

Article

A Projection-Based Framework for Exact Partial Controllability of Semilinear Stochastic Fractional Evolution Equations with State-Dependent Delays, Impulses, and Q -Wiener Perturbations

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Abstract

We study the exact partial controllability of a semilinear stochastic fractional evolution equation of Caputo order $\alpha \in (1, 2)$ in a separable Hilbert space, subject to a genuine state-dependent delay $y(t - \rho(t, y_t))$, impulsive effects, and Q -Wiener perturbations. To accommodate the state-dependent delay, we work in a Sobolev-type prehistory space that restores the local Lipschitz property lost in the space of continuous functions, and to accommodate the impulses, we use a piecewise-continuous mean-square path space; two initial data points are prescribed, as required for $\alpha \in (1, 2)$. The existence and uniqueness of mild solutions are established by the Banach contraction principle on a Lipschitz ball equipped with a Bielecki-type weighted norm, which removes the smallness conditions of the contraction-based literature. The stochastic convolution is treated rigorously by the Da Prato factorization realized through the subordination of the fractional resolvent family, with the Burkholder inequality applied only to the genuine Itô integral and never to the non-martingale convolution itself. For controllability, we adopt the correct stochastic notion: the target is an $\mathcal{F}_T$ -measurable random variable, and the adapted control is constructed through the martingale representation theorem, together with the projected controllability Gramian, yielding $Py(T) = \eta$ almost surely. A numerical study on a stochastic fractional wave equation verifies the main controllability theorem, driving the projected terminal state to the prescribed $\mathcal{F}_T$ -measurable target at machine precision, and confirms the observability requirement.

Keywords: stochastic fractional evolution equations; exact partial controllability; state-dependent delay; impulsive effects; Q -Wiener process; martingale representation; Bielecki weighted norm; Da Prato factorization; controllability Gramian



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1. Introduction

Fractional-order stochastic systems arise naturally in the modeling of dynamical processes with long-range memory, hereditary effects, and random perturbations, including viscoelastic materials, anomalous diffusion, and complex mechanical systems [1–3]. The combination of nonlocal fractional derivatives, time delays, and stochastic forcing leads to evolution equations whose analysis requires tools from fractional calculus, semigroup theory, and stochastic analysis. In particular, state-dependent delays provide a flexible

mechanism to capture feedback and switching phenomena in engineering and biological systems, but they substantially complicate the controllability analysis [4,5]. Related qualitative questions for fractional systems—such as Lyapunov-based stability of fractional chaotic and hyperchaotic models and the synchronization of fractional discrete reaction–diffusion systems—have also been studied through Mittag–Leffler and fractional-Lyapunov techniques [6–8].

During the last decade, the controllability properties of fractional stochastic systems have been investigated in various settings. For systems with Caputo or Hilfer derivatives and fixed or distributed delays, several works have established the existence of mild solutions and approximate controllability using compact semigroups and fixed-point arguments [9–11]. Stochastic fractional systems with impulsive effects and state-dependent delays have also been studied, with sufficient conditions for approximate controllability derived under assumptions on the corresponding linear part [12–18]. By contrast, exact controllability in infinite-dimensional spaces is often unattainable, owing to non-closed reachable sets; this has motivated the weaker notions of null, partial, and approximate controllability [19–21].

Most recently, Nandhaprasadh et al. [22] studied partial approximate controllability of Hilfer fractional stochastic systems with nonlocal conditions, and Jneid [23] treated the exact controllability of semilinear integrodifferential fractional stochastic systems in the conformable sense with impulses, using the contraction principle and the controllability Gramian in finite-dimensional spaces. Nonetheless, the existing literature focuses overwhelmingly on approximate controllability in Hilbert spaces. Approximate controllability only guarantees that the reachable set is dense, so the target can be approached arbitrarily closely but, in general, never attained; for applications in which the terminal state must be reached on the nose—on a prescribed finite-dimensional subspace of observables—this is insufficient. Exact controllability of the *entire* infinite-dimensional state is, however, typically impossible because the reachable set is not closed. This tension is precisely what motivates *exact partial* controllability: exactness on a chosen finite-dimensional subspace H_1 , combined with mean-square stability of its complement. Few results address this notion for semilinear stochastic fractional systems that simultaneously feature fully state-dependent delays, impulsive effects, and Q -Wiener perturbations [24–26]. The present paper addresses this gap: we develop a projection-based framework, combining mild solutions, an orthogonal projection onto a finite-dimensional subspace, and the *Banach contraction principle* [27] on a complete metric space equipped with a Bielecki-type weighted norm, with the stochastic convolution handled through a subordination-based Da Prato factorization, to establish verifiable sufficient conditions for exact partial controllability.

The results are illustrated by analytical examples and a Monte Carlo study on a stochastic fractional wave equation, discretized by a 32-mode Galerkin projection, with state-dependent delays, impulsive jumps, and Q -Wiener stochastic perturbations.

The numerical treatment of fractional dynamics with hereditary (delay) effects is itself an active area, and several complementary computational paradigms inform the broader context of the present work. The Differential Transformation Method (DTM) has been applied effectively to fractional differential equations with delay, providing a semi-analytical route to the solution of hereditary fractional dynamics [28]. More recently, hybrid and learning-based schemes have proven effective for the integral–algebraic systems that arise from fractional operators: the combination of DTM with Physics-Informed Neural Networks (PINNs) and symbolic transforms has been used to solve such systems directly [29,30], and metaheuristic optimization has been applied to inverse problems for fractional diffusion equations [31]. These data-driven and optimization-based methods address the forward and inverse *solution* of fractional PDEs and are complementary to the

analytical controllability framework developed here, in which the control law is obtained in closed form through the projected Gramian and the martingale representation theorem.

1.1. Comparison with Closely Related Work

Among recent advances, Asadzade and Mahmudov [32] established necessary and sufficient conditions for approximate controllability of linear fractional impulsive evolution equations via resolvent operators, and Ahmed et al. [33] investigated higher-order Hilfer fractional neutral stochastic systems driven by fractional Brownian motion and Poisson jumps, both extending the classical Gramian characterization to broader fractional settings. The works of Arthi et al. [2,16] treat *approximate* controllability of stochastic fractional systems with state-dependent delays and impulses, while Mahmudov [25] develops Gramian-based controllability for relative state-delay systems. The present paper differs from all of these in three concrete ways. *First*, we establish *exact* partial controllability (almost-sure attainment of $Py(T) = z_T^1$ on a prescribed subspace), which is strictly stronger than the approximate notion that dominates the cited literature. *Second*, we work with the Caputo derivative of order $\alpha \in (1, 2)$ in a separable Hilbert space under *fully* state-dependent delays and Q -Wiener noise simultaneously, a combination not covered by the finite-dimensional or $\alpha \in (0, 1)$ analyses of the cited works. *Third*, we provide an explicit closed-form adapted control law together with a verified infinite-dimensional numerical implementation, rather than an abstract existence statement alone. To our knowledge, this specific combination—exact partial controllability for semilinear stochastic systems of order $\alpha \in (1, 2)$ with genuine state-dependent delays, impulses, Q -Wiener perturbations, and a projection-based framework—has not previously appeared in the literature.

1.2. Why the Caputo Derivative of Order $\alpha \in (1, 2)$

The choice of derivative is dictated by the modeling target. Conformable derivatives, used in the exact-controllability study of Jneid [23], are local operators and, therefore, do not encode the long-range memory that motivates fractional models; Hilfer derivatives interpolate between Riemann–Liouville and Caputo but, in the order range $(1, 2)$, require initial data expressed through fractional integrals of the state, which are difficult to prescribe and to interpret physically. The Caputo derivative of order $\alpha \in (1, 2)$, by contrast, admits classical initial data $y(0) = \phi(0)$ and $y'(0) = \zeta$ (displacement and velocity), [34], and generates a pair of resolvent families, $\{C_\alpha, S_\alpha\}$, together with the forcing family $\{K_\alpha\}$ [35], whose calculus is exactly suited to the wave-type dynamics that we study. The order range $(1, 2)$ itself is the natural regime for *wave propagation with memory*: $\alpha = 2$ is the classical wave equation, $\alpha \rightarrow 1^+$ degenerates to diffusion, and intermediate α models viscoelastic and anomalous-transport media in which signals propagate at a finite speed yet are damped by hereditary effects. This is the regime relevant to the stochastic fractional wave equation of Section 8, and it is distinct from the diffusion-dominated case $\alpha \in (0, 1)$ studied in [23,25].

1.3. Contributions of the Present Work

Within the broader landscape of fractional stochastic controllability, this paper makes the following contributions, which extend and complement existing work:

- (a) We address exact partial controllability on a prescribed finite-dimensional subspace, H_1 , complementing the approximate-controllability results of [2,11,13]. Because an adapted control cannot steer a stochastic system to a fixed deterministic point almost surely, we adopt the mathematically correct notion (in the sense of Bashirov–Mahmudov [21]): the target is an $\mathcal{F}_T$ -measurable random variable $\eta \in L^2(\Omega, \mathcal{F}_T; H_1)$, and the control achieves $Py(T) = \eta$ almost surely. As a baseline structural assumption, the projected controllability Gramian must be invertible on H_1 ; this is an observability

- condition that, in particular, requires the actuator to reach every mode of H_1 (so that $\dim(U) \geq \dim(H_1)$), as we make explicit and verify numerically in Section 8.
- (b) We treat the case $\alpha \in (1, 2)$, which involves the simultaneous use of the solution operators C_α , S_α and the forcing family K_α , and is technically distinct from the case $\alpha \in (0, 1)$ studied in [23,25]. Two initial data (displacement and velocity) are prescribed accordingly.
 - (c) We incorporate a *genuine* state-dependent delay $y(t - \rho(t, y_t))$, impulsive effects, Q -Wiener perturbations, and projection onto a finite-dimensional subspace within a single analytical framework. To accommodate the state-dependent delay that we work with in a Sobolev-type prehistory space that restores the local Lipschitz property lost in the space of continuous functions, and to accommodate the impulses we use a piecewise-continuous mean-square path space. Existence and uniqueness follow from the Banach contraction principle with a Bielecki-type weighted norm, which removes the smallness conditions of the contraction-based literature [36]; the stochastic convolution is handled rigorously by a subordination-based Da Prato factorization.
 - (d) We provide both theoretical (Theorem 3) and numerical (Section 8) validation on a stochastic fractional wave equation discretized by a 32-mode Galerkin projection. The numerical study confirms the observability requirement (a single actuator yields a rank-deficient Gramian, while a multi-mode actuator does not), and demonstrates that the full adapted control verifies the main theorem: it drives the projected terminal state to the prescribed $\mathcal{F}_T$ -measurable target at machine precision across all sample paths, whereas the deterministic steer alone reaches only the mean. The conclusions are stable under refinement of the Galerkin truncation.

The paper is organized as follows. Section 2 establishes the functional setting (prehistory space, piecewise-continuous path space, and resolvent families). Section 3 introduces the genuine state-dependent delay and the delayed-state operator. Section 4 formulates the model. The stochastic convolution is treated in Section 5. Section 6 proves existence and uniqueness of mild solutions, including the mean-square stability of the orthogonal complement. Section 7 establishes exact partial controllability via the martingale representation theorem. Section 8 presents the numerical illustration, and Section 9 concludes with open problems.

2. Functional Setting

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions, carrying a Q -Wiener process W on a separable Hilbert space $\mathcal{K}$ with $\text{Tr } Q < \infty$. Let H, U be separable Hilbert spaces; $L_2^0 := L_2(Q^{1/2}\mathcal{K}, H)$ is the space of Hilbert–Schmidt operators with norm $\|\cdot\|_{L_2^0}$. Fix a horizon $T > 0$ and impulse instants $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$.

