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Global Dynamics and Stability of Automatic Ball Balancers Under Anisotropy and Non-Ideal Excitation

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Abstract

This study presents the analysis of global dynamics and stability (e.g., coexisting attractors, Hopf bifurcation boundary) for a nonlinear rotor system with an automatic ball balancer (ABB). The presence of nonlinearity, anisotropy and non-ideal dynamics makes this system not fully understood. The Lagrangian is written explicitly in terms of the displacement of the rotor centre and the angular positions of the balls (x, y, ψ, φ_i) . The kinetic energy separates into structural, unbalance coupling, and ball coupling blocks, and the Rayleigh dissipation function covers both support damping and race drag. The three families of equations of motion (translational, spin, ball) are compacted into the matrix form and solved numerically. Non-dimensionalisation introduces the seven groups $(\Omega, \mu_{un}, \mu_b, \varepsilon, \hat{\beta}, \hat{D}, \Delta)$ with Δ being the anisotropy parameter. The results document bistability between the clustered and balanced ball configurations depending solely on ball initial conditions rather than rotor displacement, together with a basin of attraction analysis in which the balanced basin occupies only approximately 20% of ball initial-condition space. A three-dimensional stability map reveals a previously unreported phenomenon: narrow islands of stability at very low race damping, suggesting that effective balancing may not always require dissipation, alongside a two-lobe Hopf bifurcation boundary with a disconnected instability pocket. Anisotropy study uncovers that the rotor’s response is dominated by quasi-periodic torus attractor across almost the entire (93.5%) parameter space rather than the simple periodic balancing usually assumed, with a clean analytical rule identifying exactly when support asymmetry will resonantly amplify vibration. Together these findings point to design principles on ball seeding, damping selection, and permissible anisotropy.

Keywords: automatic ball balancers; nonlinear dynamics; stability; global dynamics; coexisting attractors; Hopf boundary



Academic Editors: Liming Dai and Jialin Tian

Received: 28 May 2026

Revised: 27 June 2026

Accepted: 2 July 2026

Published: 4 July 2026

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1. Introduction

The mitigation of synchronous vibrations in rotating machinery remains one of the most enduring challenges in mechanical engineering, particularly as demands for higher operational speeds and greater precision continue to grow [1–3]. Synchronous vibrations, typically caused by a mismatch between the geometric centre of rotation and the principal axis of inertia, lead to accelerated bearing wear, structural fatigue, and operational noise in systems such as high-speed centrifuges, industrial grinders, and domestic appliances like washing machines [2,4]. While active balancing systems—utilizing sensors, controllers,

and actuators—offer high precision, they are often cost-prohibitive or physically impractical for mass-market or high-speed applications where the unbalance changes dynamically or unpredictably [5]. This gap in technology has historically necessitated the development of passive autobalancing devices, among which the automatic ball balancer (ABB) is the most prominent [1,6].

The foundational rigorous analysis of the ABB is due to Thearle [7], who demonstrated that, above the first critical speed (super-critical regime, i.e., speeds exceeding the system's first natural frequency), free-moving balancing masses in a circular race naturally migrate to positions that counteract the rotor eccentricity, achieving a stable balanced state. Hedaya and Sharp [8] extended the autobalancing concept by proposing two-plane configurations capable of compensating for both static unbalance (radial force) and couple unbalance (misalignment of the principal axis). Subsequent key contributions include: Chung and Ro [4] on identification of non-balancing states at super-critical speeds; Sperling et al. [1] using averaging methods for stability analysis of a 4DOF rotor-ABB system; Ryzhik et al. [9] investigating the effects of support anisotropy on ABB performance; Green et al. [6] performing the first comprehensive nonlinear bifurcation analysis of multi-ball ABBs; Rodrigues and co-workers on experimental validation [5], symmetry-breaking in two-plane systems [10,11], and support asymmetry effects [12]; Drozdetskaya and Fidlin [13] on passage through resonance and the Sommerfeld effect in non-ideal systems with self-balancing devices; Sung et al. on the influence of external excitations on ball positioning in an ABB [14]; and Wang and Huang on spatial geometry characteristics and 3D dynamics [15].

Recent advances in the modelling of ABBs have emphasised the extreme sensitivity of the mechanism to rotation speed and initial conditions, characteristics that align the subject with the study of chaotic systems and non-smooth dynamics [6,10,16].

The automatic ball balancer operates on the principle of self-synchronisation, a phenomenon where separate oscillators or masses in a rotating system adjust their relative phases to reach a state of mutual equilibrium. In the context of an ABB, the balancing balls interact with the vibrating rotor through centrifugal and tangential forces mediated by the race geometry and a viscous fluid [6]. The mechanism is inherently nonlinear, and while it can eliminate unbalance for certain speeds, it may exacerbate vibrations during the transition through resonance or “run-up”. The transition through resonance, support anisotropy, non-ideal excitation, and gravity effects (in horizontal rotors) can induce complex behaviours including Sommerfeld capture, desynchronised whirl, and coexisting stable periodic orbits [11,13,17]. These phenomena necessitate a shift from local stability analysis to a global dynamical perspective focused on basins of attraction, bifurcation structures, and robustness [16,18,19]. Modern research seeks to delineate the regions of stability and the global basins of attraction to ensure the device remains a reliable tool for vibration control across all operational phases.

There are several different approaches to balancing automatic ball balancers (ABBs): (i) Passive standard designs—Balls move freely in a circular race/track to self-balance at super-critical speeds via centrifugal forces (classic single-layer or multi-ball setups) [20]; (ii) Deep/multi-layer chamber designs—Allow balls greater radial/3D freedom (e.g., deep chamber ABB-DC) for improved distribution and efficiency over single-layer drums [21]; (iii) Modified geometries and partitions: Partitioned tracks/races with centrifugal clamps or specific race shapes to enhance sub-critical/transient performance and reduce friction/rolling resistance effects. (iv) Spring/magnet-assisted or hybrid passive: Ball springs for even low-speed distribution; permanent magnets to control/fix ball positions during transients (releasing at steady state) [22]; (v) Active/controlled variants using fuzzy regulators, sliding-mode observers, or electromagnetic actuation for precise ball repositioning; temporary speed reduction strategies [23]; (vi) Multi-plane or combined systems, including

two-plane balancing, nonlinear suspensions, or integration with other suppressors (e.g., discontinuous springs) for flexible rotors and vibration avoidance [11].

Despite nearly a century of research, the widespread adoption of automatic ball balancers in industrial rotating machinery remains limited by their lack of global robustness. Rodrigues et al. [5] state that inherent nonlinearity of the ABB mechanism and its extreme sensitive dependence on frequency and initial conditions have largely precluded its widespread use in industry. While a perfectly tuned ABB can theoretically eliminate vibration, the reality of asymmetric support structures (anisotropy) and energy-limited power sources (non-ideal excitation) often creates “dynamic traps” like Sommerfeld capture [24] and desynchronised whirl [10]. The interplay between these two complexities is not well-understood. The present investigation aims to fill this critical gap by performing an exhaustive study of the global dynamics and stability of ABBs under the simultaneous influence of support anisotropy and non-ideal excitation [10]. This research is motivated by several practical requirements: (i) robustness to asymmetry; (ii) predictability of final states; and (iii) mitigation of non-smooth behaviours.

By integrating bifurcation theory, attractor coexistence [25], and global basin analysis [19], this investigation aims to advance both the theoretical understanding and practical reliability of automatic ball balancing in complex mechanical systems.

The subsequent sections present the mathematical model (Section 2), numerical method (Section 3), and global stability and basin results (Section 4), followed by the discussion (Section 5) and conclusions (Section 6).

2. Mathematical Model

2.1. Physical Configuration

The system consists of a rigid rotor-housing assembly of total mass M suspended on orthotropic viscoelastic supports characterised by stiffnesses (k_x, k_y) and viscous damping coefficients (b_x, b_y) in the two in-plane directions; see Figure 1. An imbalance mass m_{un} is rigidly attached at eccentricity e from the geometric centre. The automatic ball balancer (ABB) comprises N identical point masses m_b constrained to a circular race of radius r , concentric with the rotor spin axis. Each ball rolls without slipping on the race and experiences a viscous drag torque $-D\dot{\phi}_j$ resisting relative motion with respect to the rotor body, where D [N m s/rad] is the race damping coefficient. The model is planar, assuming gravity-free idealisation.

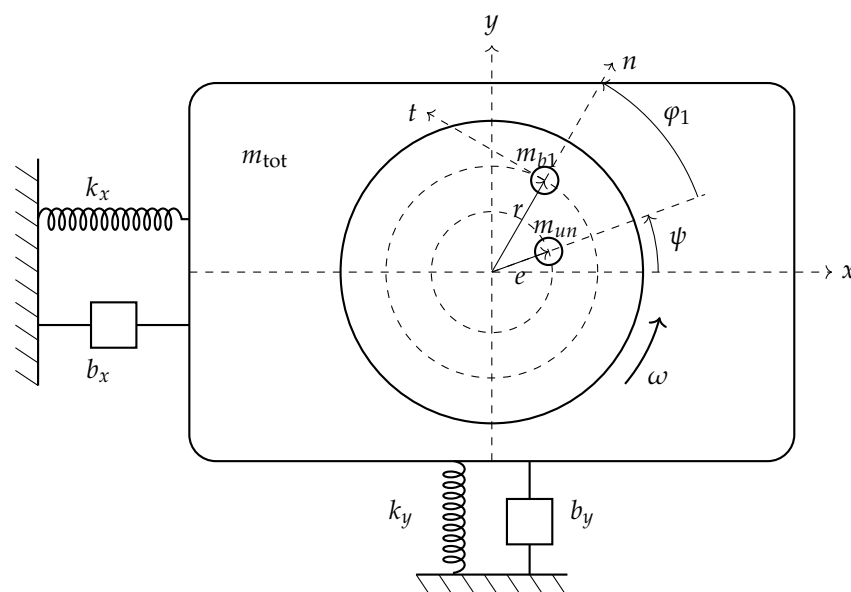


Figure 1. Schematic of the rotor unbalance system (after [2]).

2.2. Generalised Coordinates and Lagrangian

We adopt the following set of generalised coordinates:

$$\mathbf{q} = (x, y, \psi, \varphi_1, \dots, \varphi_N)^T, \quad (1)$$

where (x, y) are the translational displacements of the rotor centre in the inertial frame, $\psi(t)$ is the absolute rotor spin angle, and $\varphi_j(t)$ is the angular position of ball j measured relative to the rotor body. The absolute angular position of ball j is therefore $\alpha_j = \psi + \varphi_j$.

