

Article

Fundamental Issues in Spectrum Measurement [†]

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Abstract

Insight into spectrum measurement is provided regarding the possibility of achieving a unified definition of the measurand for important classes of phenomena, including periodic, transitory impulsive, stochastic stationary, and stochastic non-stationary ones. Furthermore, the notion of characterizing the observation window as a key feature of the measurement method is introduced, and the issue of reducing the variability of the results is discussed, with a focus on the key role that can be played by the multi-taper method.

Keywords: time series; signal analysis; spectrum estimation; dynamic measurement; stationary and non-stationary processes

1. Introduction

Spectrum measurement can be regarded as a special and particularly important type of dynamic measurement. These measurements are currently receiving significant attention due to their theoretical and practical relevance [1–6]. A dynamic measurement can be classified as direct when the measurand is the time behavior of a quantity of interest, or indirect when the measurand—representing the information of interest—is obtained through a transformation applied to the original signal [5]. Spectrum measurement is arguably the most important example of the latter. While spectrum estimation has so far been studied mainly within the field of signal processing [7–9], here, it is suggested that examining its foundations could be valuable for both theoretical (including epistemological) and practical reasons.

These topics, first analyzed in the conference paper [10], are discussed in more detail here, extending the content with regard to both general epistemological aspects (Section 2) and specific issues about linear and quadratic analyses, moving toward a general approach (Section 3). Moreover, advanced metrological issues, including the concept of characterizing window and data tapering, are introduced in Section 4.3.

The spectrum is often the final result of the measurement process; in other cases, some synthetic parameters are obtained from the spectrum for an immediate informative purpose. For example, when dealing with sea waves, the spectrum can be used to obtain the significant wave height, its period, the maximum wave height, and so on. Other examples include the areas of optics or spectroscopy [11] and medicine [12].

Here, we attempt to provide some insight into the fundamentals of spectrum measurement, including the following issues:

- The informative value of a “spectrum”, either linear or quadratic, justifying its interest as a measurable property, is discussed.



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- The possibility of achieving a unified definition of the measurand, i.e., of the spectrum, is outlined, which implies that it is possible to consider spectrum measurement as a unique task in spite of the noteworthy differences among different types of dynamic processes of interest, namely periodic, transitory impulsive, stationary stochastic, and slowly varying non-stationary stochastic.
- The structure of the measurement process, both for linear and quadratic spectra, for the different types of processes is considered, outlining two main issues in particular:
 - The concept of the “characterizing window”;
 - Advances in data tapering.

Due to the vast nature of the subject, in this paper, only theoretical and epistemological aspects are considered. Some examples of the application of the principles discussed here are mentioned in the references. They concern parallel studies carried out by the authors, mainly concerning the measurement of sea waves.

2. General Epistemological Aspects

A preliminary question is: what is the benefit of moving to spectrum measurement, rather than limiting to direct measurement, i.e., to the measurement of the time-behavior of the dynamic process under investigation? In fact, often the spectrum more effectively highlights the information of interest. The reason for this lies in the broader epistemological value of the analysis procedure.

To show this in the case of interest here, let us recall that, in the pioneering work by Fourier [13], the spectral analysis was firstly fully developed for periodic signals, and it may be useful to briefly highlight the underlying idea of that analysis.

Then, let $x(t)$ be a periodic function/signal, that is $x(t) = x(t + T_p)$ where T_p is the period and $f_0 = 1/T_p$ is the fundamental frequency. Indeed, under fairly general conditions, the following spectral representation holds:

$$x(t) = c_0 + \sum_{h=1}^{\infty} c_h \cdot \cos(2\pi h f_0 t + \varphi_h), \quad (1)$$

where the series $\{c_h\}$ and $\{\varphi_h\}$ constitute respectively the amplitude—or modulus—and phase spectra. Since (1) shows that the informative content remains exactly the same, a key question arises: what is the benefit of turning from a time domain to a frequency domain representation?

Here we suggest that the value of this spectral representation is linked to the cognitive property of virtually any form of analysis. In fact, the fundamental investigation approach of an analysis in general is the de-composition of a complex “object” into elementary components, whose properties are well-known and clearly understandable. Through this approach, it becomes possible to infer the properties of the object as an appropriate, typically “weighted”, combination of its elementary components [14,15]. A simple daily example is the analysis of drinkable water, which allows us to determine whether that type of water is, for example, better suited for the stomach or the kidneys, based on its components and of their (relative) amounts.

In our specific case, the complex “object” undergoing spectral analysis is a signal, which is decomposed into well-known elementary cosine functions at different frequencies. These frequencies are all multiple of the fundamental one, f_0 , and each one has a specific amplitude c_h —representing the amount of each component and a phase φ_h —which describes how they interact with each other. In many cases, this is precisely the information that is looked for, or, using measurement terminology, the measurand.

