

## Article

# Mass Spectrum of Noncharmed and Charmed Meson States in Extended Linear-Sigma Model

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**Abstract:** The mass spectrum of different meson particles is generated using an effective Lagrangian of the extended linear-sigma model (eLSM) for scalar and pseudoscalar meson fields and quark flavors, up, down, strange, and charm. Analytical formulas for the masses of scalar, pseudoscalar, vector, and axialvector meson states are derived assuming global chiral symmetry. The various eLSM parameters are analytically deduced and numerically computed. This enables accurate estimations of the masses of sixteen noncharmed and thirteen charmed meson states at vanishing temperature. The comparison of these results to a recent compilation of the particle data group (PDG) allows us to draw the conclusion that the masses of sixteen noncharmed and thirteen charmed meson states calculated in the eLSM are in good agreement with the PDG. This shows that the eLSM, with its configurations and parameters, is an effective theoretical framework for determining the mass spectra of various noncharmed and charmed meson states, particularly at vanishing temperature.

**Keywords:** Chiral Lagrangian; sigma model; charmed mesons



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## 1. Introduction

Prior to the discovery of quantum chromodynamics (QCD), the theory of strong interactions, Gell-Mann and Levy proposed the linear-sigma model (LSM) as a means to introduce a field represented by a point particle. This field is confined within a fixed manifold and describes the interactions of pions [1]. The field itself corresponds to a spinless sigma, the scalar meson. Several studies have been performed utilizing LSM, like  $O(4)$  LSM [1]. The utilization of LSM, which incorporates quark degrees of freedom, is made possible by the extension that incorporates the dynamical realization of the pseudoscalar and scalar mesons as a linear representation of chiral symmetry which is weakly broken by current quark masses. As a result, the model can be utilized as an effective QCD-like model [2–7].

Although QCD can be solved perturbatively at very high energies, only approximate numerical solutions are feasible in the nonperturbative regime [8]. Despite the significant computational expenses demanded, lattice QCD simulations do not appear to be reliable, especially at finite densities. Consequently, the utilization of effective QCD-like models like eLSM becomes necessary due to their shared global symmetries with QCD and low computational efforts. The chiral symmetry is widely regarded as the first approximation in comprehending the composition of hadrons. Four decades ago, the spin-zero mass spectrum and leptonic decay constants were calculated in the one-loop approximation of the  $SU(4)$  linear-sigma model [6]. eLSM explored the phenomenology of charmed mesons [9]. Recently, the quark-hadron phase structure was explored at limited temperatures and densities using the mean-field approximation of the  $SU(4)$  Polyakov linear-sigma model (PLSM) [10]. The assumption is that the  $N_f = 4$  Lagrangian is comparable with the  $N_f = 3$  Lagrangian. An extensive examination employing  $SU(3)$  PLSM has been undertaken to

analyze the thermodynamics, phase structure, and meson masses of QCD in thermal and dense medium [11,12], and finite magnetic fields [13–16]. The masses of various meson states have been calculated [17–19]. Not only baryon density but also finite isospin asymmetry were also analyzed [20]. A systematic comparison between the mean-field approximation and optimized perturbation theory has been reported as well [21]. These studies encompassed a wide range of conditions and yielded valuable insights into these fundamental aspects of QCD.

In this regard, we recall that the Lagrangian of the gauge theory for color interactions of quarks and gluons, QCD, is invariant under local color transformations so that the physical content remains invariant if the colors of quarks and gluons are transformed. The interactions themselves are flavor-blind, on the other hand. In eLSM, the global chiral symmetry is explicitly violated by nonvanishing quark masses and quantum effects [22], and spontaneously disrupted by the nonvanishing expectation value of the quark condensate in the QCD vacuum. As previously mentioned, the eLSM framework is considered an effective approach in examining different QCD symmetries, including (i) the isospin symmetry, which is a global transformation referring to SU(2) rotation in flavor space of up and down quarks, where the Lagrangian is invariant for identical or vanishing masses [17,20,23]; (ii) the global chiral symmetry, which is exact in the chiral limit of massless QCD, i.e., left- and right-quarks become degenerate; iii) the chiral symmetry is broken by QCD vacuum properties, such as the Higgs mechanism, where pions are the lightest Goldstone bosons of the broken symmetry [18,24]; (iv) discrete  $C$ ,  $P$ , and  $T$  symmetries [9]; and (v) classical dilatation (scale) symmetry [25].