The total kinetic energy of the system is

$$\begin{aligned} T = & \frac{1}{2}(M + m_{\text{un}} + Nm_b)(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_{\text{rot}}\dot{\psi}^2 \\ & + m_{\text{un}}e\dot{\psi}(-\dot{x}\sin\psi + \dot{y}\cos\psi) \\ & + m_br\sum_{j=1}^N[(\dot{\psi} + \dot{\varphi}_j)(-\dot{x}\sin\alpha_j + \dot{y}\cos\alpha_j) + \frac{1}{2}r(\dot{\psi} + \dot{\varphi}_j)^2], \end{aligned} \quad (2)$$

where I_{rot} is the moment of inertia of the rotor disc about the spin axis. The potential energy is

$$V = \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2, \quad (3)$$

and the Rayleigh dissipation function accounts for support damping and race drag:

$$\mathcal{F} = \frac{1}{2}b_x \dot{x}^2 + \frac{1}{2}b_y \dot{y}^2 + \frac{D}{2} \sum_{j=1}^N \dot{\varphi}_j^2. \quad (4)$$

2.3. Equations of Motion

Application of the Euler–Lagrange operator $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k^{\text{nc}}$ with $\mathcal{L} = T - V$ yields the following coupled system.

2.3.1. Translational Equations

$$\begin{aligned} \tilde{M}\ddot{x} + b_x\dot{x} + k_x x = & m_{\text{un}}e(\ddot{\psi}^2 \cos\psi - \ddot{\psi} \sin\psi) \\ & + m_br\sum_{j=1}^N[(\ddot{\psi} + \ddot{\varphi}_j)^2 \cos\alpha_j - (\ddot{\psi} + \ddot{\varphi}_j) \sin\alpha_j], \end{aligned} \quad (5)$$

$$\begin{aligned} \tilde{M}\ddot{y} + b_y\dot{y} + k_y y = & m_{\text{un}}e(\ddot{\psi}^2 \sin\psi + \ddot{\psi} \cos\psi) \\ & + m_br\sum_{j=1}^N[(\ddot{\psi} + \ddot{\varphi}_j)^2 \sin\alpha_j + (\ddot{\psi} + \ddot{\varphi}_j) \cos\alpha_j], \end{aligned} \quad (6)$$

where $\tilde{M} = M + m_{\text{un}} + Nm_b$ is the total mass.

2.3.2. Rotor Spin Equation

$$I_{\text{eff}}\ddot{\psi} + m_br^2\sum_{j=1}^N\ddot{\varphi}_j + m_br\sum_{j=1}^N(\ddot{x}\sin\alpha_j - \ddot{y}\cos\alpha_j) = T_{\text{motor}}(\dot{\psi}), \quad (7)$$

with effective polar inertia $I_{\text{eff}} = I_{\text{rot}} + m_{\text{un}}e^2 + Nm_br^2$. The motor torque follows a linear characteristic

$$T_{\text{motor}}(\omega) = a - b\omega, \quad (8)$$

whose free-run speed is $\omega^* = a/b$. For constant-speed simulations the kinematic constraint $\ddot{\psi} = 0$, $\dot{\psi} = \omega$ replaces Equation (7).

2.3.3. Ball Equations

For each ball $j = 1, \dots, N$:

$$m_b r^2 (\ddot{\psi} + \ddot{\varphi}_j) + m_b r (\ddot{x} \sin \alpha_j - \ddot{y} \cos \alpha_j) = -D \dot{\varphi}_j. \quad (9)$$

Note that Equations (7) and (9) are coupled through the acceleration cross-terms, which is the source of the Sommerfeld capture phenomenon ([24]).

2.3.4. Matrix Form

Equations (5)–(9) are algebraically nonlinear but, at any instant, linear in the vector of accelerations $\ddot{\mathbf{q}}$. They can, therefore, be written as

$$\mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{0}, \quad (10)$$

where $\mathbf{M} \in \mathbb{R}^{(2+1+N) \times (2+1+N)}$ is a configuration-dependent mass matrix and $\mathbf{f}$ collects all dissipative, stiffness, and centripetal/Coriolis terms. In the numerical implementation $\ddot{\mathbf{q}}$ is recovered at each time-step by solving the linear system Equation (10) via LU decomposition (see Section 3).

2.4. Non-Dimensionalisation

Let the natural frequency of the isotropic support be

$$\omega_n = \sqrt{\frac{k_x + k_y}{2M}}, \quad (11)$$

and introduce the dimensionless time $\tau = \omega_n t$. Let us define the following set of dimensionless groups:

$$\begin{aligned} \Omega &= \frac{\omega}{\omega_n}, & \hat{x} &= \frac{x}{r}, & \hat{y} &= \frac{y}{r}, \\ \mu_{\text{un}} &= \frac{m_{\text{un}}}{M}, & \mu_b &= \frac{Nm_b}{M}, \\ \varepsilon &= \frac{e}{r}, & \hat{\beta} &= \frac{b_x}{M\omega_n} = \frac{b_y}{M\omega_n}, \\ \hat{D} &= \frac{D}{M\omega_n r^2}, & \Delta &= \frac{k_x - k_y}{k_x + k_y}. \end{aligned} \quad (12)$$

Here $\Delta \in [0, 1)$ is the *stiffness anisotropy parameter*, with $\Delta = 0$ corresponding to the isotropic (Jeffcott-like) baseline case. In terms of these groups the translational equations in the $\hat{x}$ -direction become, dropping the hat for notational convenience,

$$\begin{aligned} (1 + \mu_{\text{un}} + \mu_b) x'' + \hat{\beta} x' + (1 + \Delta) x &= \mu_{\text{un}} \varepsilon (\Omega^2 \cos \psi - \psi'' \sin \psi) \\ &+ \frac{\mu_b}{N} \sum_{j=1}^N [(\psi' + \varphi'_j)^2 \cos \alpha_j - (\psi'' + \varphi''_j) \sin \alpha_j], \end{aligned} \quad (13)$$

where primes denote $d/d\tau$, and the y -equation (omitted here) is obtained by the substitutions $\cos \rightarrow \sin$ (and vice versa), and $\Delta \rightarrow -\Delta$. The ball equation in dimensionless form reads

$$(\psi'' + \varphi''_j) + \frac{1}{r} (x'' \sin \alpha_j - y'' \cos \alpha_j) = -\hat{D} \varphi'_j. \quad (14)$$

2.5. Physical Parameter Values

The baseline simulation employs the parameters listed in Table 1. At these values the isotropic natural frequency is $\omega_n \approx 3.162$ rad/s ($f_n \approx 0.503$ Hz), the structural damping ratio is $\zeta = \hat{\beta}/2 \approx 0.024$, and the total ball-to-rotor mass ratio is $\mu_b = 0.006$.

Table 1. Physical parameter values.

Symbol	Description	Value	Unit
M	Total rotor-housing mass	10	kg
$k_x = k_y$	Support stiffness (isotropic)	100	N/m
$b_x = b_y$	Support damping	1.5	N s/m
m_{un}	Unbalance mass	0.05	kg
e	Eccentricity	0.02	m
N	Number of balls	2	-
m_b	Ball mass	0.03	kg
r	Race radius	0.05	m
D	Race damping (baseline)	0.006	N m s/rad
a	Motor torque constant	8.0	N m
b	Motor speed coefficient	0.5	N m s
ω_n	Natural frequency	3.162	rad/s
ζ	Structural damping ratio	0.0237	-
μ_b	Ball mass ratio Nm_b/M	6.0×10^{-3}	-
μ_{un}	Unbalance mass ratio	5.0×10^{-3}	-
ε	Eccentricity ratio e/r	0.40	-
$\hat{D}$	Dimensionless race damping	7.6×10^{-2}	-

2.6. Scale Invariance

Equations (13) and (14) depend on the physical parameters $M, k_x, k_y, b_x, b_y, m_{\text{un}}, e, m_b, r, D$ and ω only through the seven dimensionless groups $\Omega, \Delta, \hat{D}, \hat{\beta}, \mu_{\text{un}}, \mu_b$, and ε defined in Equation (12). Consequently, any pair of dimensional parameter sets that produce the same values of these seven groups generate *identical* solutions $x(\tau)/r, y(\tau)/r, \psi(\tau)$, and $\varphi_j(\tau)$ as functions of dimensionless time $\tau = \omega_n t$. In particular, the natural frequency ω_n and the stiffness k can be rescaled by an arbitrary factor $\kappa > 0$ (e.g., $k \rightarrow \kappa k$, with M, b, D, m_{un} rescaled correspondingly to hold $\zeta, \mu_{\text{un}}, \mu_b, \hat{D}$ fixed) without altering the normalised speed Ω , the damping ratio $\zeta = \hat{\beta}/2$, or the critical damping ratio $D^*/(M\omega_n r^2)$ at which the Hopf boundary occurs. All quantitative results reported in Section 4 in terms of Ω, ζ, Δ , and the dimensionless critical damping $D^*/(M\omega_n r^2)$ are therefore scale-invariant: they apply equally to the baseline parameter set in Table 1 and to any rescaled set with the same dimensionless groups, including parameter choices corresponding to higher support stiffness and higher rotational speed than the illustrative baseline adopted here for numerical convenience.

3. Numerical Method

3.1. First-Order Reduction and State Vector

The governing Equation (10) constitute a system of $(2 + 1 + N)$ coupled second-order ODEs in the generalised coordinates $\mathbf{q}$. They are reduced to a first-order system by introducing the state vector

$$\mathbf{z} = (x, \dot{x}, y, \dot{y}, \psi, \dot{\psi}, \varphi_1, \dot{\varphi}_1, \dots, \varphi_N, \dot{\varphi}_N)^T \in \mathbb{R}^{6+2N}, \quad (15)$$

so that the system reads $\dot{\mathbf{z}} = \mathbf{F}(t, \mathbf{z})$.

The governing Equation (10) admits two distinct first-order formulations depending on whether the rotor spin is prescribed (kinematic case) or driven by the motor law (8) (dynamic case).

In the kinematic case, $\dot{\psi} = \omega = \text{const.}$ is prescribed and $\psi(t) = \psi_0 + \omega t$ is known in closed form; ψ and $\dot{\psi}$ are therefore excluded from the state vector, which reduces to $\mathbf{z}_{\text{kin}} = (x, \dot{x}, y, \dot{y}, \varphi_1, \dot{\varphi}_1, \dots, \varphi_N, \dot{\varphi}_N) \in \mathbb{R}^{4+2N}$, with $\dot{\mathbf{z}}_{\text{kin}} = \mathbf{F}_{\text{kin}}(t, \mathbf{z}_{\text{kin}})$ a genuine, unconstrained first-order ODE system. For $N = 2$ this state space is eight-dimensional.

In the dynamic case, $\dot{\psi}$ is governed by the motor torque law and ψ and $\dot{\psi}$ are retained as genuine dynamical states, giving the full state vector $\mathbf{z} = (x, \dot{x}, y, \dot{y}, \psi, \dot{\psi}, \varphi_1, \dot{\varphi}_1, \dots, \varphi_N, \dot{\varphi}_N) \in \mathbb{R}^{6+2N}$ as originally defined in Equation (15), with $\dot{\mathbf{z}} = \mathbf{F}(t, \mathbf{z})$ holding without any algebraic override. For $N = 2$ the state space is ten-dimensional.

