

Article

Ulam's Type Stability of a Discrete Matrix Equation with Second-Order Difference and Single Delay

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Abstract

In this paper, we investigate the Ulam stability problem for a second-order discrete matrix difference equation with a single delay. The primary aim of this work is to establish rigorous Ulam-type stability criteria for this class of delayed matrix difference systems. Firstly, we formally define Hyers–Ulam stability (HUS) and Hyers–Ulam–Rassias stability (HURS) tailored to delayed discrete matrix equations, and derive an essential norm estimate for the delayed discrete matrix function (DDMF). Relying on this critical norm inequality, we further derive sufficient conditions to guarantee the Hyers–Ulam and Hyers–Ulam–Rassias stability of the addressed equation, which constitute the core theoretical contributions of this study. Compared with the existing literature, the obtained stability judgments are less conservative and applicable to second-order matrix delayed difference models rarely discussed in previous works. Finally, several numerical examples are presented to demonstrate the correctness and effectiveness of the derived theoretical conclusions.

Keywords: discrete matrix equation; second-order difference; single delay; Ulam's type stability

1. Introduction

In 1940, Ulam [1], affiliated with the University of Wisconsin, proposed the famous stability problem related to functional equations: under what circumstances is it possible to approximate an approximately additive mapping with a strictly additive one? One year later, Hyers [2] provided a solution to this problem in the setting of Banach spaces, thereby introducing the notion of Hyers–Ulam stability. In 1978, Rassias [3] developed a generalized form of Hyers–Ulam stability for mappings, which greatly promoted the progress of this research area. Over the years, the theory of Ulam's stability has undergone continuous development within the contexts of operator theory, functional equations, differential equations, and difference equations. As is widely known, stability is a crucial qualitative property of differential systems. There are various types of stability, but recently researchers have focused on Ulam's stability.

It is worth noting that Popa [4,5] pioneered investigations into the HUs of difference equations back in 2005. Following this foundational work, a wealth of subsequent studies have extended this stability framework to various classes of difference equations, leading to a sustained and rapid expansion of the literature on HUs for discrete systems (see [6–20]). Notably, delay is a pervasive feature in real-world engineering and



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physical systems, which renders the analysis of delayed difference equations particularly meaningful and practically relevant. Therefore, numerous researchers have directed their efforts toward investigating the stability of time-delayed discrete systems. However, obtaining explicit solution representations for delay difference systems via conventional approaches poses notable challenges. Fortunately, Shuklin and Khusainov [21] addressed linear autonomous delay systems with commutable matrices by introducing the concept of a delayed exponential matrix function (DEMF). Inspired by this idea, Diblík and Khusainov [22] extended the methodology to characterize solutions for discrete delay systems. Later on, the DEMF approach has been broadly adopted to derive solution representations, stability results, and relative controllability criteria for a wide range of such systems [23–36]. It is noteworthy that Wang et al. [27] explored the Ulam-type stability of second-order difference discrete delay systems. Additionally, the Hyers–Ulam stability of impulsive time-delay semi-linear systems and the Ulam stability of non-singular delay differential equations have been analyzed via delayed exponential matrix techniques [37,38]. Later, Rahmat et al. [39] extended the results from [38] to investigate the HUs of non-singular delay difference equations. Moonswan et al. [40] further investigated the HUs, exponential stability, and relative controllability of non-singular delay difference systems, thereby extending the research outcomes presented in [38,39]. This demonstrates that DDMF serve as an effective analytical framework for discrete delay systems. As a prominent and active research area, Hyers–Ulam stability has been widely investigated by numerous scholars, who primarily employ the delayed exponential matrix technique to derive relevant results.

In the context of discrete systems, the modeling of delayed dynamics frequently relies on difference equations, which describe how a system's state evolves from one time step to the next. Among these, second-order difference equations are particularly relevant for capturing phenomena involving acceleration-like changes (e.g., vibrations, rate-of-change of rates), while matrix-valued formulations extend applicability to multi-variable systems, where interactions between components are inherently coupled (e.g., multi-axis control systems, networked nodes, or multi-sector economic models) [41,42]. Recently, Diblík [43,44] derived explicit solutions for linear discrete matrix delay equations and second-order difference delayed discrete matrix equations, utilizing a novel, delayed exponential matrix function. The stability analysis of discrete-time systems has attracted considerable attention due to its wide applications in control theory, numerical computation, and engineering practice. The work of Chen et al. [45] investigated the Hyers–Ulam stability and relative controllability of a linear matrix discrete system with single delay, which offers a useful theoretical basis for analyzing delayed discrete matrix equations. Furthermore, Elshenhab et al. [46] studied the existence of exact solutions and the stability of first-order linear discrete matrix equations with multiple delays and non-permutable matrices, further extending the theoretical foundations for more complex discrete systems. Shankar et al. [47] presented a comprehensive survey on the Ulam–Hyers stability of fractional-order differential equations, which not only covers fundamental results in the field but also provides a valuable reference for stability analysis involving differential and difference equations. Although the above studies have laid a solid foundation for stability analysis of discrete and differential systems, further investigation is still needed for the Hyers–Ulam stability of second-order discrete matrix equations with time delays. As far as we are aware, the Ulam's stability of delayed discrete matrix equations particularly second-order delayed discrete matrix equations involving second-order difference sremains unexplored in the existing literature. Driven by this research gap, the present work

focuses on the Ulam's stability of a second-order difference discrete matrix equation with single delay.