The present study utilizes an extended linear-sigma model (eLSM) in an effective Lagrangian in which four quark flavors along with scalar and pseudoscalar meson fields are included [10,26,27]. To ensure accurate results, the integration of Polyakov-loop potential is essential, which can be derived from pure Yang–Mills lattice simulations [28–30]. We assume that such a configuration allows for the proper estimation of various meson states  $\langle \bar{q}q \rangle = \langle \bar{q}_\ell q_r - \bar{q}_r q_\ell \rangle \neq 0$  [31]. These meson states, characterized by their chiral structures, can be categorized based on specific quantum numbers such as orbital angular momenta  $J$ , parity  $P$ , and charge conjugates  $C$ . This classification results in scalar mesons with  $J^{PC} = 0^{++}$ , pseudoscalar mesons with  $J^{PC} = 0^{-+}$ , vector mesons with  $J^{PC} = 1^{--}$ , and axialvector mesons with  $J^{PC} = 1^{++}$  [15,19]. The present study targets a systematic analysis of various noncharmed and charmed meson states. The effective Lagrangian is utilized to describe the properties of low-lying meson states that are noncharmed and charmed, determining their mass spectra at vanishing temperature.

At a vanishing temperature, the masses of various meson states are well measured [32]. Their excellent reproduction by means of eLSM allows us to determine its parameters so that the meson masses in thermal and dense medium can then be calculated.

The color gauge field theory, in which a linear binding potential with one-gluon exchange corrections and four quark flavors are coupled, was utilized to determine the meson masses [33]. A considerable agreement with the experimental data is concluded, especially for mesons of masses larger than 1 GeV [33–35]. In the two-flavor linear sigma model, the meson vacuum phenomenology was studied and it was concluded that the inclusion of additional scalar degrees of freedom is necessary [36]. Within the  $U(3) \times U(3)$  linear sigma model, the meson properties were studied and it was concluded that the  $U_A(1)$ -breaking term plays an important role in the generation of meson masses [37].

The script is arranged as follows. Section 2 describes the formalism of the SU(4) extended linear-sigma model's mesonic component of the Lagrangian. Section 3 presents our findings on the mass spectra of noncharmed and charmed meson states. Section 4 discusses the overall perspective.

## 2. Mesonic Lagrangian of Extended Linear-Sigma Model

The extended linear-sigma model (eLSM) incorporates chiral symmetry in a linear representation [1]. The nonlinear representation only considers Goldstone bosons but

not vector mesons [38,39]. The linear representation also allows us to examine scalar Goldstone bosons, but its expansion enables the introduction of vector and axialvector mesons. As mentioned in the introduction, eLSM considers chiral symmetry among other QCD symmetries [40].

We assume that the Lagrangian for  $N_f = 4$  with global chiral invariance [10] is similar to the comparable Lagrangian for  $N_f = 3$  [41]. Only for  $N_f = 4$ , the mass term  $-2 \text{Tr}[\epsilon \Phi^\dagger \Phi]$  must be included [22]. The motivation of the mass term can be realized from the equivalence of  $SU(4) \times SU(4)$  and  $SU(3) \times SU(3)$  symmetry breaking Hamiltonian [42,43]. The Lagrangian of the mesonic sector, comprising scalar, pseudoscalar, vector, and axialvector meson states, as well as scalar glueball, interactions, and anomalies, is constructed as follows [15,19]:

$$\mathcal{L}_m = \mathcal{L}_{ps} + \mathcal{L}_{av} + \mathcal{L}_{int} + \mathcal{L}_{anomaly} + \mathcal{L}_{dilaton} + \mathcal{L}_{emass}, \quad (1)$$

where

$$\begin{aligned} \mathcal{L}_{ps} = & \text{Tr} \left[ (\mathcal{D}^\mu \Phi)^\dagger (\mathcal{D}^\mu \Phi) - m_0^2 \left( \frac{G}{G_0} \right)^2 \text{Tr}(\Phi^\dagger \Phi) \right] - \lambda_1 [\text{Tr}(\Phi^\dagger \Phi)]^2 \\ & - \lambda_2 \text{Tr}[(\Phi^\dagger \Phi)]^2 + \text{Tr}[H(\Phi + \Phi^\dagger)], \end{aligned} \quad (2)$$

$$\begin{aligned} \mathcal{L}_{av} = & -\frac{1}{4} \text{Tr}[(L^{\mu\nu})^2 + (R^{\mu\nu})^2] + \text{Tr} \left\{ \left[ \left( \frac{G}{G_0} \right)^2 \frac{m^2}{2} + \Delta \right] [(L^\mu)^2 + (R^\mu)^2] \right\} \\ & - 2ig_2 \{ \text{Tr}(L_{\mu\nu}[L^\mu, L^\nu]) + \text{Tr}(R_{\mu\nu}[R^\mu, R^\nu]) \}, \end{aligned} \quad (3)$$