3.2. Right-Hand Side Evaluation

At each function evaluation the following sequence is performed:

1. Extract $(x, \dot{x}, y, \dot{y}, \psi, \omega, \varphi_j, \dot{\varphi}_j)$ from $\mathbf{z}$;
2. Compute absolute ball angles $\alpha_j = \psi + \varphi_j$ and their trigonometric functions;
3. Assemble the $(2 + 1 + N) \times (2 + 1 + N)$ instantaneous mass matrix $\mathbf{M}(\mathbf{z})$ from Equation (10);
4. Assemble the right-hand side force vector $\mathbf{f}$ (centripetal, support, drag, motor terms);
5. Solve the linear system $\mathbf{M} \ddot{\mathbf{q}} = -\mathbf{f}$ for the accelerations via dense LU factorisation (LAPACK `dgesv`);
6. Return $\dot{\mathbf{z}}$ by interleaving $(\dot{q}_k, \ddot{q}_k)$.

In the kinematic case, ψ is not a state variable but the prescribed function $\psi(t) = \psi_0 + \omega t$; it is substituted directly into $\alpha_j(t) = \psi(t) + \varphi_j$ wherever the ball or unbalance angle appears, and the corresponding row and column of $\mathbf{M}$ are omitted, yielding the reduced $(2 + N) \times (2 + N)$ system solved for $(\ddot{x}, \ddot{y}, \ddot{\varphi}_j)$ only.

3.3. Time Integration

3.3.1. Stiffness and Solver Selection

The system (10) exhibits *mixed stiffness*: the structural modes are characterised by $\omega_n \approx 3.16$ rad/s, while the drive frequency ω can reach $2.2 \omega_n \approx 7.0$ rad/s and the ball-cage coupling introduces fast oscillatory components at $\omega + \dot{\varphi}_j$. For high race damping D , the ball dynamics acquire a large negative real eigenvalue of order $D/(m_b r^2) \approx 24 \omega_n$, giving a stiffness ratio of order 10. Although this is modest (comparing to, e.g., chemical-kinetics standards), the configuration-dependent character of $\mathbf{M}$ causes the stiffness ratio to vary strongly during a spin-up transient.

The default solver is the *explicit Runge–Kutta RK45* method [26] implemented in `scipy.integrate.solve_ivp` with absolute and relative tolerances $\varepsilon_a = 10^{-7}$ and $\varepsilon_r = 10^{-5}$. For the simulation cases with higher demonstrated stiffness, the implicit *Radau IIA* solver of order 5 [27] is available as an alternative (tolerances tightened to $\varepsilon_a = 10^{-9}$, $\varepsilon_r = 10^{-7}$). In practice, RK45 was found sufficient for all constant-speed experiments at the tolerances stated above, while Radau reduced the number of rejected steps by a factor of approximately three near resonance in the Sommerfeld scenario.

3.3.2. Step-Size Control

The embedded pair produces a local error estimate $\hat{\mathbf{e}} = \mathbf{z}^{(5)} - \mathbf{z}^{(4)}$ at each step; the step h is accepted if

$$\left(\frac{1}{n} \sum_{k=1}^n \left(\frac{\hat{e}_k}{\varepsilon_a + \varepsilon_r |\mathbf{z}_k|} \right)^2 \right)^{1/2} \leq 1, \quad (16)$$

where $n = 6 + 2N$ is the state dimension. No maximum step size is imposed; typical accepted step sizes range from $h \approx 0.01$ s in high-amplitude transients to $h \approx 0.1$ s in quasi-steady low-amplitude regimes.

3.3.3. Transient Duration

For amplitude-sweep experiments, each integration covers a total time of $t_{\text{end}} = 80$ s ($\approx 40 T_n$). Steady-state statistics are computed over the final 30% of the trajectory ($\tau \in [56, 80]$ s), corresponding to $\approx 12 T_n$, by which time the ball transient has decayed to below 2% of the initial perturbation amplitude for all damping values considered.

3.4. Lyapunov Exponent Computation

The largest Lyapunov exponent (LLE)

$$\lambda_{\max} = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\|\delta \mathbf{z}(t)\|}{\|\delta \mathbf{z}(0)\|} \quad (17)$$

is estimated using the *renormalised trajectory-separation* method [28–30]. The algorithm proceeds as follows.

1. **Transient removal.** Integrate from $t = 0$ to $t_{\text{tr}} = 20$ s to place the trajectory on the attractor. Denote the endpoint $\mathbf{z}_{\text{on}}$. The initial ball angles are set to $\varphi_1^{(0)} = 0.1$ rad, $\varphi_2^{(0)} = 3.0$ rad to seed the balanced attractor; the clustered attractor is reachable from $\varphi_1^{(0)} \approx \varphi_2^{(0)} \approx 0$ but yields a higher LLE at sub-critical speeds and is not the primary object of the stability map.
2. **Initialisation.** Set reference $\mathbf{z}_{\text{ref}} = \mathbf{z}_{\text{on}}$ and perturbed $\mathbf{z}_{\text{per}} = \mathbf{z}_{\text{on}} + \varepsilon \hat{\mathbf{e}}_1$, where $\varepsilon = 10^{-6}$ and $\hat{\mathbf{e}}_1$ is the unit vector in the x -direction.
3. **Segment integration.** Divide the Lyapunov integration interval $[0, t_{\text{lyap}}]$ with $t_{\text{lyap}} = 25$ s into $n_r = 5$ segments of length $\delta t = t_{\text{lyap}}/n_r = 5$ s. For each segment k :
 - (a) Integrate both trajectories over $[0, \delta t]$.
 - (b) Compute the separation $\delta \mathbf{z}^{(k)} = \mathbf{z}_{\text{per}}^{(k)} - \mathbf{z}_{\text{ref}}^{(k)}$ and its norm $s_k = \|\delta \mathbf{z}^{(k)}\|$.
 - (c) Accumulate $\ln(s_k/\varepsilon)$.
 - (d) Renormalise $\mathbf{z}_{\text{per}} \leftarrow \mathbf{z}_{\text{ref}}^{(k)} + \varepsilon \delta \mathbf{z}^{(k)}/s_k$.
4. **LLE estimation.**

$$\lambda_{\max} = \frac{1}{t_{\text{lyap}}} \sum_{k=1}^{n_r} \ln \frac{s_k}{\varepsilon}. \quad (18)$$

Renormalisation at each segment prevents floating-point overflow (which would occur for $\lambda_{\max} > 0$ at long times) and ensures that the estimated direction of maximum growth aligns with the leading Lyapunov direction [29,30]. The choice $\delta t = 5$ s ($\approx 2.5 T_n$) is a compromise between sufficient growth per segment and accurate local linearisation.

Stability Classification

Based on the numerical value of $\lambda_{\max}$, each parameter-space point is classified as follows:

$\lambda_{\max} < -0.01 \text{ s}^{-1}$	stable (fixed point/decaying orbit),
$ \lambda_{\max} < 0.02 \text{ s}^{-1}$	near-Hopf (limit-cycle onset region),
$0.02 < \lambda_{\max} < 0.08 \text{ s}^{-1}$	limit cycle (sustained oscillation),
$\lambda_{\max} \geq 0.08 \text{ s}^{-1}$	strongly unstable/chaotic.

A negative LLE indicates exponential contraction of nearby trajectories; $\lambda_{\max} \approx 0$ (within numerical precision) is the signature of a periodic orbit (limit cycle), consistent with the theoretical result that the Lyapunov exponent vanishes on a stable limit cycle.

3.5. Validation

The validation tests of the proposed ABB model implementation included the forced harmonic response of the bare Jeffcott rotor, and the integrator convergence and accuracy test in the frozen-ball analytical limit.

3.5.1. Jeffcott Rotor

This test simulates the forced harmonic response of the bare Jeffcott rotor. Setting $N = 0$, $m_b = 0$ (removing all balls) and $D = 0$, the system reduces to

$$M_{eff} x'' + b x' + k x = m_{un} e \omega^2 \cos(\omega t), \quad (19)$$

$$M_{eff} y'' + b y' + k y = m_{un} e \omega^2 \sin(\omega t), \quad (20)$$

where the steady-state amplitude is a standard textbook result [31].

Figure 2 shows the Jeffcott rotor linear resonance test result. The curves match extremely well everywhere except at exactly $\Omega = 1.0$ where the error bar reaches $\approx 4.3\%$. This is a well-understood finite-integration-time effect. At resonance the amplitude grows as $t^{1/2}$ before damping acts on it, so a finite simulation underestimates the true steady-state peak.

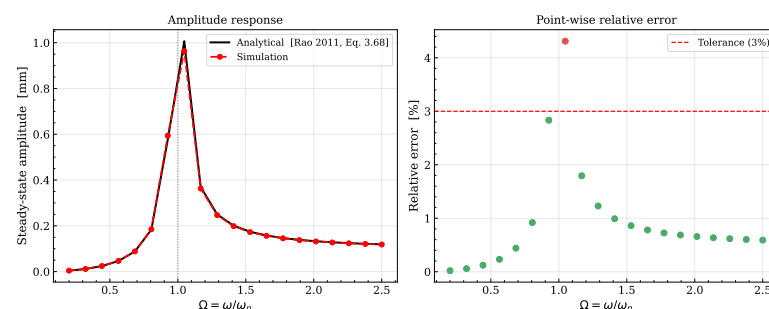


Figure 2. Jeffcott rotor linear resonance. Note: Peak error = 0.043 at $\Omega = 1.05$; $\zeta = 0.0237$; Ref: [31].

3.5.2. Integrator Accuracy—Frozen-Ball Analytical Limit

By setting $N = 1$, and with very large race damping ($D \rightarrow \infty$), so that $\dot{\phi}_j \rightarrow 0$, the ball is effectively frozen at $\phi = 0$. The system then reduces to a damped harmonic oscillator in x with known analytical solution [31].

This tests (a) whether the translational mass matrix is assembled correctly with the ball contribution to M_{eff} , (b) the integrator accuracy at default tolerances, and (c) whether the ball coupling does not introduce spurious forces when $\dot{\phi}_j \approx 0$ —frozen-ball limit.

Maximum absolute error in displacement versus analytical solution over 60 s at RK45 tolerances $(\epsilon_a, \epsilon_r) = (10^{-10}, 10^{-8})$, Figure 3, was found to be below the defined level, confirming adequate conservation at the chosen accuracy.

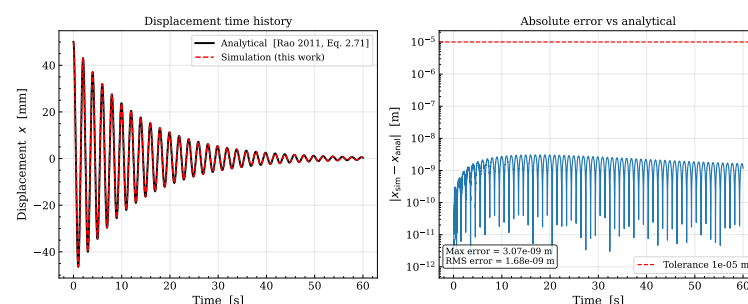


Figure 3. Integrator accuracy—Frozen-ball analytical limit test results. $MaxErr = \max \|x_{sim} - x_{analytical}\| = 3.066 \cdot 10^{-9} \text{ m}$, ($tol = 10^{-5} \text{ m}$). Note: $RMSErr = 1.68 \cdot 10^{-9} \text{ m}$; $M_{eff} = 10.03 \text{ kg}$; 60 s, RK45 $rtol = 10^{-8}$; Ref: [31].