$$\begin{cases} \Delta^2 x(n) + Mx(n - \tau) + x(n - \tau)N = f(n), & n \in \mathbb{Z}_0^T, \\ x(n) = \varphi(n), & n \in \mathbb{Z}_{-\tau}^1, \end{cases} \quad (1)$$

where $\tau \in \mathbb{Z}_1^\infty$ denotes the time delay, and Δ^2 represents the second-order forward difference operator satisfying $\Delta^2 x(n) = x(n + 2) - 2x(n + 1) + x(n)$. The unknown matrix-valued function $x : \mathbb{Z}_{-\tau}^T \rightarrow \mathbb{R}^{r \times r}$ satisfies (1), where M, N are given constant $r \times r$ real matrices ($M \in \mathbb{R}^{r \times r}$, $N \in \mathbb{R}^{r \times r}$). The initial matrix function $\varphi : \mathbb{Z}_{-\tau}^1 \rightarrow \mathbb{R}^{r \times r}$ is provided, with $\Delta\varphi(n) = \varphi(n + 1) - \varphi(n)$ for all $n \in \mathbb{Z}_{-\tau}^1$. For integers $s, t \in \mathbb{Z}$ with $s \leq t$, we define the finite integer set $\mathbb{Z}_s^t := \{s, s + 1, \dots, t\}$ and the infinite set $\mathbb{Z}_s^\infty := \{s, s + 1, \dots\}$. Additionally, $f : \mathbb{Z}_0^T \rightarrow \mathbb{R}^{r \times r}$ is a given matrix function, with T being a fixed positive integer.

Studying such equations is critical for understanding and regulating delayed multi-variable discrete systems across engineering (e.g., networked control, power grid stability), physics (e.g., discrete lattice dynamics), and economics (e.g., multi-sector growth models), where addressing delay-induced complexities is key to ensuring system performance and stability.

The core contribution of this work lies in establishing sufficient conditions for the Ulam-type stability of a second-order difference discrete matrix equation with single delay, leveraging a key norm estimation for the DEMF, and extending the scope of systems considered in Ref. [27], it poses greater challenges in terms of key techniques. In this process, we further apply and extend the framework of special delayed matrix functions. The obtained findings serve as a valuable complement to the existing theory on the stability of delayed discrete matrix equations.

The structure of this paper is outlined as follows: Section 2 introduces the auxiliary matrix functions ($\mathcal{N}_c(\cdot, M, N, \tau)$ and $\mathcal{N}_s(\cdot, M, N, \tau)$), along with their key properties, and presents the explicit solution representation for the second-order difference discrete matrix equation with single delay, alongside the formal definitions of Ulam-type stability. Section 3 establishes new results concerning the Ulam-type stability of the target delayed discrete matrix equation, where sufficient stability conditions are rigorously proven via the auxiliary matrix functions and the derived solution representation. Section 4 provides illustrative examples to validate the practical applicability of the main theoretical findings.

2. Preliminaries

In this section, we present the formal definitions and fundamental concepts relevant to the subsequent analysis. For a constant matrix $H \in \mathbb{R}^{r \times r}$, we define $\|H\| = \max_{\|x\|=1} \|Hx\|$, where $\|\cdot\|$ denotes the vector norm. At the same time, for an unknown matrix $M(\cdot) \in \mathbb{R}^{r \times r}$, we define $\|M(\cdot)\|_F = (\sum_{i=1}^r \sum_{j=1}^r |a_{ij}|^2)^{\frac{1}{2}}$. The binomial coefficients being defined as

$$\binom{p}{q} =: \begin{cases} \frac{p!}{q!(p-q)!} & \text{if } p \geq q \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $p, q \in \mathbb{Z}$ and $0!$ is defined as 1. We use Θ and I to represent the $r \times r$ zero matrix and identity matrix, respectively. Based on the results from [44], we introduce two delayed discrete matrix functions $\mathcal{N}_c(\cdot, M, N, \tau)$ and $\mathcal{N}_s(\cdot, M, N, \tau)$ as follows:

$$\mathcal{N}_c(n, M, N, \tau) = \begin{cases} \Theta, \forall n \in \mathbb{Z}_{-\infty}^{-\tau-1}, \\ I, \forall n \in \mathbb{Z}_{-\tau}^1, \\ \dots \\ I - \frac{n!}{2!(n-2)!} (M + N) + \frac{(n-\tau)!}{4!(n-\tau-4)!} (M^2 + 2MN + N^2) + \dots \\ + (-1)^\ell \frac{(n-(\ell-1)\tau)!}{(2\ell)! (t-2\ell-(\ell-1)\tau)!} \sum_{s=0}^{\ell} M^{\ell-s} N^s, \\ \forall n \in \mathbb{Z}_{(\ell-1)(\tau+2)+2}^{\ell(\tau+2)+1}, \ell = 0, 1, 2, \dots, \end{cases} \quad (2)$$

and

$$\mathcal{N}_s(n, M, N, \tau) = \begin{cases} \Theta, \forall n \in \mathbb{Z}_{-\infty}^{-\tau}, \\ I \frac{(n+\tau)!}{1!(n+\tau-1)!}, \forall n \in \mathbb{Z}_{-\tau+1}^2, \\ \dots \\ I \frac{(n+\tau)!}{1!(n+\tau-1)!} - \frac{n!}{3!(n-3)!} (M + N) + \frac{(n-\tau)!}{5!(n-\tau-5)!} (M^2 + 2MN + N^2) + \dots \\ + (-1)^\ell \frac{(n-(\ell-1)\tau)!}{(2\ell+1)! (t-2\ell-1-(\ell-1)\tau)!} \sum_{s=0}^{\ell} M^{\ell-s} N^s, \\ \forall n \in \mathbb{Z}_{(\ell-1)(\tau+2)+3}^{\ell(\tau+2)+2}, \ell = 0, 1, 2, \dots. \end{cases} \quad (3)$$

For simplicity $\mathcal{N}_c(\cdot) := \mathcal{N}_c(\cdot, M, N, \tau)$ and $\mathcal{N}_s(\cdot) := \mathcal{N}_s(\cdot, M, N, \tau)$, ℓ is a natural number, representing the power of the matrix.