As a practical example, consider the balancing of a rotating machine: the amplitude of the fundamental frequency provides information about the eccentricity unbalance

and its phase indicates its angular position. Both pieces of information are essential for balancing [16]. More generally, spectrum measurement is a crucial source of information for technical diagnostics [17]. Another example is sea state estimation and coastal monitoring, where spectrum measurement proves highly informative [18,19]. Further applications regard medicine [20,21] and materials characterization by nanoscopy [22].

3. Modeling the Measurand

3.1. Dynamic Phenomena and Related Spectra

Each measurement process is founded on a model of the *measurand* (the quantity to be measured). A dynamic phenomenon, or process, can be described in terms of one (or more) quantity, whose value varies over time. Dynamic measurement aims at its characterization by measuring the time-varying quantity that we can call x .

Then, in direct dynamic measurement, the measurand is the time behavior of x , either as a continuous function of time, $x(t)$, or as a discrete time history (or time series), x_i . If the time series is obtained by uniform sampling, we obtain the following:

$$x_i = x(t_i) = x(i\Delta t), \quad (2)$$

where Δt is the sampling interval, which will be assumed to be constant from here on.

Dynamic phenomena can be classified in different types, including the following:

- Periodic;
- Transitory impulsive;
- Stationary stochastic;
- Non-stationary (weakly varying) stochastic.

All of them deserve specific scientific and technical interest; moreover, for all these cases, a spectral representation does exist, although with remarkable differences. Before briefly reviewing them, we outline a few key methodological points.

First, a clear distinction must be drawn between stationary and non-stationary ones. Informally, a stationary process is one that maintains some “properties” invariant in time. A non-stationary process, instead, is simply one that does not have such a characteristic.

Furthermore, in regards of the approach to spectral analysis, in some (authoritative) texts, a distinction is made between deterministic and stochastic spectral analysis [8,9], intending that in the case of periodic and transient processes, the spectrum can be defined without the notion of probability, which is instead necessary when the concept of the stochastic process is introduced. We suggest that this distinction, although useful for didactic purposes, seems somewhat questionable from a theoretical standpoint and is therefore avoided here.

Lastly, another important difference can be determined between linear (or direct) and quadratic (or “average”) analysis. This distinction, instead, when properly specified, may be important, both from mathematical and metrological standpoints, and it will thus be maintained and examined as follows.

After this premise, we briefly review the spectral representations of the above-mentioned phenomena—at least as they are commonly presented—with the aim of highlighting their cognitive and informative relevance and presenting a unified perspective.

3.2. Periodic Phenomena

A spectral representation for periodic phenomena has already been presented in (1). An alternative, equivalent, formulation is provided by

$$x(t) = \sum_{h=-\infty}^{+\infty} X_h \cdot \exp(j2\pi h f_0 t), \quad (3)$$

where the spectrum is now complex-valued and is given by Fourier Series (FS):

$$X_h = \frac{1}{T_p} \int_{-T_p/2}^{+T_p/2} x(t) \exp(-j2\pi h f_0 t) dt = |X_h| \exp(j\theta_k). \quad (4)$$

Again, the general interpretation of the above representation as a form of analysis is possible, wherein the “components” upon which such analysis is based are now the well-known complex exponential functions $\{\exp(j2\pi h f_0 t), h = -\infty, \dots, +\infty\}$ and amplitude and phases spectra are given by the series $\{|X_h|\}$ and $\{\theta_k\}$, respectively. The equivalence between representations (1) and (3) can be proved using simple trigonometric identities. In fact, in actual measurement, such series will be necessarily limited to the minimum number of terms necessary to provide a representation compatible with the uncertainty limits that are acceptable for the measurement process.

3.3. Interlude: Linear and Quadratic Spectral Analysis

So far, only linear spectral analysis has been considered. Before proceeding to more complex phenomenal structures, it may be useful to outline the basic features of linear and quadratic spectra, since their characters are significantly different.

From an application perspective, linear analysis is necessary only when phase information is required—such as in system balancing—or when evaluating system frequency response via impulse testing. Conversely, quadratic analysis constitutes a measurand in its own right, as it quantifies the power or energy distribution as a function of frequency. Furthermore, linear analysis cannot be directly applied to stochastic or erratic processes, as it would yield non-robust results characterized by the same degree of randomness as the input signal.

Continuing with periodic phenomena, a linear spectrum is defined through (3), which, in this context, may be considered a synthesis formula/procedure, since it allows a reconstruction of the original signal, and through (4), which similarly acts as its analysis counterpart, as discussed in Section 2.