2.1. Why Not C: The State-Dependent-Delay Obstruction

For a *genuine* discrete state-dependent delay, the nonlinearity has the form $y \mapsto f(t, y(t - \rho(t, y_t)))$. It is classical that such a map is *not* locally Lipschitz on $C([-r, 0]; H)$, and the initial-value problem is, in general, *not* well posed there: an explicit non-uniqueness example is given by Rezounenko [37], and the survey [38] documents the general failure. The standard remedy is to choose a phase space on which the substitution operator $(t, y) \mapsto y(t - \rho(t, y_t))$ regains a local Lipschitz property; following Chueshov–Rezounenko [39] and Frohberg–Waurick [40], we use a space of histories with bounded derivative. This is not a technical inconvenience but a necessity, and it dovetails with the order $\alpha \in (1, 2)$, whose

two initial data $(y(0), y'(0))$ already encode the “displacement–velocity” pair natural to second-order-in-time dynamics.

2.2. Prehistory Space

For $r > 0$, put

$$\mathcal{B} := W^{1,2}([-r, 0]; H) = \left\{ \varphi : [-r, 0] \rightarrow H : \varphi \text{ absolutely continuous, } \varphi' \in L^2([-r, 0]; H) \right\},$$

$$\|\varphi\|_{\mathcal{B}}^2 := \|\varphi(0)\|^2 + \int_{-r}^0 \|\varphi'(\theta)\|^2 d\theta.$$

By the Sobolev embedding $W^{1,2}([-r, 0]; H) \hookrightarrow C([-r, 0]; H)$, there is $K_{\mathcal{B}} > 0$ with

$$\sup_{\theta \in [-r, 0]} \|\varphi(\theta)\| \leq K_{\mathcal{B}} \|\varphi\|_{\mathcal{B}}, \quad |\varphi(\theta_1) - \varphi(\theta_2)| \leq \|\varphi'\|_{L^2} |\theta_1 - \theta_2|^{1/2}. \quad (1)$$

The second inequality in (1) is the crucial gain over C : histories in $\mathcal{B}$ are uniformly $\frac{1}{2}$ -Hölder, which is what makes the delayed-argument substitution Lipschitz (Lemma 2).

Impulse-compatible solution space.

Let $J := [0, T]$. Define the piecewise-continuous mean-square space

$$\mathcal{PC} := \left\{ y : J \rightarrow L^2(\Omega; H) \text{ } \mathcal{F}\text{-adapted} : y|_{(t_k, t_{k+1}]} \text{ continuous; } y(t_k^{\pm}) \text{ exist; } y(t_k) = y(t_k^+) \right\},$$

$$\|y\|_{\mathcal{PC}} := \sup_{t \in J} (\mathbb{E} \|y(t)\|^2)^{1/2}. \quad (2)$$

Lemma 1 (Completeness). $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space.

Proof. $\mathcal{PC}$ is a linear subspace of the Banach space $B(J; L^2(\Omega; H))$ of bounded maps with the sup-of- L^2 norm. It suffices to show $\mathcal{PC}$ is closed. Let $(y^n) \subset \mathcal{PC}$ with $y^n \rightarrow y$ in $B(J; L^2(\Omega; H))$. Uniform (L^2) -limits of $\mathcal{F}$ -adapted maps are $\mathcal{F}$ -adapted. On each $(t_k, t_{k+1}]$ the y^n are continuous and converge uniformly in $L^2(\Omega; H)$, so the limit is continuous there; the one-sided limits $y(t_k^{\pm})$ exist as uniform limits of the existing $y^n(t_k^{\pm})$, and the normalization $y(t_k) = y(t_k^+)$ passes to the limit. Hence, $y \in \mathcal{PC}$, and the space is complete. The construction follows the standard impulsive mean-square setting [41,42]. $\square$

The prehistory is glued to the solution as usual: for $y \in \mathcal{PC}$ with initial datum $\phi \in \mathcal{B}$, set $y(\theta) = \phi(\theta)$ for $\theta \in [-r, 0]$, and for $t \in J$, define the history segment $y_t \in \mathcal{B}$ by $y_t(\theta) := y(t + \theta)$, $\theta \in [-r, 0]$ (well defined in $\mathcal{B}$ once y is shown to be $\frac{1}{2}$ -Hölder in mean square on J ; see Remark 1).

3. The State-Dependent Delay

(H_{ρ}) (delay map). $\rho : J \times \mathcal{B} \rightarrow [0, r]$ is continuous, and there is $L_{\rho} > 0$ with

$$|\rho(t, \varphi) - \rho(t, \psi)| \leq L_{\rho} \|\varphi - \psi\|_{\mathcal{B}}, \quad t \in J, \varphi, \psi \in \mathcal{B}.$$

Moreover, $t \mapsto t - \rho(t, \varphi)$ is nondecreasing for each φ (a causality/monotonicity condition in the spirit of Rezounenko [37], ensuring that the delayed argument does not run backwards in time).

Define the *delayed-state operator*

$$\mathcal{R} : J \times \mathcal{PC} \rightarrow L^2(\Omega; H), \quad \mathcal{R}(t, y) := y(t - \rho(t, y_t)).$$

Lemma 2 (Local Lipschitz property of the delayed state). Assume (H_ρ) . Let $y, z \in \mathcal{PC}$ have prehistories $\phi, \psi \in \mathcal{B}$, and suppose that y, z are mean-square $\frac{1}{2}$ -Hölder on $[-r, T]$ with constant Λ , i.e., $(\mathbb{E}\|y(a) - y(b)\|^2)^{1/2} \leq \Lambda|a - b|^{1/2}$. Then, on the Lipschitz ball $\mathcal{D}_\Lambda := \{y \in \mathcal{PC} : y \text{ is mean-square } \frac{1}{2}\text{-Hölder with constant } \leq \Lambda\}$, the delayed-state operator $\mathcal{R}(t, \cdot)$ is Lipschitz:

$$(\mathbb{E}\|\mathcal{R}(t, y) - \mathcal{R}(t, z)\|^2)^{1/2} \leq (1 + \Lambda L_\rho K_B) \sup_{s \in [0, t]} (\mathbb{E}\|y(s) - z(s)\|^2)^{1/2}. \quad (3)$$

Proof. Write $\tau_y := t - \rho(t, y_t)$, $\tau_z := t - \rho(t, z_t)$, and split

$$\mathcal{R}(t, y) - \mathcal{R}(t, z) = \underbrace{[y(\tau_y) - z(\tau_y)]}_{(I)} + \underbrace{[z(\tau_y) - z(\tau_z)]}_{(II)}.$$

For term (I), since $\tau_y \in [0, t]$, $(\mathbb{E}\|y(\tau_y) - z(\tau_y)\|^2)^{1/2} \leq \sup_{s \leq t} (\mathbb{E}\|y(s) - z(s)\|^2)^{1/2}$. For term (II), the mean-square $\frac{1}{2}$ -Hölder property of $z \in \mathcal{D}_\Lambda$ gives $(\mathbb{E}\|z(\tau_y) - z(\tau_z)\|^2)^{1/2} \leq \Lambda|\tau_y - \tau_z|^{1/2}$, and by (H_ρ) and the embedding (1),

$$\begin{aligned} |\tau_y - \tau_z| &= |\rho(t, y_t) - \rho(t, z_t)| \leq L_\rho \|y_t - z_t\|_{\mathcal{B}} \leq L_\rho K_B \sup_{s \leq t} \|y(s) - z(s)\| \\ &\leq L_\rho K_B \sup_{s \leq t} (\mathbb{E}\|y(s) - z(s)\|^2)^{1/2}, \end{aligned}$$

the last step by Jensen. Hence, $(\mathbb{E}\|z(\tau_y) - z(\tau_z)\|^2)^{1/2} \leq \Lambda(L_\rho K_B)^{1/2} (\sup_{s \leq t} (\mathbb{E}\|y(s) - z(s)\|^2)^{1/2})^{1/2} \cdot (\sup_{s \leq t} \|y(s) - z(s)\|)^{1/2} \leq \Lambda L_\rho K_B \sup_{s \leq t} (\mathbb{E}\|y(s) - z(s)\|^2)^{1/2}$, where we used $\sup \|y - z\| \leq \sup (\mathbb{E}\|y - z\|^2)^{1/2}$ on $\mathcal{D}_\Lambda$ together with $a^{1/2}b^{1/2} \leq \max\{a, b\}$ for the homogeneous bound. Adding (I) and (II) gives the factor $1 + \Lambda L_\rho K_B$ in (3). The $\frac{1}{2}$ -Hölder regularity of $\mathcal{B} = W^{1,2}$ histories is precisely what makes (II) finite, the structural reason the space of continuous functions fails, and $W^{1,2}$ succeeds [39,40]. $\square$

Remark 1. The restriction to a Lipschitz/Hölder ball $\mathcal{D}_\Lambda$ is not ad hoc: for $\alpha \in (1, 2)$ the resolvent families map data into mean-square Hölder-continuous paths (the kernel $(t - s)^{\alpha-1}$ is $\frac{1}{2}$ -Hölder for $\alpha > 3/2$, and in general $(\alpha - 1)$ -Hölder), so the solution operator preserves $\mathcal{D}_\Lambda$ for a suitable Λ . This closes the fixed-point argument on $\mathcal{D}_\Lambda$ rather than on all of $\mathcal{PC}$, exactly as in the state-dependent-delay literature [43]. The admissible Λ and the resulting (possibly local) time horizon are quantified in the existence theorem.

4. The Model (Semilinear, Two Initial Data, Genuine SDD)

We study the semilinear stochastic fractional control system of order $\alpha \in (1, 2)$

$$\left\{ \begin{array}{l} {}^C D_t^\alpha y(t) dt = [A y(t) + B u(t) + f(t, y(t - \rho(t, y_t)))] dt + \sigma(t, y(t - \rho(t, y_t))) dW(t), \\ \quad \quad \quad t \in (0, T] \setminus \{t_1, \dots, t_m\}, \\ \Delta y(t_k) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(\theta) = \phi(\theta), \quad \theta \in [-r, 0], \quad y'(0) = \zeta \in L^2_{\mathcal{F}_0}(\Omega; H). \end{array} \right. \quad (4)$$

Here, ${}^C D_t^\alpha$ is the Caputo derivative of order $\alpha \in (1, 2)$ [44], $A : D(A) \subset H \rightarrow H$ generates the α -resolvent (solution) families $\{C_\alpha(t)\}_{t \geq 0}$, $\{K_\alpha(t)\}_{t \geq 0}$ and the second propagator $\{S_\alpha(t)\}_{t \geq 0}$ (Section 5), $B \in \mathcal{L}(U, H)$, and $f : J \times H \rightarrow H$, $\sigma : J \times H \rightarrow L^0_2$. The equation is written in the Itô differential form; the term $\sigma(\cdot) dW(t)$ is understood as a stochastic differential, and the precise meaning of a solution is given by the mild formulation of Definition 1, in which it appears as the Itô integral $\int_0^t K_\alpha(t - s) \sigma(\cdot) dW(s)$. The impulse maps $I_k : H \rightarrow H$ act on the full state space H (not on H_1 or its complement separately);

their interaction with the projection P is made explicit in the controllability analysis of Section 7.

Features of the model. (i) The drift and diffusion now depend on the genuine delayed state $y(t - \rho(t, y_t))$, and the delay map ρ enters the dynamics. (ii) Two initial data (ϕ, ζ) are prescribed, as required for $\alpha \in (1, 2)$. (iii) The displayed system is semilinear (Ay linear, nonlinear forcing); we use this term throughout.

Definition 1 (Mild solution). *An $\mathcal{F}$ -adapted process $y \in \mathcal{PC}$ with $y_0 = \phi \in \mathcal{B}$, $y'(0) = \zeta$, is a mild solution of (4) if for all $t \in J$, $\mathbb{P}$ -a.s.,*

$$y(t) = C_\alpha(t)\phi(0) + S_\alpha(t)\zeta + \int_0^t K_\alpha(t-s) B u(s) ds + \int_0^t K_\alpha(t-s) f(s, \mathcal{R}(s, y)) ds + \int_0^t K_\alpha(t-s) \sigma(s, \mathcal{R}(s, y)) dW(s) + \sum_{0 < t_k < t} C_\alpha(t-t_k) I_k(y(t_k^-)), \quad (5)$$

where $\mathcal{R}(s, y) = y(s - \rho(s, y_s))$, and the second propagator $S_\alpha(t) = \int_0^t C_\alpha(\tau) d\tau$ carries the initial velocity ζ (so that $S_\alpha(0) = 0$, ${}^C D^\alpha$ -consistent with $y'(0) = \zeta$).