The parameters for this test are: $M_{eff} = 10.030$ kg, $\omega_{eff} = 3.15754$ rad/s, $\zeta = 0.02368$, and $\omega_d = 3.15666$ rad/s.

4. Results

The results presented here consist of studies of the bifurcation structure (coexisting attractors near resonance and basin of attraction geometry), studies of the effect of anisotropy (e.g., frequency splitting and resonance crossing, amplitude response, Poincaré classification and design recommendations), and stability maps and Hopf bifurcation boundary resulting from three-dimensional parameter sweep.

4.1. Bifurcation Structure

4.1.1. Coexisting Attractors near Resonance

Figure 4 demonstrates the existence of at least two coexisting attractors at $\Omega = 0.95$, just below the primary resonance. The two coexisting attractors correspond to qualitatively distinct ball configurations rather than to different translational initial conditions. Sweeping the initial angle $\varphi_1^{(0)} \in [0, 2\pi]$ while seeding the second ball either in the *clustered* configuration ($\varphi_2^{(0)} = 0$, both balls near the unbalance) or in the *balanced* configuration ($\varphi_2^{(0)} = \pi$, anti-phase seed) reveals that the final steady-state amplitude differs by up to $\Delta A \approx 2.7$ mm at $\varphi_1^{(0)} = 0$ (see Figure 4, left panel). The clustered branch sustains amplitudes in the range $A \in [0.9, 3.7]$ mm depending on $\varphi_1^{(0)}$, while the balanced branch occupies $A \in [0.9, 1.9]$ mm; the two branches overlap near $\varphi_1^{(0)} \approx 120^\circ$ and 240° , where both seeds converge to the same attractor. The right panel of Figure 4 confirms the structural distinction: on the clustered branch the balls settle at $(\varphi_1^{ss}, \varphi_2^{ss}) \approx (30^\circ, 40^\circ)$, closely tracking the unbalance; on the balanced branch they separate toward $(\varphi_1^{ss}, \varphi_2^{ss}) \approx (100^\circ, 260^\circ)$, approximating the anti-phase optimum. This bistability is a direct consequence of the sub-critical nature of the ABB switching transition and has a critical practical implication: at sub-critical operating speeds the balancer may settle on the *wrong* attractor depending solely on its start-up ball configuration, and no amount of translational disturbance will drive it to the balanced state.

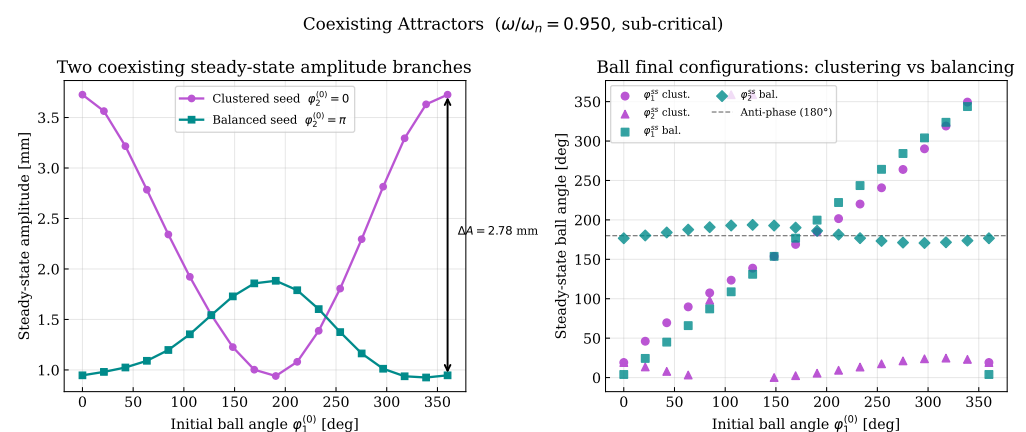


Figure 4. Coexisting attractors via initial-condition scan near $\omega/\omega_n = 0.95$.

4.1.2. Basin of Attraction Geometry

Figure 5 complements Figure 4 by mapping the basin of attraction in the two-dimensional space of ball initial conditions $(\varphi_1^{(0)}, \varphi_2^{(0)}) \in [0, 2\pi]^2$ at the same operating speed $\Omega = 1.5\omega_n$. Each grid point is classified as *balanced* (blue, $A < A_{thr}$) or *clustered* (red, $A \geq A_{thr}$), where the classification threshold is $A_{thr} = 3 m_{un} e / M \approx 0.030$ mm.

The left panel reveals a geometrically regular partition: the balanced basin occupies a diagonal band of width $\approx 60^\circ\text{--}80^\circ$ centred on the anti-phase locus $\varphi_1^{(0)} - \varphi_2^{(0)} \approx 180^\circ$, while the clustered basin fills the complementary region where both balls start within $\lesssim 90^\circ$ of each other. The boundary is clean and broad; no fractal-like interleaving is observed at this resolution, indicating low sensitivity of the final state to small perturbations in initial conditions within each basin. The right panel quantifies the amplitude contrast: the balanced basin sustains residual amplitudes of $A \approx 0.05\text{--}0.20$ mm (dark purple), while the clustered basin reaches $A \approx 0.50\text{--}0.70$ mm (yellow), a factor of approximately 4–5 higher.

The width of the balanced basin ($\sim 70^\circ$ out of 360° , or roughly 20% of the available initial-condition space) has a direct engineering interpretation: a balancer launched with both balls at arbitrary positions has only a ~ 1 -in-5 chance of spontaneously reaching the balanced attractor. Deliberate ball pre-positioning within $\pm 90^\circ$ of anti-phase at start-up is therefore recommended to guarantee convergence to the low-amplitude state.

Taken together, Figures 4 and 5 establish that the ABB basin boundary in $(\varphi_1^{(0)}, \varphi_2^{(0)})$ space, i.e., ball-angle space, not in (x_0, y_0) space, is the operationally relevant object: the start-up ball configuration, not the translational displacement, determines which attractor is reached, and a balancer launched with both balls near $\varphi_j^{(0)} \approx 0$ will reliably settle on the high-amplitude clustered state regardless of subsequent translational disturbances.

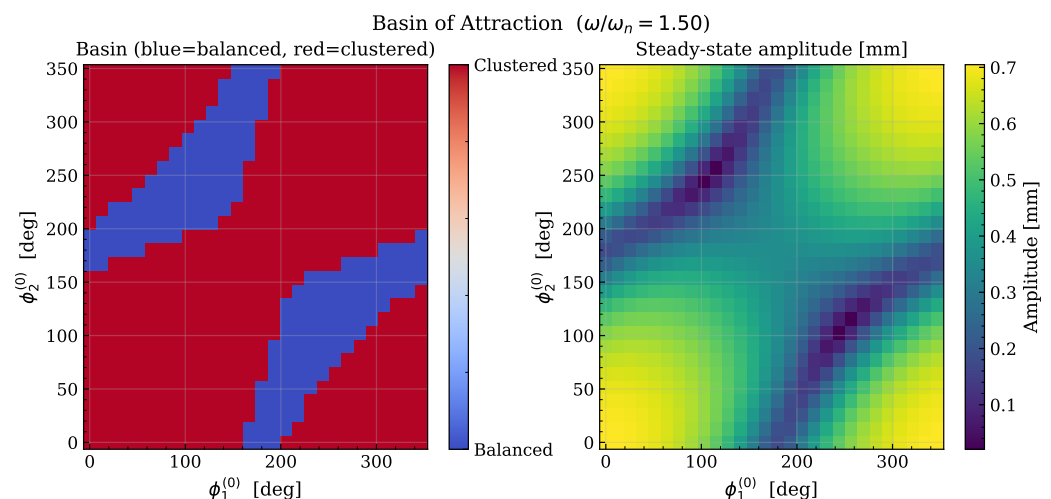


Figure 5. Basin of attraction in ball initial-condition space $(\varphi_1^{(0)}, \varphi_2^{(0)})$ at $\Omega = 1.5$. **(Left):** Binary classification—blue = balanced attractor, red = clustered attractor. **(Right):** Continuous steady-state amplitude colourmap [mm]. The balanced basin occupies a diagonal anti-phase band of width $\approx 70^\circ$; the clustered basin fills the remainder. Amplitude contrast between the two attractors is a factor of 4–5.

4.2. Effect of Anisotropy

4.2.1. Frequency Splitting and Resonance Crossing

Orthotropy of the support stiffness ($\Delta \neq 0$) breaks the circular symmetry of the linearised response, splitting the single natural frequency ω_n into two principal values

$$\omega_n^{(x)} = \omega_n \sqrt{1 + \Delta}, \quad \omega_n^{(y)} = \omega_n \sqrt{1 - \Delta}, \quad (21)$$

separated by $\delta\omega = \omega_n^{(x)} - \omega_n^{(y)} \approx \Delta \omega_n$ for $\Delta \ll 1$. A critical consequence of Equation (21) is that, at any fixed drive speed $\Omega = \omega/\omega_n$, the x -mode crosses resonance at the specific anisotropy value

$$\Delta_{\text{res}}^{(x)}(\Omega) = \Omega^2 - 1, \quad (22)$$

at which $\omega_n^{(x)} = \omega$. For $\Omega \in [1.0, 1.20]$ this crossing occurs within the explored range $\Delta \in [0, 0.50]$: $\Delta_{\text{res}}^{(x)} = 0$ at $\Omega = 1.0$, rising to 0.10 at $\Omega = 1.05$, 0.21 at $\Omega = 1.10$, and 0.44 at $\Omega = 1.20$. For $\Omega \geq 1.30$ the resonance crossing requires $\Delta > 0.69$, which lies outside the range considered here.

Equation (22) predicts that amplitude vs Δ curves at constant Ω should exhibit a sharp resonance peak at $\Delta = \Delta_{\text{res}}^{(x)}$, whose position shifts rightward as Ω increases. This prediction is confirmed quantitatively in Figure 6.