Yet, there is another important approach which consists of the measurement of the quadratic spectrum. This quantity is relevant in various aspects. The most immediate interpretation is in terms of energy/power. In fact, the overall “power” associated with the (periodic) phenomenon is given by the following:

$$P_x = \frac{1}{T_p} \int_{-T_p/2}^{+T_p/2} x^2(t) dt, \quad (5)$$

and an equivalent characterization is provided by the well-known and extensively used root mean square value:

$$x_{rms} = \sqrt{P_x}. \quad (6)$$

In many cases, the above-defined power has a physical interpretation as an actual “physical” power, expressed in watts, but in general, it constitutes a convenient indicator of the overall intensity/strength of the phenomenon, since it properly accounts for the presence of both positive and negative variations, which, in the general case, do not cancel, but rather enforce themselves. A similar interpretation holds true for the rms value, with the advantage that it has the same measurement unit as the measurand, and thus its physical interpretation is more straightforward, but with the disadvantage of not satisfying some useful additive properties, as will be shown below.

Then, quadratic spectrum analysis corresponds (in this case) to the power spectrum, which can be expressed, with reference to (1) and assuming¹ $c_0 = 0$, in the simple form:

$$P_h = \frac{c_h^2}{2}, \quad (7)$$

and provides the distribution of the overall power in the spectral domain, according to Parseval's property,

$$P_x = \sum_h P_h, \quad (8)$$

which highlights the additive character of the power spectrum.

The case of $c_0 \neq 0$ can be treated easily, but it is often less significant than the case where it is null. Furthermore—as will become clearer in the following—power spectra provide a way for performing some averaging (ultimately related to the addition property (8)), which will be necessary when dealing with phenomena with remarkably erratic behavior.

3.4. Transitory Impulsive Phenomena

The basic property ensuring the possibility of spectral analysis of periodic phenomena is that their “power”, expressed by (5), is finite. On the other hand, transitory impulsive phenomena allow a spectral representation, if they are characterized by a finite “energy”, i.e., a finite (overall) quadratic value [23]:

$$E = \int_{-\infty}^{+\infty} x^2(t) dt < \infty, \quad (9)$$

and the spectral representation is now based on

$$x(t) = \int_{-\infty}^{+\infty} X(f) \exp(+j2\pi ft) df, \quad (10)$$

where $X(f)$ is a continuous function of frequency, defined by the Fourier Transform (FT), or Fourier integral:

$$X(f) = \int_{-\infty}^{+\infty} x(t) \exp(-j2\pi ft) dt, \quad (11)$$

and properly called a “spectral density”, due to its continuous character.

Again, a representation in terms of energy is possible, with the energy spectral density being

$$\eta(f) = |X(f)|^2, \quad (12)$$

which shows how the overall energy of the phenomenon is (continuously) distributed in the spectral domain:

$$E = \int_{-\infty}^{+\infty} \eta(f) df, \quad (13)$$

which constitutes a reformulation of Parseval's theorem.

Again, the benefit of this representation consists of the possibility of expressing the signal as a combination—now being a continuous function, the spectral density—of elementary well-known functions, the imaginary exponentials $\exp(+j2\pi ft)$, but there are also other remarkable aspects. Consider, for example, the special but important case of shock testing of mechanical systems, such as motorcycle helmets, or the testing of vehicle suspensions. In this case, the forcing action is a sharp impulse—ideally a delta-Dirac pulse—and the FT of the response would be the frequency response of the system, a highly informative feature.

In the continuous case, the units of measurement (UM) are given in Table 1. Conversely, in the discrete case (DFT), normalization must always be verified, as implementation conventions vary and no universal standard exists. The normalization of windows used

in PSD analysis requires a unitary area, whereas windows employed in the analysis of periodic signals must exhibit a unitary amplitude in their central region.

Table 1. Overview of the measurement units of possible spectra.

Analysis Type	Result Measurement Units
FS	same units of the time signal
FT	same units of the time signal
DFT normalized as a sampling of the FT	discretization of the FT has the same units
DFT normalized as a sampling of the FS ¹	discretization of the FS has the same units
Energy spectral density	energy unit divided by frequency
PSD	power units divided by f

¹ in this case it should be properly called DFS.

3.5. A Unified Approach

So far, the differences between the spectral representations of periodic and transitory impulsive phenomena have been shown. Yet, since the basic ideas and the application perspective is very similar, it is desirable to move toward a unified approach, since unification may be, in general, considered value in science [24,25].

As demonstrated below, despite the notable differences between the classes of phenomena under consideration, the same conceptual framework holds true in all cases; i.e., the possibility of expressing the cognitive value of spectrum measurement as an analysis process aiming to reduce the observed phenomena to a composition of well-known sine and cosine functions.

In fact, this is possible, at the price of somewhat more sophisticated mathematics. In fact, this requires moving from the familiar Fourier integral to a generalization of it, namely the Fourier-Stieltjes integral. Although a detailed presentation of this issue is far beyond the scope of this paper, a simple introduction of the underlying basic idea can be beneficial, in view of the above-mentioned value of reaching a unified approach. The mentioned Fourier-Stieltjes integral reads:

$$x(t) = \int_{-\infty}^{+\infty} \exp(+j2\pi ft) d\zeta(f), \quad (14)$$

where $\zeta(f)$ is the integrated spectrum. Intuitively, the integrated spectrum can be understood as the integral of the spectrum, up to the considered frequency f , i.e., $\zeta(f) = \int_{-\infty}^f X(\phi) d\phi$. If ζ is differentiable, i.e., $d\zeta(f) = X(f)df$, (14) reduces to (10), where $d\zeta(f) = O(df)$.