Lemma 1 ([44], Theorem 6). Let $M, N \in \mathbb{R}^{r \times r}$ be constant matrices. Assume further that the commutativity conditions $\varphi(n)N = N\varphi(n)$ for all $n \in \mathbb{Z}_{-\tau}^0$ and $Nf(n) = f(n)N$ for all $n \in \mathbb{Z}_{\infty}^0$ hold. Under these assumptions, the matrix function

$$\begin{aligned} x(n) &= \mathcal{N}_c(n)\varphi(-\tau) + \mathcal{N}_s(n)\Delta\varphi(-\tau) + \sum_{i=-\tau+1}^0 \mathcal{N}_s(n-\tau-i)\Delta^2\varphi(i-1) \\ &\quad + \sum_{i=0}^{n-2} \mathcal{N}_s(n-1-\tau-i)f(i), \quad n \in \mathbb{Z}_{-\tau}^{\infty}, \end{aligned} \quad (4)$$

solves the problem (1).

Now, we present the formal definition of Ulam's stability for the considered system.

Definition 1. System (1) is said to possess HUs if there exists a constant $K > 0$ with the property that, for any $\varepsilon > 0$, every matrix-valued function $\chi(\cdot) : \mathbb{Z}_0^T \rightarrow \mathbb{R}^{r \times r}$ satisfying the inequalities

$$\begin{cases} \|\Delta^2\chi(n) + M\chi(n-\tau) + \chi(n-\tau)N - f(n)\| \leq \varepsilon, \quad n \in \mathbb{Z}_0^T, \tau \in \mathbb{Z}_1^\infty, \\ \|\chi(n) - \varphi(n)\| \leq \varepsilon, \quad n \in \mathbb{Z}_{-\tau}^1, \end{cases} \quad (5)$$

admits an exact solution $x(\cdot) : \mathbb{Z}_0^T \rightarrow \mathbb{R}^{r \times r}$ of (1) such that

$$\|\chi(n) - x(n)\| \leq K\varepsilon, \quad n \in \mathbb{Z}_0^T.$$

Definition 2. System (1) exhibits HURs relative to Φ if there exists $K_\Phi > 0$ such that, for all $\varepsilon > 0$, $\psi > 0$, and any nondecreasing $\Phi(\cdot) : \mathbb{Z}_0^T \rightarrow \mathbb{R}$, any matrix function $\chi(\cdot) : \mathbb{Z}_0^T \rightarrow \mathbb{R}^{r \times r}$ satisfying

$$\begin{cases} \|\Delta^2\chi(n) + M\chi(n-\tau) + \chi(n-\tau)N - f(n)\| \leq \varepsilon\Phi(n), \quad n \in \mathbb{Z}_0^T, \tau \in \mathbb{Z}_1^\infty, \\ \|\chi(n) - \varphi(n)\| \leq \varepsilon\psi, \quad n \in \mathbb{Z}_{-\tau}^1, \end{cases} \quad (6)$$

has a corresponding exact solution $x(\cdot) : \mathbb{Z}_0^T \rightarrow \mathbb{R}^{r \times r}$ of (1) with

$$\|\chi(n) - x(n)\| \leq K_\Phi \varepsilon \Phi(n), \quad n \in \mathbb{Z}_0^T.$$

The constant K_Φ is termed the Hyers–Ulam–Rassias constant for system (1).

Remark 1. It follows from (5) that there exists a matrix function $g(\cdot) : \mathbb{Z}_0^T \rightarrow \mathbb{R}^{r \times r}$ such that

$$\begin{cases} \Delta^2 \vartheta(n) + M\vartheta(n - \tau) + \vartheta(n - \tau)N - f(n) = g(n), & n \in \mathbb{Z}_0^T, \tau \in \mathbb{Z}_1^\infty, \\ \vartheta(n) = \varphi(n), & n \in \mathbb{Z}_{-\tau}^1, \end{cases}$$

where $\|g(n)\| \leq \varepsilon$ for all $n \in \mathbb{Z}_0^T$.

Remark 2. Based on relation (6), one can find a matrix function $G(\cdot) : \mathbb{Z}_0^T \rightarrow \mathbb{R}^{r \times r}$ satisfying

$$\begin{cases} \Delta^2 \vartheta(n) + M\vartheta(n - \tau) + \vartheta(n - \tau)N - f(n) = G(n), & n \in \mathbb{Z}_0^T, \tau \in \mathbb{Z}_1^\infty, \\ \vartheta(n) = \varphi(n), & n \in \mathbb{Z}_{-\tau}^1, \end{cases}$$

where $\|G(n)\| \leq \varepsilon \Phi(n)$ for $n \in \mathbb{Z}_0^T$.

Next, we establish the boundedness of $\mathcal{N}_s(\cdot)$, which is essential for the subsequent proof of the main results.

Lemma 2. For every $n \in \mathbb{Z}_0^\infty$, the estimate

$$\|\mathcal{N}_c(n)\| \leq e^{\max(\|M\| + \|N\|)n} \quad (7)$$

is valid.