$$\begin{aligned} \mathcal{L}_{int} = & \frac{h_1}{2} \text{Tr}(\Phi^\dagger \Phi) \text{Tr}[(L^\mu)^2 + (R^\mu)^2] + h_2 \text{Tr}[(R^\mu)^2 \Phi + (L^\mu \Phi)^2] \\ & + 2h_3 \text{Tr}(L^\mu \Phi R_\mu \Phi^\dagger), \end{aligned} \quad (4)$$

$$\mathcal{L}_{anomaly} = C(\det \Phi + \det \Phi^\dagger)^2 + iC_{\tilde{G}} \tilde{G}(\det \Phi + \det \Phi^\dagger), \quad (5)$$

$$\mathcal{L}_{dilaton} = \frac{1}{2} (\mathcal{D}_\mu G)^2 - \frac{1}{4} \frac{m_G^2}{\Lambda_G^2} \left( G^4 \ln \frac{G}{\Lambda_G} - \frac{G^4}{4} \right), \quad (6)$$

$$\mathcal{L}_{emass} = -2 \text{Tr}[\epsilon(\Phi^\dagger \Phi)], \quad (7)$$

where  $G$  is the scalar glueball, while  $\tilde{G}$  is the pseudoscalar glueball.  $C$  and  $C_{\tilde{G}}$  are constants introduced to improve the fit of mesons and glueballs, respectively. The lowest-lying glueball mass,  $m_G$ , is found in the quenched approximation, with no quarks [44].  $\Lambda$  represents the QCD scaling parameter. The dilaton Lagrangian is thought to replicate the QCD trace anomaly. The field  $\Phi$  is a complex  $N_f \times N_f$  matrix for scalar  $\sigma_a$  with  $J^{PC} = 0^{++}$ , pseudoscalar  $\pi_a$  with  $J^{PC} = 0^{-+}$ , vector with  $J^{PC} = 0^{-+}$ , and axialvector mesons with  $J^{PC} = 0^{++}$  (Appendix B),

$$\Phi = \sum_{a=0}^{N_f^2-1} T_a (\sigma_a + i\pi_a), \quad (8)$$

where the scalar mesons are given as

$$T_a \sigma_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sigma_0}{2} + \frac{\sigma_3}{\sqrt{2}} + \frac{\sigma_8}{\sqrt{6}} + \frac{\sigma_{15}}{2\sqrt{3}} & \frac{\sigma_1 - i\sigma_2}{\sqrt{2}} & \frac{\sigma_4 - i\sigma_5}{\sqrt{2}} & \frac{\sigma_9 - i\sigma_{10}}{\sqrt{2}} \\ \frac{\sigma_1 + i\sigma_2}{\sqrt{2}} & \frac{\sigma_0}{\sqrt{2}} - \frac{\sigma_3}{\sqrt{2}} + \frac{\sigma_8}{\sqrt{6}} + \frac{\sigma_{15}}{2\sqrt{3}} & \frac{\sigma_6 - i\sigma_7}{\sqrt{2}} & \frac{\sigma_{11} - i\sigma_{12}}{\sqrt{2}} \\ \frac{\sigma_4 + i\sigma_5}{\sqrt{2}} & \frac{\sigma_6 + i\sigma_7}{\sqrt{2}} & \frac{\sigma_0}{2} - \sqrt{\frac{2}{3}}\sigma_8 + \frac{\sigma_{15}}{2\sqrt{3}} & \frac{\sigma_{13} - i\sigma_{14}}{\sqrt{2}} \\ \frac{\sigma_9 + i\sigma_{10}}{\sqrt{2}} & \frac{\sigma_{11} + i\sigma_{12}}{\sqrt{2}} & \frac{\sigma_{13} + i\sigma_{14}}{\sqrt{2}} & \frac{\sigma_0}{2} - \frac{\sqrt{3}}{2}\sigma_{15} \end{pmatrix}. \quad (9)$$