Furthermore, this generalization holds true also in the case of the Fourier series, where $\zeta(f)$ is a step function, with a step occurring in correspondence to each discrete frequency present in the spectrum, in which $d\zeta(f) = O(1)$, and $d\zeta(f) = 0$ in all other frequencies. Finally, and most significantly, this formulation also includes the case of stationary stochastic phenomena and can be extended to non-stationary processes, as will be shown below.

3.6. Stationary Stochastic Phenomena

Informally, stationary stochastic phenomena manifest themselves with erratic signals that maintain some properties that are invariant in time. A stochastic process can be modeled as a collection of time histories that constitute realizations of the overall process. Each realization is characterized by a finite “power” that is a finite mean quadratic value:

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt < \infty, \quad (15)$$

for which, as anticipated, a spectral representation is possible through the Fourier-Stieltjes integral:

$$x(t) = \int_{-\infty}^{+\infty} \exp(+j2\pi ft) d\zeta(f). \quad (16)$$

In this case, for the integrated spectrum $\zeta(f)$, $d\zeta(f) = O(\sqrt{f})$, which is an intermediate condition between what happens for periodic and transitory impulsive phenomena, as mentioned in the previous subsection, since now $\zeta(f)$ is neither a step function, nor is it differentiable. A proof of expression (16) is provided in [8], where mathematical details are highlighted. This is of high theoretical value, since it also confirms that these signals result from a composition of imaginary exponentials, and thus of “sine and cosine waves with varying frequencies and (random) amplitudes and phases” [8]. Furthermore, this representation founds the possibility of describing the process in terms of distribution of the power P in the frequency domain, through a Power Spectral Density, PSD, or $S(f)$, such that

$$P_{f_1 f_2} = \int_{f_1}^{f_2} S(f) df, \quad (17)$$

where $P_{f_1 f_2}$ denotes the power in the frequency band (f_1, f_2) . In fact, it can be proved that in the case $S(f)$ is a continuous function,

$$S(f) df = E[|d\zeta(f)|^2], \quad (18)$$

where E denotes the expectation operator [8].

Apart from mathematical details, the important points here are that:

- Even in the case of stationary stochastic processes, it is possible to interpret signals as resulting from the combination of sine and cosine components, i.e., the notion of a spectrum is well-founded.
- It is necessary to consider quadratic, rather than linear, spectra, which have an “average” character, due to the associated expectation operator, since the linear spectrum is very erratic and of limited practical interest.

This last consideration supports the discussion in Section 3.3, where it was noted that quadratic spectra not only provide information on the “energy” associated with the process, but also constitute the preferred approach in the case of processes with erratic realizations.

3.7. Non-Stationary Phenomena with Time-Varying Spectra

A special case of non-stationarity has already been considered, regarding the transitory impulsive phenomena. In that case, however, the entire phenomena can be described by a single spectrum. Here, instead, we consider the case where the spectrum itself varies through time. Noteworthy, there are two main representations, one for linear and the other for quadratic spectra.

The former is provided by observing the signal through a moving Gaussian window, $w(t)$. The corresponding representation is thus given by

$$x(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(f, \tau) \exp(+j2\pi f t) df d\tau, \quad (19)$$

where

$$X(f, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(\tau) w(t - \tau) \exp(-j2\pi f t) d\tau, \quad (20)$$

is the Gabor transform of $x(t)$ [26]. This is essentially a deterministic formulation, appropriate to describe essentially regular processes, such as the vibration in slow run up of a rotating machine, mainly due to residual unbalance, and is strictly related to the rotation speed.

For more irregular phenomena, the model of the nonstationary stochastic process is needed. The point here is to replace the “basic elements” $\phi(f, t) = \exp(+j2\pi ft)$ appearing in (19) with elements that “although non-stationary, have an oscillatory form and in which the notion of frequency is still dominant” [8]. One proper representation can be thus:

$$x(t) = \int_{-\infty}^{+\infty} A(f, t) \exp(+j2\pi ft) d\zeta(f), \quad (21)$$

where $A(f, t)$ is an amplitude modulated sine wave [8,27,28]. In terms of power, the evolutionary power spectral density (EPSD) comes to play, defined by

$$S(f, t) df = [A(f, t)]^2 E[|d\zeta(f)|^2], \quad (22)$$

which constitutes a generalization of (18).

3.8. Conclusion About the Model of the Measurand

From the above consideration, spectrum measurement emerges as highly valuable information-gathering process. For the four main classes of dynamic processes examined, it aims to perform an analysis by interpreting them as a composition of basic well-known functions, namely sines and cosines, and determining this composition for any phenomena of interest through a measurement process.

