

Article

Observation and Control of Sharp Many-Body Localization in Cold-Atom Optical Lattices

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Abstract

We investigate the localization transition in a one-dimensional extended Aubry–André–Harper (AAH) model, emphasizing its implementation as a quantum simulator in ultracold atomic optical lattices. In the single-particle case, quasi-periodic hopping modulation allows the localization transition point to be clearly observed even for small system sizes. By driving the system into strongly Anderson localized states immediately after the transition, we observe a sharp many-body localization (MBL) transition upon introducing interactions, with the MBL transition point closely approaching that of Anderson localization. To demonstrate the effects of interactions, we map out a global phase diagram and find that critical states in this model are easily thermalized. Contrary to previous studies where enhanced interactions significantly promoted thermalization, we find that increasing interactions does not notably shift the MBL transition point. Importantly, our setup offers a practical and experimentally accessible platform for studying sharp MBL transitions using ultracold atoms in optical lattices, bridging MBL physics with advances in quantum optics and cold-atom technologies.

Keywords: ultracold atomic optical lattices; many-body localization; Anderson localization; interactions; critical states

1. Introduction

Anderson localization is a cornerstone of condensed matter physics [1], where strong disorder drives a metal–insulator transition and profoundly influences theories of the quantum Hall effect and electron transport [2]. Crucially, this phenomenon is not confined to electronic systems. Over the past two decades, photonics and quantum optics have emerged as powerful and highly controllable platforms for exploring wave transport in disordered and interacting systems, offering unprecedented access to the coherent manipulation, detection, and engineering of quantum states of light and matter. In particular, quantum optical platforms such as photonic lattices, cavity and waveguide quantum electrodynamics systems, and ultracold atoms coupled to optical fields provide exquisite control over disorder strength, interaction parameters, and measurement backaction, as well as direct real-space imaging of wavefunction dynamics [3,4]. These capabilities make quantum optics uniquely suited for simulating Anderson localization and related interference phenomena in regimes that are difficult to access in traditional condensed matter settings.

Many-body localization (MBL) is a frontier topic exploring the interplay between disorder and interactions [5–7]. It is an extension of Anderson localization to the many-body regime. A key feature of MBL is the preservation of local information from the initial state



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over long-time evolution [8], alongside approximate integrability [9,10]. These properties have broad implications for nonequilibrium dynamics [8,11], quantum information [12], random matrix theory [9,10], and quantum entanglement [13,14], reflecting its broad relevance across quantum physics, especially in the study of disordered systems [15,16].

Experimentally, ultracold atomic systems, particularly Bose–Einstein condensates [17,18], provide a highly controllable platform for studying MBL [8,19]. From the perspective of quantum optics, these systems realize hybrid light matter quantum simulators in which laser fields both confine and coherently control atomic ensembles, while simultaneously enabling the synthesis of effective disorder, interactions, and driving at the quantum level. Laser-based techniques, including optical lattices, optical tweezers, Raman transitions, and spin-dependent potentials, enable precise engineering of both the spatial structure and the interaction landscape of atomic ensembles [20]. Optical lattices create periodic or quasi-periodic potentials whose depth and geometry can be dynamically tuned, allowing direct emulation of disordered or quasi-disordered lattice Hamiltonians [8,19]. Optical tweezers allow the deterministic preparation and rearrangement of individual atoms, providing single-site resolution essential for probing local observables and correlations [21,22]. Coherent laser driving, such as Raman transitions or microwave-assisted manipulations, enables the realization of spin-dependent interactions and Floquet-engineered Hamiltonians [23–25], further enriching the accessible parameter space. These capabilities enable the direct observation of hallmark MBL phenomena, including the preservation of local memory [8], the coexistence of thermalizing and localized regions [26], and the measurement of entanglement growth exhibiting logarithmic spreading [19]. Beyond probing MBL, these laser-mediated and quantum optical control platforms unify disordered many-body physics with tools from quantum optics, enabling the study of coherent dynamics, measurement-induced effects, and engineered dissipation in strongly interacting settings. They provide exceptional versatility for investigating complex quantum systems with disorder and interactions that can be precisely tuned, all under well-controlled and experimentally accessible conditions [27,28]. Despite these advances, understanding the MBL transition remains challenging [29,30]. Consequently, identifying well-defined MBL transition points is crucial, both for providing reliable theoretical benchmarks and for guiding future experiments in ultracold atomic and hybrid quantum optical systems [31,32].

To identify a sharp MBL transition within current computational and experimental capabilities, we propose the method illustrated in Figure 1, where the leftmost axis corresponds to the non-interacting system. Previous studies have shown that single-particle systems with quasi-periodic potentials, such as the Aubry–André–Harper (AAH) model, exhibit exact localization transition points due to self-duality [33–35], marked here by the green dot. Introducing interactions leads to the thermalization of non-localized and partially Anderson localized states [36–40], while strongly Anderson localized states remain stable and transform into MBL phases [8,41]. Intuitively, if non-localized states are thermalized while the thermalization of Anderson localized states is suppressed (yellow arrow with red cross), the MBL transition point becomes well-defined and coincides with the transition point of Anderson localization. Therefore, the focus of this method is on inducing thermalization of non-localized states while maintaining the stability of localized states. To implement this experimentally, we propose a feasible strategy suitable for cold-atom optical lattice platforms. That is, before introducing interactions, we first induce a sharp Anderson localization transition in the single-particle case to make the non-interacting system enter strongly Anderson localized phases rapidly after the localization transition. Here, “strongly” denotes states that are robust enough to suppress thermalization effectively (yellow arrow with red cross). Such a procedure can be realized using ultracold atoms trapped in optical lattices, where quasi-periodic potentials and hopping modulations can be

engineered with high precision, directly linking theoretical predictions with experimentally controllable systems.