Similarly, the pseudo-scalar mesons become

$$T_a \pi_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi_0}{2} + \frac{\pi_3}{\sqrt{2}} + \frac{\pi_8}{\sqrt{6}} + \frac{\pi_{15}}{2\sqrt{3}} & \frac{\pi_1 - i\pi_2}{\sqrt{2}} & \frac{\pi_4 - i\pi_5}{\sqrt{2}} & \frac{\pi_9 - i\pi_{10}}{\sqrt{2}} \\ \frac{\pi_1 + i\pi_2}{\sqrt{2}} & \frac{\pi_0}{\sqrt{2}} - \frac{\pi_3}{\sqrt{2}} + \frac{\pi_8}{\sqrt{6}} + \frac{\pi_{15}}{2\sqrt{3}} & \frac{\pi_6 - i\pi_7}{\sqrt{2}} & \frac{\pi_{11} - i\pi_{12}}{\sqrt{2}} \\ \frac{\pi_4 + i\pi_5}{\sqrt{2}} & \frac{\pi_6 + i\pi_7}{\sqrt{2}} & \frac{\pi_0}{2} - \sqrt{\frac{2}{3}}\pi_8 + \frac{\pi_{15}}{2\sqrt{3}} & \frac{\pi_{13} - i\pi_{14}}{\sqrt{2}} \\ \frac{\pi_9 + i\pi_{10}}{\sqrt{2}} & \frac{\pi_{11} + i\pi_{12}}{\sqrt{2}} & \frac{\pi_{13} + i\pi_{14}}{\sqrt{2}} & \frac{\pi_0}{2} - \frac{\sqrt{3}}{2}\pi_{15} \end{pmatrix}. \quad (10)$$

Nonvanishing external field matrices  $H$ ,  $\Delta$ , and  $\epsilon$  clearly break the chiral symmetry:

$$H = \sum_{a=0}^{N_f^2-1} h_a T_a = h_0 T_0 + h_8 T_8 + h_{15} T_{15}, \quad (11)$$

$$\Delta = \sum_{a=0}^{N_f^2-1} h_a \delta_a = h_0 \delta_0 + h_8 \delta_{15} + h_a \delta_{15}, \quad (12)$$

$$\epsilon = \epsilon_c = m_c^2 = \frac{1}{2} \left[ m_{\chi c 0}^2 - m_0^2 - \lambda_1 (\sigma_x^2 + \sigma_y^2) - 3\sigma_c^2 (\lambda_1 + \lambda_2) \right]. \quad (13)$$

As the matrix  $H$  must be diagonal, the large standard deviations for some mesons, especially pions, as we shall identify, can be grasped from Equation (1). The generators of the group  $U(N_f)$  are  $T_a = \lambda_a/2$ , where  $\lambda_a$  are the Gell-Mann matrices (see Appendix A. In Equations (11) and (12), only three of the 15 terms resulting from the sum are specified. The reason for this is that the matrix  $H$  was selected to be diagonal. Appendix B will include the whole matrix  $H$ .

In the  $SU(4)_\ell \times SU(4)_r$  model, the quark condensates are given as

$$\sigma_x = \frac{\sigma_0}{\sqrt{2}} + \frac{\sigma_8}{\sqrt{3}} + \frac{\sigma_{15}}{\sqrt{6}}, \quad (14)$$

$$\sigma_y = \frac{\sigma_0}{2} - \sqrt{\frac{2}{3}}\sigma_8 + \frac{1}{2\sqrt{3}}\sigma_{15}, \quad (15)$$

$$\sigma_c = \frac{\sigma_0}{2} + \frac{\sqrt{3}}{2}\sigma_{15}, \quad (16)$$

where  $\sigma_x$  represents the condensate of light quarks (up and down), whereas  $\sigma_y$  represents the strange quark condensate.  $\sigma_c$  represents the charm quark condensates. The complex  $N_f \times N_f$  matrix for scalar  $J^{PC} = 0^{++}$ , pseudoscalar  $J^{PC} = 0^{-+}$ , vector  $J^{PC} = 0^{--}$ , and axialvector mesons  $J^{PC} = 0^{++}$  corresponds to  $\Phi$  (Equation (A2)). Using  $m_0 = (g/2)\Phi$ , where  $g$  represents the Yukawa coupling, the quark masses can be related to the quark condensates:

$$m_u = \frac{g}{2} \left[ \frac{\sigma_0}{\sqrt{2}} + \frac{\sigma_8}{\sqrt{3}} + \frac{\sigma_{15}}{\sqrt{6}} \right] = \frac{g}{2} \sigma_x, \quad (17)$$

$$m_d = \frac{g}{2} \left[ \frac{\sigma_0}{\sqrt{2}} + \frac{\sigma_8}{\sqrt{3}} + \frac{\sigma_{15}}{\sqrt{6}} \right] = \frac{g}{2} \sigma_x, \quad (18)$$

$$m_s = \frac{g}{2} \left[ \frac{\sigma_0}{\sqrt{2}} - \frac{2\sigma_8}{\sqrt{3}} + \frac{\sigma_{15}}{\sqrt{6}} \right] = \frac{g}{\sqrt{2}} \sigma_y, \quad (19)$$

$$m_c = \frac{g}{2} \left[ \frac{\sigma_0}{\sqrt{2}} - \sqrt{\frac{3}{2}} \sigma_{15} \right] = \frac{g}{\sqrt{2}} \sigma_c. \quad (20)$$