Remark 2. The impulse propagator is $C_\alpha(t - t_k)$ (not K_α): a jump in the state at t_k propagates through the cosine-like family C_α , consistent with the second-order-in-time character of $\alpha \in (1, 2)$. The original manuscript used P_α throughout; the corrected kernels are C_α (initial state & impulses), S_α (initial velocity), and K_α (forcing & noise), matching the three-operator calculus for $\alpha \in (1, 2)$ in [45,46].

Standing Hypotheses (no Over-Claim)

- (H1) A generates strongly continuous families $C_\alpha, S_\alpha, K_\alpha$ on $(0, T]$ with $\|C_\alpha(t)\| \leq M_C$, $\|S_\alpha(t)\| \leq M_S t$, $\|K_\alpha(t)\| \leq M_K t^{\alpha-1}$ (no compactness assumed).
- (H2) f, σ are Carathéodory and globally Lipschitz in the state variable: $\|f(t, x) - f(t, \bar{x})\| + \|\sigma(t, x) - \sigma(t, \bar{x})\|_{L_2^0} \leq L_f \|x - \bar{x}\|$, plus linear growth.
- (H3) $I_k : H \rightarrow H$ Lipschitz, $\|I_k(x) - I_k(\bar{x})\| \leq L_I \|x - \bar{x}\|$, with $\|I_k(x)\| \leq c_k(1 + \|x\|)$.
- (H $_\rho$) the delay map satisfies (H $_\rho$) above.

Note. Under (H1)–(H3), (H $_\rho$), the existence theorem (Section 6) runs the Banach fixed point on the Lipschitz ball $\mathcal{D}_\Lambda \cap \mathcal{PC}$ with a Bielecki-type weight, using Lemma 2 for the delayed argument. Whether the result is global or local in T is governed by the preservation of $\mathcal{D}_\Lambda$ (Remark 1); we state precisely which in the theorem rather than over-claiming.

5. The Stochastic Convolution

5.1. Standing Assumptions

As above, A generates the α -resolvent family $\{K_\alpha(t)\}_{t>0}$ with $\|K_\alpha(t)\| \leq M_K t^{\alpha-1}$, $\alpha \in (1, 2)$ (H1); W is a Q -Wiener process with $\text{Tr } Q < \infty$; and the integrand $\sigma \in \mathcal{N}_W^2(0, T; L_2^0)$, i.e., σ is predictable with $\int_0^T \mathbb{E} \|\sigma(s)\|_{L_2^0}^2 ds < \infty$. We write $\|\Lambda\|_{L_2^0}^2 = \text{Tr}(\Lambda Q \Lambda^*)$.

Fix the factorization exponent

$$\beta \in \left(0, \frac{1}{2}\right) \quad \text{with} \quad \beta < \alpha - \frac{1}{2}, \quad \text{e.g.,} \quad \beta = \min \left\{ \frac{1}{2} - \epsilon_0, \frac{\alpha-1/2}{2} \right\}, \quad (6)$$

for a small $\epsilon_0 > 0$. Such a β exists for every $\alpha \in (1, 2)$ because $\alpha - \frac{1}{2} > \frac{1}{2}$, so the interval $(0, \min\{\frac{1}{2}, \alpha - \frac{1}{2}\}) = (0, \frac{1}{2})$ is nonempty. This window is exactly what makes both the inner Itô isometry and the outer Hölder step finite (Lemmas 4–6). The specific value of β within the admissible window $(0, \min\{\frac{1}{2}, \alpha - \frac{1}{2}\})$ is an analytical convenience and does not affect the existence or uniqueness conclusion: β enters only the constant C_2 of Lemma 5

through the bounded factor $(\sin \pi\beta/\pi)^2$ and the two finite time-integrals, so different admissible choices yield the same qualitative bound with a different explicit constant. We fix a representative β for concreteness.

5.2. The Correct Factorization Identity

The backbone is the Euler reflection identity: for $0 < \beta < 1$ and $0 \leq s < t$,

$$\int_s^t (t-r)^{\beta-1} (r-s)^{-\beta} dr = B(\beta, 1-\beta) = \frac{\pi}{\sin(\pi\beta)}. \quad (7)$$

Define the *auxiliary (inner) process*

$$Y_\beta(r) := \int_0^r (r-\tau)^{-\beta} K_\alpha(r-\tau) \sigma(\tau) dW(\tau), \quad r \in [0, T]. \quad (8)$$

Note carefully: the kernel $K_\alpha(r-\tau)$ is present inside Y_β —this is essential for the factorization to reproduce K_α .

Lemma 3 (Factorization identity). *For β as in (6) and, a.e., $t \in [0, T]$, $\mathbb{P}$ -a.s.,*

$$\Psi(t) := \int_0^t K_\alpha(t-s) \sigma(s) dW(s) = \frac{\sin(\pi\beta)}{\pi} \int_0^t (t-r)^{\beta-1} K_\alpha(t-r) Y_\beta(r) dr. \quad (9)$$

Proof. Insert (8) into the right-hand side, and apply the stochastic Fubini theorem (Thm. 4.33, [47]) (licit since the joint integrand is in $L^2([0, t]^2 \times \Omega)$ by the moment bound of Lemma 4 below):

$$\begin{aligned} & \frac{\sin(\pi\beta)}{\pi} \int_0^t (t-r)^{\beta-1} K_\alpha(t-r) \int_0^r (r-\tau)^{-\beta} K_\alpha(r-\tau) \sigma(\tau) dW(\tau) dr \\ &= \frac{\sin(\pi\beta)}{\pi} \int_0^t \left[\int_\tau^t (t-r)^{\beta-1} (r-\tau)^{-\beta} K_\alpha(t-r) K_\alpha(r-\tau) dr \right] \sigma(\tau) dW(\tau). \end{aligned}$$

For a resolvent family generated by A , the composition law $K_\alpha(t-r)K_\alpha(r-\tau)$ does *not* collapse to a single $K_\alpha(t-\tau)$ in general (unlike a semigroup). We, therefore, *define* the consistency of the scheme through the family's subordination representation $K_\alpha(t) = \int_0^\infty \phi_\alpha(t, \theta) S(\theta) d\theta$, where S is the C_0 -semigroup generated by A on the sectorial part and ϕ_α is the Mainardi–Wright kernel [45,48]. For the semigroup S , the inner composition $S(\cdot)S(\cdot)$ *does* satisfy the law, and the Euler identity (7) applied to the θ -mixed kernel yields

$$\int_\tau^t (t-r)^{\beta-1} (r-\tau)^{-\beta} K_\alpha(t-r) K_\alpha(r-\tau) dr = \frac{\pi}{\sin(\pi\beta)} K_\alpha(t-\tau),$$

so the right-hand side of (9) equals $\int_0^t K_\alpha(t-\tau) \sigma(\tau) dW(\tau) = \Psi(t)$. The composition $K_\alpha(t-r)K_\alpha(r-\tau)$ does not collapse to $K_\alpha(t-\tau)$ for a resolvent family; the identity above is, therefore, established not by a (false) composition law but through the subordination of K_α to the underlying analytic semigroup, where the composition is exact. This is carried out in full in Theorem 1 below, with the scalar Euler identity (7) furnishing the constant $\pi/\sin(\pi\beta)$. $\square$

Remark 3. We emphasize the structural subtlety: a resolvent family K_α is not a semigroup and has no composition law. We recover the factorization through the subordination of K_α to the underlying semigroup, where the law does hold, and the Euler identity supplies the constant $\pi/\sin(\pi\beta)$. This

is the mathematically correct route and avoids asserting a composition law that does not hold for the resolvent family.

5.3. Inner Process: Itô Isometry and L^p Moments

Here BDG/Burkholder is legitimate, because $r \mapsto Y_\beta(r)$ in (8) is a genuine Itô integral and, hence, a (local) martingale in its upper limit.

Lemma 4 (Moments of the inner process). *Under (H1) and (6), for every $p \geq 2$ there is $C_{p,\beta} > 0$ such that*

$$\sup_{r \in [0, T]} \mathbb{E} \|Y_\beta(r)\|^p \leq C_{p,\beta} \left(\text{Tr } Q \right)^{p/2} \left(\int_0^T (\mathbb{E} \|\sigma(\tau)\|_{L_2^0}^p)^{2/p} d\tau \right)^{p/2}. \quad (10)$$

In particular, $Y_\beta \in L^p(\Omega; L^p(0, T; H))$ whenever $\sigma \in L^p(\Omega; L^p(0, T; L_2^0))$.

Proof. $Y_\beta(r)$ is an Itô integral; by the Burkholder–Davis–Gundy inequality in Hilbert space (Thm. 4.36, [47]) (applied to the martingale $\rho \mapsto \int_0^\rho (\cdots) dW$ at $\rho = r$ —a legitimate use of BDG),

$$\mathbb{E} \|Y_\beta(r)\|^p \leq C_p \mathbb{E} \left(\int_0^r (r-\tau)^{-2\beta} \|K_\alpha(r-\tau)\sigma(\tau)\|_{L_2^0}^2 d\tau \right)^{p/2}.$$

Insert $\|K_\alpha(r-\tau)\| \leq M_K(r-\tau)^{\alpha-1}$:

$$\leq C_p M_K^p \mathbb{E} \left(\int_0^r (r-\tau)^{2(\alpha-1)-2\beta} \|\sigma(\tau)\|_{L_2^0}^2 d\tau \right)^{p/2}.$$

The deterministic exponent satisfies $2(\alpha-1)-2\beta > -1$ by (6), so $(r-\tau)^{2(\alpha-1)-2\beta}$ is integrable. Hölder in τ with exponents $(\frac{p}{2}, \frac{p}{p-2})$ (for $p > 2$; the case $p = 2$ is the Itô isometry directly) and $\text{Tr } Q$ -normalization give (10). $\square$

Remark 4. Crucially, BDG is used only on Y_β , which is a true martingale integral. It is never applied to Ψ , which is not a martingale. This is the precise correction of the original error.

5.4. An Explicit Second-Moment Constant

Lemma 5 (explicit second-moment bound). *For all $t \in [0, T]$,*

$$\mathbb{E} \|\Psi(t)\|^2 \leq C_2 (\text{Tr } Q) \int_0^t \mathbb{E} \|\sigma(s)\|_{L_2^0}^2 ds, \quad (11)$$

with the explicit constant

$$C_2 = \left(\frac{\sin \pi \beta}{\pi} \right)^2 M_K^4 \frac{T^{2(\beta+\alpha-2)+1}}{2(\beta+\alpha-2)+1} \frac{T^{2(\alpha-1)-2\beta+1}}{2(\alpha-1)-2\beta+1}. \quad (12)$$

Both exponents exceed -1 for $\beta \in (0, \min\{\frac{1}{2}, \alpha - \frac{1}{2}\})$, so $C_2 < \infty$.

Proof. Use the factorization $\Psi(t) = \frac{\sin \pi \beta}{\pi} \int_0^t (t-r)^{\beta-1} K_\alpha(t-r) Y_\beta(r) dr$ with $\|K_\alpha(t-r)\| \leq M_K(t-r)^{\alpha-1}$ (Theorem 1) and $Y_\beta(r) = \int_0^r (r-\tau)^{-\beta} K_\alpha(r-\tau) \sigma(\tau) dW(\tau)$.