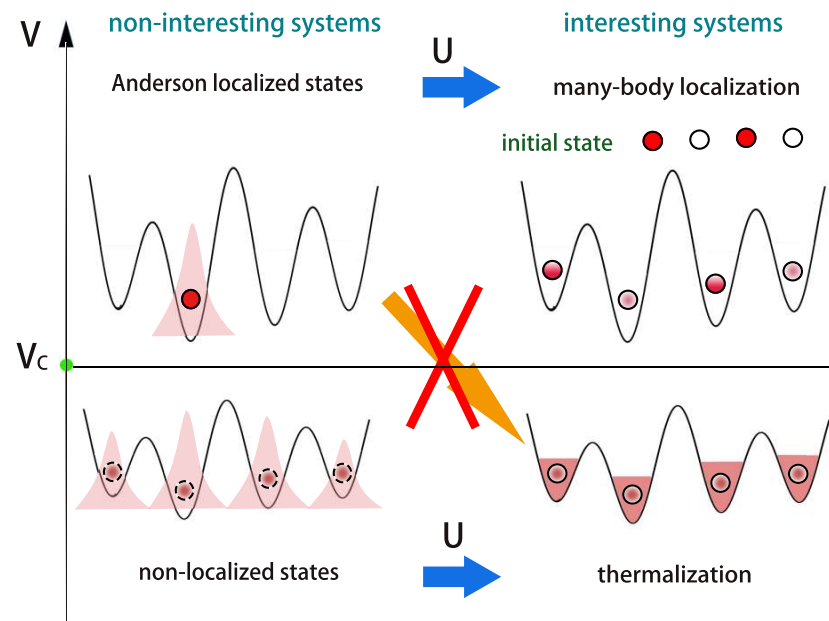


Figure 1. Schematic illustration of the transition from thermalization to MBL in a system of ultracold atoms trapped in an optical lattice. The MBL transition is induced by adding the interaction U to a non-interacting system with self-duality. The black curve represents a schematic of the optical lattice. The color inside the circles represents the probability of particles at the corresponding site. The more vivid the color, the higher the probability. In the absence of interactions, the shading illustrates the distribution of single-particle wave functions, while in the presence of interactions, the system performs time evolution with the Néel state as the initial state [8]. In the MBL phase, the system retains information of the initial state, whereas in the thermalization phase, the information of the initial state is erased. V_c indicates the Anderson localization transition point. The interaction not only induces the thermalization of non-localized and partially Anderson localized states, but also causes Anderson localization to be MBL. When the thermalization of Anderson localized states is blocked, the transition point of MBL is the same as that of Anderson localization.

2. Model

We investigate a many-body extended AAH model in a chain with L sites [42], which reads

$$\hat{H} = - \sum_j t_j (\hat{c}_j^\dagger \hat{c}_{j+1} + h.c.) + \sum_j V_j \hat{n}_j + \sum_j U \hat{n}_j \hat{n}_{j+1}, \quad (1)$$

where $\hat{c}_j^\dagger$ ($\hat{c}_j$) creates (annihilates) a spinless fermion at the site j and $\hat{n}_j = \hat{c}_j^\dagger \hat{c}_j$ is the fermion number operator. The hopping term $t_j = t + \mu \cos[2\pi\alpha(j + \frac{1}{2}) + \phi]$ and the on-site potential term $V_j = V \cos(2\pi\alpha j + \phi)$ have amplitudes t , μ and V , respectively. α is an irrational number and ϕ denotes the phase factor. U represents the strength of the nearest-neighbor interaction. In the non-interacting case, this model can be derived from a 2D Hofstadter model [43,44], which can be realized in ultracold atomic systems [45]. Equation (1) reduces to the AAH model for $U = 0$ and $\mu = 0$, where an extended-localized transition appears at $V_c/t = 2.0$ [33,34]. Here, V_c denotes the critical potential amplitude at which the localization transition occurs. By utilizing self-dual transformation, a critical-localized transition is found and the transition point is at $V_c/\mu = 2.0$ for $\mu > t$ [46]. All parameters in this model can be independently tuned in experiments by adjusting the laser intensity, Raman coupling strengths, or the depth of the main optical lattice. Thus, scanning

the dimensionless ratios V/μ or V/t across the critical values $V_c/\mu = 2.0$ or $V_c/t = 2.0$ provides a direct experimental route to probe the localization transition. Furthermore, this model is also a good platform for observing many-body critical phases [42].

To find a sharp Anderson localization transition, we introduce the inverse participation ratio (IPR), which is defined as $\text{IPR}^{(m)} = \sum_j |\psi_j^{(m)}|^4$ [35,47], where $\psi_j^{(m)}$ is the eigenstate with the energy level m and site j . After the localization transition, a larger value of IPR indicates that the wave function is more localized. In the many-body case, the core quantity we used is the averaged ratio of adjacent gaps $\langle r \rangle = \langle \min\{s_m, s_{m+1}\} / \max\{s_m, s_{m+1}\} \rangle$ [9,10,41], where $\langle \cdot \rangle$ means taking the average and $s_m = E_{m+1} - E_m$ is the energy gap between adjacent eigenvalues. In the framework of random matrix theory, $\langle r \rangle \approx 0.53$ corresponds to the Gaussian orthogonal ensemble (GOE) statistics characteristic of the thermalization phase, whereas $\langle r \rangle = 2 \ln 2 - 1 \approx 0.386$ follows Poisson statistics, indicating the MBL phase [6,48]. To study systems with large sizes, we adopt the shift-invert diagonalization method [49]. The target energy is fixed at $E = (E_{\max} + E_0)/2$, where E_0 and $E_{\max}$ denote the ground eigenvalue and the highest excited eigenvalue of the spectrum, respectively. In this work, we use 100 realizations at least and retain about 500 states near the target energy. Boundary conditions are set to be periodic boundary conditions. We fill $N = (L - 1)/2$ spinless fermions in a chain with an odd number of sites. This setting allows the system size to take Fibonacci numbers $L = 13$ and $L = 21$, which makes the potential manifest as an approximate cycle [50]. The Fibonacci numbers (e.g., $L = 13, 21$) are chosen because they provide the optimal rational approximants to the incommensurate frequency α (the inverse golden ratio $\alpha = (\sqrt{5} - 1)/2$), which minimizes finite-size mismatches under periodic boundary conditions. While these specific sizes are employed for numerical accuracy, the underlying physics, particularly the sharpness of the transition, is generic and applies to other system sizes as well, albeit with increased finite-size effects for off-Fibonacci numbers.