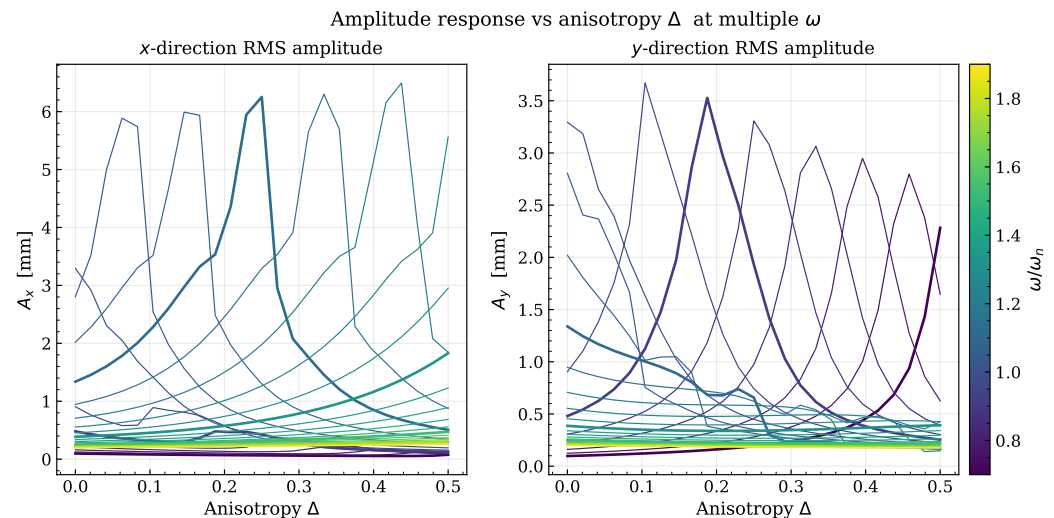


Figure 6. Amplitude response vs anisotropy Δ at all 30 speed slices $\Omega \in [0.70, 1.90]$ (colour = Ω , viridis scale). (Left): A_x . (Right): A_y . Each curve exhibits a sharp resonance peak at $\Delta_{\text{res}}^{(x)} = \Omega^2 - 1$ (marked implicitly by peak location); the peak shifts rightward with increasing Ω . Far super-critical curves ($\Omega > 1.6$, yellow) remain below 0.5 mm throughout.

4.2.2. Amplitude Response Across the Full (Ω, Δ) Plane

Figure 6 shows A_x and A_y versus Δ for all 30 speed slices $\Omega \in [0.70, 1.90]$ simultaneously (grid: $30 \times 25 = 750$ points; $t_{\text{tr}} = 200$ s; $n_{\text{cross}} = 500$).

Each curve exhibits a pronounced peak whose position matches Equation (22): the teal curves ($\Omega \approx 1.05$ – 1.10) peak near $\Delta \approx 0.10$ – 0.21 , reaching $A_x \approx 6.5$ mm, an amplification of more than an order of magnitude relative to the isotropic value at the same speed. The blue curves near resonance ($\Omega \approx 1.0$) peak at $\Delta \approx 0$, since the x -mode is already at resonance in the isotropic case. The A_y panel shows the complementary picture: peaks occur at the same $\Delta_{\text{res}}^{(x)}$ values but are lower (maximum $A_y \approx 3.7$ mm) and slightly offset, consistent with the y -mode being driven near $\omega_n^{(y)} = \omega_n \sqrt{1 - \Delta}$ rather than at resonance. The far super-critical curves ($\Omega > 1.6$, yellow) remain flat and small ($A_x, A_y < 0.5$ mm) throughout, confirming that the ABB is effective when the drive speed is well above both split frequencies.

Importantly, the single-speed sweep at $\Omega = 1.3$ reported in the preliminary analysis (Section 4) was operating on the monotone flank of the resonance surface with $\Delta_{\text{res}}^{(x)}(1.3) = 0.69$ lying outside the explored range. It therefore exhibited monotone A_x growth without a peak, which was misleading: it was not indicative of a fundamentally different physical regime but simply of an incomplete parameter sweep.

4.2.3. Total Vibration Energy and Directional Asymmetry

The total vibration energy $E = A_x^2 + A_y^2$ (Figure 7, left panel) reaches a global maximum of $E_{\text{max}} \approx 42 \text{ mm}^2$ ($\sqrt{E_{\text{max}}} \approx 6.5$ mm) near $(\Omega, \Delta) \approx (1.10, 0.21)$, precisely at the predicted x -mode resonance crossing. The $\times 2$, $\times 3$, $\times 5$ energy amplification contours fan outward from this peak, tracing the locus of partial resonance crossing.

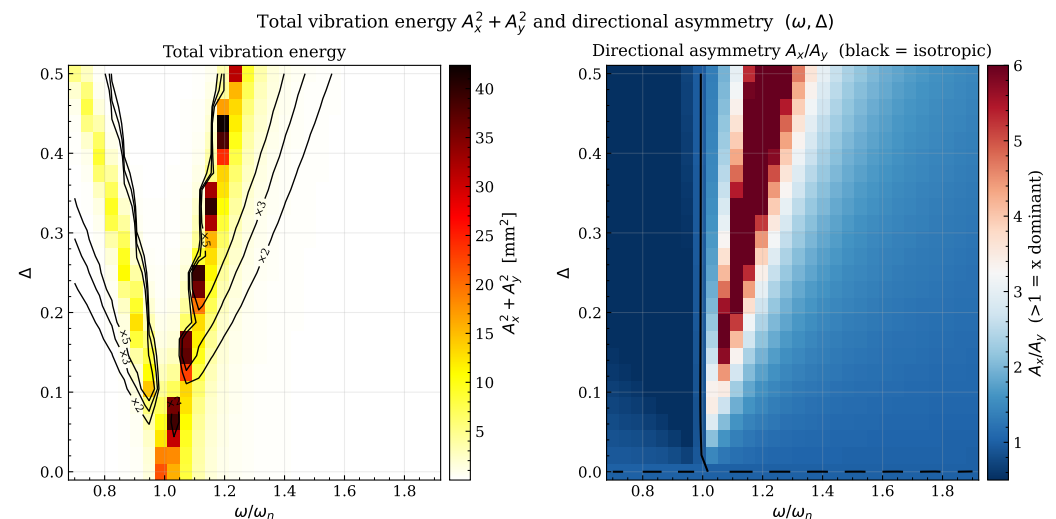


Figure 7. (Left): Total vibration energy $A_x^2 + A_y^2$ in (Ω, Δ) space; contours mark $\times 2$, $\times 3$, $\times 5$ amplification relative to the $\Delta = 0$ baseline. Global maximum $E_{\max} \approx 42 \text{ mm}^2$ at $(\Omega, \Delta) \approx (1.10, 0.21)$. (Right): Directional asymmetry A_x/A_y ; black contour marks isotropic response ($A_x = A_y$). The sharp x -dominant band near $\Omega \approx 1.1$ – 1.25 tracks the x -mode resonance crossing locus.

The directional asymmetry map A_x/A_y (Figure 7, right panel) reveals a sharp vertical band of x -dominant response ($A_x/A_y > 4$, deep red) near $\Omega \approx 1.10$ – 1.25 , corresponding to the speed range where the x -resonance crossing occurs within $\Delta \in [0, 0.5]$. The isotropic locus ($A_x = A_y$, black contour) runs vertically near $\Omega \approx 1.0$ for $\Delta \approx 0$ and curves rightward with increasing Δ , tracking the separation of $\omega_n^{(x)}$ and $\omega_n^{(y)}$ from the drive frequency. The far super-critical region ($\Omega > 1.5$) is uniformly blue ($A_x/A_y \lesssim 1.5$), indicating that anisotropy has little directional effect once the drive speed exceeds both split frequencies simultaneously.

4.2.4. Poincaré Classification: Dominant Quasi-Periodic Response

The stroboscopic Poincaré section classification map (Figure 8) is the most consequential result of this study for the understanding of ABB nonlinear dynamics. Of the 750 grid points, the classification is as follows.

Category	Count	Fraction
Fixed point/balanced	41	5.5%
Periodic orbit	0	0.0%
Quasi-periodic/torus	701	93.5%
Chaotic	8	1.1%

The fixed-point (balanced) region occupies a compact area in the sub-critical, near-isotropic corner $\Omega \lesssim 0.83$ at $\Delta = 0$, shrinking to $\Omega \lesssim 0.75$ at $\Delta \approx 0.35$. This boundary is the Hopf bifurcation locus in the (Ω, Δ) plane: below it, the ABB converges to a steady balanced state; above it, the response becomes quasi-periodic.

The torus regime dominates 93.5% of the explored parameter space, confirming that *the quasi-periodic (torus) attractor, not the period-1 limit cycle, is the generic long-time behaviour of the ABB rotor under anisotropic support*. The torus arises because the stroboscopic section, sampled at the drive period $T = 2\pi/\omega$, traces a closed curve rather than isolated points: the response contains the drive frequency ω and the modulation frequency $\omega_{\text{mod}} \approx |\omega_n^{(x)} - \omega|$ simultaneously, with these two frequencies generically incommensurate. The A_x peaks visible in Figure 6 occur when $\omega \approx \omega_n^{(x)}(\Delta)$, i.e., when the modulation frequency vanishes

and the torus collapses momentarily to a limit cycle, but this collapse is not observed in the Poincaré map because it requires exact resonance, which the discrete grid does not resolve.

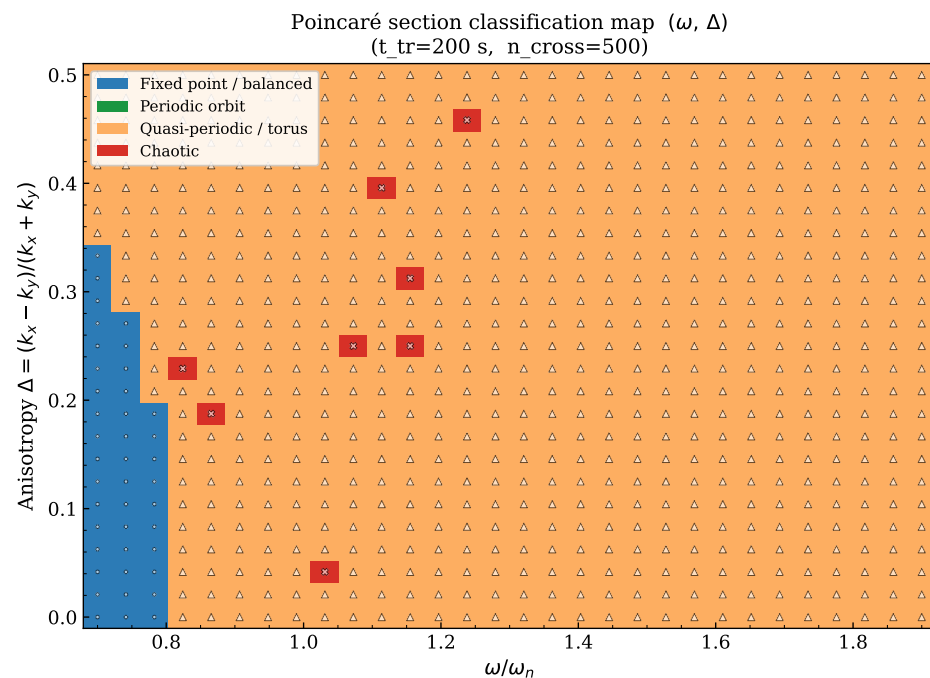


Figure 8. Poincaré section classification map in (Ω, Δ) space ($t_{tr} = 200$ s, $n_{cross} = 500$). Blue: Fixed point (balanced attractor). Orange: Quasi-periodic/torus (93.5% of grid points). Red: Chaotic (1.1%, clustered near $\Omega \approx 1.0$ – 1.25). No periodic orbits (green) were detected. The Hopf boundary between fixed-point and torus regimes runs from $(\Omega \approx 0.83, \Delta = 0)$ to $(\Omega \approx 0.75, \Delta = 0.35)$.