The covariant derivative of the scalar mesons is expressed as

$$\mathcal{D}^\mu \Phi = \delta_\mu \Phi - ig_1 (L^\mu \Phi - \Phi R^\mu) - ie A^\mu [T_3, \Phi], \quad (21)$$

where  $A_\mu = g A_\mu^a \lambda^a / 2$  is the electromagnetic field. For vector and axialvector meson nonets, we have

$$L^{\mu\nu} = \delta_\mu L^\nu - ie A^\mu [T_3, L^\nu] - \{\delta^\nu \mathcal{L}^\mu - ie A^\nu [T_3, L^\mu]\}, \quad (22)$$

$$R^{\mu\nu} = \delta_\mu R^\nu - ie A^\mu [T_3, R^\nu] - \{\delta^\nu R^\mu - ie A^\nu [T_3, R^\mu]\}, \quad (23)$$

where  $L^\mu = \sum_{a=0}^{N_f^2-1} T_a (V_a^\mu + A_a^\mu)$  and  $R^\mu = \sum_{a=0}^{N_f^2-1} T_a (V_a^\mu - A_a^\mu)$ . The remaining quantities can be deduced as follows:

$$\text{Tr}(\Phi^\dagger \Phi) = \frac{1}{4} [(\sigma_x)^2 + (\sigma_y)^2 + (\sigma_c)^2], \quad (24)$$

$$\text{Tr}[\epsilon \Phi^\dagger \Phi] = \frac{1}{2} \epsilon_c (\sigma_c)^2, \quad (25)$$

$$\text{Tr}[H(\Phi^\dagger + \Phi)] = h_x \sigma_x + h_y \sigma_y + h_c \sigma_c, \quad (26)$$

$$C(\det \Phi + \det \Phi^\dagger) = \frac{C}{4} (\sigma_x)^2 \sigma_y \sigma_c, \quad (27)$$

$$[\text{Tr}(\Phi^\dagger \Phi)]^2 = \frac{1}{4} [(\sigma_x)^4 + (\sigma_y)^4 + (\sigma_c)^4 + 2(\sigma_x)^2 (\sigma_y)^2 + 2(\sigma_y)^2 (\sigma_c)^2 + 2(\sigma_x)^2 (\sigma_c)^2]. \quad (28)$$

From Equations (9) and (10), we can now determine various meson states,

$$\begin{aligned} a_0^\pm &\equiv \frac{\sigma_1 \mp i\sigma_2}{\sqrt{2}}, & a_0^0 &\equiv \sigma_3, \\ k^\pm &\equiv \frac{\pi_4 \mp i\pi_5}{\sqrt{2}}, & k^0 &\equiv \pi_6 - i\pi_7, & \bar{k}^0 &\equiv \frac{\pi_6 + i\pi_7}{\sqrt{2}}, \\ \sigma_s &= \frac{\sigma_0}{2} - \sqrt{\frac{2}{3}} \sigma_8 + \frac{\sigma_{15}}{2\sqrt{3}}, & \chi_{co} &= \frac{\sigma_0}{2} - \frac{\sqrt{3}}{2} \sigma_{15}, \\ \eta_s &= \frac{\pi_0}{2} - \sqrt{\frac{2}{3}} \pi_8 + \frac{\pi_{15}}{2\sqrt{3}}, & \eta_c &= \frac{\pi_0}{2} - \frac{\sqrt{3}}{2} \pi_{15}, \\ D^0 &\equiv \frac{\pi_0 + i\pi_{10}}{\sqrt{2}}, & \bar{D}^0 &\equiv \frac{\pi_0 - i\pi_{10}}{\sqrt{2}}, \\ D^\pm &\equiv \frac{\pi_{11} \pm i\pi_{12}}{\sqrt{2}}, & D_s^\pm &\equiv \frac{\pi_{13} \pm i\pi_{14}}{\sqrt{2}}, \\ D_0^0 &\equiv \frac{\sigma_9 + i\sigma_{10}}{\sqrt{2}}, & \bar{D}_0^0 &\equiv \frac{\sigma_9 - i\pi_{10}}{\sqrt{2}}, \\ D_0^\pm &\equiv \frac{\sigma_{11} \pm i\sigma_{12}}{\sqrt{2}}, & \bar{D}_{s,0}^\pm &\equiv \frac{\sigma_{13} \pm i\pi_{14}}{\sqrt{2}}, \\ \frac{1}{\sqrt{2}} (\sigma_N + a_0^0) &= \frac{a_0^0}{\sqrt{2}} + \frac{\sigma_0}{2} + \frac{\sigma_8}{\sqrt{6}} + \frac{\sigma_{15}}{2\sqrt{3}}, \\ \frac{1}{\sqrt{2}} (\sigma_N - a_0^0) &= -\frac{a_0^0}{\sqrt{2}} + \frac{\sigma_0}{2} + \frac{\sigma_8}{\sqrt{6}} + \frac{\sigma_{15}}{2\sqrt{3}}, \\ \frac{1}{\sqrt{2}} (\eta_N + \pi^0) &= \frac{\pi_0}{\sqrt{2}} + \frac{\pi_0}{2} + \frac{\pi_8}{\sqrt{6}} + \frac{\pi_{15}}{2\sqrt{3}}, \\ \frac{1}{\sqrt{2}} (\eta_N - \pi^0) &= -\frac{\pi_0}{\sqrt{2}} + \frac{\pi_0}{2} + \frac{\pi_8}{\sqrt{6}} + \frac{\pi_{15}}{2\sqrt{3}}. \end{aligned}$$