3. Sharp Anderson Localization Transition

In Figure 2a, we show the mean IPR over the entire spectrum as a function of V for different t and μ . We denote the energy unit by X . For $t = 1$ and $\mu = 0$, we take $X = t$ as the unit, whereas for $t = 0$ and $\mu = 1$, we take $X = \mu$ as the unit. It can be found that the case of $t = 0.0$ and $\mu = 1.0$ exhibits a sharper localization transition. This is also the case where the sharpest localization transition occurs for the combination of t and μ (IPR of other combinations not shown). Here, we use a system size of $L = 21$ and one can find that $\langle \text{IPR} \rangle$ changes drastically near the transition point at $V_c/\mu = 2$ for $t = 0.0$ and $\mu = 1.0$. In contrast, $\langle \text{IPR} \rangle$ for $t = 1.0$ and $\mu = 0.0$, corresponding to the AAH model, does not show abrupt behavior near the transition point for $L = 21$. This indicates that the AAH model requires a larger system size to clearly show the localization transition point, whereas the quasi-periodic hopping modulation can present the localization transition point at a smaller size. For potential amplitudes below 2, $\langle \text{IPR} \rangle$ for $t = 0.0$ and $\mu = 1.0$ is markedly larger than that for $t = 1.0$ and $\mu = 0.0$. This difference arises because the former undergoes a critical-localized transition [46], whereas the latter exhibits an extended-localized transition [33,34]. In Figure 2b, we show $\langle \text{IPR} \rangle$ as a function of ϕ for $V/\mu = 2.0$, which reveals a clear ϕ -dependence. In the following many-body section, we demonstrate that the value of $\langle \text{IPR} \rangle$ provides insight into whether critical states can be thermalized in finite systems. By fitting localized states with $\psi_j \propto e^{-|j-j_0|/\xi}$, where j_0 is the localization center, we extract the Anderson localization length ξ . Figure 2c,d show ξ as a function of V for $t = 1.0, \mu = 0.0$ and for $t = 0.0, \mu = 1.0$, respectively. For $t = 1.0$ and $\mu = 0.0$ [Figure 2c], the Anderson localization length is given by $\xi = 1 / \ln |V/2t|$, consistent with previous results for the AAH model [51]. For $t = 0.0$ and $\mu = 1.0$ [Figure 2d], the Anderson

localization length ξ can be derived from the Lyapunov exponent γ [52–54]. The detailed derivation is presented below.

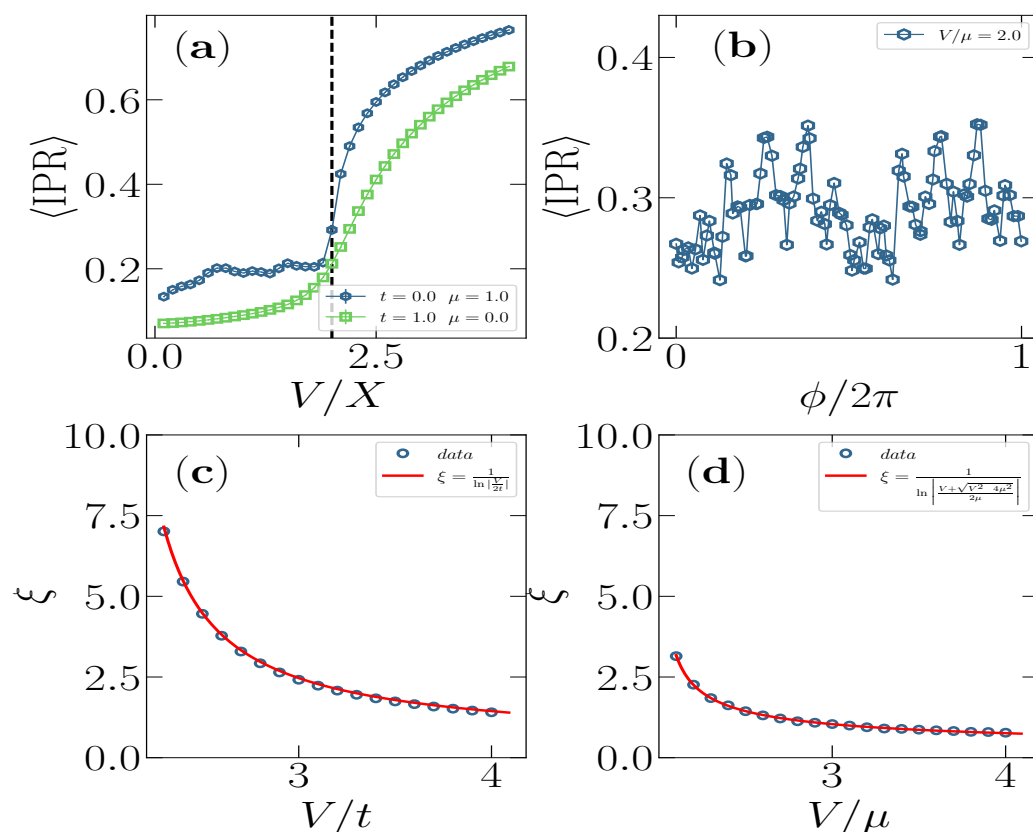


Figure 2. The mean IPR as a function of V for different t and μ in (a) and as a function of ϕ for $t = 0$, $\mu = 1$, and $V/\mu = 2.0$ in (b). Anderson localization length ξ as a function of V for $t = 1.0$, $\mu = 0.0$ in (c) and for $t = 0.0$, $\mu = 1.0$ in (d). In (a), the black dashed line indicates the Anderson localization transition point at $V/t = 2.0$ or $V/\mu = 2.0$. Here we use $L = 21$ in (a,b) and $L = 987$ in (c,d).