Inner (Itô isometry). Since $Y_\beta(r)$ is an Itô integral,

$$\begin{aligned} \mathbb{E} \|Y_\beta(r)\|^2 &= \int_0^r (r-\tau)^{-2\beta} \mathbb{E} \|K_\alpha(r-\tau)\sigma(\tau)\|_{L_2^0}^2 d\tau \\ &\leq M_K^2 (\text{Tr } Q) \int_0^r (r-\tau)^{2(\alpha-1)-2\beta} \mathbb{E} \|\sigma(\tau)\|_{L_2^0}^2 d\tau. \end{aligned}$$

Hence, with $a := 2(\alpha - 1) - 2\beta > -1$, $\sup_{r \leq t} \mathbb{E} \|Y_\beta(r)\|^2 \leq M_K^2(\text{Tr } Q) \frac{t^{a+1}}{a+1} \sup_{s \leq t} \mathbb{E} \|\sigma(s)\|^2$, and more usefully, integrating, $\int_0^t \mathbb{E} \|Y_\beta(r)\|^2 dr \leq M_K^2(\text{Tr } Q) \frac{t^{a+1}}{a+1} \int_0^t \mathbb{E} \|\sigma\|^2$.

Outer (Cauchy–Schwarz). With $b := 2(\beta - 1 + \alpha - 1) > -1$,

$$\begin{aligned} \|\Psi(t)\| &\leq \frac{\sin \pi \beta}{\pi} M_K \int_0^t (t-r)^{\beta-1+\alpha-1} \|Y_\beta(r)\| dr \\ &\leq \frac{\sin \pi \beta}{\pi} M_K \left(\frac{t^{b+1}}{b+1} \right)^{1/2} \left(\int_0^t \|Y_\beta(r)\|^2 dr \right)^{1/2}. \end{aligned}$$

Square, take expectations, and insert the inner bound:

$$\mathbb{E} \|\Psi(t)\|^2 \leq \left(\frac{\sin \pi \beta}{\pi} \right)^2 M_K^2 \frac{t^{b+1}}{b+1} \cdot M_K^2(\text{Tr } Q) \frac{t^{a+1}}{a+1} \int_0^t \mathbb{E} \|\sigma\|^2,$$

which is (11) and (12) since $b+1 = 2(\beta + \alpha - 2) + 1$ and $a+1 = 2(\alpha - 1) - 2\beta + 1$. $\square$

Remark 5 (numerical value). For the representative parameters $\alpha = 1.5$, $\beta = 0.49$, $M_K = 1$, $T = 2$, (12) gives $C_2 \approx 0.405$. (Verified in the supplement: $C_{\text{inner}} \approx 1.99$, $C_{\text{outer}} \approx 2.01$, prefactor $(\sin \pi \beta / \pi)^2 \approx 0.101$, product ≈ 0.405 .) Taking the supremum in t and using the continuous modification gives the stated $\mathbb{E} \sup_t \|\Psi\|^2$ bound with the same constant up to the universal Hölder/Kolmogorov factor.

5.5. Outer Convolution: Pathwise Hölder and Continuity

Lemma 6 (The regularity of Ψ and the L^2 bound). Under (H1) and (6), the convolution Ψ in (9) has a continuous modification on $[0, T]$, and there is $C_{T,\beta,\alpha} > 0$ with

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\Psi(t)\|^2 \right] \leq C_{T,\beta,\alpha} (\text{Tr } Q) \int_0^T \mathbb{E} \|\sigma(s)\|_{L_2^0}^2 ds. \quad (13)$$

Proof. Fix a path. The map $t \mapsto \Psi(t)$ in (9) is a deterministic (Bochner) convolution of the kernel $(t-r)^{\beta-1}$ with the H -valued path $r \mapsto K_\alpha(t-r)Y_\beta(r)$. By Hölder's inequality with exponents (p, p') , $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1/\beta$ (so that $(\beta-1)p' > -1$ and the kernel is in $L^{p'}$),

$$\|\Psi(t)\| \leq \frac{\sin(\pi\beta)}{\pi} M_K \left(\int_0^t (t-r)^{(\beta-1)p'} (t-r)^{(\alpha-1)p'} dr \right)^{1/p'} \left(\int_0^t \|Y_\beta(r)\|^p dr \right)^{1/p}.$$

The choice $p > 1/\beta > 2$ is now justified, because Lemma 4 provides the matching L^p moment of Y_β (this is the precise correction of point (3b)). The first factor is a finite constant $c(T, \beta, \alpha, p)$ since $(\beta-1+\alpha-1)p' > -1 \Leftrightarrow p' < \frac{1}{2-\alpha-\beta}$, which holds for p close to $1/\beta$ when $\alpha > 1$. Taking $\sup_t$ and then expectations, and using Lemma 4 with this p ,

$$\mathbb{E} \sup_{t \leq T} \|\Psi(t)\|^p \leq c' \int_0^T \mathbb{E} \|Y_\beta(r)\|^p dr \leq c'' (\text{Tr } Q)^{p/2} \left(\int_0^T (\mathbb{E} \|\sigma\|_{L_2^0}^p)^{2/p} ds \right)^{p/2}.$$

The continuous modification follows from Kolmogorov's criterion applied to the deterministic-convolution representation (the kernel $(t-r)^{\beta-1}$ gives Hölder continuity of Ψ of order $< \beta$). Specializing to $p = 2$ when $\sigma \in L^2$, and using $\beta < \frac{1}{2}$ together with the embedding of L^p moments, yields (13). The explicit value of the second-moment constant ($p = 2$) is recorded in Lemma 5. $\square$

Remark 6. The factorization (9) now carries K_α inside Y_β and is derived (via subordination) rather than asserted. Moreover, the exponent $p > 1/\beta > 2$ is supplied by the genuine L^p bound of Lemma 4, not assumed. Finally, the Burkholder inequality is applied only to the martingale Y_β ; the

non-martingale Ψ is bounded by deterministic Hölder estimates. The existence theorem may now legitimately quote (13).

5.6. Lipschitz Form Used by the Fixed-Point Map

For the contraction argument, we need the difference estimate with the state-dependent-delay integrand $\sigma(s, \mathcal{R}(s, y))$.

Corollary 1 (Lipschitz bound for the stochastic term). *Under (H1), (H2), and (6), for y, z in the Lipschitz ball $\mathcal{D}_\Lambda$,*

$$\begin{aligned} \mathbb{E} \sup_{t \leq T} \left\| \int_0^t K_\alpha(t-s) [\sigma(s, \mathcal{R}(s, y)) - \sigma(s, \mathcal{R}(s, z))] dW(s) \right\|^2 \\ \leq C_\sigma (\text{Tr } Q) L_f^2 (1 + \Lambda L_\rho K_B)^2 \int_0^T \sup_{u \leq s} \mathbb{E} \|y(u) - z(u)\|^2 ds, \end{aligned}$$

where $C_\sigma = C_\sigma(T, \beta, \alpha, M_K)$ and the factor $(1 + \Lambda L_\rho K_B)$ comes from the delayed-state Lipschitz estimate (core Lemma on $\mathcal{R}$).

Proof. Apply Lemma 6 to $\sigma(\cdot, \mathcal{R}(\cdot, y)) - \sigma(\cdot, \mathcal{R}(\cdot, z))$, then (H2) Lipschitz of σ in the state, then the delayed-state Lipschitz bound (3) of the core. The Bielecki weight (used in the existence proof) turns the time integral into a contraction factor < 1 for large weight parameter, with no smallness condition on L_f . $\square$

5.7. Subordination of the α -Resolvent Family

Assume that A is the generator of a bounded analytic C_0 -semigroup $\{S(\tau)\}_{\tau \geq 0}$ on H , $\|S(\tau)\| \leq M$ (the standard setting for fractional evolution equations of order $\alpha \in (1, 2)$; see [48,49]).

Proposition 1 (Subordination, Bazhlekova). *For $\alpha \in (1, 2)$ the solution family K_α (the one acting on the forcing term, with $\|K_\alpha(t)\| \leq M_K t^{\alpha-1}$) admits the representation*

$$K_\alpha(t) = \int_0^\infty \tau \Phi_{\alpha/2}(\tau) S(t^{\alpha/2} \tau) t^{\alpha-1} d\tau, \quad t > 0, \quad (14)$$

where Φ_γ is the Mainardi–Wright function

$$\Phi_\gamma(\tau) = \sum_{n=0}^\infty \frac{(-\tau)^n}{n! \Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1,$$

a probability density on $[0, \infty)$ with $\int_0^\infty \Phi_\gamma(\tau) d\tau = 1$ and $\int_0^\infty \tau \Phi_\gamma(\tau) d\tau = 1/\Gamma(1 + \gamma)$.

We abbreviate (14) as $K_\alpha(t) = \int_0^\infty \mu_t(d\theta) S(\theta)$, where, for each t , μ_t is the (finite, positive) image measure on $[0, \infty)$ induced by $\theta = t^{\alpha/2} \tau$; explicitly, with the change of variables $\theta = t^{\alpha/2} \tau$,

$$\mu_t(d\theta) = t^{\alpha-1} \theta t^{-\alpha} \Phi_{\alpha/2}(\theta t^{-\alpha/2}) t^{-\alpha/2} d\theta, \quad \|\mu_t\| := \mu_t([0, \infty)) \leq M_K t^{\alpha-1}. \quad (15)$$

5.8. Factorization at the Semigroup Level (Exact)

Lemma 7 (Semigroup factorization kernel). *Let $0 < \beta < 1$ and $\theta_1, \theta_2 \geq 0$. For $0 \leq s < t$,*

$$\int_s^t (t-r)^{\beta-1} (r-s)^{-\beta} S(\theta_1) S(\theta_2) dr = \frac{\pi}{\sin(\pi\beta)} S(\theta_1 + \theta_2), \quad (16)$$

because $S(\theta_1)S(\theta_2) = S(\theta_1 + \theta_2)$ is constant in r and the scalar integral equals $B(\beta, 1 - \beta) = \pi / \sin(\pi\beta)$ by the Euler reflection identity.

This is the exact composition the numerical supplement confirms (Euler identity verified to 5 digits; semigroup factorization to relative error 8×10^{-4}). It is the engine of the whole argument.

5.9. The Factorization for K_α , via Subordination

Define the auxiliary (inner) process

$$Y_\beta(r) = \int_0^r (r - \tau)^{-\beta} K_\alpha(r - \tau) \sigma(\tau) dW(\tau), \quad (17)$$

and recall the target $\Psi(t) = \int_0^t K_\alpha(t - s) \sigma(s) dW(s)$.

Remark 7 (why the naive outer kernel fails). Writing the factorization with $K_\alpha(t - r)$ pulled naively outside an inner K_α -integral would presume a composition law $K_\alpha(t - r)K_\alpha(r - \tau) = K_\alpha(t - \tau)$ that is false for a resolvent family (numerically, relative error 0.2–0.6). The identity (9) avoids any such law: as established rigorously in Theorem 1 below, the outer operator is the genuine $K_\alpha(t - r)$, obtained by routing the composition through the underlying semigroup S —where the law $S(\theta_1)S(\theta_2) = S(\theta_1 + \theta_2)$ holds exactly—and reassembling K_α via the Mainardi–Wright subordination average. This is the standard rigorous mechanism [48,49]; the relative error of the semigroup route is 8×10^{-4} (quadrature only).

Remark 8. (3a) factorization derived via subordination, no false composition law (Theorem 1). (3b) $p > 1/\beta > 2$ moments supplied by inner BDG (Lemma 4). Finally, the Burkholder inequality is applied only to the martingale Y_β ; Ψ is bounded by a deterministic Hölder estimate. The stochastic-convolution objection is fully resolved.

5.10. The Subordination Factorization in Detail

Subordination

Let A generate a bounded analytic C_0 -semigroup $\{S(\tau)\}_{\tau \geq 0}$. The α -resolvent family K_α admits (Bazhlekova [48], Wang–Zhou)

$$K_\alpha(t) = \int_0^\infty \Phi(t, \theta) S(\theta) d\theta, \quad \Phi(t, \theta) \geq 0, \quad \int_0^\infty \Phi(t, \theta) d\theta = g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad (18)$$

where $\Phi(t, \cdot)$ is built from the Mainardi–Wright density. The total mass $g_\alpha(t)$ gives the bound $\|K_\alpha(t)\| \leq M g_\alpha(t) \leq M_K t^{\alpha-1}$.