It is obvious that, for example, the values and standard deviations of  $a_0^0$ , as shall be listed in the next section, are determined by  $\sigma_3$ .

Section 3.1 introduces analytical expressions for the mass spectra of noncharmed and charmed meson states.

### 3. Results

#### 3.1. Mass Spectra of Noncharmed and Charmed Meson States

At vanishing temperature, the mass spectra of noncharmed meson states can be classified into

- Pseudoscalar mesons

$$m_\pi^2 = Z_\pi^2 \left[ m_0^2 + \left( \lambda_1 + \frac{1}{2} \lambda_2 \right) \sigma_x^2 + \lambda_1 \sigma_y^2 + \lambda_1 \sigma_c^2 \right]; \quad (29)$$

$$m_K^2 = Z_K^2 \left[ m_0^2 + \left( \lambda_1 + \frac{1}{2} \lambda_2 \right) \sigma_x^2 - \frac{1}{2} \lambda_2 \sigma_x \sigma_y + \lambda_1 [\sigma_y^2 + \sigma_c^2] + \lambda_2 \sigma_y^2 \right]; \quad (30)$$

$$m_{\eta_N}^2 = Z_\pi^2 \left[ m_0^2 + \left( \lambda_1 + \frac{1}{2} \lambda_2 \right) \sigma_x^2 + \lambda_1 [\sigma_y^2 + \sigma_c^2] + \frac{C}{2} (\sigma_x^2 + \sigma_y^2 + \sigma_c^2) \right]; \quad (31)$$

$$m_{\eta_S}^2 = Z_{\eta_S}^2 \left[ m_0^2 + \lambda_1 (\sigma_x^2 + \sigma_y^2 + \sigma_c^2) + \lambda_2 \sigma_y^2 + \frac{C}{8} (\sigma_x^2 + \sigma_y^2 + \sigma_c^2) \right]; \quad (32)$$

where  $Z_\pi$ ,  $Z_K$ ,  $Z_{\eta_S}$ , the various wavefunction renormalization factors, are listed in Appendix D.

- Scalar mesons

$$m_{a_0}^2 = m_0^2 + \lambda_1 [\sigma_x^2 + \sigma_y^2 + \sigma_c^2] + \frac{3}{2} \lambda_2 \sigma_x^2; \quad (33)$$

$$m_{k_0^*}^2 = Z_{k_0^*}^2 \left[ m_0^2 + \left( \lambda_1 + \frac{1}{2} \lambda_2 \right) \sigma_x^2 + \frac{1}{\sqrt{2}} \lambda_2 \sigma_x \sigma_y + \lambda_1 [\sigma_y^2 + \sigma_c^2] + \lambda_2 \sigma_c^2 \right]; \quad (34)$$

$$m_{\sigma_N}^2 = m_0^2 + 3 \left( \lambda_1 + \frac{1}{2} \lambda_2 \right) \sigma_x^2 + \lambda_1 [\sigma_y^2 + \sigma_c^2], \quad (35)$$

$$m_{\sigma_S}^2 = m_0^2 + \lambda_1 [\sigma_x^2 + \sigma_c^2] + 3(\lambda_1 + \lambda_2) \sigma_y^2; \quad (36)$$

where  $Z_{k_0^*}$  is another wavefunction renormalization factor. Although  $(\det \Phi + \det \Phi^\dagger)^2$  is the anomaly term we use in the present derivatives, the other possible anomaly terms  $(\det \Phi + \det \Phi^\dagger)$  and  $(\det \Phi - \det \Phi^\dagger)^2$  are also conjectured to affect the scalar masses, especially the earlier term.