In Equation (1), the Hamiltonian for $t = 0.0$ and $\mu = 1.0$ can be rewritten as

$$-\mu \cos\left(2\pi\alpha\left(j + \frac{1}{2}\right) + \phi\right)\psi_{j+1} - \mu \cdot \cos\left(2\pi\alpha\left(j - \frac{1}{2}\right) + \phi\right)\psi_{j-1} + V \cos(2\pi\alpha j + \phi)\psi_j = E\psi_j, \quad (2)$$

where all labels are the same as those in Equation (1). It is the real-space Schrödinger equation for the single-particle Hamiltonian, corresponding to Equation (1) for $t = 0.0$ and $\mu = 1.0$. To analytically derive the localization length, we construct the transfer matrix T_j from Equation (2). The corresponding transfer matrix T_j satisfies

$$\begin{pmatrix} \psi_{j+1} \\ \psi_j \end{pmatrix} = T_j \begin{pmatrix} \psi_j \\ \psi_{j-1} \end{pmatrix} = \begin{pmatrix} \frac{E - V \cos(\theta + \phi)}{-\mu \cos(\theta + \phi + \delta)} & \frac{\cos(\theta + \phi - \delta)}{-\cos(\theta + \phi + \delta)} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_j \\ \psi_{j-1} \end{pmatrix}, \quad (3)$$

where $\theta = 2\pi\alpha j$ and $\delta = \pi\alpha$. T_j can be decomposed into two parts $T_j = A_j B_j$, where

$$A_j = \frac{1}{-\mu \cos(\theta + \phi + \delta)}, \quad (4)$$

and

$$B_j = \begin{pmatrix} E - V \cdot \cos(\theta + \phi) & \mu \cdot \cos(\theta + \phi - \delta) \\ -\mu \cdot \cos(\theta + \phi + \delta) & 0 \end{pmatrix}. \quad (5)$$

Now, the Lyapunov exponent reads $\gamma = \lim_{L \rightarrow \infty} \frac{1}{L} \ln \|\prod_{j=1}^L T_j\| = \gamma^A + \gamma^B$, where

$$\gamma^A = \lim_{L \rightarrow \infty} \frac{1}{L} \ln \left| \prod_{j=1}^L A_j \right|, \quad (6)$$

and

$$\gamma^B = \lim_{L \rightarrow \infty} \frac{1}{L} \ln \left| \prod_{j=1}^L B_j \right|. \quad (7)$$

Using ergodic theory, we map $(\theta + \phi + \delta) \rightarrow \phi$,

$$\begin{aligned} \gamma^A &= \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{j=1}^L \ln |A_j| \\ &= \frac{1}{2\pi} \int_0^{2\pi} \ln \frac{1}{|\mu \cos \phi|} d\phi \\ &= \ln \left| \frac{2}{\mu} \right|. \end{aligned} \quad (8)$$

To obtain γ^B , we complexify the phase $\phi \rightarrow \phi + i\varepsilon$ and let ε tend to infinity. The matrix B_j becomes

$$B_j(\phi + i\varepsilon) = e^{-i(\theta+\phi)} e^\varepsilon \begin{bmatrix} -\frac{V}{2} & \frac{\mu e^{i\delta}}{2} \\ \frac{-\mu e^{-i\delta}}{2} & 0 \end{bmatrix} + \mathcal{O}(1). \quad (9)$$

This gives $\gamma^B(\varepsilon) = \varepsilon + \ln \left| \frac{V + \sqrt{V^2 - 4\mu^2}}{4} \right|$. By the global theory [53,55], we obtain $\gamma^B = \gamma^B(\varepsilon = 0) = \ln \left| \frac{V + \sqrt{V^2 - 4\mu^2}}{4} \right|$. Thus, the Lyapunov exponent is given by

$$\gamma = \ln \left| \frac{V + \sqrt{V^2 - 4\mu^2}}{2\mu} \right|. \quad (10)$$

The vanishing of the Lyapunov exponent signals the localization transition, which occurs at $V_c/\mu = 2$. The Anderson localization length ξ is the reciprocal of the Lyapunov exponent, which reads

$$\xi = \frac{1}{\gamma} = \frac{1}{\ln \left| \frac{V + \sqrt{V^2 - 4\mu^2}}{2\mu} \right|}. \quad (11)$$

Then we set $V = V_c + \delta V$, with the localization transition point $V_c = 2\mu$, and obtain

$$\sqrt{V^2 - 4\mu^2} = \sqrt{(2\mu + \delta V)^2 - 4\mu^2} = \sqrt{4\mu \delta V + (\delta V)^2}. \quad (12)$$

Substituting this into γ gives

$$\gamma = \ln \left(1 + \frac{\delta V}{2\mu} + \sqrt{\frac{\delta V}{\mu} + \frac{(\delta V)^2}{4\mu^2}} \right). \quad (13)$$

For $\delta V \rightarrow 0^+$, γ becomes

$$\gamma \approx \ln \left(1 + \frac{\delta V}{2\mu} + \sqrt{\frac{\delta V}{\mu}} \right) \approx \sqrt{\frac{\delta V}{\mu}}. \quad (14)$$

Thus, we have

$$\gamma \sim \left| \frac{V - V_c}{\mu} \right|^{1/2}, \quad (15)$$

and the localization length diverges as

$$\xi \sim \left| \frac{V - V_c}{\mu} \right|^{-\nu} \quad \text{with} \quad \nu = \frac{1}{2}. \quad (16)$$

The critical exponent $\nu = \frac{1}{2}$ here is different from the result of the AAH model with $\nu = 1$ [51], indicating that the critical exponent is model-dependent.