4. The Spectrum Measurement Process

4.1. General Unified Spectrum Measurement Procedure

Let us now consider the spectrum measurement process, which, as anticipated, can be considered as a kind of indirect measurement [29,30], including two phases:

- Direct measurement of the time behavior of some quantity, $x(t)$, representative of the process under investigation.
- Estimation of the spectrum associated with $x(t)$.

Direct dynamic measurement has been treated regarding its probabilistic modeling [31], dynamic compensation [32] and uncertainty evaluation elsewhere. Spectrum estimation can instead be seen as a part of the restitution phase of the whole measurement according to the terminology proposed, e.g., in [5], and it will be discussed here.

4.1.1. Linear Spectra

With current instrumentation, direct measurement is made in digital form; thus, the discrete time series

$$\{x_i = x(i\Delta t), \quad i = 1, \dots, N, \quad T = N\Delta t\}, \quad (23)$$

is firstly obtained, where T generically denotes the observation time, and the sampling is information-preserving, provided that the Nyquist condition is satisfied. The Discrete Fourier Transform (DFT) constitutes an alternative representation of the observations:

$$X_k = X(k\Delta f) = \Delta t \sum_{i=1}^N x_i \exp\left(-jik\frac{2\pi}{N}\right), \quad (24)$$

where $\Delta f = 1/N\Delta t$ is the spectral resolution. The transformation is reversible, since the original time series can be recovered by the inverse transform (IDFT):

$$x_i = \Delta f \sum_{k=1}^N X_k \exp\left(+jik \frac{2\pi}{N}\right), \quad (25)$$

and thus, it is information-preserving and constitutes a sufficient statistic of the data.

Note that (24) is the discretized version of (11), the FT, but the discretized version of the FS, for $T \geq T_p$ and considering the whole frequency set, only differs from (24) in that division for $T = N\Delta t$ is required. Therefore, up to a scaling factor, both discrete transformations can be unified as follows:

$$X_k \propto \sum_{i=1}^N x_i \exp\left(-jik \frac{2\pi}{N}\right). \quad (26)$$

Note now that if the signal is null outside the observation window $[0, T]$, the integrated spectrum would be

$$\zeta_k = \sum_{l=1}^k X_l \Delta f. \quad (27)$$

And thus, simply, for $k > 1$:

$$X_k = \frac{\zeta_k - \zeta_{k-1}}{\Delta f}. \quad (28)$$

Therefore, (25) also constitutes the discretization of (14), and the same holds for (4) and for $d\zeta(f)$ appearing in (16). Thus, it constitutes a unified formula for linear spectral estimation, for periodic, transitory impulsive and stochastic stationary phenomena. Indeed, it also appears in (21), where it is multiplied by an amplitude modulated function, A . This could be taken as the general expression for linear spectral analysis, since (26) can be obtained by simply taking $A = 1$ everywhere.

Coming back to the issue of the limited observation time, if instead the signal is non null outside the observation interval, a systematic effect occurs, since the observed spectrum is the convolution of the original spectrum with the FT of the “rectangular window”, $W_R(k)$, that is

$$\hat{X}_k = \sum_l X_l W_R(k-l). \quad (29)$$

This is the so-called window effect, and it can be mitigated by pre-processing the data and by multiplying them for a proper data taper, w_i , i.e., by replacing (24) with

$$X_k \propto \sum_{i=1}^N w_i x_i \exp\left(-jik \frac{2\pi}{N}\right). \quad (30)$$

Criteria for the proper choice of data taper have been discussed elsewhere and will be assumed to be known in the following. A quick review of them can be found in [33].

4.1.2. Quadratic Spectra

A quadratic spectrum, from a deterministic standpoint, is simply the square modulus of the linear one, that is as follows:

$$Q_k = |X_k|^2. \quad (31)$$

Based on the discussion above, depending on how X_k has been scaled, it may represent either a power or an energy spectrum. However, the key point is that a quadratic spectrum can be considered as an average spectrum. This is why it definitely replaces the linear spectrum in the case of the stochastic process. In addition to providing the power distribution in the frequency domain, quadratic analysis—in the case of cross spectra—produces information on phase, coherence, and transfer functions.

To highlight, the averaging property, an expectation operator is included here, as in (18) and in (22).

We thus define, from now on, a quadratic spectrum as follows:

$$S_k = E(|X_k|^2) \quad (32)$$

where the corresponding symbol has been changed to S , due to the presence of the expectation operation, but it can still be interpreted as either a power or an energy spectrum, depending upon the way X_k has been calculated.