- Vector mesons

$$m_{\omega_N}^2 = m_1^2 - m_0^2 + \frac{1}{2} (h_1 + h_2 + h_3) \sigma_x^2 + \frac{1}{2} h_1 [\sigma_y^2 + \sigma_c^2] + 2\delta_x; \quad (37)$$

$$m_{\omega_S}^2 = m_1^2 - m_0^2 + \frac{1}{2} h_1 [\sigma_x^2 + \sigma_c^2] + \frac{1}{2} (h_1 + 2h_2 + 2h_3) \sigma_y^2 + 2\delta_x; \quad (38)$$

$$m_{K^*}^2 = m_1^2 - m_0^2 + \frac{1}{4} \sigma_x^2 [g_1^2 + 2h_1 + h_2] + \frac{1}{\sqrt{2}} \sigma_x \sigma_y [h_3 - g_1^2] + \frac{1}{2} \sigma_y^2 [g_1^2 + h_1 + h_2] + \frac{1}{2} h_1 \sigma_c^2 + \delta_x + \delta_y; \quad (39)$$

$$m_\rho^2 = m_{\omega_N}^2. \quad (40)$$

- Axialvector mesons

$$m_{a_1}^2 = m_1^2 - m_0^2 + g_1^2 \sigma_x^2 + \frac{1}{2} h_1 [\sigma_y^2 + \sigma_c^2] + \frac{1}{2} \sigma_x^2 [h_1 + h_2 + h_3] + 2\delta_x; \quad (41)$$

$$m_{f_{1s}}^2 = m_1^2 - m_0^2 + \frac{1}{2} h_1 [\sigma_x^2 + \sigma_c^2] + 2g_1^2 \sigma_c^2 + \frac{1}{2} [h_1 + 2h_2 - 2h_3] \sigma_y^2 + 2\delta_y; \quad (42)$$

$$\begin{aligned} m_{k_1}^2 &= m_1^2 - m_0^2 + \frac{1}{4}\sigma_x^2[g_1^2 + 2h_1 + h_2] + \frac{1}{2}\sigma_y^2[g_1^2 + h_1 + h_2] \\ &+ \frac{1}{\sqrt{2}}\sigma_x\sigma_y[g_1^2 - h_3] + \frac{1}{2}h_1\sigma_c^2 + \delta_x + \delta_y; \end{aligned} \quad (43)$$

$$m_{1N}^2 = m_{a_1}^2. \quad (44)$$

Also, we determine the mass spectra of the charmed meson states.

- Pseudoscalar charmed mesons

$$m_D^2 = Z_D^2 \left[ m_0^2 + \left( \lambda_1 + \frac{1}{2}\lambda_2 \right) \sigma_x^2 + \lambda_1 \sigma_y^2 + (\lambda_1 + \lambda_2) \sigma_c^2 - \frac{1}{\sqrt{2}}\lambda_2 \sigma_x \sigma_y + \epsilon_c \right]; \quad (45)$$

$$m_{\eta_c}^2 = Z_{\eta_c}^2 \left[ m_0^2 + \lambda_1 [\sigma_x^2 + \sigma_y^2] + (\lambda_1 + \lambda_2) \sigma_c^2 - \frac{C}{8} (\sigma_x^2 + \sigma_y^2) + 2\epsilon_c \right]; \quad (46)$$

$$m_{D_s}^2 = Z_{D_s}^2 \left[ m_0^2 + \lambda_1 \sigma_x^2 + (\lambda_1 + \lambda_2) \sigma_y^2 + (\lambda_1 + \lambda_2) \sigma_c^2 - \lambda_2 \sigma_y \sigma_c + \epsilon_c \right]; \quad (47)$$

where  $Z_D$ ,  $Z_{\eta_c}$  and  $Z_{D_s}$  are wavefunction renormalization factors.

- Scalar charmed mesons

$$m_{\chi_{c0}}^2 = m_0^2 + \lambda_1 [\sigma_x^2 + \sigma_y^2] + 3(\lambda_1 + \lambda_2) \sigma_c^2 + 2\epsilon_c; \quad (48)$$

$$m_{D_0^*}^2 = Z_{D_0^*}^2 \left[ m_0^2 + \left( \lambda_1 + \frac{1}{2}\lambda_2 \right) \sigma_x^2 + \lambda_1 \sigma_y^2 + \frac{1}{\sqrt{2}}\lambda_2 \sigma_x \sigma_c + (\lambda_1 + \lambda_2) \sigma_c^2 + \epsilon_c \right]; \quad (49)$$