4. Phase Diagram

Next, we focus on the case of $t = 0$ and $\mu = 1$ and introduce interactions. In Figure 3, we show a global $U - V$ phase diagram, in which the interaction varies from 0.01 to 6.0. In the non-interacting limit, the system undergoes a critical-localized transition as V increases and the transition point is at $V/\mu = 2.0$ [46] (black dashed line). The red markers indicate MBL transition points obtained from finite-size scaling analysis of $\langle r \rangle$ (typical examples are shown in Figures 4 and 5). Here, we simultaneously present both the single-particle and many-particle transition points in the phase diagram to highlight the influence of interactions on the localization transition point. Different from Ref. [42] in which critical states are transformed into many-body critical states, this work shows that critical states in the limit case of $t = 0$ are so sensitive that even a very weak interaction ($U/\mu = 0.01$) can thermalize them. This difference may arise from the fact that, in a finite system, efficiently incommensurately distributed zeros (IDZs) cannot be realized [54,56]. Here, IDZs are defined as spatial locations with zero-amplitude hoppings, which play a crucial role in the formation of critical properties in single-particle systems [54]. It is worth pointing out that very weak interactions ($U \ll 1$) are not a favorable condition for thermalization [38–40]. However, in this work, we find that even a weak interaction, e.g., $U/\mu = 0.01$, can effectively thermalize the critical states. For $0.1 \lesssim U/\mu \lesssim 5.0$, transition points of MBL are at $V/\mu \approx 2.3 \sim 2.5$, indicating that increased interaction does not significantly promote thermalization. It stems from the fact that Equation (1) provides a nice platform to realize that the non-interacting system enters strongly Anderson localized states rapidly after the localization transition. Stable Anderson localized states require large interactions to achieve thermalization, which results in strongly Anderson localized states that cannot be thermalized while the interaction is smaller than the critical one U_c (here, U_c is defined as a critical interaction strength, above which the interaction begins to promote thermalization). Furthermore, one can also find that MBL transition points are slightly larger than the Anderson localization transition point. This is due to the fact that the Anderson localized states near the transition point have larger localization lengths than those in the large- V region [51], making them more unstable and thus more easily thermalized by interactions. For large- U , the transition point where thermalization is disrupted decreases as the interaction increases, resulting from the strong interaction effect [39]. In the limit $U \rightarrow \infty$, strong interactions break the spectrum into different sectors, and Equation (1) maps onto a non-interacting Hamiltonian and the thermalization does not occur for any V [37–40].

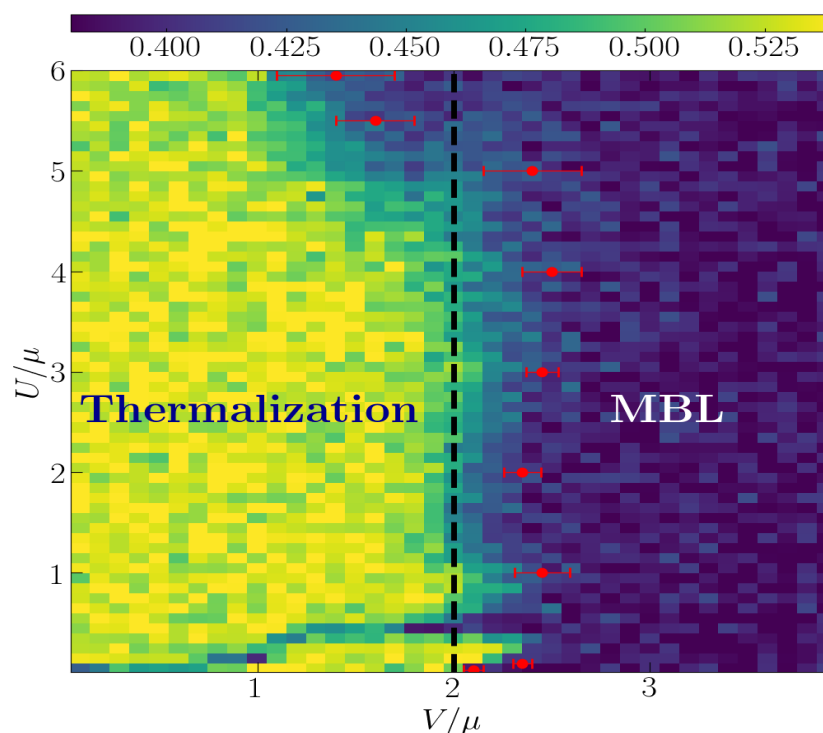


Figure 3. The $U - V$ phase diagram. Color represents the value of $\langle r \rangle$. The black dashed line marks the Anderson localization transition point at $V/\mu = 2.0$. Red markers indicate many-body localization transition points V_c drawn from the analysis of $\langle r \rangle$ for $L = 13 \sim 21$ by Ansatz $f[(V - V_c)L^{1/\nu}]$ and error-bars report an interval of confidence of V_c .

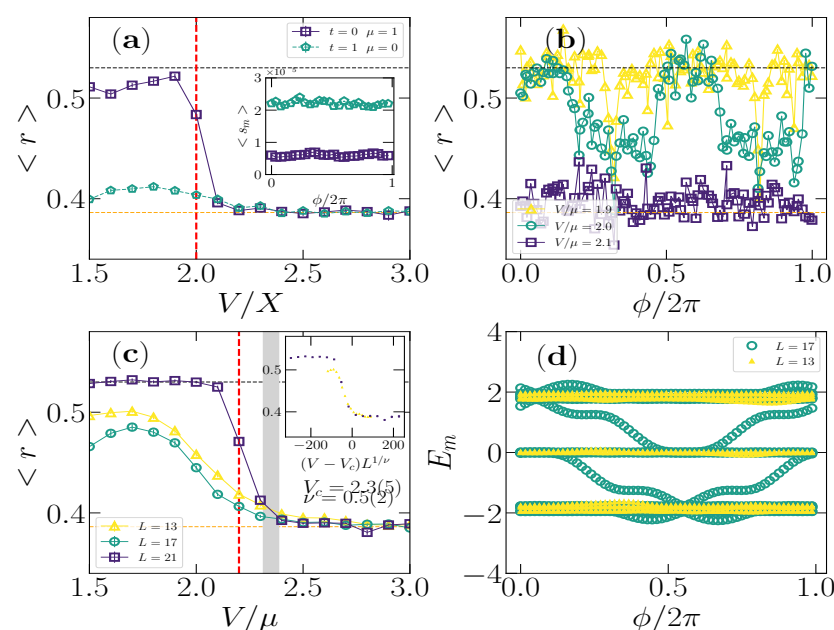


Figure 4. Statistical analysis for perturbative weak interactions. (a,c) Averaged ratio of adjacent gaps $\langle r \rangle$ as a function of V for $U/\mu = 0.01$ ($U/t = 0.01$) and $U/\mu = 0.1$, respectively. (b,d) Averaged ratio of adjacent gaps $\langle r \rangle$ and single-particle eigenvalues E_m as a function of the phase factor ϕ (100 samples), respectively. $L = 21$ in (a,b). $t = 0$ and $\mu = 1$ in (b–d). $U/\mu = 0.01$ and $V/\mu = 2.0$ in (b). $V/\mu = 1.8$ in (d). The insert in (a) shows the averaged energy gaps $\langle s_m \rangle$ for $V/\mu = 1.8$. The black and orange lines in (a–c) indicate $\langle r \rangle$ of GOE and Poisson statistics, respectively. The red dotted lines in (a,c) indicate the statistical mutation point of $L = 21$. In (c), the gray line marks the transition point obtained by the Ansatz $f[(V - V_c)L^{1/\nu}]$ from the data of $L = 13$ and $L = 21$, as shown in the insert.