Proof. For any integer $\ell \geq 0$ and $n \in \mathbb{Z}_{(\ell-1)(\tau+2)+2}^{\ell(\tau+2)+1}$, we take the matrix norm of Equation (2) to obtain

$$\begin{aligned} \|\mathcal{N}_c(n)\| &= \|I\| + \binom{n}{2} (\|M\| + \|N\|) + \binom{n-\tau}{4} (\|M\|^2 + 2\|M\|\|N\| + \|N\|^2) + \dots \\ &\quad + \binom{n-(\ell-1)\tau}{2\ell} (\|M\|^\ell + \binom{1}{\ell} \|M\|^{\ell-1}\|N\| + \dots + \binom{\ell-1}{\ell} \|M\|\|N\|^{\ell-1} \\ &\quad + \binom{\ell}{\ell} \|N\|^\ell) \\ &\leq 1 + \frac{n!}{2!(n-2)!} (\|M\| + \|N\|) + \frac{(n-\tau)!}{4!(n-\tau-4)!} (\|M\| + \|N\|)^2 + \dots \\ &\quad + \frac{(n-(\ell-1)\tau)!}{(2\ell)!(n-2\ell-(\ell-1)\tau)!} (\|M\| + \|N\|)^\ell \\ &\leq 1 + \frac{n^2}{2!} (\|M\| + \|N\|) + \frac{n^4}{4!} (\|M\| + \|N\|)^2 + \dots + \frac{n^{2\ell}}{(2\ell)!} (\|M\| + \|N\|)^\ell \\ &\leq 1 + n(\|M\| + \|N\|) + \frac{n^2}{2!} (\|M\| + \|N\|)^2 + \frac{n^3}{3!} (\|M\| + \|N\|)^3 \\ &\quad + \dots + \frac{n^{2\ell}}{(2\ell)!} (\|M\| + \|N\|)^{2\ell}, \end{aligned}$$

when $\|M\| + \|N\| > 1$, we get

$$\|\mathcal{N}_c(n)\| \leq e^{(\|M\| + \|N\|)n},$$

when $\|M\| + \|N\| \leq 1$, we have

$$\|\mathcal{N}_c(n)\| \leq e^n.$$

From the above, we can obtain

$$\|\mathcal{N}_c(n)\| \leq e^{(\|M\| + \|N\|)n}.$$

□

Lemma 3. For every $n \in \mathbb{Z}_0^\infty$, the estimate

$$\|\mathcal{N}_s(n)\| \leq n + \tau - 1 + e^{\max(1, \|M\| + \|N\|)n} \quad (8)$$

is valid.

Proof. For any integer $\ell \geq 0$ and $n \in \mathbb{Z}_{(\ell-1)(\tau+2)+3'}^{\ell(\tau+2)+2}$, we take the matrix norm of Equation (3) to obtain

$$\begin{aligned} \|\mathcal{N}_s(n)\| &= \|(n+m)I - \binom{n}{3}(M+N) + \binom{n-\tau}{5}(M+N)^2 + \dots \\ &\quad + (-1)^\ell \binom{n-(\ell-1)m}{2\ell+1}(A+B)^\ell\| \\ &\leq (n+\tau)\|I\| + \binom{n}{3}(\|M\| + \|N\|) + \binom{n-\tau}{5}(\|M\| + \|N\|)^2 + \dots \\ &\quad + \binom{n-(\ell-1)\tau}{2\ell+1}(\|M\| + \|N\|)^\ell \\ &\leq (n+\tau-1) + 1 + (\|M\| + \|N\|)n + \frac{n(n-1)(n-2)}{3!}(\|M\| + \|N\|) \\ &\quad + \frac{(n-\tau)!}{5!(n-\tau-5)!}(\|M\| + \|N\|)^2 + \dots + \frac{(n-(\ell-1)\tau)!}{(2\ell+1)!(n-2\ell-1-(\ell-1)\tau)!}(\|M\| + \|N\|)^\ell \\ &\leq (n+\tau-1) + 1 + (\|M\| + \|N\|)n + \frac{n^3}{3!}(\|M\| + \|N\|) + \frac{n^5}{5!}(\|M\| + \|N\|)^2 + \dots \\ &\quad + \frac{n^{2\ell+1}}{(2\ell+1)!}(\|M\| + \|N\|)^\ell \\ &\leq (n+\tau-1) + \left[1 + t \max\left(1, (\|M\| + \|N\|)\right) + \frac{t^3 \max\left(1, (\|M\| + \|N\|)^3\right)}{3!} \right. \\ &\quad \left. + \frac{n^5 \max\left(1, (\|M\| + \|N\|)^5\right)}{5!} + \dots + \frac{n^{2\ell+1} \max\left(1, (\|M\| + \|N\|)^{2\ell+1}\right)}{(2\ell+1)!} \right] \\ &\leq (n+\tau-1) + \left[1 + n \max\left(1, (\|A\| + \|B\|)\right) + \frac{t^2 \max\left(1, (\|A\| + \|B\|)^2\right)}{2!} \right. \\ &\quad + \frac{n^3 \max\left(1, (\|M\| + \|N\|)^3\right)}{3!} + \frac{n^4 \max\left(1, (\|M\| + \|N\|)^4\right)}{4!} \\ &\quad + \frac{n^5 \max\left(1, (\|M\| + \|N\|)^5\right)}{5!} + \dots + \frac{n^{2\ell} \max\left(1, (\|M\| + \|N\|)^{2\ell}\right)}{(2\ell)!} \\ &\quad \left. + \frac{n^{2\ell+1} \max\left(1, (\|M\| + \|N\|)^{2\ell+1}\right)}{(2\ell+1)!} \right] \\ &= (n+\tau-1) + \sum_{j=0}^{\infty} \frac{\left(\max(1, (\|M\| + \|N\|))n\right)^j}{j!} \\ &= (n+\tau-1) + e^{\max(1, \|M\| + \|N\|)n}. \end{aligned}$$

□

Remark 3. Based on Lemmas 2 and 3, we can obtain for $n \in [0, T]$,

$$\sup_{n \in [0, T]} \|\mathcal{N}_c(n)\| \leq e^{(\|M\| + \|N\|)T}, \quad T \in \mathbb{Z}^+$$

and

$$\sup_{n \in [0, T]} \|\mathcal{N}_s(n)\| \leq (T + \tau - 1) + e^{\max(1, \|M\| + \|N\|)T}, \quad T \in \mathbb{Z}^+,$$

where $\mathbb{Z}^+ \in \mathbb{Z}_1^\infty$ denotes the set of positive integers.