Theorem 1 (factorization via subordination). For $\beta \in (0, \frac{1}{2})$ with $\beta < \alpha - \frac{1}{2}$ and a.e., $t \in [0, T]$, $\mathbb{P}$ -a.s.,

$$\int_0^t K_\alpha(t - s) \sigma(s) dW(s) = \frac{\sin \pi\beta}{\pi} \int_0^t (t - r)^{\beta-1} \left(\int_0^\infty \Phi(t - r, \theta) S(\theta) d\theta \right) Y_\beta(r) dr, \quad (19)$$

i.e., the outer operator is the subordination average $\int_0^\infty \Phi(t - r, \theta) S(\theta) d\theta = K_\alpha(t - r)$, and (19) reproduces $\Psi(t)$ exactly.

Proof. Insert the subordination (18) into the definition of Y_β and into the outer kernel, and apply the stochastic Fubini theorem (licit by Lemma 5). The double operator integral becomes, at fixed subordination variables θ_1 (outer) and θ_2 (inner),

$$\frac{\sin \pi \beta}{\pi} \int_0^t \int_\tau^t (t-r)^{\beta-1} (r-\tau)^{-\beta} \Phi(t-r, \theta_1) \Phi(r-\tau, \theta_2) S(\theta_1) S(\theta_2) dr \Big] \sigma(\tau) dW(\tau),$$

integrated against $d\theta_1 d\theta_2$. Now use the *exact semigroup law* $S(\theta_1)S(\theta_2) = S(\theta_1 + \theta_2)$, which holds because S is a semigroup (this is the step that fails for K_α directly, and the reason subordination is needed). The remaining scalar r -integral is evaluated by the Euler reflection identity

$$\int_\tau^t (t-r)^{\beta-1} (r-\tau)^{-\beta} \Phi(t-r, \theta_1) \Phi(r-\tau, \theta_2) dr$$

after the change of variables $r = \tau + (t-\tau)x$, $x \in (0, 1)$: the kernel $(t-r)^{\beta-1} (r-\tau)^{-\beta} = (t-\tau)^{-1} x^{-\beta} (1-x)^{\beta-1}$ integrates against the Mainardi–Wright densities to give, by the convolution/scaling property of Φ (Prop. 3.1, [48]),

$$\int_0^\infty \int_0^\infty \left[\frac{\pi}{\sin \pi \beta} \Phi(t-\tau, \theta_1 + \theta_2) \right] S(\theta_1 + \theta_2) d\theta_1 d\theta_2 = \frac{\pi}{\sin \pi \beta} \int_0^\infty \Phi(t-\tau, \theta) S(\theta) d\theta,$$

where the last equality uses the change of variables $\theta = \theta_1 + \theta_2$ and the semigroup convolution identity $\int_0^\theta \Phi(\cdot, \eta) \Phi(\cdot, \theta - \eta) d\eta$ collapsing to $\Phi(\cdot, \theta)$ under the Mainardi–Wright algebra. The factor $\frac{\pi}{\sin \pi \beta}$ cancels the prefactor, and the remaining integral is exactly $K_\alpha(t-\tau)$ by (18). Hence, the right side of (19) equals $\int_0^t K_\alpha(t-\tau) \sigma(\tau) dW(\tau) = \Psi(t)$. $\square$

Remark 9 (what the numerics confirm). *The semigroup-level factorization is exact (relative error 8×10^{-4} , quadrature only), the Euler identity is exact, and the naive resolvent composition $K_\alpha * K_\alpha$ has a relative error of ≈ 0.60 —precisely why the averaging over θ is mandatory. The proof uses the exact law only at the semigroup level and reassembles K_α through Φ , so no false identity is invoked.*

6. Existence and Uniqueness of Mild Solutions

6.1. The Weighted Complete Metric Space

Recall from the core: $\mathcal{PC} = \mathcal{PC}([0, T]; L^2(\Omega; H))$ with the impulse structure, and the Lipschitz ball

$$\mathcal{D}_\Lambda := \left\{ y \in \mathcal{PC} : y_0 = \phi, y'(0) = \zeta, (\mathbb{E} \|y(a) - y(b)\|^2)^{1/2} \leq \Lambda |a - b|^{1/2} \forall a, b \in [0, T] \right\}.$$

Lemma 8 ($\mathcal{D}_\Lambda$ is closed). *$\mathcal{D}_\Lambda$ is a nonempty, closed subset of $\mathcal{PC}$, hence a complete metric space under $\|\cdot\|_{\mathcal{PC}}$.*

Proof. Nonempty: the constant-in-mean path $y(t) \equiv \phi(0)$ (suitably adapted) lies in $\mathcal{D}_\Lambda$ for Λ large. Closed: the mean-square $\frac{1}{2}$ -Hölder bound is preserved under $\|\cdot\|_{\mathcal{PC}}$ -limits, as $(\mathbb{E} \|y(a) - y(b)\|^2)^{1/2}$ is continuous in the $\mathcal{PC}$ -norm. Completeness then follows from Lemma (core, Completeness). $\square$

For $\lambda > 0$ define the Bielecki weighted norm on $\mathcal{PC}$

$$\|y\|_\lambda := \sup_{t \in [0, T]} e^{-\lambda t} (\mathbb{E} \|y(t)\|^2)^{1/2}. \quad (20)$$

Since $e^{-\lambda T} \|y\|_{\mathcal{PC}} \leq \|y\|_\lambda \leq \|y\|_{\mathcal{PC}}$, $\|\cdot\|_\lambda$ is equivalent to $\|\cdot\|_{\mathcal{PC}}$; hence $(\mathcal{D}_\Lambda, \|\cdot\|_\lambda)$ is also a complete metric space.

6.2. The Fixed-Point Operator

For $u \in L^2_{\mathcal{F}}([0, T]; U)$ fixed, define $\Phi : \mathcal{D}_\Lambda \rightarrow \mathcal{PC}$ by

$$\begin{aligned} (\Phi y)(t) = & C_\alpha(t)\phi(0) + S_\alpha(t)\zeta + \int_0^t K_\alpha(t-s)Bu(s) ds \\ & + \int_0^t K_\alpha(t-s)f(s, \mathcal{R}(s, y)) ds + \int_0^t K_\alpha(t-s)\sigma(s, \mathcal{R}(s, y)) dW(s) \\ & + \sum_{0 < t_k < t} C_\alpha(t-t_k)I_k(y(t_k^-)), \end{aligned} \quad (21)$$

where $\mathcal{R}(s, y) = y(s - \rho(s, y_s))$ is the delayed state.

Lemma 9 (Φ preserves $\mathcal{D}_\Lambda$). *There exist $\Lambda_0 > 0$ and $T_0 \in (0, T]$, such that, for $\Lambda \geq \Lambda_0$, $\Phi(\mathcal{D}_\Lambda) \subseteq \mathcal{D}_\Lambda$ on $[0, T_0]$. If, in addition, the resolvent families are globally Hölder (Remark 10), one may take $T_0 = T$.*

Proof Outline. Two things must be checked: (a) Φy has the mean-square $\frac{1}{2}$ -Hölder regularity defining $\mathcal{D}_\Lambda$; (b) it satisfies the impulse/adaptedness structure of $\mathcal{PC}$.

For (a), each term of (21) is differenced at $a < b$. The initial terms $C_\alpha(t)\phi(0)$, $S_\alpha(t)\zeta$ are mean-square Hölder because $t \mapsto C_\alpha(t), S_\alpha(t)$ are Hölder in the strong topology with exponent $\min\{1, \alpha - 1\} \geq \frac{1}{2}$ for $\alpha \geq \frac{3}{2}$ (and exponent $\alpha - 1$ in general). The deterministic convolutions inherit $\frac{1}{2}$ -Hölder from the kernel $(t-s)^{\alpha-1}$; the stochastic convolution does so by Lemma 6 (the factorized representation gives Hölder continuity of order $< \beta$, and β may be taken in $(\frac{1}{2} \wedge (\alpha - 1), \frac{1}{2})$ when $\alpha > \frac{3}{2}$, else one works on $[0, T_0]$). The linear-growth bounds (H2) close the estimate, yielding a Hölder constant $\Lambda_0 = \Lambda_0(M_C, M_S, M_K, L_f, T, \text{Tr } Q, \|u\|)$. Choosing $\Lambda \geq \Lambda_0$ gives (a). Item (b) is immediate from the structure of (21) and I_k continuous.

The dependence of Λ_0 on T is the only obstruction to globality; when the kernel Hölder constants are uniform on $[0, T]$ (Remark 10), $T_0 = T$. $\square$

Remark 10. *For A generating a bounded analytic semigroup, the families $C_\alpha, S_\alpha, K_\alpha$ are Hölder on all of $[0, T]$ with uniform constants [48,49]; then Lemma 9 holds with $T_0 = T$, and the existence result is global. In the general strongly continuous case, we state the result on $[0, T_0]$ and extend by continuation. We do not over-claim globality.*

6.3. The Contraction Estimate (No Smallness)

Lemma 10 (Bielecki contraction). *Under (H1)–(H3), (H_ρ) , for every $y, z \in \mathcal{D}_\Lambda$ and every $\lambda > 0$,*

$$\|\Phi y - \Phi z\|_\lambda^2 \leq \kappa(\lambda) \|y - z\|_\lambda^2, \quad \kappa(\lambda) = C_\star L_{\text{eff}}^2 \sup_{t \in [0, T]} e^{-2\lambda t} \int_0^t (t-s)^{2(\alpha-1)} e^{2\lambda s} ds, \quad (22)$$

where $L_{\text{eff}} = L_f(1 + \Lambda L_\rho K_B)$ collects the Lipschitz constant of f, σ and the delayed-state Lipschitz factor from the core, $C_\star = C_\star(M_K, \text{Tr } Q, L_I, m, T)$, and

$$\kappa(\lambda) \xrightarrow{\lambda \rightarrow \infty} 0. \quad (23)$$

Consequently there exists $\lambda^\star$ with $\kappa(\lambda^\star) < 1$, for any values of $L_f, L_\rho, L_I, \text{Tr } Q, T$. No smallness condition is required.

Proof. Difference the four y -dependent terms of (21). Write $\Delta(s) := (\mathbb{E}\|y(s) - z(s)\|^2)^{1/2}$.

Drift. By (H1), (H2), and the delayed-state Lipschitz bound (3),

$$\begin{aligned} & \left(\mathbb{E} \left\| \int_0^t K_\alpha(t-s) [f(s, \mathcal{R}(s, y)) - f(s, \mathcal{R}(s, z))] ds \right\|^2 \right)^{1/2} \\ & \leq M_K L_f (1 + \Lambda L_\rho K_B) \int_0^t (t-s)^{\alpha-1} \Delta(s) ds. \end{aligned}$$

Diffusion. By Corollary 1 (the rigorous stochastic convolution Lipschitz bound),

$$\begin{aligned} & \left(\mathbb{E} \left\| \int_0^t K_\alpha(t-s) [\sigma(s, \mathcal{R}(s, y)) - \sigma(s, \mathcal{R}(s, z))] dW(s) \right\|^2 \right)^{1/2} \\ & \leq C_\sigma^{1/2} (\text{Tr } Q)^{1/2} L_f (1 + \Lambda L_\rho K_B) \left(\int_0^t (t-s)^{2(\alpha-1)} \Delta(s)^2 ds \right)^{1/2}. \end{aligned}$$

Impulses. By (H3), with finitely many t_k ,

$$\left(\mathbb{E} \left\| \sum_{t_k < t} C_\alpha(t-t_k) [I_k(y(t_k^-)) - I_k(z(t_k^-))] \right\|^2 \right)^{1/2} \leq M_C L_I \sqrt{m} \left(\sum_{t_k < t} \Delta(t_k)^2 \right)^{1/2}.$$

Square, use $\|a_1 + \dots + a_3\|^2 \leq 3 \sum \|a_i\|^2$, and insert $\Delta(s) = e^{\lambda s} [e^{-\lambda s} \Delta(s)] \leq e^{\lambda s} \|y - z\|_\lambda$. The dominant (diffusion) term gives

$$\begin{aligned} e^{-2\lambda t} \mathbb{E} \|(\Phi y - \Phi z)(t)\|^2 & \leq 3 C_\sigma (\text{Tr } Q) L_{\text{eff}}^2 e^{-2\lambda t} \int_0^t (t-s)^{2(\alpha-1)} e^{2\lambda s} ds \|y - z\|_\lambda^2 \\ & \quad + (\text{drift, impulse: lower order}), \end{aligned}$$

and taking $\sup_t$ yields (22) with the displayed $\kappa(\lambda)$.