4.1.3. To Sum Up

The estimation of linear and quadratic spectra has been introduced by proposing unified formulae, the main phenomena under consideration here. They are, for linear spectra:

$$X_k \propto \sum_{i=1}^N A(k, i) w_i x_i \exp\left(-jik \frac{2\pi}{N}\right), \quad (33)$$

and, for quadratic spectra,

$$S_k = E(|X_k|^2). \quad (34)$$

Proper spectral estimation requires us to optimize the analysis parameter (e.g., the observation time, the data tapering function. . .) with criteria that depends upon the type of phenomenon under consideration. Such aspects have been addressed, to some extent, e.g., in [10] and will not be recalled here.

Instead, two main fundamental issues, namely the characteristic data window (to be defined soon) and some aspects of advanced data tapering, which up to now have not received, to the authors' knowledge, great attention in the scientific/technical literature, will be treated in the following.

4.2. The Characterizing Window

We begin by considering periodic phenomena. For mathematical simplicity, the model of such phenomena is defined across the entire time domain, i.e., from $-\infty$ to $+\infty$. Clearly, no real phenomenon can satisfy this infinite duration; rather, the model serves as an appropriate representation when observed in a suitable time window. For example, in the case of a rotating machine, such as a turbine in a thermal energy production plant, its vibration can be considered periodic (with a period associated with the rotation speed of the machine) in a time window included in the regime functioning of the machine/plant [34]. We propose to call that a characterizing window for the phenomenon under consideration, with duration T_c .

But similar ideas also apply to transitory impulsive phenomena: here, T_c would represent the overall duration of the phenomena. Indeed, this is an important feature for the design of the measuring system, which would aim to capture the phenomenon, typically including some kind of triggering and dual buffering.

In the case of stochastic stationary phenomena, consider for example the soil vibration transmitted to some accurate measurement apparatus. Since the main source of such noise is the effect of urban traffic, the spectrum would be stationary during the time in a day when the traffic is approximately constant, but there would be significant difference between the situation, say, near midday or near midnight. Thus, a proper characterizing window must also be identified.

Finally, consider again the case of a rotating machine, during start-up or shut-down phases. Since the rotation speed variation is controlled and known, the duration of the characterizing window is selected such that speed variation within the window is consider-

ably negligible, rendering the phenomenon approximately periodic. The identification of the characterizing window is clearly an important step when moving from a stationary to a non-stationary spectrum condition.

To further illustrate this important point, let us consider the following simple model:

$$x(t) = \cos(2\pi f_0 t) \cos(2\pi f_1 t) + 0.5 \cos(2\pi f_2 t), \quad (35)$$

with $f_0 = 0.01$ Hz, $f_1 = 0.1$ Hz, $f_2 = 0.2$ Hz, $\Delta t = 1$ s and a total recording time $T = 200$ s, shown in Figure 1. The aim is to consider a process where there coexists a component which is clearly periodic, the one at frequency f_2 , with another that, being amplitude-modulated, can be regarded as a special (simple) case of a non-stationary phenomenon. The question is: will the analysis be able to distinguish the stationary and the non-stationary component?

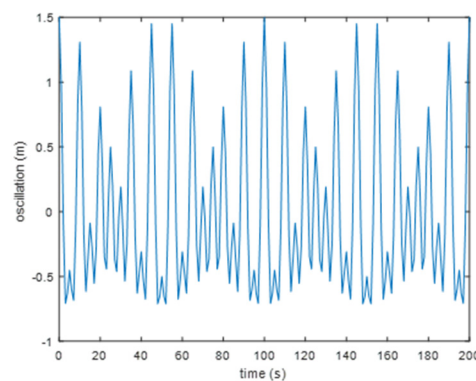


Figure 1. The test signal under consideration.

Due to the “deterministic” character of the phenomenon, a STFT analysis has been performed. Different values of the characterizing duration T_0 have been selected and the results are presented in Figure 2.

The results show what follows.

For $T_0 = 40$ s (possibly the optimal choice), the modulated (non-stationary) and the periodic (stationary) components of the phenomenon are clearly distinguishable and quantifiable.

On the other hand, for $T_0 = 160$ s, the phenomenon appears stationary, although with two close components, due to the modulation condition.

The other solutions are, for different reasons, somewhat critical.

In fact, for $T_0 = 20$ s, the two components interact, due to insufficient spectral resolution.

For $T_0 = 80$ s, the lower component presents a somewhat ambiguous (for the untrained eye) pattern, characterized by “holes” in the spectral profile.

Therefore, to sum up, a first-attempt, tentative conclusion could be:

- The distinction between stationary and non-stationary phenomena is perhaps more academical than operative.
- The characterizing window (duration) of the process is a key analysis feature.

As previously discussed, the identification of such a window can be based on previous knowledge about the phenomenon, such as stationarity tests, acceptable drift of spectral peaks, variance reduction, or time–frequency resolution requirements. Alternatively, when prior information is unavailable, a trial-and-error approach may be adopted.