3. Main Results

In this section, we establish the HUs and HURs for the difference discrete matrix equation.

Theorem 1. The delayed discrete matrix Equation (1) is Hyers–Ulam stable on the interval $[0, T]$.

Proof. Let $\varepsilon > 0$ be arbitrary, and suppose that a matrix function $\chi(\cdot) : \mathbb{Z}_0^T \rightarrow \mathbb{R}^{r \times r}$ satisfies

$$\begin{cases} \|\Delta^2 \chi(n) + M\chi(n - \tau) + \chi(n - \tau)N - f(n)\| \leq \varepsilon, & n \in \mathbb{Z}_0^T, \tau \in \mathbb{Z}_1^\infty, \\ \|\chi(n) - \varphi(n)\| \leq \varepsilon, & n \in \mathbb{Z}_{-\tau}^1. \end{cases}$$

In view of Remark 1, there exists a matrix function $g(\cdot) : \mathbb{Z}_0^T \rightarrow \mathbb{R}^{r \times r}$ such that

$$\begin{cases} \Delta^2 \chi(n) + M\chi(n - \tau) + \chi(n - \tau)N - f(n) = g(n), & n \in \mathbb{Z}_0^T, \tau \in \mathbb{Z}_1^\infty, \\ \chi(t) = \varphi(t), & t \in \mathbb{Z}_{-\tau}^1, \end{cases} \quad (9)$$

where $\|g(n)\| \leq \varepsilon$. By virtue of Equations (1) and (4), (9) admits a solution

$$\begin{aligned} \chi(n) &= \mathcal{N}_c(n)\varphi(-\tau) + \mathcal{N}_s(n)\Delta\varphi(-\tau) + \sum_{i=-\tau+1}^0 \mathcal{N}_s(n - \tau - i)\Delta^2\varphi(i - 1) \\ &\quad + \sum_{i=0}^{n-2} \mathcal{N}_s(n - 1 - \tau - i)(f(i) + g(i)), \end{aligned}$$

therefore, we can get

$$\begin{aligned} \|\chi(n) - x(n)\| &= \left\| \sum_{i=0}^{n-2} \mathcal{N}_s(n - 1 - \tau - i)g(i) \right\| \\ &\leq \sum_{i=0}^{n-2} \|\mathcal{N}_s(n - 1 - \tau - i)\| \|g(i)\| \\ &\leq \sum_{i=0}^{n-2} \|\mathcal{N}_s(n - 1 - \tau - i)\| \varepsilon. \end{aligned} \quad (10)$$

For $n \in [0, T]$, by virtue of Remark 3, we obtain

$$\begin{aligned} \|\chi(n) - x(n)\| &\leq \sum_{i=0}^{T-2} \|\mathcal{N}_s(n - 1 - \tau - j)\| \varepsilon \\ &\leq (T - 1)(T + \tau - 1 + e^{\max(1, \|M\| + \|N\|)T}) \varepsilon \\ &\leq K_1 \varepsilon, \end{aligned}$$

namely,

$$\|\chi(n) - x(n)\| \leq K_1 \varepsilon,$$

where $K_1 = (T - 1)(T + \tau - 1 + e^{\max(1, \|M\| + \|N\|)T})$. In accordance with Definition 1, we conclude that Equation (1) is Hyers–Ulam stable. $\square$

Theorem 2. *The delayed discrete matrix Equation (1) is Hyers–Ulam–Rassias stable on the interval $[0, T]$.*

Proof. Let $\varepsilon > 0$ and $\psi > 0$ be arbitrary, and suppose that a matrix function $\chi(\cdot) : \mathbb{Z}_0^T \rightarrow \mathbb{R}^{r \times r}$ satisfies

$$\begin{cases} \|\Delta^2 \chi(n) + M\chi(n - \tau) + \chi(n - \tau)N - f(n)\| \leq \varepsilon \Phi(n), & n \in \mathbb{Z}_0^T, \tau \in \mathbb{Z}_1^\infty, \\ \|\chi(n) - \varphi(n)\| \leq \varepsilon \psi, & n \in \mathbb{Z}_{-\tau}^1. \end{cases}$$

In view of Remark 2, there exists a matrix function $G(\cdot) : \mathbb{Z}_0^T \rightarrow \mathbb{R}^{r \times r}$ such that

$$\begin{cases} \Delta^2 \chi(n) + M\chi(n - \tau) + \chi(n - \tau)N - f(n) = G(n), & n \in \mathbb{Z}_0^T, \tau \in \mathbb{Z}_1^\infty, \\ \chi(n) = \varphi(n), & n \in \mathbb{Z}_{-\tau}^1, \end{cases} \quad (11)$$

where $\|G(n)\| \leq \varepsilon \Phi(n)$. Based on Lemma 1, (11) has solution

$$\begin{aligned} \chi(n) &= \mathcal{N}_c(n)\varphi(-\tau) + \mathcal{N}_s(n)\Delta\varphi(-\tau) + \sum_{i=-\tau+1}^0 \mathcal{N}_s(n - \tau - i)\Delta^2\varphi(i - 1) \\ &\quad + \sum_{i=0}^{n-2} \mathcal{N}_s(n - 1 - \tau - i)(f(i) + G(i)). \end{aligned}$$