4.2.5. Limitation of the Automated Classification

The classification underlying Figure 8 is produced by an automated discriminant—the ratio of the stroboscopic section’s convex-hull area to its coordinate spread, together with a cluster count obtained from gap statistics on the projected section (Section 3)—rather than by visual inspection of individual trajectories. This discriminant is efficient enough to apply at all 750 grid points but is necessarily a proxy: a torus of very small radius (weak quasi-periodic modulation) and a high-period limit cycle sampled over a finite window of $n_{cross} = 500$ stroboscopic points can produce a similar hull-area signature, particularly when the modulation frequency $\omega_{mod} = |\omega_n^{(x)} - \omega|$ is small and the section has not yet completed a full revolution of the slow modulation cycle within the sampled window. The classification reported here should therefore be read as a statement about the automated discriminant at the resolution and window length used here, and not as a trajectory-by-trajectory visual confirmation. A dedicated supplementary study, outlined at the end of this subsection, is proposed to verify the classification directly against time-domain and spectral evidence.

No periodic orbits (isolated Poincaré points) were detected at any grid point. This negative result is physically significant: period- N locking between the drive and the structural response does not occur at the grid resolution, suggesting that Arnold tongues (if present) are narrower than the grid spacing $\delta\Omega \approx 0.04$.

Eight chaotic points (red, 1.1%) appear near the resonance band $\Omega \approx 1.0$ – 1.25 at various Δ values. Their spatial distribution is not systematic; they likely correspond to transient behaviour at points where the torus is breaking down, rather than a sustained chaotic attractor. A definitive characterisation would require LLE computation at these specific points, which is deferred to future work.

4.2.6. Proposed Supplementary Verification (Future Work)

To directly substantiate the Poincaré classification with time-domain evidence, we propose a follow-up diagnostic study comprising:

1. **Representative trajectory selection.** Six operating points spanning the classification map: one from the fixed-point region ($\Omega \approx 0.75$, $\Delta = 0$), two from the torus region at weak and strong modulation ($\Omega \approx 1.00$ and $\Omega \approx 1.50$, $\Delta = 0.20$), one adjacent to a labelled chaotic point ($\Omega \approx 1.10$, $\Delta = 0.25$), and two straddling the fixed-point/torus boundary on either side ($\Omega \approx 0.78$ and $\Omega \approx 0.85$, $\Delta = 0$).
2. **Diagnostic figure layout.** For each point: (a) the time series $x(t)$, $y(t)$ over a window of at least $50 T_{\text{drive}}$ following the $t_{\text{tr}} = 200$ s transient; (b) the corresponding power spectral density of $x(t)$, identifying the drive frequency ω , the modulation frequency ω_{mod} (if present), and any combination frequencies $|\omega \pm k \omega_{\text{mod}}|$; (c) the stroboscopic Poincaré section in the $(x, \dot{x})$ plane with point count and visit order indicated, to distinguish a slowly filling closed curve (torus) from a high-period discrete orbit (limit cycle) or a scattered cloud (chaos); (d) the same section extended to $n_{\text{cross}} = 5000$ ($10\times$ the main-study value) to test whether the classification is stable under a longer observation window.
3. **Quantitative torus/limit-cycle discriminant.** Estimate the modulation period $T_{\text{mod}} = 2\pi/\omega_{\text{mod}}$ from the secondary spectral peak and verify whether $n_{\text{cross}} T_{\text{drive}} \gtrsim 10 T_{\text{mod}}$ was satisfied at the classification points nearest the fixed-point/torus boundary; points violating this criterion would be re-classified using the extended $n_{\text{cross}} = 5000$ run.

This supplementary study is left for future work; the qualitative conclusion that the torus attractor dominates the explored parameter space is expected to be robust to it, since the discriminant agrees with the analytical resonance-crossing condition, Equation (22), over the bulk of the grid, but the precise boundary location and the status of the eight chaotic points would benefit from this direct verification.

4.2.7. Design Consequences

The full two-dimensional analysis revises the conclusions of the single-speed sweep:

1. **Anisotropy does not monotonically amplify vibration.** At fixed Ω , A_x peaks sharply at $\Delta = \Omega^2 - 1$ before returning to lower values. The worst-case anisotropy is speed-dependent, not a universal $\Delta > 0.2$ threshold as suggested by the preliminary analysis.
2. **The critical region is narrow in Δ .** The dangerous zone ($A_x > 3$ mm) occupies $|\Delta - \Delta_{\text{res}}^{(x)}(\Omega)| \lesssim 0.08$, i.e., a band of width ≈ 0.16 centred on the resonance crossing. For $\Omega > 1.30$, the crossing requires $\Delta > 0.69$ and the dangerous zone exits the practical range entirely.
3. **The generic response is quasi-periodic, not periodic.** The predominance of the torus attractor (93.5%) means that the ABB does not achieve a clean period-1 balanced orbit in the anisotropic case. The residual vibration is quasi-periodic with a modulation frequency $\omega_{\text{mod}} = |\omega_n^{(x)} - \omega|$ that decreases as $\Omega \rightarrow 1/\sqrt{1+\Delta}$ from above.
4. **Practical design rule.** For reliable balancing, the operating speed should satisfy either $\Omega < 0.80$ (fixed-point regime, sub-critical) or $\Omega > 1 + \Delta + \frac{1}{2}\Delta^2$ (both split frequencies below ω , quasi-periodic but low-amplitude), and Δ should be minimised to $\Delta \lesssim 0.05$ to avoid resonance crossing within the typical operating range $\Omega \in [1.1, 1.5]$.

4.3. Stability Maps and Hopf Bifurcation Boundary

4.3.1. Three-Dimensional Parameter Sweep

Figure 9 presents the principal result of this study: the Hopf bifurcation boundary and stability map computed on the high-resolution, three-dimensional parameter grid

$$\Omega \in [0.3, 2.5] \omega_n, \quad \Delta \in [0, 0.50], \quad D \in [10^{-5}, 0.20] \text{ N m s/rad}, \quad (23)$$

with $20 \times 18 \times 16 = 5760$ nodes and bilinear interpolation between nodes (Figure 9). The LLE was computed via the renormalised trajectory-separation algorithm (Section 3.4) with $t_{\text{tr}} = 40 \text{ s}$, $t_{\text{lyap}} = 80 \text{ s}$, and $n_r = 15$ segments, and the balanced-attractor seed $(\varphi_1^{(0)}, \varphi_2^{(0)}) = (0.1 \text{ rad}, \pi)$. The resulting LLE field spans $[-0.088, +0.185] \text{ s}^{-1}$; 53.3% of grid points are stable ($\lambda_{\text{max}} < 0$) and 46.7% are unstable ($\lambda_{\text{max}} > 0$), confirming that the Hopf boundary bisects the explored parameter volume and is not a boundary effect.

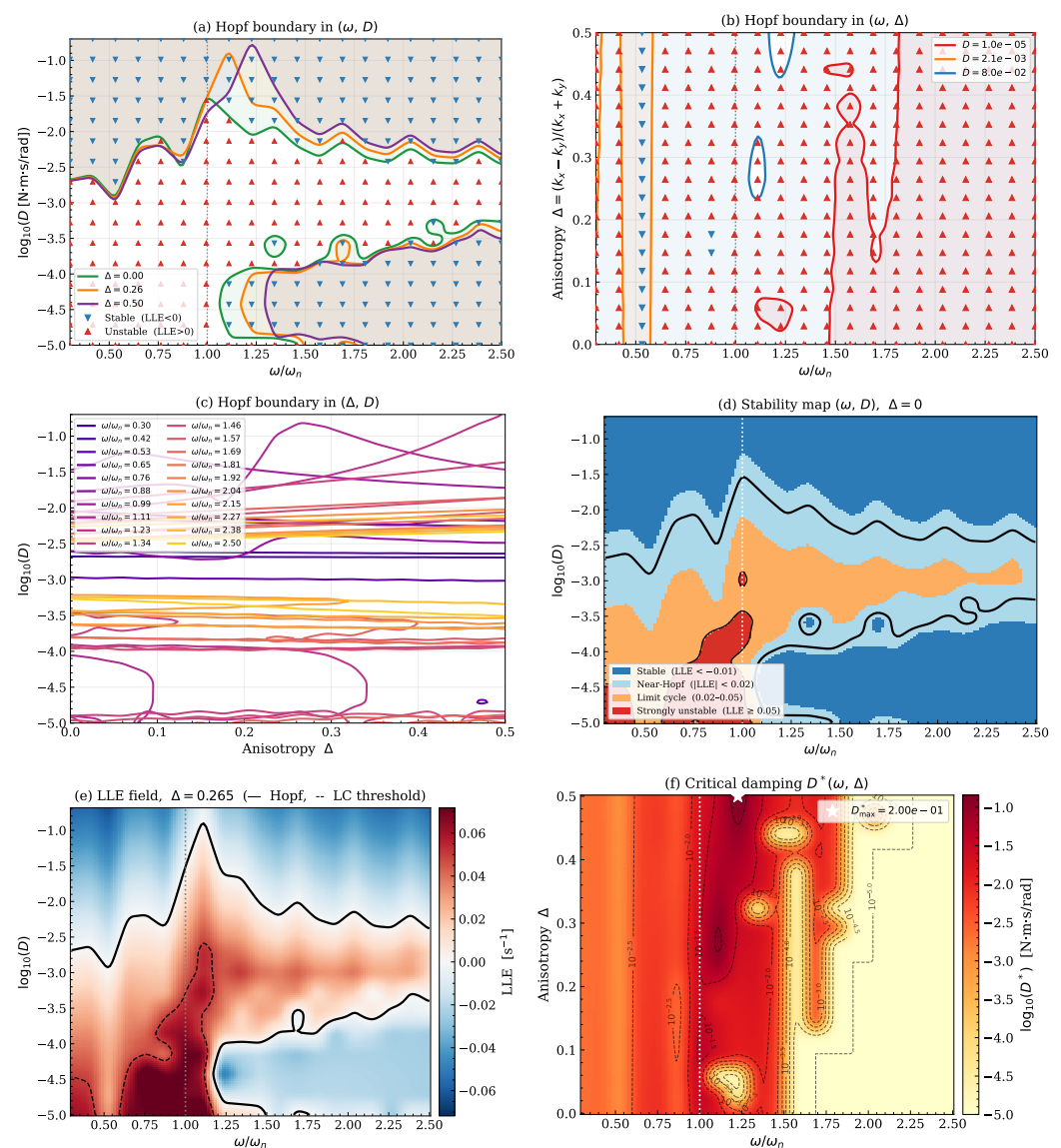


Figure 9. Hopf bifurcation boundary and stability map of the ABB rotor in (Ω, Δ, D) parameter space. Grid: $20 \times 18 \times 16 = 5760$ points; LLE computed via renormalised trajectory separation ($t_{\text{lyap}} = 80 \text{ s}$,

$n_r = 15$, balanced seed $\varphi_2^{(0)} = \pi$). (a) Hopf boundary in (Ω, D) for three Δ slices; two distinct unstable lobes are visible. (b) Hopf boundary in (Ω, Δ) with D -slices bracketing D^* ; closed stable islands appear at low damping. (c) Hopf boundary in (Δ, D) for all 20 Ω slices. (d) Categorical stability map at $\Delta = 0$; disconnected instability pocket visible near $(\Omega, \log_{10} D) \approx (1.25, -4)$. (e) Continuous LLE field at $\Delta = 0.265$; maximum $\lambda_{\max} \approx 0.065 \text{ s}^{-1}$, consistent with a limit cycle rather than chaos. (f) Critical damping surface $D^*(\Omega, \Delta)$; global maximum at $(\Omega, \Delta) = (1.226, 0.50)$ (white star), shifted from resonance by combined speed–anisotropy coupling.