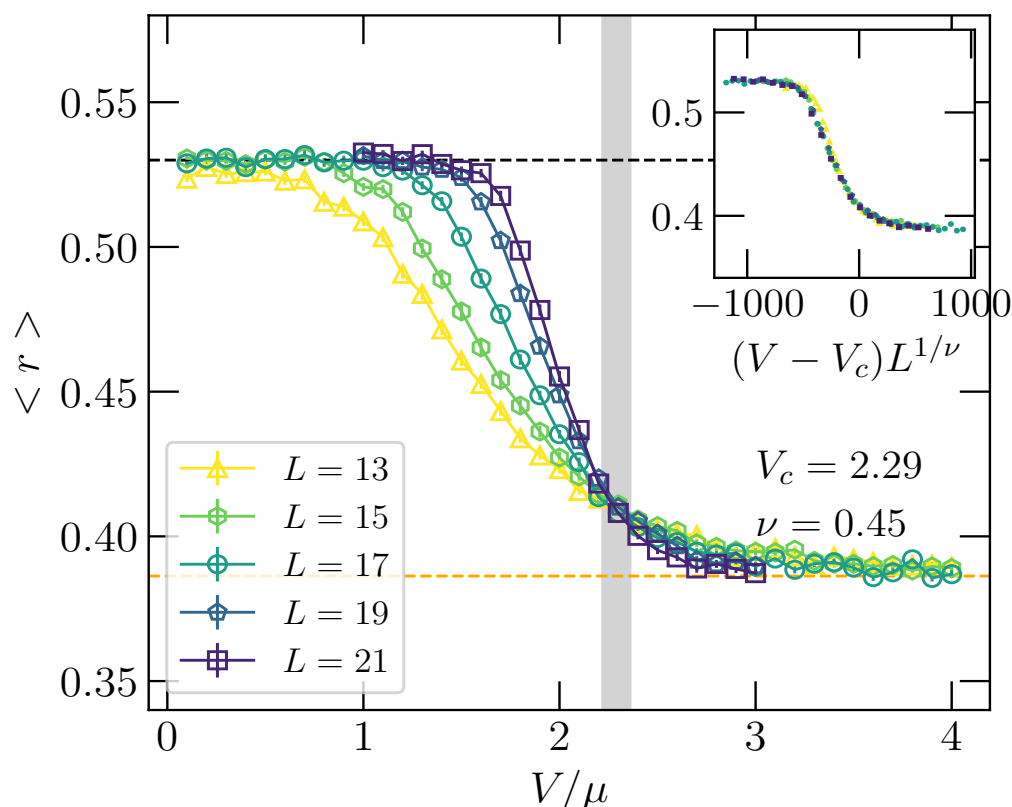


Figure 5. Averaged ratio of adjacent gaps $\langle r \rangle$ as a function of V for $U/\mu = 2.0$. The gray line indicates the transition point obtained from the Ansatz $f[(V - V_c)L^{1/\nu}]$, as shown in the inset. The black and orange lines indicate $\langle r \rangle$ of GOE and Poisson statistics, respectively.

5. Perturbative Weak Interactions for $U/\mu \ll 1$

In Figure 4a, we add the perturbative weak interaction with a strength of 0.01 to two non-interacting systems with an extended-localized transition ($t = 1$ and $\mu = 0$) and a critical-localized transition ($t = 0$ and $\mu = 1$). Intuitively, extended states tend to be more distributed in the whole space [50], thus they are more prone to thermalization than critical states [42,57]. However, Figure 4a shows a counterintuitive result, in which critical states are more easily thermalized than extended states. By investigating averaged energy gaps $\langle s_m \rangle$ at target energy in the inset of Figure 4a, we determine that this is due to the fact that critical states have relatively small energy gaps, which makes them prone to hybridization after the interaction perturbation is introduced. In Figure 4b, we focus on the case of $t = 0$ and $\mu = 1$. By carefully inspecting the statistics near $V/\mu \approx 2.0$, we find that the statistics for $V/\mu = 2.0$ are highly ϕ -dependent. The system exhibits thermalization with $\langle r \rangle \approx 0.53$ for some phase factors, while it shows non-thermalization with $\langle r \rangle < 0.53$ for other phase factors. The behavior of $\langle r \rangle$ is associated with $\langle \text{IPR} \rangle$ in Figure 2b. Specifically, the regions that do not experience thermalization correspond to those with higher $\langle \text{IPR} \rangle$ values. Continuing to keep $t = 0$ and $\mu = 1$, and slightly increasing the interaction to $U/\mu = 0.1$, the statistics for $L = 21$ in Figure 4c show that $\langle r \rangle \approx 0.53$ for $V/\mu = 2.0$ and $V/\mu = 2.1$. This indicates that the self-dual point and localized states close to it are thermalized. To find the critical amplitude V_c and the critical exponent ν , we use the finite-size scaling Ansatz for the level statistics $\langle r \rangle = f[(V - V_c)L^{1/\nu}]$, where f is an unknown universal scaling function. For each candidate pair of (V_c, ν) , we plot the data of $\langle r \rangle$ for different system sizes L against the scaled variable $x = (V - V_c)L^{1/\nu}$. The correct V_c and ν are determined by systematically tuning the parameters until all data points collapse onto a single universal curve (i.e., minimizing the spread between different L curves).