Now, we have

$$\begin{aligned} \|\chi(n) - x(n)\| &= \left\| \sum_{i=0}^{n-2} \mathcal{N}_s(n - 1 - \tau - i)G(i) \right\| \\ &\leq \sum_{i=0}^{n-2} \|\mathcal{N}_s(n - 1 - \tau - i)\| \|G(i)\| \\ &\leq \sum_{i=0}^{n-2} \|\mathcal{N}_s(n - 1 - \tau - i)\| \varepsilon \Phi(i) \\ &\leq \|\mathcal{N}_s(n - 1 - \tau) + \mathcal{N}_s(n - 2 - \tau) + \cdots + \mathcal{N}_s(1 - \tau)\| (n - 1) \varepsilon \Phi(n) \\ &\leq (\|\mathcal{N}_s(n - 1 - \tau)\| + \|\mathcal{N}_s(n - 2 - \tau)\| + \cdots + \|\mathcal{N}_s(1 - \tau)\|) (n - 1) \varepsilon \Phi(n). \end{aligned}$$

By virtue of Remark 3, we further obtain for all $n \in [0, T]$

$$\begin{aligned} \|\chi(n) - x(n)\| &\leq (T - 1)^2 (T + \tau - 1 + e^{\max(1, \|M\| + \|N\|)T}) \varepsilon \Phi(n) \\ &= \varepsilon K_\Phi \Phi(n), \end{aligned}$$

which implies that

$$\|\chi(n) - x(n)\| \leq \varepsilon K_\Phi \Phi(n),$$

where

$$K_\Phi = (T - 1)^2 (T + \tau - 1 + e^{\max(1, \|M\| + \|N\|)T}) > 0.$$

Consequently, Equation (1) is Hyers–Ulam–Rassias stable. $\square$

The main theoretical achievements of this paper are summarized in Theorems 1 and 2, which deliver the essential sufficient conditions for Ulam-type stability of second-order discrete matrix difference equations with single delay. These two theorems lay a solid theoretical foundation for subsequent numerical verification and extended research.

4. Examples

Example 1. Set $r = 2$, $T = 5$, $\tau = 1$. We now examine the following second-order linear discrete delayed matrix difference equation:

$$\begin{cases} \Delta^2 x(n) + Mx(n-1) + x(n-1)N = f(n), & n \in \mathbb{Z}_0^5, \\ x(n) = \varphi(n), & n \in \mathbb{Z}_{-1}^1, \end{cases} \quad (12)$$

with inequality

$$\begin{cases} \|\Delta^2 x(n) + Mx(n-1) + x(n-1)N - f(n)\| \leq 0.8, & n \in \mathbb{Z}_0^5, \\ \|x(n) - \varphi(n)\| \leq 1, & n \in \mathbb{Z}_{-1}^1, \end{cases} \quad (13)$$

and

$$\begin{cases} \|\Delta^2 x(n) + Mx(n-1) + x(n-1)N - f(n)\| \leq 0.8\Phi(t), & n \in \mathbb{Z}_0^5, \\ \|x(n) - \varphi(n)\| \leq 0.5, & n \in \mathbb{Z}_{-1}^1, \end{cases} \quad (14)$$

where $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $f(t) = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$, $\varphi(n) = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$, $\psi = 0.5$, $\Phi(n) = e^n$.

If a matrix function $\chi(\cdot) : \mathbb{Z}_0^\infty \rightarrow \mathbb{R}^{2 \times 2}$ satisfies (13), then in light of Remark 1, there exists a matrix function $g(\cdot) : \mathbb{Z}_0^\infty \rightarrow \mathbb{R}^{2 \times 2}$ such that $\|g(n)\| \leq \varepsilon$, and

$$\begin{cases} \Delta^2 \chi(n) + M\chi(n-1) + \chi(n-1)N - f(n) = g(n), & n \in \mathbb{Z}_0^5, \\ \chi(n) = \varphi(n), & n \in \mathbb{Z}_{-1}^1, \end{cases} \quad (15)$$

Furthermore, the solution of Equation (12) is given by

$$\begin{aligned} x(n) &= \mathcal{N}_c(n)\varphi(-1) + \mathcal{N}_s(n)\Delta\varphi(-1) + \sum_{i=-1}^0 \mathcal{N}_s(n-1-i)\Delta^2\varphi(i-1) \\ &\quad + \sum_{i=0}^{n-2} \mathcal{N}_s(n-2-i)f(i). \end{aligned}$$

Let $\varepsilon = 0.8$, and define a matrix function $g(\cdot) : \mathbb{Z}_0^\infty \rightarrow \mathbb{R}^{2 \times 2}$ by

$$g(n) = \begin{pmatrix} 0.6 \cos(n + \pi/2) & 0 \\ 0 & 0.6 \sin(n + \pi/2) \end{pmatrix},$$

then it is clear that

$$\|g(n)\|_F = \left((0.6 \cos(n + \pi/2))^2 + 0^2 + 0^2 + (0.6 \sin(n + \pi/2))^2 \right)^{\frac{1}{2}} = (0.6^2)^{\frac{1}{2}} = 0.6 \leq \varepsilon = 0.8.$$

We now present the solution to the perturbed delayed discrete matrix Equation (15):

$$\begin{aligned} \chi(n) &= \mathcal{N}_c(n)\varphi(-1) + \mathcal{N}_s(n)\Delta\varphi(-1) + \sum_{i=-1}^0 \mathcal{N}_s(n-1-i)\Delta^2\varphi(i-1) \\ &\quad + \sum_{i=0}^{n-2} \mathcal{N}_s(n-2-i)(f(i) + g(i)). \end{aligned}$$

According to (3), it can be obtained

$$\mathcal{N}_s(n) = \begin{cases} \Theta, & n \in \mathbb{Z}_{-\infty}^{-1}, \\ (n+1)I, & n \in \mathbb{Z}_0^2, \\ (n+1)I - \frac{n!}{3!(n-3)!}(M+N), & t \in \mathbb{Z}_3^5. \end{cases}$$

Then we have

$$\begin{aligned} \|\chi(n) - x(n)\| &= \left\| \sum_{i=0}^{n-2} \mathcal{N}_s(n-2-i)g(i) \right\| \\ &= \|\mathcal{N}_s(n-2)g(0) + \mathcal{N}_s(n-3)g(1) + \cdots + \mathcal{N}_s(0)g(n-2)\|. \end{aligned}$$

For $n \in \mathbb{Z}_0^5$, the values of $\|\chi(n) - x(n)\|$ are presented in Table 1.