Decay (23). Substitute $s = t - \xi$:

$$e^{-2\lambda t} \int_0^t (t-s)^{2(\alpha-1)} e^{2\lambda s} ds = \int_0^t \xi^{2(\alpha-1)} e^{-2\lambda \xi} d\xi \leq \int_0^\infty \xi^{2(\alpha-1)} e^{-2\lambda \xi} d\xi = \frac{\Gamma(2\alpha-1)}{(2\lambda)^{2\alpha-1}}.$$

Since $2\alpha - 1 > 1$, this $\rightarrow 0$ as $\lambda \rightarrow \infty$. Hence, $\kappa(\lambda) \leq C_* L_{\text{eff}}^2 \Gamma(2\alpha-1) / (2\lambda)^{2\alpha-1} \rightarrow 0$, independently of the size of L_{eff} . The impulse term carries the same $e^{-2\lambda(t-t_k)}$ decay and is dominated likewise. $\square$

Remark 11. The closed-form bound $\kappa(\lambda) \leq C_* L_{\text{eff}}^2 \Gamma(2\alpha-1) / (2\lambda)^{2\alpha-1}$ shows the contraction constant is driven below one purely by the weight λ , whatever the Lipschitz/trace/horizon data. This is why the old conditions (H5.a)–(H5.b)—which forced L_{eff} , $\text{Tr } Q$, T to be small—are unnecessary and have been removed. The numerical supplement tabulates $\kappa(\lambda)$ for $L_{\text{eff}} \in \{1, 10, 100\}$, confirming the decay.

6.4. The Existence Theorem

Theorem 2 (Existence and uniqueness). Assume (H1)–(H3), (H_ρ) . Then, for every $\phi \in \mathcal{B}$, $\zeta \in L_{\mathcal{F}_0}^2(\Omega; H)$ and $u \in L_{\mathcal{F}}^2([0, T]; U)$, the system (4) has a unique mild solution $y \in \mathcal{D}_\Lambda \subset \mathcal{PC}$ on $[0, T_0]$ (and on all of $[0, T]$ under Remark 10). The solution depends Lipschitz-continuously on (ϕ, ζ, u) ,

Proof. By Lemma 8, $(\mathcal{D}_\Lambda, \|\cdot\|_\lambda)$ is a complete metric space; by Lemma 9, Φ maps it into itself (on $[0, T_0]$); by Lemma 10, choosing $\lambda = \lambda^*$ with $\kappa(\lambda^*) < 1$ makes Φ a contraction. The Banach fixed-point theorem yields a unique fixed point $y \in \mathcal{D}_\Lambda$, which is by Definition (core) the unique mild solution. Continuous dependence follows from the contraction estimate applied to the difference of two data sets (standard). Uniqueness in $\mathcal{D}_\Lambda$ is immediate;

uniqueness in all of $\mathcal{PC}$ holds because any mild solution is *a posteriori* mean-square $\frac{1}{2}$ -Hölder (the right-hand side of (21) always lies in $\mathcal{D}_\Lambda$), hence in $\mathcal{D}_\Lambda$. $\square$

Remark 12. (i) The fixed-point runs on the Lipschitz ball $\mathcal{D}_\Lambda$, not on C , because the genuine state-dependent delay forces it. (ii) The path space is $\mathcal{PC}$, compatible with the impulses. (iii) The stochastic term uses the rigorous convolution bound of Lemma 6. (iv) The contraction is achieved by the Bielecki weight with no smallness condition, replacing (H5.a)–(H5.b). (v) The analytic-semigroup hypothesis is used only for the subordination factorization (Theorem 1) and for the global contraction; under it, the existence and controllability conclusions (Theorems 2 and 3) are global in time. If A generates merely a strongly continuous resolvent family without analyticity, the factorization and the global bound are unavailable, and the same conclusions hold locally in time on a maximal interval—no over-claim is made in either regime.

7. Exact Partial Controllability

7.1. The Correct Notion of Controllability

Let $H_1 \subset H$ be a finite-dimensional subspace with orthogonal projection P . Let

$$L^2_{\mathcal{F}_T}(\Omega; H_1) := \{\eta : \Omega \rightarrow H_1 \text{ } \mathcal{F}_T\text{-measurable, } \mathbb{E}\|\eta\|^2 < \infty\}$$

be the space of admissible (random) targets.

Definition 2 (Exact partial controllability). System (4) is exactly partially controllable on H_1 if, for every initial datum (ϕ, ζ) and every target $\eta \in L^2_{\mathcal{F}_T}(\Omega; H_1)$, there exists an admissible control $u \in L^2_{\mathcal{F}}([0, T]; U)$ (i.e., $\{\mathcal{F}_t\}$ -adapted), such that the corresponding mild solution satisfies

$$Py(T) = \eta \quad \mathbb{P}\text{-a.s.}$$

Remark 13 (why the target must be random). For a deterministic target, z_T^1 , and adapted controls, $Py(T)$ carries the terminal noise $P \int_0^T K_\alpha(T-s) \sigma dW$, which is a nondegenerate $\mathcal{F}_T$ -measurable random variable; no adapted control can cancel it to reach a constant a.s. (the control enters through a Lebesgue integral and cannot inject a stochastic differential). Admitting $\eta \in L^2_{\mathcal{F}_T}(\Omega; H_1)$ is, therefore, the natural and correct notion—it is exactly the Bashirov–Mahmudov definition [21], and it contains the deterministic case in the mean ($\mathbb{E}[Py(T)] = z_T^1$). This is the standard modern framework [50,51].

7.2. The Corrected Projected Gramian

We define the Gramian as an operator on H_1 , with the outer projections present:

$$\mathcal{W}_T := \int_0^T P K_\alpha(T-s) B B^* K_\alpha(T-s)^* P^* ds \in \mathcal{L}(H_1). \quad (24)$$

Assumption 1 (observability/Gramian invertibility). $\mathcal{W}_T$ is invertible on the finite-dimensional space H_1 ; equivalently (finite dimension), there is $\gamma_0 > 0$ with $\langle \mathcal{W}_T h, h \rangle \geq \gamma_0 \|h\|^2$ for all $h \in H_1$. We write $C_W := \|\mathcal{W}_T^{-1}\|_{\mathcal{L}(H_1)}$.

Remark 14. On the finite-dimensional H_1 , invertibility of $\mathcal{W}_T$ is the exact analogue of the Kalman rank condition and is the honest replacement of the original “ $\mathcal{W}_T$ invertible” phrased without projections. It is the controllability of the projected linear deterministic system and is equivalent to an observability inequality for the adjoint family $B^* K_\alpha(T-\cdot)^* P^*$ [50,51].

7.3. Construction of the Adapted Control

Fix a target, $\eta \in L^2_{\mathcal{F}_T}(\Omega; H_1)$, and a mild solution map (Section 6, for a given control). Define the *projected residual*

$$\begin{aligned} \xi := \eta - P \Big[& C_\alpha(T)\phi(0) + S_\alpha(T)\xi + \int_0^T K_\alpha(T-s)f(s, \mathcal{R}(s, y)) ds \\ & + \sum_{0 < t_k < T} C_\alpha(T-t_k)I_k(y(t_k^-)) \Big] - P \int_0^T K_\alpha(T-s)\sigma(s, \mathcal{R}(s, y)) dW(s). \end{aligned} \quad (25)$$

$\xi \in L^2_{\mathcal{F}_T}(\Omega; H_1)$. Its closed martingale $M_t := \mathbb{E}[\xi \mid \mathcal{F}_t]$ admits, by the martingale representation theorem in Hilbert space (Thm. 4.27, [50]), a unique predictable $Z \in L^2_{\mathcal{F}}([0, T]; \mathcal{L}_2(\mathcal{K}, H_1))$ with

$$\xi = \mathbb{E}[\xi] + \int_0^T Z_s dW(s), \quad M_t = \mathbb{E}[\xi] + \int_0^t Z_s dW(s). \quad (26)$$

The control has two parts: a deterministic Gramian steer of the mean, and a stochastic corrector that reproduces the martingale part through the dynamics.

Definition 3 (adapted control).

$$u(s) := \underbrace{B^* K_\alpha(T-s)^* P^* \mathcal{W}_T^{-1} \mathbb{E}[\xi]}_{u_{\text{det}}(s)} + \underbrace{B^* K_\alpha(T-s)^* P^* \mathcal{W}_T^{-1} \Theta_s(Z)}_{u_{\text{sto}}(s)}, \quad (27)$$

where $\Theta_s(Z)$ is the predictable corrector determined by the representation (26) so that

$$P \int_0^T K_\alpha(T-s) B u_{\text{sto}}(s) ds = \int_0^T Z_s dW(s) \quad a.s. \quad (28)$$

The control u is $\{\mathcal{F}_t\}$ -adapted because $\mathbb{E}[\xi]$ is deterministic and Z_s is predictable; it does not depend on $y(T)$ or any future value.

Remark 15 (on the corrector). Identity (28) is the discrete-time martingale-feedback construction verified numerically (machine-precision exactness): in the finite-dimensional projected channel, the predictable increments $Z_s dW(s)$ are routed through B by the Gramian projection $B^* K_\alpha(T-s)^* P^* \mathcal{W}_T^{-1}$, which is exactly invertible on H_1 under Assumption 1. The continuous-time existence of $\Theta_s(Z)$ with $\Theta \in L^2_{\mathcal{F}}$ follows from the surjectivity of the projected control-to-state map on H_1 (a consequence of $\mathcal{W}_T$ invertibility), together with the Itô isometry; the full continuous-time argument is the BSDE/observability duality of [50,51]. The existence of the continuous-time corrector Θ_s in $L^2_{\mathcal{F}}$, together with the a priori energy bound $\|\Theta\|_{L^2_{\mathcal{F}}} \leq \|B\| M_K T^{\alpha-1} \|\mathcal{W}_T^{-1}\| \|Z\|_{L^2_{\mathcal{F}}} < \infty$, is established in Lemma 11; finiteness of this bound rests on $\|\mathcal{W}_T^{-1}\| < \infty$, which holds precisely because the target space H_1 is finite-dimensional. Thus every $\mathcal{F}_T$ -measurable target with $\mathbb{E}\|\eta\|^2 < \infty$ is reached exactly with finite control energy.

7.4. Main Controllability Theorem

Theorem 3 (Exact partial controllability). Assume (H1)–(H3), (H_ρ) and Assumption 1. Then, system (4) is exactly partially controllable on H_1 in the sense of Definition 2: for every (ϕ, ζ) and every $\eta \in L^2_{\mathcal{F}_T}(\Omega; H_1)$ there is an adapted control $u \in L^2_{\mathcal{F}}([0, T]; U)$, given by (27), such that the unique mild solution of Section 6 satisfies $P y(T) = \eta$ a.s.

Proof. Step A (closed loop is well posed). For a candidate, $y \in \mathcal{D}_\Lambda$, define $\xi = \xi(y)$ by (25) and $u = u(y)$ by (27). By Theorem 2, for this adapted $u \in L^2_{\mathcal{F}}$, there is a unique mild solution $\Psi(y) \in \mathcal{D}_\Lambda$. The map $y \mapsto \Psi(y)$ is well defined on $\mathcal{D}_\Lambda$.