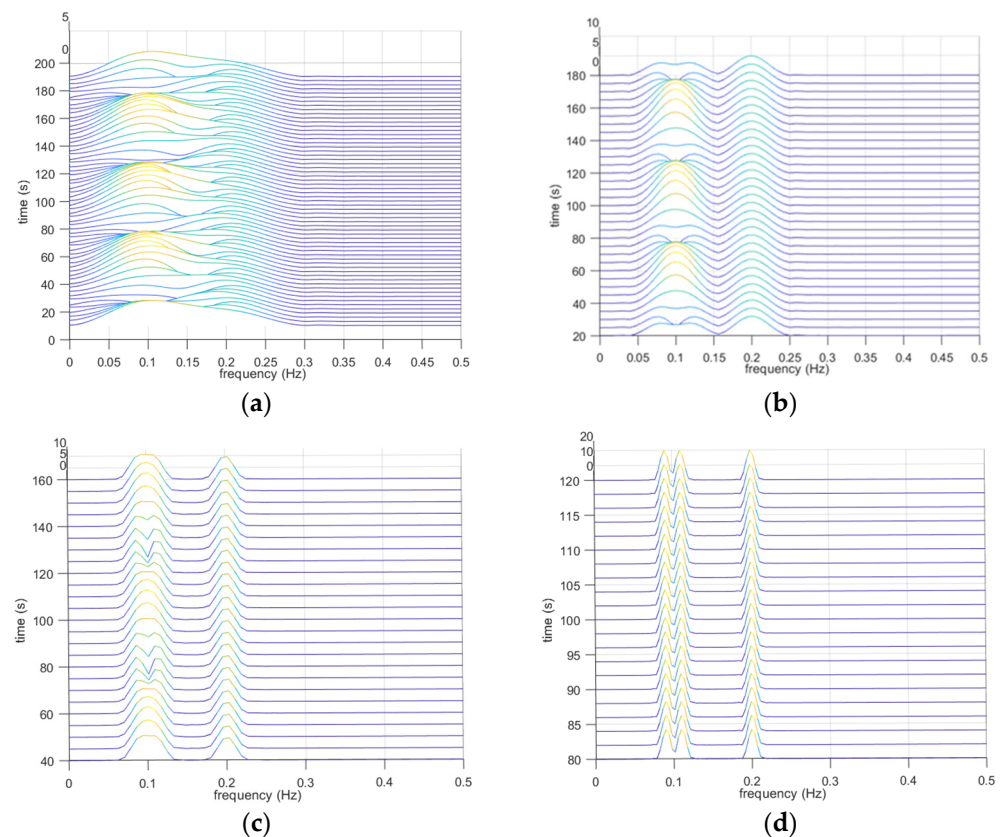


Figure 2. Results of the STFT of the test signal for various values of the duration T_0 of the observation window: (a) $T_0 = 20$ s; (b) $T_0 = 40$ s; (c) $T_0 = 80$ s; (d) $T_0 = 160$ s. Waterfall plot of the signal, where the spectral amplitude is represented by line height and color scale.

4.3. Variability of the Phenomenon and Advanced Data Tapering

4.3.1. Introductory Notes

The variability observed in measured dynamic phenomena includes two main sources: the inherent, intrinsic variability of the phenomenon itself and the measurement noise. The goal of the analysis is to extract a mean behavior and reduce the effect of noise. Thus, for processes characterized by stochastic realizations and/or a non-negligible amount of noise, quadratic spectra represent the preferred analysis, which includes an expectation operation, as indicated in (34). The question then becomes how to implement such an expectation operation.

4.3.2. Traditional Approach to Variability Reduction

In the case of stationary stochastic processes, the expectation operation in (34) can be implemented by parsing the whole record of duration, T , in n segments of duration $T_0 = N_0 \Delta t$, with $T_0 \ll T$, without or with partial overlap (typically 50%), as proposed by Welch [33]. This procedure, sometimes called Weighted Overlapped Segments Averaging (WOSA), can be formulated as follows [9]. Let $w_1, \dots, w_{N_0}$ be a data taper, applied to reduce spectral leakage, then the modified periodogram for the l -th segment is

$$\hat{S}_l(f) = \Delta t \left| \sum_{i=1}^{N_0} w_i x_{i+l-1} e^{-j2\pi f i \Delta t} \right|^2, \quad (36)$$

and the final estimator reads as follows:

$$\hat{S}(f) = \frac{1}{n} \sum_{k=0}^{n-1} \hat{S}_{k+1}(f), \quad (37)$$

where n is the number of segments and m is an integer-valued shift factor, satisfying $0 < m \leq N_0$ and $m(n-1) = N - N_0$. Therefore, for a fixed total observation time, T , with $T \leq T_c$, typically dictated by both the need of having a quasi-real-time monitoring and/or to the fact that the process can be considered only locally stationary, a trade-off is needed between the conflicting requirements of having a good spectral resolution, for which T_0 should be large, and thus n small, and having a low uncertainty due to the intrinsic variability of the phenomenon, which would require n large.