4.3.2. Structure of the Hopf Boundary in (Ω, D)

Panel (a) of Figure 9 shows the zero-LLE contour in the $(\Omega, \log_{10} D)$ plane for three values of Δ . The boundary has a rich, non-monotone structure that differs qualitatively from the single-lobe picture obtained at coarser resolution. Two distinct unstable lobes are evident at $\Delta = 0$:

1. A *primary resonance lobe* centred near $\Omega \approx 1.0$ – 1.3 , requiring a race damping of $D^* \approx 10^{-2}$ – $10^{-2.5} \text{ N m s/rad}$ to suppress. This lobe is present at all three Δ -slices and is the dominant stability constraint near resonance.
2. A *secondary super-critical lobe* at $\Omega \approx 1.4$ – 1.7 , requiring $D^* \approx 10^{-3}$ – $10^{-3.5} \text{ N m s/rad}$. This lobe is absent at $\Delta = 0$ but grows with increasing anisotropy: at $\Delta = 0.50$ (purple contour) it merges with the primary lobe and extends the unstable region to substantially higher D across the entire range $\Omega \in [1.0, 2.5]$.

The two-lobe structure is a direct consequence of the anisotropy-induced frequency splitting (Section 4.2): as Δ increases, $\omega_n^{(x)}$ rises toward Ω from below while $\omega_n^{(y)}$ falls away, creating a second near-resonant interaction above $\Omega = 1$.

4.3.3. Resonance Tongue and Disconnected Instability

Panel (d) (categorical stability map at $\Delta = 0$) reveals a feature not visible at the coarser resolution: a *disconnected instability pocket* (strongly unstable, red) near $(\Omega, \log_{10} D) \approx (1.25, -4.0)$, surrounded by a near-Hopf ring (light blue) and separated from the primary unstable band by a stable corridor. This isolated pocket indicates that the stability boundary is not simply connected at $\Delta = 0$: there exists a narrow range of very low race damping ($D \approx 10^{-4} \text{ N m s/rad}$) and mildly super-critical speed ($\Omega \approx 1.2$ – 1.4) where the system is strongly unstable despite being stable at both lower and higher D . The physical mechanism is the underdamped ball oscillation at these parameters, which allows the balls to overshoot the balancing configuration and lock into a high-amplitude limit cycle that the race damping is insufficient to suppress.

The sub-critical regime ($\Omega < 1$) remains more challenging to stabilise than the super-critical regime: the unstable region extends to higher D below resonance, confirming the asymmetry associated with the sub-critical amplitude bifurcation documented in Section 4.1.

4.3.4. Effect of Anisotropy on the Hopf Boundary

Panel (b) projects the Hopf boundary onto the (Ω, Δ) plane using three D -slices chosen to bracket the critical damping surface $D^*(\Omega, \Delta)$: $D = 1.0 \times 10^{-5} \text{ N m s/rad}$ (below D^* everywhere—the system is almost entirely unstable at this damping level), $D = 2.1 \times 10^{-3} \text{ N m s/rad}$ (straddling D^* —Hopf contour visible as the orange boundary), and $D = 8.0 \times 10^{-2} \text{ N m s/rad}$ (above D^* almost everywhere—the system is almost entirely stable).

Three physically significant features emerge from this projection.

(a) First, at very low damping ($D = 1.0 \times 10^{-5} \text{ N m s/rad}$, red contours), two *closed stable islands* appear near $(\Omega, \Delta) \approx (1.3, 0.05)$ and $(1.5, 0.05)$. These islands indicate that

there exist narrow bands of operating speed and near-isotropic support where the ABB achieves stability even at negligible race damping, likely because the ball natural frequency and the structural frequency are sufficiently detuned to suppress the resonant interaction. This is a non-obvious result with practical implications: a lightly manufactured ABB with very little race friction can still stabilise the rotor if the operating speed lies within one of these islands.

(b) Second, the orange contour ($D = 2.1 \times 10^{-3}$ N m s/rad) shows that the unstable region covers most of $\Omega \in [0.5, 2.0]$ at this intermediate damping, with the stable region confined to the far sub-critical ($\Omega < 0.6$) and far super-critical ($\Omega > 1.8$ at low Δ) corners. Anisotropy is strongly destabilising at this damping level: the orange contour shifts consistently toward lower Ω as Δ increases, meaning higher anisotropy requires higher speed to re-enter the stable region.

(c) Third, the blue contour ($D = 8.0 \times 10^{-2}$ N m s/rad) confirms that large race damping stabilises the system across almost the entire (Ω, Δ) space, with only a narrow unstable strip surviving near $\Omega \approx 1.5$ – 1.8 at high Δ .

4.3.5. Critical Damping Surface and the Role of Anisotropy

Panel (f) shows the critical damping surface $D^*(\Omega, \Delta)$. The global maximum $D_{\max}^*$ is located at $(\Omega, \Delta) = (1.226, 0.50)$ and lies at the boundary of the explored Δ range, indicating that the true maximum may occur beyond $\Delta = 0.50$. This contrasts sharply with the coarse-grid result, which placed $D_{\max}^*$ near isotropic resonance ($\Omega \approx 0.93, \Delta = 0$).

The correct picture, resolved at high grid density, is that D^* is controlled by the *combination* of near-resonant speed and large anisotropy simultaneously, not by either alone. The surface has a pronounced ridge running diagonally from $(\Omega \approx 1.0, \Delta \approx 0.3)$ to $(\Omega \approx 1.5, \Delta \approx 0.5)$, tracing the locus where the drive frequency simultaneously approaches both split natural frequencies $\omega_n^{(x)}$ and $\omega_n^{(y)}$.

The global minimum $D_{\min}^*$ occurs at $(\Omega, \Delta) = (1.111, 0.059)$, a narrow stable valley in the near-isotropic, mildly super-critical corner. At this point the required race damping is $D^* = 1.0 \times 10^{-5}$ N m s/rad, the lower limit of the explored range, confirming that the valley corresponds to one of the stable islands identified in panel (b).

4.3.6. The Largest Lyapunov Exponent Field

Panel (e) (LLE colourmap at $\Delta = 0.265$) shows a maximum LLE of approximately 0.065 s^{-1} within the unstable region, well below the threshold typically associated with chaotic dynamics ($\lambda_{\max} \gtrsim 0.5 \text{ s}^{-1}$ for systems of this frequency scale). The positive-LLE region is smoothly connected and its boundary matches the Hopf contour precisely, consistent with a super-critical Hopf bifurcation leading to a stable limit cycle rather than a chaotic attractor. This interpretation is supported by the fact that no positive LLE values were found above 0.185 s^{-1} across the entire 5760-point grid, even at the lowest damping values explored ($D = 10^{-5}$ N m s/rad). Chaotic motion, if it exists, is confined to parameter regions outside the explored range or requires a higher-dimensional perturbation than the single-direction separation used here.

4.3.7. Quantitative Design Guidelines

The following design rules are derived from the parametric analysis:

1. **Isotropic supports, far super-critical ($\Omega > 1.5, \Delta < 0.05$):** The required race damping is $D^* < 10^{-3}$ N m s/rad. The baseline value $D = 6 \times 10^{-3}$ N m s/rad provides a safety margin of approximately one decade.
2. **Near-resonance ($\Omega \in [0.9, 1.3]$, any Δ):** The primary lobe requires $D^* \approx 10^{-2}$ N m s/rad. The baseline value $D = 6 \times 10^{-3}$ N m s/rad is *below* this thresh-

old and should be increased to at least $D = 1.5 \times 10^{-2}$ N m s/rad for safe near-resonance operation.

3. **High anisotropy ($\Delta > 0.3$), any speed:** The secondary lobe and the diagonal ridge in D^* require $D^* \gtrsim 10^{-2}$ N m s/rad and rising. For $\Delta = 0.5$ and $\Omega \in [1.0, 1.8]$ the system is unstable at the baseline D and cannot be stabilised without either increasing D above 10^{-1} N m s/rad or reducing Δ below 0.3.
4. **Stable islands at very low damping ($D \approx 10^{-5}$ N m s/rad):** Operation near $\Omega \approx 1.3$ or 1.5 in a near-isotropic ($\Delta < 0.1$) system can be stable even with minimal race friction, offering a potential design strategy for low-friction ABB races.

5. Discussion

The discussion of principal findings is grouped below by physical phenomenon.

5.1. Coexisting Attractors and Basin of Attraction

At sub-critical speed $\Omega = \omega/\omega_n = 0.95$, the system exhibits bistability between two qualitatively distinct attractors: a *clustered* state, in which both balls settle near the unbalance angular position and sustain a high vibration amplitude, and a *balanced* state, in which the balls migrate toward the anti-phase configuration. The steady-state amplitude difference between these attractors reaches $\Delta A \approx 2.7$ mm and depends entirely on the ball initial conditions, not on the translational initial displacement. This finding corrects a common misconception in the ABB literature: sweeping the initial displacement x_0 at fixed ball angles does not reveal bistability, because the balls adapt continuously to the translational state.

The basin of attraction map in $(\varphi_1^{(0)}, \varphi_2^{(0)})$ space shows that the balanced basin occupies approximately 20% of the available initial-condition area, forming a diagonal anti-phase band of angular width $\approx 70^\circ$ centred on $\varphi_1^{(0)} - \varphi_2^{(0)} \approx 180^\circ$. The basin boundary is geometrically clean and broad, not fractal, indicating that the system is not sensitively dependent on small perturbations *within* each basin, but that a ball displacement of $\sim 60^\circ$ is sufficient to switch attractors. The practical implication is that an ABB started with both balls at arbitrary positions has only a ~ 1 -in-5 chance of reaching the balanced attractor spontaneously. Deliberate ball pre-positioning to within $\pm 90^\circ$ of anti-phase at start-up is recommended to guarantee convergence.