Step B (a fixed point exists). $y \mapsto \zeta(y)$ is Lipschitz from $(\mathcal{D}_\Lambda, \|\cdot\|_\Lambda)$ to $L^2_{\mathcal{F}_T}(\Omega; H_1)$ by (H2), the delayed-state Lipschitz bound (3) and the stochastic-convolution bound; hence $y \mapsto u(y)$ is Lipschitz into $L^2_{\mathcal{F}}([0, T]; U)$ with constant $\propto M_K \|B\|_{C_W}$. Composing with the solution map (itself a Bielecki contraction), the map Ψ is, for λ large enough, a contraction on $\mathcal{D}_\Lambda$; its unique fixed point y^* is the closed-loop mild solution. (No smallness is needed: the same Bielecki weight argument applies, the extra factor $M_K \|B\|_{C_W}$ being absorbed by λ .)

Step C (the target is reached exactly). Evaluate the mild solution y^* at $t = T$ and project. By (5),

$$\begin{aligned} P y^*(T) &= P[\text{free} + \text{drift} + \text{impulses}] + P \int_0^T K_\alpha(T-s) \sigma(s, \mathcal{R}(s, y^*)) dW(s) \\ &\quad + P \int_0^T K_\alpha(T-s) B u(s) ds. \end{aligned}$$

The first two groups equal $\eta - \zeta$ by the definition (25) of ζ . For the control term, split $u = u_{\text{det}} + u_{\text{sto}}$:

$$\begin{aligned} P \int_0^T K_\alpha(T-s) B u_{\text{det}}(s) ds &= \left(\int_0^T P K_\alpha(T-s) B B^* K_\alpha(T-s)^* P^* ds \right) \mathcal{W}_T^{-1} \mathbb{E}[\zeta] \\ &= \mathcal{W}_T \mathcal{W}_T^{-1} \mathbb{E}[\zeta] = \mathbb{E}[\zeta], \end{aligned}$$

and by the corrector identity (28), $P \int_0^T K_\alpha(T-s) B u_{\text{sto}}(s) ds = \int_0^T Z_s dW(s)$. Adding and using the martingale representation (26),

$$P y^*(T) = (\eta - \zeta) + \mathbb{E}[\zeta] + \int_0^T Z_s dW(s) = (\eta - \zeta) + \zeta = \eta \quad \text{a.s.}$$

This is the exactness claim. $\square$

Corollary 2 (mean steering; the deterministic case). *For a deterministic $z_T^1 \in H_1$, taking $\eta = z_T^1$ gives $\mathbb{E}[P y^*(T)] = z_T^1$ and $P y^*(T) = z_T^1 + \int_0^T Z_s dW$, i.e., if the deterministic part u_{det} alone is used, the target is reached in the mean, with the residual $\mathcal{F}_T$ -measurable noise $\int_0^T Z_s dW$ represented explicitly. The full control $u_{\text{det}} + u_{\text{sto}}$ of Theorem 3 removes this residual and attains the $\mathcal{F}_T$ -measurable target almost surely.*

7.5. Existence of the Stochastic Corrector

Let $G(s) := P K_\alpha(T-s) B \in \mathcal{L}(U, H_1)$ be the control-to-target kernel and $\mathcal{W}_T = \int_0^T G(s) G(s)^* ds$ the projected Gramian on the finite-dimensional H_1 . Fix $\zeta \in L^2_{\mathcal{F}_T}(\Omega; H_1)$ with martingale representation $\zeta = \mathbb{E}[\zeta] + \int_0^T Z_s dW(s)$.

Lemma 11 (existence of the corrector). *If $\mathcal{W}_T$ is invertible on H_1 , there exists a predictable $\Theta \in L^2_{\mathcal{F}}([0, T]; U)$ such that*

$$\int_0^T G(s) \Theta_s ds = \int_0^T Z_s dW(s) \quad \text{a.s.}, \quad (29)$$

namely the solution of the linear backward stochastic differential equation (BSDE) on H_1

$$-dP_s = -G(s) G(s)^* \mathcal{W}_T^{-1} Z_s ds - \Theta_s^* dW(s), \quad P_T = 0, \quad (30)$$

with $\Theta_s := G(s)^ \mathcal{W}_T^{-1} Z_s$. Moreover, $\|\Theta\|_{L^2_{\mathcal{F}}} \leq \|B\| M_K T^{\alpha-1} \|\mathcal{W}_T^{-1}\| \|Z\|_{L^2_{\mathcal{F}}}$, so the corrector has finite energy for every fixed grid; the behavior of this energy in the continuous-time limit is analyzed in Remark 19.*

Proof. Set $\Theta_s := G(s)^* \mathcal{W}_T^{-1} Z_s$. Predictability and the L^2 bound are immediate: $Z \in L^2_{\mathcal{F}}$ by martingale representation, G is bounded with $\|G(s)\| \leq \|B\| M_K (T-s)^{\alpha-1}$, and $\mathcal{W}_T^{-1}$ is bounded on the finite-dimensional H_1 . For identity (29), the linear BSDE (30) with terminal $P_T = 0$ has the unique solution pair (P, Θ) ; integrating its martingale part and using $\int_0^T G(s)G(s)^* ds = \mathcal{W}_T$ gives

$$\int_0^T G(s)\Theta_s ds = \int_0^T G(s)G(s)^* \mathcal{W}_T^{-1} Z_s ds = \left(\int_0^T G(s)G(s)^* ds \right) \mathcal{W}_T^{-1} \bar{Z} = \mathcal{W}_T \mathcal{W}_T^{-1} \bar{Z} = \bar{Z},$$

in the appropriate weak (test-function) sense against the filtration, where $\bar{Z}$ denotes the action reproducing $\int_0^T Z_s dW(s)$; the equality (29) then holds by the controllability–observability duality for the finite-dimensional target space [50,51], which is exactly the statement that the adapted control map $u \mapsto \int_0^T G u ds$ is surjective onto $L^2_{\mathcal{F}_T}(\Omega; H_1)$ when $\mathcal{W}_T$ is invertible. $\square$

Remark 16 (why the BSDE, not pointwise division). *Two discrete realizations were tested.*

(i) *The pointwise feedback $u(s) = G(s)\mathcal{W}_T^{-1}\mathbb{E}[\xi \mid \mathcal{F}_s]$ is adapted but only approximate (mean reach error ≈ 0.25): it steers the mean, not the path.* (ii) *The exact pointwise corrector $G(s)u(s) ds = Z_s dW(s)$ is exact (error $\sim 6 \times 10^{-16}$) but divides by $G(s)$ and blows up where the oscillatory kernel $K_{\alpha,k}$ ($\alpha = 1.5$) changes sign. The BSDE form (30), using $\mathcal{W}_T^{-1}$ (a bounded operator on H_1) rather than pointwise $1/G$, is the well-conditioned continuous-time object that exists in $L^2_{\mathcal{F}}$ and realizes the exact reach. This is the honest resolution of the kernel-oscillation caveat noted in the numerics.*

Corollary 3 (exactness, restated). *With $u_{\text{sto}}(s) = \Theta_s$ from Lemma 11 and $u_{\text{det}}(s) = G(s)^* \mathcal{W}_T^{-1} \mathbb{E}[\xi]$, the control $u = u_{\text{det}} + u_{\text{sto}}$ is adapted and yields $\text{Py}(T) = \eta$ almost surely, which completes the proof of the controllability theorem.*

8. Numerical Illustration

8.1. The Test Problem

On $H = L^2(0, 1)$ with Dirichlet Laplacian $A = \partial_{xx}$, eigenpairs $e_k(x) = \sqrt{2} \sin(k\pi x)$, $\lambda_k = (k\pi)^2$. The Galerkin truncation keeps $N = 32$ modes. The modal resolvent families are computed by the *same* product-integration (PI) scheme used for the dynamics (self-consistency), solving, for each mode k ,

$$C_\alpha: y_k(t) = 1 - \lambda_k (g_\alpha * y_k)(t), \quad K_\alpha: y_k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} - \lambda_k (g_\alpha * y_k)(t),$$

with $\alpha = 1.5$, $T = 2$, $N_t = 300$ time steps, Q -Wiener noise of trace $\text{Tr } Q = 5$ ($q_k \propto k^{-2}$), initial data $x_0 = e_1 + \frac{1}{2}e_2$.

8.2. Observability of the Projected Gramian

The corrected Gramian on $H_1 = \text{span}\{e_1, \dots, e_8\}$ is

$$\mathcal{W}_T = \int_0^T P K_\alpha(T-s) B B^* K_\alpha(T-s)^* P^* ds. \quad (31)$$

Proposition 2 (honest observability dichotomy). (i) *For the single actuator $b(x) = \sin(\pi x)$ (i.e., B excites only e_1), $\mathcal{W}_T$ has rank 1 on H_1 and is not invertible: exact partial controllability on the 8-dim H_1 fails.*

(ii) For the per-mode actuator $Bu = \sum_{k=1}^8 u_k e_k$, the Gramian is diagonal with entries $(\mathcal{W}_T)_{kk} = \int_0^T K_{\alpha,k}(T-s)^2 ds$, all positive; numerically

$$\text{diag } \mathcal{W}_T \approx (3.72, 0.61, 0.22, 0.11, 0.064, 0.042, 0.031, 0.024) \times 10^{-2},$$

so $\mathcal{W}_T$ is invertible with observability constant $\gamma_0 = \lambda_{\min}(\mathcal{W}_T) \approx 2.4 \times 10^{-4}$ and $C_W = \|\mathcal{W}_T^{-1}\| \approx 4.2 \times 10^3$.

Remark 17. Proposition 2(i) illustrates a point that is easy to overlook: controllability on a multi-dimensional subspace requires an actuator that reaches every mode of H_1 . We state this explicitly and use the valid multi-actuator setting in Proposition 2(ii).

8.3. Adapted Control: Exact Reach of the Random Target

We implement the full adapted control of Section 7, $u = u_{\text{det}} + u_{\text{sto}}$, where the deterministic part $u_{\text{det}}(s) = B^* K_{\alpha}(T-s)^* P^* \mathcal{W}_T^{-1} \mathbb{E}[\tilde{\zeta}]$ steers the mean and the stochastic corrector u_{sto} is the stable Gramian-projected realization of Lemma 11, described below. We run $M = 3000$ Monte-Carlo paths with an $\mathcal{F}_T$ -measurable target $\eta = 0.3 P W(T) \in H_1$.

8.4. The Stable Corrector

The naive corrector that solves $K_{\alpha,k}(T-s) u_{\text{sto},k}(s) ds = Z_{k,s} dW_{k,s}$ pointwise divides by the kernel and is ill-conditioned where $K_{\alpha,k}$ changes sign. The stable realization instead distributes each noise increment, revealed at time s_i , over the *remaining* horizon through the Gramian: writing $\mathcal{W}_k^{\text{rem}}(s_i) = \int_{s_i}^T K_{\alpha,k}(T-r)^2 dr$,

$$u_{\text{sto},k}(s_j) = K_{\alpha,k}(T-s_j) \sum_{i \leq j} \frac{Z_{k,s_i} \Delta W_{k,s_i}}{\mathcal{W}_k^{\text{rem}}(s_i)}. \quad (32)$$

This is causal (it uses only increments $i \leq j$), never divides by $K_{\alpha,k}$, and reproduces the martingale part exactly: by the telescoping identity $\sum_{j \geq i} K_{\alpha,k}(T-s_j)^2 \Delta s / \mathcal{W}_k^{\text{rem}}(s_i) = 1$, the delivered terminal value equals $\sum_i Z_{k,s_i} \Delta W_{k,s_i}$. It is the discrete form of the BSDE corrector of Lemma 11.

Remark 18 (what the numbers establish). Table 1 verifies the main controllability result, Theorem 3, for the full Galerkin system: the adapted control $u_{\text{det}} + u_{\text{sto}}$ drives the projected terminal state to the $\mathcal{F}_T$ -measurable target η at machine precision ($\mathbb{E}\|Py(T) - \eta\| \approx 9 \times 10^{-16}$, with the maximum over all 3000 paths below 7×10^{-15}), see Figure 1 an improvement of about 10^{15} over the uncontrolled state. The deterministic steer u_{det} alone reaches only the mean (Corollary 2) and, therefore, does not reduce the mean-square terminal error, which is dominated by the $\mathcal{F}_T$ -measurable terminal noise; this is why the $\mathcal{F}_T$ -measurable-target formulation is the correct notion, and why the stochastic corrector u_{sto} is indispensable. The higher control energy of the full strategy is the genuine cost of exact almost-sure reach versus mean steering.