This procedure can also be applied to periodic phenomena if the phase information is not of interest.

In the case of slowly varying nonstationary processes, two levels of tapering are involved: at each time instant t , a PSD is to be calculated in the neighborhood of t , i.e., in the interval $\left[t - \frac{T_v}{2}, t + \frac{T_v}{2}\right]$, where T_v is the duration of a local, running, observation window, which in turn has to be parsed into segments of duration T_0 to allow averaging, as in the case of the PSD. Thus, a more complex trade-off is required, considering that:

- T_0 should be large enough to guarantee appropriate spectral resolution.
- $T_v > T_0$ should be large enough to reduce the intrinsic uncertainty of the phenomenon.
- T_v should be not too large, to ensure proper time resolution.

In this case, $T_v \leq T_c$ is required.

Let us now discuss the impact of advanced tapering strategies in the estimation process.

4.3.3. Advanced Data Tapering

Data tapering is a crucial aspect in spectral analysis and it is actually used, with the criteria mentioned above, in a wide range of processes. Therefore, it has been the subject of close investigation and of a generalization based on a seminal idea by Thomson [35]. To briefly outline such idea, let us consider a discrete-time version of (16) that is as follows:

$$x_i = \int_{-f_s/2}^{+f_s/2} \exp(+j2\pi f i \Delta t) d\zeta(f), \quad (38)$$

where $f_s = \frac{1}{\Delta t}$. To simplify notation and without loss of generality, let us now assume that we can fix the origin of time in the middle of the observation record, i.e.,

$$\{x_i = x(i\Delta t), \quad i = -M, \dots, 0, \dots, +M, \quad T = N\Delta t = (2M+1)\Delta t\}. \quad (39)$$

If Equation (29) is also rewritten in a continuous frequency domain, we obtain

$$X(f) = \sum_l X_l W_R(f-l), \quad (40)$$

and combining (38) and (40) and explicating $W_R(f)$ leads to

$$X(f) = \int_{-f_s/2}^{+f_s/2} \frac{\sin N\pi(f-v)}{\sin \pi(f-v)} d\zeta(v), \quad (41)$$

which can be recognized as a linear Fredholm integral equation of the first kind.

Then, it makes sense to look for an approximate solution of such an equation, by projecting the observations on a set of K orthonormal function, such as the discrete prolate spheroidal functions, that now acts as tapers, acting on the entire data record, before taking their (quadratic) FT. Then, the FT can be applied to the tapered record, thus obtaining an estimator of the spectrum:

$$\hat{S}_m(f) = \Delta t \left| \sum_{i=1}^N h_{i,m} x_{i+l-1} e^{-j2\pi f i \Delta t} \right|^2, \quad (42)$$

where $h_{i,m}$, with $m = 1, \dots, K$, denotes the m -th taper, and the result can be obtained by the sum of such estimators:

$$\hat{S}(f) = \frac{1}{K} \sum_{m=0}^{K-1} \hat{S}_m(f). \quad (43)$$

Again, this is a trade-off between spectral resolution and variance of the estimate and spectral resolution, based on the value of K .

In the authors' view, this approach provides a more sound foundation than the previous one, since tapering is no longer "heuristic" but rather based on the approximate solution of (41). This method is known as the multi-taper approach and a thorough comparison of it with the WOSA method has been recently made by the authors, in the case of both unimodal and bimodal spectral analysis of sea waves [36].

Concerning the metrological performance of the WOSA algorithm, when the original Welch framework is adopted (cosine window with 50% overlapping between segments), formulas are available both for uncertainty and spectral resolution: the effective bandwidth is related both to the taper (w , as "window") and the duration of the individual segments; $T_0 \Delta f_e = \alpha_w T_0^{-1} \alpha_w$ is a factor that depends on the kind of the selected taper and on how bandwidth is defined. In the case of the Hanning window and considering a half-power bandwidth, $\alpha_w = 1.44$. Concerning the variance of the estimator with a 50% overlap, a relative standard deviation $u_S/S = \sqrt{(11/18)N_0 N^{-1}}$ can be assumed, where S denotes the spectrum and u_S denotes its absolute standard deviation. For additional information on the performance of different algorithms, see [36].

5. Conclusions

Insights into spectrum measurement were provided, with special regard for the following three key points.

- The possibility of achieving a unified definition of the measurand for important classes of phenomena, including periodic, transitory impulsive, stochastic stationary and stochastic nonstationary ones, through the Fourier-Stieltjes integral.
- The importance of the characterizing observation window as a key feature for the measurement method.
- The issue of reducing estimator variability and the key role played by the multi-taper method.

It is hoped that these results will prove useful and inspiring for the operators in charge of spectrum measurement, both in scientific and technological areas.

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Note

- ¹ The case of a non-null mean value can be treated easily, but it is often less significant than the case where it is null.

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