In Figure 4c, a transition point (gray line) occurs at $V/\mu = 2.3(5)$, which is slightly larger than the mutation point of statistics (red dotted line). A simple understanding is that this is due to the inability of localized states to be fully thermalized in small system sizes. In the thermodynamic limit, both should indicate the same point. Note that there is an anomaly in Figure 4c: contrary to the usual expectation that larger systems approach the GOE statistics more closely, the data for $L = 17$ deviate more from the GOE limit than those for $L = 13$. This counterintuitive behavior is attributed to the localized edge states emerging for $L = 17$ (see Figure 4d), which interfere with the statistical properties. The origin of the localized edge states can be proven by mapping Equation (1) for $U = 0$ into a two-dimensional model. The non-interacting Hamiltonian in Equation (1) is ϕ -dependent, thus it can be rewritten as

$$\hat{H}(\phi) = - \sum_j t_j(\phi) (\hat{c}_j^\dagger \hat{c}_{j+1} + h.c.) + \sum_j V_j(\phi) \hat{n}_j. \quad (17)$$

We define a 2D Hamiltonian $\hat{H}^{2D} = \int_0^{2\pi} (d\phi/2\pi) \hat{H}(\phi)$ [45], in which the operator $\hat{c}_j$ is replaced by $\hat{c}_{j,\phi}$. Now, the hopping term with $t = 0$ in Equation (1) reads

$$\begin{aligned} \hat{H}_t(\phi) &= - \sum_{j,\phi} t_j(\phi) (\hat{c}_{j,\phi}^\dagger \hat{c}_{j+1,\phi} + h.c.) \\ &= - \mu \sum_{j,\phi} \cos \left[2\pi\alpha \left(j + \frac{1}{2} \right) + \phi \right] \hat{c}_{j,\phi}^\dagger \hat{c}_{j+1,\phi} + h.c. \\ &= - \frac{\mu}{2} \sum_{j,\phi} (e^{i(2\pi\alpha)(j+\frac{1}{2})} e^{i\phi} + h.c.) \hat{c}_{j,\phi}^\dagger \hat{c}_{j+1,\phi} + h.c. \end{aligned} \quad (18)$$

By applying the Fourier transform $\hat{c}_{j,\phi} = \sum_m e^{-i\phi m} \hat{c}_{j,m}$ to the hopping term, we obtain

$$\begin{aligned} \hat{H}_t^{2D} &= - \frac{\mu}{2} \sum_{j,m} e^{i(2\pi\alpha)(j+\frac{1}{2})} \hat{c}_{j,m}^\dagger \hat{c}_{j+1,m+1} \\ &\quad - \frac{\mu}{2} \sum_{j,m} e^{-i(2\pi\alpha)(j+\frac{1}{2})} \hat{c}_{j,m+1}^\dagger \hat{c}_{j+1,m} \\ &\quad + h.c. \end{aligned} \quad (19)$$

Similarly, the on-site potential term reads

$$\begin{aligned} \hat{H}_v(\phi) &= V \sum_{j,\phi} \cos(2\pi\alpha j + \phi) \hat{n}_{j,\phi} \\ &= \frac{V}{2} \sum_{j,\phi} (e^{i(2\pi\alpha j)} e^{i\phi} + h.c.) \hat{c}_{j,\phi}^\dagger \hat{c}_{j,\phi}, \end{aligned} \quad (20)$$

after the Fourier transform, it becomes

$$\hat{H}_v^{2D} = \frac{V}{2} \sum_{j,m} e^{i(2\pi\alpha j)} \hat{c}_{j,m}^\dagger \hat{c}_{j,m+1} + h.c. \quad (21)$$

Finally, we obtain

$$\begin{aligned} \hat{H}_{2D} &= - \frac{\mu}{2} \sum_{j,m} e^{i(2\pi\alpha)(j+\frac{1}{2})} \hat{c}_{j,m}^\dagger \hat{c}_{j+1,m+1} + h.c. \\ &\quad - \frac{\mu}{2} \sum_{j,m} e^{-i(2\pi\alpha)(j+\frac{1}{2})} \hat{c}_{j,m+1}^\dagger \hat{c}_{j+1,m} + h.c. \\ &\quad + \frac{V}{2} \sum_{j,m} e^{i(2\pi\alpha j)} \hat{c}_{j,m}^\dagger \hat{c}_{j,m+1} + h.c. \end{aligned} \quad (22)$$

This is a two-dimensional model with magnetic flux, which is strictly equivalent to Equation (1) for $t = 0$ and $U = 0$. In Equation (22), the magnetic flux is encoded in the complex Peierls phase factors $e^{\pm i(2\pi\alpha j)}$ [43,44], accompanying the hopping terms along both j - and m - directions. These position-dependent phases accumulate around each plaquette, yielding a uniform effective magnetic flux determined by α . This establishes the exact mapping between our 1D incommensurate model and a 2D Hofstadter-type model under a uniform magnetic field. This 2D model can form localized edge states when L is not approximately equal to the period of the magnetic field. To show the localized edge states, we plot the single-particle eigenvalues as a function of ϕ for $L = 13$ and $L = 17$ in Figure 4d, in which one can find clear localized edge states between different sub-bands for $L = 17$. Conversely, For $L = 13$, when the lattice length is a Fibonacci number approximately matching the period of the magnetic field, no obvious localized edge states are observed.