Table 1. Norm of the difference between approximate and exact solutions for Equation (12).

n	$\ \chi(n) - x(n)\ $	$\ \chi(n) - x(n)\ \leq K\varepsilon$
$n = 1$	0.0000	$\ \chi(n) - x(n)\ \leq \frac{1}{10}\varepsilon$
$n = 2$	0.6000	$\ \chi(n) - x(n)\ \leq \varepsilon$
$n = 3$	1.6056	$\ \chi(n) - x(n)\ \leq 3\varepsilon$
$n = 4$	2.6930	$\ \chi(n) - x(n)\ \leq 4\varepsilon$
$n = 5$	3.7250	$\ \chi(n) - x(n)\ \leq 5\varepsilon$

Choose $K = 5$, we observe that Theorem 1, we have

$$\|\chi(n) - x(n)\| \leq K\varepsilon.$$

Thus, (12) is Hyers–Ulam stable on $\mathbb{Z}_0^5$.

Figure 1 presents the dynamic response curves of the solution to Equation (12), clearly demonstrating that the considered delayed second-order discrete matrix equation satisfies Hyers–Ulam stability under the selected parameter values.

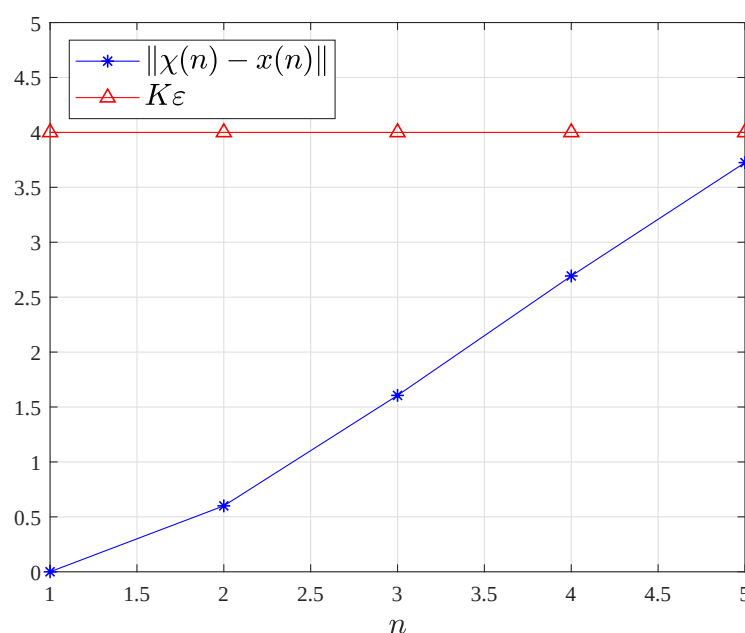


Figure 1. State trajectory of the delayed discrete matrix system verifying the Hyers–Ulam stability.

Example 2. Set $r = 2$, $T = 5$, $\tau = 1$. We now examine the following second-order linear discrete delayed matrix difference equation:

$$\begin{cases} \Delta^2 x(n) + Mx(n-1) + x(n-1)N = f(n), & n \in \mathbb{Z}_0^5, \\ x(n) = \varphi(n), & n \in \mathbb{Z}_{-1}^1, \end{cases} \quad (16)$$

with inequality

$$\begin{cases} \|\Delta^2 x(n) + Mx(n-1) + x(n-1)N - f(n)\| \leq 0.8, & n \in \mathbb{Z}_0^5, \\ \|x(n) - \varphi(n)\| \leq 1, & n \in \mathbb{Z}_{-1}^1, \end{cases} \quad (17)$$

and

$$\begin{cases} \|\Delta^2 x(n) + Mx(n-1) + x(n-1)N - f(n)\| \leq 0.8\Phi(t), & n \in \mathbb{Z}_0^5, \\ \|x(n) - \varphi(n)\| \leq 0.5, & n \in \mathbb{Z}_{-1}^1, \end{cases} \quad (18)$$

where $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $f(t) = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$, $\varphi(n) = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$, $\psi = 0.5$, $\Phi(n) = e^n$.

If a matrix function $\chi(\cdot) : \mathbb{Z}_0^\infty \rightarrow \mathbb{R}^{2 \times 2}$ satisfies (18), then in light of Remark 2, there exists a matrix function $G(\cdot) : \mathbb{Z}_0^\infty \rightarrow \mathbb{R}^{2 \times 2}$ such that $\|G(n)\| \leq \varepsilon\Phi(n)$, and

$$\begin{cases} \Delta^2 \chi(n) + M\chi(n-1) + \chi(n-1)N - f(n) = G(n), & t \in \mathbb{Z}_0^5, \\ \chi(n) = \varphi(n), & n \in \mathbb{Z}_{-1}^1. \end{cases} \quad (19)$$

Furthermore, the solution of Equation (19) is given by

$$\begin{aligned} \chi(n) &= \mathcal{N}_c(n)\varphi(-1) + \mathcal{N}_s(n)\Delta\varphi(-1) + \sum_{i=-1}^0 \mathcal{N}_s(n-1-i)\Delta^2\varphi(i-1) \\ &\quad + \sum_{i=0}^{n-2} \mathcal{N}_s(n-2-i)(f(i) + G(i)). \end{aligned}$$