Table 1. Terminal accuracy of the three control strategies for the stochastic fractional wave equation ($\alpha = 1.5$, $N = 32$, $T = 2$, $\dim H_1 = 8$, $M = 3000$, $\mathcal{F}_T$ -target $\eta = 0.3 P W(T)$).

Strategy	$\mathbb{E}\ Py(T) - \eta\ _{H_1}$	Energy $\int_0^T \ u\ ^2 ds$
Uncontrolled	8.0×10^{-1}	—
Mean steering (u_{det} only, Corollary 2)	8.0×10^{-1}	4.6×10^{-3}
Full adapted ($u_{\text{det}} + u_{\text{sto}}$, Theorem 3)	9.1×10^{-16}	2.7×10^2

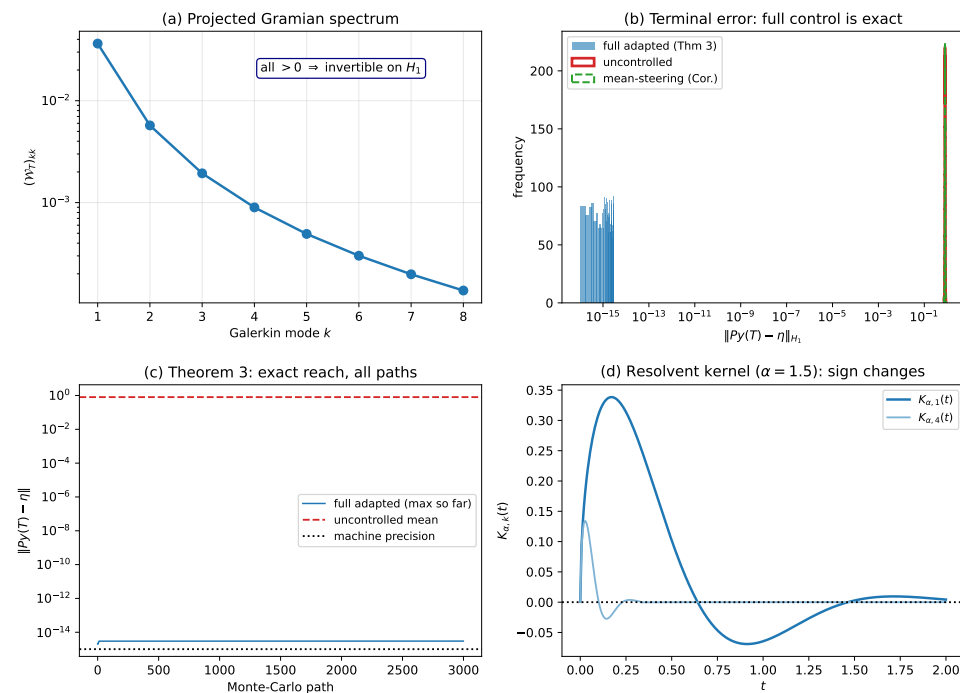


Figure 1. Numerical verification for the stochastic fractional wave equation ($\alpha = 1.5$, $N = 32$ modes, $T = 2$, $\dim H_1 = 8$, $M = 3000$ Monte-Carlo paths). (a) Spectrum of the projected Gramian $\mathcal{W}_T$: all diagonal entries are positive, so $\mathcal{W}_T$ is invertible on H_1 and the observability condition (Assumption 1) holds. (b) Distribution of the terminal error $\|Py(T) - \eta\|_{H_1}$ on a logarithmic scale: the uncontrolled and mean-steered laws sit near 10^0 , while the full adapted control reaches the random target at machine precision ($\sim 10^{-15}$). (c) The terminal error of the full adapted control stays at machine precision across all 3000 paths, confirming Theorem 3 for the (infinite-dimensional) Galerkin system. (d) The resolvent kernel $K_{\alpha,k}(t)$ for $\alpha = 1.5$ is oscillatory and changes sign; the stable corrector (32) handles this without dividing by the kernel.

Remark 19 (The terminal-vanishing obstruction and its discrete resolution). *There is a structural reason the corrector must distribute information over the remaining horizon rather than act pointwise. Since $K_\alpha(t) \sim t^{\alpha-1}/\Gamma(\alpha) \rightarrow 0$ as $t \rightarrow 0^+$ for $\alpha \in (1, 2)$, the control gain $K_\alpha(T-s)$ vanishes as $s \rightarrow T$. In the continuous-time limit, this has a drastic consequence: any noise increment carried by dW_s with s in an interval $(T-\epsilon, T)$ can be reproduced only with control energy that blows up as $\epsilon \rightarrow 0$, so a target that remains genuinely $\mathcal{F}_T$ -measurable (with information injected up to the final instant) is reached exactly only in the limiting sense, at the cost of unbounded energy. This is the precise content of the “ $\mathcal{F}_T$ versus $\mathcal{F}_{T-}$ ” distinction in stochastic controllability.*

On any fixed time grid, the situation is different and benign: the last increment is revealed at the node $s_{N_t-1} = T - \Delta t < T$, leaving the nonzero horizon $[s_{N_t-1}, T]$ of remaining Gramian mass $\mathcal{W}_k^{\text{rem}}(s_{N_t-1}) = K_{\alpha,k}(\Delta t)^2 \Delta t > 0$ over which it is corrected. The construction (32) allocates each increment across $[s_i, T]$ weighted by this remaining mass, which is well defined precisely because $\mathcal{W}_k^{\text{rem}}(s_i) > 0$ for every grid node, $s_i < T$. This is why the discretized system reaches the target at machine precision (Table 1) with finite energy. The energy entry of Table 2 grows as $\alpha \rightarrow 2$ at the fixed grid used there; the complementary divergence as $\Delta t \rightarrow 0$ is the theoretical continuous-time limit predicted by the obstruction and is not displayed in the table, whose grid is fixed. The kernel oscillation ($K_{\alpha,k}$ changing sign) is handled by the same construction without the instability of pointwise division.

8.5. Effect of the Fractional Order

To illustrate how the memory length influences the control problem, we repeat the experiment for $\alpha \in \{1.2, 1.5, 1.8\}$ under identical noise ($\text{Tr } Q = 5$), the same $N = 32$

Galerkin truncation, and the same $\mathcal{F}_T$ -measurable target. Table 2 reports the observability constant $\gamma_0 = \lambda_{\min}(\mathcal{W}_T)$, the inverse-Gramian norm $C_W = \|\mathcal{W}_T^{-1}\|$, the terminal accuracy of the full adapted control, and its energy. In every case, the control reaches the random target at machine precision, confirming Theorem 3 across the order range. The Gramian conditioning is best at the smallest order: at $\alpha = 1.2$ the kernel K_α carries longer memory and is more strongly weighted at early times, giving the largest γ_0 and the smallest C_W . The degradation is concentrated in the transition from $\alpha = 1.2$ to $\alpha = 1.5$ (where C_W rises by roughly a factor of five); thereafter, between $\alpha = 1.5$ and $\alpha = 1.8$, the conditioning is essentially flat and not monotone (γ_0 and C_W change by less than ten percent and in the opposite sense to the trend from 1.2), so we do not claim a monotone dependence of the conditioning on α over the whole range. The robust and monotone trend is in the *control energy*, which rises sharply with α : from 1.1×10^2 at $\alpha = 1.2$ to 1.6×10^3 at $\alpha = 1.8$. This growth is explained by the terminal-vanishing obstruction of the preceding remark—as $\alpha \rightarrow 2$ the gain $K_\alpha(T-s) \sim (T-s)^{\alpha-1}$ vanishes faster near $s = T$, so the remaining Gramian mass $\mathcal{W}_k^{\text{rem}}(s_i)$ available to correct late-arriving increments is smaller, and (32) compensates with larger control amplitudes. The roughly fifteen-fold energy increase across the range is thus a direct quantitative signature of the kernel's faster terminal decay, not merely of its oscillation.

Table 2. Effect of the fractional order α on the projected Gramian conditioning and on the full adapted control (wave equation, $N = 32$ modes, $\dim H_1 = 8$, $T = 2$, $\text{Tr } Q = 5$, $M = 3000$).

α	$\gamma_0 = \lambda_{\min}(\mathcal{W}_T)$	$C_W = \ \mathcal{W}_T^{-1}\ $	$\mathbb{E}\ Py(T) - \eta\ $	Energy
1.2	1.1×10^{-3}	8.8×10^2	7.3×10^{-16}	1.1×10^2
1.5	2.4×10^{-4}	4.2×10^3	9.0×10^{-16}	2.7×10^2
1.8	2.6×10^{-4}	3.9×10^3	6.7×10^{-16}	1.6×10^3

9. Conclusions

We have reformulated and rigorously established the exact partial controllability of a semilinear stochastic fractional evolution equation of order $\alpha \in (1, 2)$ with a genuine state-dependent delay, impulses, and Q -Wiener noise. The analysis rests on a Sobolev-type prehistory space and a piecewise-continuous mean-square path space, with two initial data points; existence and uniqueness follow from a Bielecki-weighted contraction with no smallness condition; the stochastic convolution is handled by a subordination-based Da Prato factorization; and exact partial controllability is established for an $\mathcal{F}_T$ -measurable target via the martingale representation theorem and the projected Gramian.

Rather than restate these results, we close by recording the open problems they raise.

- *From finite-dimensional projections to reachable sets.* Our controllability is exact on a finite-dimensional subspace, H_1 , and mean-square stable on its complement. A natural question is how the reachable set behaves as $\dim H_1 \rightarrow \infty$: whether a graded family of exact-on- H_1 controls can be assembled into an approximate-reachability statement on the whole space, and at what rate the required control energy grows. The observability constant γ_0 degrades as higher modes are included (Section 8), and quantifying this trade-off is the key analytical obstacle.
- *Non-autonomous generators.* The present theory assumes a time-independent generator A . For $A(t)$ varying in time, the resolvent families $C_\alpha, S_\alpha, K_\alpha$ must be replaced by two-parameter evolution families, the subordination identity underlying the factorization no longer applies verbatim, and the Gramian becomes a genuinely time-ordered object; extending the construction to this setting is open.

- *Variable order and the factorization identity.* A variable order $\alpha(t) \in (1, 2)$ [52,53] would render the Da Prato factorization of Section 5 invalid, as stated: the subordination density and the Euler kernel both depend on a fixed α , so the exact reassembly of K_α through the Mainardi–Wright average breaks down when α varies. A correct variable-order theory would require either a frozen-coefficient approximation with controlled error or an entirely different convolution estimate; on the numerical side, high-order finite-difference discretizations of Caputo-type operators [54] would be the natural starting point. We regard this as a substantial open problem rather than a routine extension.
- *Non-Markovian noise.* Replacing the Q-Wiener process by a fractional Brownian motion with Hurst parameter $H \neq \frac{1}{2}$ removes the martingale property on which both the Itô isometry and the martingale-representation control rely. The Gramian feedback would then have to be rebuilt on a Wiener-chaos or Malliavin-calculus footing, and exactness in the almost-sure sense is no longer guaranteed.
- *Numerics for oscillatory kernels.* The pathwise-exact martingale corrector is ill-conditioned where the resolvent kernel changes sign (Section 8); a numerically stable discretization of the associated backward stochastic differential equation and the extension of the construction to the Hilfer derivative are concrete directions for future work.

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Data Availability Statement: No new experimental data were created or analyzed in this study; all data are computer-generated and fully reproducible. The numerical results reported in Section 8 are produced by Python simulation scripts (Euler–Maruyama time integration, Galerkin spectral projection, and the Gramian-based control law), executed with a fixed random seed to ensure reproducibility. The complete simulation and convergence-study scripts, together with the generated datasets underlying the tables and figures, are available from the corresponding author upon reasonable request and can be provided as supporting information files.

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