6. Strong Interactions for $U \sim \mathcal{O}(1)$

By further increasing the interaction strength, the anomaly observed in Figure 4c disappears, as shown in Figure 5. This suggests that the localized edge states displayed in Figure 4d have become thermalized. Comparing Figure 4a,c with Figure 5, we also note that $\langle r \rangle$ does not exhibit a sharp jump near the localization transition point. This indicates that, as interactions strengthen, larger system sizes are required to observe discontinuities in $\langle r \rangle$. By using the Ansatz $f[(V - V_c)L^{1/\nu}]$, Figure 5 identifies a transition point at $V/\mu = 2.29$, which is close to that in Figure 4c, demonstrating that increasing the interaction strength has little effect on the transition point. In Figure 6a–c, we also show the MBL transition points for other cases with $U \sim \mathcal{O}(1)$. All cases indicate that the MBL transition occurs at $V/\mu \approx 2.3$ – 2.5 . This behavior contrasts with previous studies, which reported that increasing interactions can significantly shift the MBL transition point [37–40]. In those systems, there is a competition between interaction and the disordered potential. The stronger the interaction, the more easily the localized states are destroyed, and consequently the critical amplitude for the occurrence of MBL increases with interaction strength. In our work, however, for the single-particle case, the system rapidly enters the strongly Anderson localized regime immediately after the localization transition. Stable strongly Anderson localized states require sufficiently strong interactions to enable thermalization. As a result, interactions do not significantly promote thermalization. Notably, in our work, within the scaling framework of the Ansatz $f[(V - V_c)L^{1/\nu}]$, the critical exponent is $\nu \approx 0.5$, differing from $\nu \approx 1$ reported elsewhere [6]. It is worth noting that for the single-particle case, the critical exponent of the present model is half that of the AA model in Equation (16). Upon introducing interactions, the critical exponent remains approximately half that of the many-body AA model. When the interaction is further strengthened, strong interactions disrupt the spectrum structure, leading to a lower transition point. A typical example is shown in Figure 6d for $U/\mu = 5.5$, where the localization transition occurs at $V/\mu \approx 1.7$, smaller than the transition points observed in Figures 5 and 6a–c.

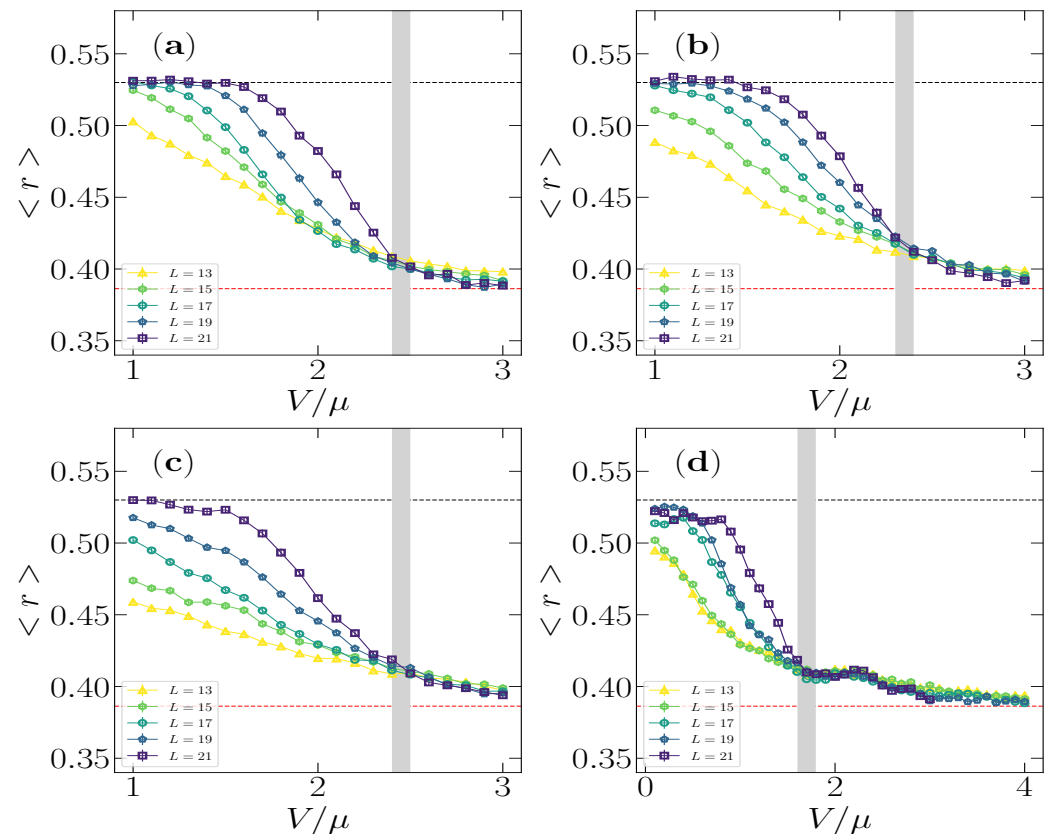


Figure 6. Averaged ratio of adjacent gaps $\langle r \rangle$ as a function of V for $U/\mu = 1.0$ in (a), $U/\mu = 3.0$ in (b), $U/\mu = 4.0$ in (c) and $U/\mu = 5.5$ in (d). Other parameters are the same as those in Figure 5.

7. Conclusions

In this work, we study the localization transition in the one-dimensional extended AAH model. By inducing a sharp Anderson localization transition in a single-particle model, we successfully realize a correspondingly sharp MBL transition upon the introduction of interactions. As the interaction strength increases, the sharpness of the transition gradually diminishes. However, within the parameter range $U/\mu < 5$, the MBL transition point remains largely unaffected. This robustness arises because strong Anderson localization requires sufficiently large interactions to be destabilized. When the interaction strength is below this critical threshold, the localization transition point is not significantly perturbed. Furthermore, our scaling analysis reveals that the critical exponent is approximately $\nu = 0.5$ for both the single-particle and many-body models, indicating a universal behavior in the transition.

These findings provide new insights into the interplay between disorder and interactions, highlighting that sharp MBL transitions can be engineered and remain robust under experimentally relevant interaction strengths. Importantly, our results are directly applicable to quantum optics platforms, including ultracold atoms in optical lattices, photonic lattices in coupled waveguide arrays, and cavity QED or Rydberg atom arrays, where disorder, tunneling, and interactions can be precisely engineered via laser fields and external driving. In such systems, localization physics manifests as controllable suppression of wavepacket spreading or photonic transport due to engineered interference, offering a direct optical analog of the AAH and MBL transitions. This enables experimental access to sharp localization phenomena through measurements of coherence decay, intensity correlations, and matter-wave interference patterns. Looking forward, coherent laser driving and Floquet engineering provide additional quantum optical knobs to extend the acces-

sible parameter space, enabling the study of driven localization, synthetic gauge fields, and spin-dependent interactions in disordered settings. Moreover, controlled dissipation in cavity-mediated or open quantum optical systems can be used to probe the interplay between decoherence and localization, offering a versatile platform for investigating the fundamental mechanisms governing MBL and its breakdown in highly tunable, optically controlled quantum many-body systems.

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