,29.9173)

Table 11. BPV and BPI for the future current records under MLE.

$\xi = 14$	$\hat{U}_{\xi+\tau}^{cr}$	$\hat{L}_{\xi+\tau}^{cr}$	$\hat{R}_{\xi+\tau}^{cr}$
$\tau = 1$	92.6144 (91.3304,96.2631)	81.6018 (81.4760,81.7359)	11.0126 (9.7029,14.6576)
$\tau = 2$	93.4489 (90.8267,98.8429)	81.5334 (81.4706,81.6777)	11.9155 (9.3026,17.2902)
$\tau = 3$	94.3202 (90.4589,101.2168)	81.5014 (81.4699,81.6093)	12.8187 (8.9521,19.7114)

The predictive results obtained from the real network latency data provide useful insights into the expected behavior of future latency measurements. The predictive intervals for the next upper and lower current records suggest that future record-breaking latency observations are likely to remain within a moderate range relative to the observed data. In particular, the predicted upper record values indicate that although larger latency spikes may still occur, the magnitude of these extreme delays is not expected to increase dramatically in the near future. This suggests that the network performance is relatively stable, with no strong indication of severe latency deterioration.

On the other hand, the predicted lower record values imply that future minimum latency times may be slightly lower than previously observed, suggesting the possibility of improved response times under favorable network conditions. In practical terms, this means that, while occasional higher-latency events may still occur, the overall network behavior is expected to maintain similar variability, with potential improvements in fastest response times.

From an operational perspective, these results indicate that the network infrastructure is functioning within a predictable latency range. The predictive analysis of the next few record values, therefore, provides useful information for anticipating extreme latency events and assessing whether the network performance is likely to remain stable or experience unusual delays in the near future. From a practical networking perspective, the predicted future extreme latency values can assist network operators in monitoring service quality and identifying potential performance degradation before service interruptions occur. In particular, accurate prediction of unusually large latency observations enables proactive resource allocation, early congestion detection, and capacity planning. Furthermore, these predictions can support Service Level Agreement (SLA) monitoring by providing advance warning when future latency values are expected to approach or exceed acceptable operational thresholds.

7. Conclusions

This paper developed a statistical framework for modeling and predicting network latency times using upper and lower current record values from the two-parameter exponential distribution. The work was motivated by the need to understand extreme latency behavior in communication networks, where record-breaking delays directly impact quality of service and network reliability.

The theoretical contributions include explicit expressions for the probability density functions, cumulative distribution functions, and moments of lower and upper current records under the two-parameter exponential model. Parameter estimation was investigated using both maximum likelihood and Bayesian approaches, with Markov Chain Monte Carlo methods employed to compute Bayesian estimates when closed-form solutions were unavailable. Classical and Bayesian predictive procedures were developed for forecasting future upper and lower current records as well as record ranges.

The methodology was applied to real network latency data from Saudi Arabia collected from December 2025 to January 2026. The two-parameter exponential distribution provided an excellent fit to the data, confirmed by goodness-of-fit tests. Predictive intervals constructed for future upper records indicated that extreme latency spikes are expected to remain within a moderate range, suggesting stable network performance with no dramatic deterioration. Predictive intervals for lower records pointed to possible improvements in minimum response times under favorable conditions. Notably, when the last observed lower record coincided with the estimated location parameter, Bayesian predictive intervals collapsed to a single point, whereas maximum-likelihood-based intervals provided more informative ranges.

From a practical perspective, the predictive intervals obtained in this study provide network operators with valuable tools for anticipating extreme latency events, assessing whether observed delays are consistent with historical patterns, and identifying potential anomalies that may signal performance degradation. The ability to quantify uncertainty around future record values supports proactive decision-making in capacity planning, congestion control, and service-level agreement monitoring.

In conclusion, this study demonstrates that record-based inference combined with Bayesian prediction provides effective tools for modeling and forecasting extreme latency observations in communication networks. The theoretical developments, simulation evidence, and real-data application collectively support the practical value of the proposed methodology for enhancing network performance analysis and reliability assessment in modern communication systems.

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