Let $\varepsilon = 0.8$, and define a matrix function $G(\cdot) : \mathbb{Z}_0^\infty \rightarrow \mathbb{R}^{2 \times 2}$ by

$$G(n) = \begin{pmatrix} 0.7 \cos(n + \pi/2) & 0 \\ 0 & 0.7 \sin(n + \pi/2) \end{pmatrix},$$

it is clear that

$$\|G(n)\| = \left((0.7 \cos(n + \pi/2))^2 + 0^2 + 0^2 + (0.7 \sin(n + \pi/2))^2 \right)^{\frac{1}{2}} = (0.7^2)^{\frac{1}{2}} = 0.7 \leq \varepsilon e^n = 0.8e^n = \varepsilon\Phi(n).$$

Consequently, we obtain

$$\begin{aligned} \|\chi(n) - x(n)\| &= \left\| \sum_{i=0}^{n-2} \mathcal{N}_s(n-2-i)G(i) \right\| \\ &= \|\mathcal{N}_s(n-2)G(0) + \mathcal{N}_s(n-3)G(1) + \cdots + \mathcal{N}_s(0)G(n-2)\|, \end{aligned}$$

for $n \in \mathbb{Z}_0^5$, the values of $\|\chi(n) - x(n)\|$ are presented in Table 2.

Table 2. Norm of the difference between approximate and exact solutions for Equation (16).

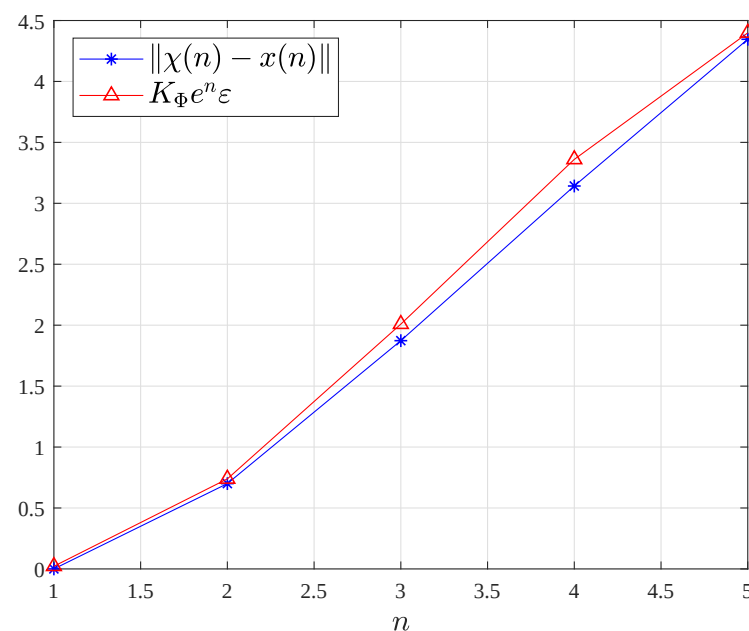
n	$\ \chi(n) - x(n)\ $	$\ \chi(t) - x(t)\ \leq K_{\Phi} \varepsilon \Phi(n)$
$n = 1$	0.0000	$\ \chi(n) - x(n)\ \leq \frac{1}{99} e \varepsilon$
$n = 2$	0.7000	$\ \chi(n) - x(n)\ \leq \frac{1}{8} e^2 \varepsilon$
$n = 3$	1.8730	$\ \chi(n) - x(n)\ \leq \frac{1}{8} e^3 \varepsilon$
$n = 4$	3.1421	$\ \chi(n) - x(n)\ \leq \frac{1}{13} e^4 \varepsilon$
$n = 5$	4.3460	$\ \chi(n) - x(n)\ \leq \frac{1}{27} e^5 \varepsilon$

Choose $K_{\Phi} = \frac{1}{8}$, we observe from Theorem 2, we have

$$\|\chi(n) - x(n)\| \leq K_{\Phi} \varepsilon \Phi(n).$$

Consequently, (16) is Hyers–Ulam–Rassias stable on $\mathbb{Z}_0^5$.

Figure 2 presents the dynamic response curves of the solution to Equation (16), clearly demonstrating that the considered delayed second-order discrete matrix equation satisfies Hyers–Ulam–Rassias stability under the selected parameter values.

**Figure 2.** State trajectory of the delayed discrete matrix system verifying the Hyers–Ulam–Rassias stability.

5. Conclusions

In this paper, we rigorously established the HUs and HURs of second-order delayed discrete matrix equations by systematically leveraging the newly constructed DDMFs along with their corresponding bounded estimation techniques under well-defined regularity conditions. Furthermore, we constructed and developed a novel, unified methodological framework tailored specifically for theoretically analyzing and verifying the Hyers–Ulam stability of such second-order delay-involved discrete matrix equations, which provides structured analytical procedures and verifiable criteria for stability assessments of the targeted equation class. Notably, this present work marks the first pioneering endeavor to adopt the proposed discrete delayed matrix functions as core analytical tools to rigorously establish Ulam-type stability, comprehensively covering both classic Hyers–Ulam stability and generalized Hyers–Ulam–Rassias stability, within the research scope of second-order

delayed discrete matrix equations, thereby filling the existing research gap in the application of DDMFs for Ulam stability of second-order discrete matrix systems with delay effects. Overall, the specific discrete delayed matrix functions introduced and utilized throughout this study are not only effectively applied to conduct quantitative stability analysis for the discussed second-order delayed discrete matrix equations but also further generalized to accommodate broader discrete delay-driven matrix equation structures. Beyond the immediate stability results derived herein, this innovative methodological approach opens up promising theoretical avenues and feasible research directions for future in-depth investigations into higher-order delayed discrete matrix equations, as well as more complex discrete matrix system problems incorporating multiple time delays, variable delays, or coupled delay structures, laying a solid analytical foundation for subsequent extended stability studies in discrete matrix dynamical systems with delay characteristics.

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