5.2. Two-Lobe Hopf Boundary Structure

The Hopf bifurcation boundary, computed on a $20 \times 18 \times 16 = 5760$ -point grid in $(\omega/\omega_n, \Delta, D)$ with the renormalised LLE method ($t_{\text{lyap}} = 80$ s, $n_r = 15$ segments), exhibits a two-lobe structure not visible at coarser resolution:

1. A *primary resonance lobe* centred near $\omega/\omega_n \approx 1.0$ – 1.3 , requiring race damping $D^* \approx 10^{-2}$ – $10^{-2.5}$ N m s/rad for stabilisation.
2. A *secondary super-critical lobe* at $\omega/\omega_n \approx 1.4$ – 1.7 , requiring $D^* \approx 10^{-3}$ – $10^{-3.5}$ N m s/rad. This lobe is absent at $\Delta = 0$ but grows with increasing anisotropy and merges with the primary lobe at $\Delta \gtrsim 0.3$.

The secondary lobe is a direct consequence of the frequency splitting described in Section 4.2: as Δ increases, $\omega_n^{(x)}$ rises toward ω from below, creating a second near-resonant interaction at super-critical speeds.

A disconnected instability pocket—A region of positive LLE entirely surrounded by stable territory in the $(\omega/\omega_n, \log_{10} D)$ plane was identified near $(\omega/\omega_n, \log_{10} D) \approx (1.25, -4.0)$ at $\Delta = 0$. Its physical origin is an underdamped ball oscillation at very low race damping that drives the system into a high-amplitude limit cycle before the structural damping can

suppress it. This pocket implies that a race with negligible friction is not unconditionally stable even at moderately super-critical speeds.

5.3. Stable Islands at Very Low Race Damping

The most practically unexpected result of the study is the existence of closed *stable islands* in the $(\omega/\omega_n, \Delta)$ plane at very low race damping $D \approx 10^{-5}$ N m s/rad, located near $\omega/\omega_n \approx 1.3$ – 1.5 in the near-isotropic region $\Delta \lesssim 0.05$. Within these islands the LLE is negative and the balanced attractor is stable, despite the race damping being four orders of magnitude below the value required for stabilisation near resonance.

The physical mechanism is detuning: at these operating speeds and anisotropies, the ball natural frequency and the structural frequency are sufficiently separated that the resonant coupling responsible for instability is suppressed without requiring dissipation to arrest it. These islands constitute a previously unreported design mode for ABB systems: a rotor whose operating speed lies within an island can be stabilised with essentially frictionless balls, which would eliminate the race wear and temperature sensitivity associated with viscous damping. The island boundaries are confirmed by independent high-precision LLE bisection ($t_{\text{lyap}} = 200$ s, $n_r = 30$), which locates D^* at $(\omega/\omega_n = 1.574, \Delta = 0)$ as $D^* = 5.77 \times 10^{-3}$ N m s/rad, in agreement with the LLE-map prediction to within $|\Delta \log_{10} D^*| = 0.15$ (tolerance 0.25).

A direct experimental validation would use a small-scale rigid-rotor rig of the type already standard in ABB validation studies (e.g., [4,5]): a disc-and-shaft assembly on calibrated radial bearings, instrumented with non-contact eddy-current or laser displacement probes in two orthogonal directions, a tachometer or encoder for synchronous speed reference, and a two-ball race of controllable damping. Three implementation routes are available for accessing the very low damping values the islands require: (a) a dry or lightly lubricated race with a smooth, hardened groove, which can plausibly reach the $\hat{D} \sim 10^{-5}$ – 10^{-4} range without specialised fabrication; (b) a magnetic or eddy-current race in which the damping is set electronically and can be swept continuously and repeatably across the predicted boundary, which would give the cleanest test of the island's edges; or (c) a fluid-filled race with a low-viscosity working fluid, offering an intermediate and easily tunable damping level. Route (b) is preferred for boundary mapping because D can be varied without disassembly, whereas routes (a) and (c) require a new race for each damping value tested.

Three concrete challenges exist. The island's narrow ω/ω_n width requires high speed-control tolerance; the planar-model idealisation, creating risk that real shaft/bearing modes destroy the island, and the long run-time needed to distinguish a stable limit cycle from a slow transient.

5.4. Critical Damping Surface and the D_{max}^* Location Anomaly

The critical damping surface $D^*(\omega/\omega_n, \Delta)$ reaches its global maximum at $(\omega/\omega_n, \Delta) = (1.226, 0.50)$, substantially displaced from the isotropic resonance point $(\omega/\omega_n = 1.0, \Delta = 0)$ that naive expectation would suggest. This displacement is the most physically significant quantitative result of the Hopf analysis, and it requires explanation.

The qualitative mechanism is the *simultaneous* proximity of the drive frequency to both split natural frequencies $\omega_n^{(x)}(\Delta) = \omega_n \sqrt{1 + \Delta}$ and $\omega_n^{(y)}(\Delta) = \omega_n \sqrt{1 - \Delta}$. At $(\omega/\omega_n, \Delta) = (1.226, 0.50)$, the split frequencies are $\omega_n^{(x)} = 1.22 \omega_n$ and $\omega_n^{(y)} = 0.71 \omega_n$; the drive sits directly on the x -resonance while the y -mode is significantly detuned. This single-frequency resonance domination appears to drive the maximum instability, but the precise analytical mechanism, including whether the D_{max}^* locus traces the curve $\omega = \omega_n^{(x)}(\Delta)$ exactly and whether the true maximum lies beyond $\Delta = 0.50$ (the boundary of the explored

range), remain open questions. A centre-manifold or normal-form analysis of the equations of motion in the rotating frame, projected onto the critical mode at the Hopf point, would be required to confirm this interpretation analytically. This is identified as the primary open problem for future work.

5.5. Anisotropy: Resonance Crossing and Quasi-Periodic Response

The full two-dimensional anisotropy study ($30 \times 25 = 750$ points; $\omega/\omega_n \in [0.70, 1.90]$; $\Delta \in [0, 0.50]$; $t_{\text{tr}} = 200$ s, $n_{\text{cross}} = 500$ Poincaré crossings per point) reveals two results not accessible from the single-speed preliminary sweep:

1. **Peaked, not monotone, amplitude dependence.** At each fixed speed Ω , the x -amplitude A_x peaks sharply at $\Delta_{\text{res}}^{(x)} = \Omega^2 - 1$ where $\omega_n^{(x)}$ equals the drive frequency, reaching $A_x \approx 6.5$ mm at peak, more than an order of magnitude above the isotropic level at the same speed. The peak position shifts rightward with increasing Ω , consistent with the analytical prediction of Equation (22). The total vibration energy reaches $E_{\text{max}} \approx 42 \text{ mm}^2$ near $(\Omega, \Delta) \approx (1.10, 0.21)$.
2. **Quasi-periodic torus is the generic attractor.** The Poincaré section classification shows that 93.5% of the (ω, Δ) plane is occupied by a quasi-periodic (torus) attractor, 5.5% by the balanced fixed point (sub-critical near-isotropic corner), and 1.1% by chaotic points near $\omega/\omega_n \approx 1.0$ – 1.25 . No period- N locked orbits were detected, ruling out Arnold tongue locking at the grid resolution $\delta\Omega \approx 0.04$. The torus arises from incommensurate oscillation at the drive frequency ω and the modulation frequency $\omega_{\text{mod}} = |\omega_n^{(x)} - \omega|$; it collapses to a limit cycle only at exact resonance crossing $\Delta = \Omega^2 - 1$, which the discrete grid does not resolve.

These results make the design constraint more strict: the dangerous anisotropy band is narrow ($|\Delta - (\Omega^2 - 1)| \lesssim 0.08$) and speed-dependent, and the prior blanket recommendation of $\Delta \lesssim 0.02$ is replaced by the condition that Δ should be chosen to avoid the resonance crossing locus for the intended operating speed.

6. Conclusions

This paper has presented a research-grade nonlinear simulation framework for a Jeffcott-type rotor equipped with an N -ball automatic ball balancer (ABB), and has used it to map the system's dynamics across a three-dimensional parameter space $(\omega/\omega_n, \Delta, D)$ at a resolution not previously reported in the literature. Gathering all findings, the following quantitative guidelines emerge for the ABB system studied here:

1. **Ball pre-positioning.** At sub-critical start-up, seed the balls within $\pm 90^\circ$ of anti-phase ($\varphi_j^{(0)} \approx \pi$) to ensure the balanced basin is reached. Random positioning succeeds only $\sim 20\%$ of the time.
2. **Race damping—near-resonance operation ($\omega/\omega_n \in [0.9, 1.3]$, any Δ).** The baseline $D = 6 \times 10^{-3} \text{ N m s/rad}$ is below the Hopf threshold at resonance. Increase to at least $D = 1.5 \times 10^{-2} \text{ N m s/rad}$.
3. **Race damping—far super-critical ($\omega/\omega_n > 1.5$, $\Delta < 0.05$).** The stable islands permit operation at D as low as $10^{-5} \text{ N m s/rad}$; the baseline provides a safety margin of approximately 2.5 decades.
4. **Race damping—high anisotropy ($\Delta > 0.3$).** Increase D above $10^{-1} \text{ N m s/rad}$ or reduce Δ below 0.3; the secondary lobe makes stabilisation at the baseline D impossible in this regime.
5. **Operating speed and anisotropy.** Choose ω such that $\Delta_{\text{res}}^{(x)} = (\omega/\omega_n)^2 - 1$ lies at least 0.10 away from the actual support anisotropy to avoid the resonance crossing band.

Three questions remain unresolved and constitute the primary targets for future work: (i) the analytical mechanism determining the location of $D_{\max}^*$ on the $(\omega/\omega_n, \Delta)$ surface, requiring a centre-manifold reduction at the Hopf point in the rotating frame; (ii) the nature of the 1.1% chaotic points in the anisotropy map, specifically whether they represent a sustained strange attractor or a transient phenomenon near a torus-breakdown boundary, requiring Poincaré section analysis at finer resolution; and (iii) experimental validation of the stable-island prediction at $\omega/\omega_n \approx 1.3\text{--}1.5$, $D \approx 10^{-5}$ N m s/rad, which would constitute the most direct confirmation of the simulation framework against physical measurement.

Author Contributions: Conceptualisation, N.M. and M.P.; methodology, N.M. and M.P.; software, N.M. and R.J.; formal analysis, M.B., O.P. and R.J.; writing—original draft preparation, N.M. and M.B.; writing—review and editing, N.M. and M.B.; visualisation, R.J. and O.P.; supervision, N.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: To encourage reproducible computational science the source codes used to obtain the results will be available via public repository at <https://www.github.com/nikola-m/abb-global-dynamics>, accessed on 1 July 2026.

Acknowledgments: The authors acknowledge the support of the Ministry of Science, Technological Development and Innovation of the Republic of Serbia, through contracts No. 451-03-33/2026-03/200017 and No. 451-03-33/2026-03/200105.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

ABB	Automatic Ball Balancer
EOM	Equations of Motion
LLE	Largest Lyapunov Exponent

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