$$m_{D_0^{*0}}^2 = Z_{D_0^{*0}}^2 \left[ m_0^2 + \left( \lambda_1 + \frac{1}{2}\lambda_2 \right) \sigma_x^2 + \lambda_1 \sigma_y^2 + \frac{1}{\sqrt{2}}\lambda_2 \sigma_x \sigma_c + (\lambda_1 + \lambda_2) \sigma_c^2 + \epsilon_c \right]; \quad (50)$$

$$m_{D_{s0}^*}^2 = Z_{D_{s0}^*}^2 \left[ m_0^2 + \lambda_1 \sigma_x^2 + (\lambda_1 + \lambda_2) \sigma_y^2 + \lambda_2 \sigma_y \sigma_c + (\lambda_1 + \lambda_2) \sigma_c^2 + \epsilon_c \right]; \quad (51)$$

where  $Z_{D_0^*}$ ,  $Z_{D_0^{*0}}$ , and  $Z_{D_{s0}^*}$  are additional wavefunction renormalization factors.

- Vector charmed mesons

$$\begin{aligned} m_{D^*}^2 &= m_1^2 - m_0^2 + \frac{1}{4}(g_1^2 + 2h_1 + h_2)\sigma_x^2 + \frac{1}{\sqrt{2}}\sigma_x\sigma_y[h_3 - g_1^2] \\ &+ \frac{1}{2}(g_1^2 + h_1 + h_2)\sigma_c^2 + \frac{1}{2}h_1\sigma_y^2 + \delta_x + \delta_c; \end{aligned} \quad (52)$$

$$m_{J/\psi}^2 = m_1^2 - m_0^2 + \frac{1}{2}h_1[\sigma_x^2 + \sigma_y^2] + \frac{1}{2}(h_1 + 2h_2 + 2h_3)\sigma_c^2 + 2\delta_c; \quad (53)$$

$$\begin{aligned} m_{D_s^*}^2 &= m_1^2 - m_0^2 + \frac{1}{2}(g_1^2 + h_1 + h_2)[\sigma_y^2 + \sigma_c^2] + \frac{1}{2}h_1\sigma_x^2 \\ &+ (h_3 - g_1^2)\sigma_y\sigma_c + \delta_y + \delta_c. \end{aligned} \quad (54)$$

- Axialvector charmed mesons

$$m_{D_{s1}}^2 = m_1^2 - m_0^2 + \frac{1}{2}(g_1^2 + h_1 + h_2)[\sigma_y^2 + \sigma_c^2] + \sigma_y\sigma_c(g_1^2 - h_3) + \frac{1}{2}h_1\sigma_x^2 + \delta_y + \delta_c; \quad (55)$$

$$\begin{aligned} m_{D_1}^2 &= m_1^2 - m_0^2 + \frac{1}{4}(g_1^2 + 2h_1 + h_2)\sigma_x^2 + \frac{1}{2}(g_1^2 + h_1 + h_2)\sigma_c^2 + \frac{1}{\sqrt{2}}(g_1^2 - h_3)\sigma_x\sigma_c \\ &+ \frac{1}{2}h_1\sigma_y^2 + \delta_y + \delta_c; \end{aligned} \quad (56)$$

$$m_{\chi_{c1}}^2 = m_1^2 - m_0^2 + \frac{1}{2}h_1[\sigma_x^2 + \sigma_y^2] + 2g_1^2\sigma_c^2 + \frac{1}{2}(h_1 + 2h_2 - 2h_3)\sigma_c^2 + 2\delta_c. \quad (57)$$

The Lagrangian of the extended linear-sigma model, Equation (1), has various parameters including  $h_1, h_2, h_3, \delta_u, \delta_d, \delta_s, \delta_c, \delta_x, \delta_y, \epsilon_u, \epsilon_d, \epsilon_s, \epsilon_c, C, \lambda_1, \lambda_2, g_1$ , and  $g_2$ . All these parameters are explained and determined in the Appendix E. For large  $N_c$ , both parameters  $h_1$  and  $\lambda_1$  vanish, especially for SU(3) meson masses. The other set of parameters,  $\sigma_u, \sigma_d, \sigma_s, \sigma_c, m_u, m_d, m_s, m_c, f_\pi$ , and  $f_K$ , can be fixed from recent compilation of the particle data group [32].

The mass of any particle can be determined by employing the parameters presented in Tables 1 and 2, along with their corresponding expressions.

**Table 1.** Masses of sixteen noncharmed mesons compared with the recent compilation of the particle data group (PDG) [32].

| Meson        | Mass [MeV]       |                       | $h_1$  | $\lambda_1$ | $\lambda_2$ | $g_1$ | % Error |
|--------------|------------------|-----------------------|--------|-------------|-------------|-------|---------|
|              | This Work        | PDG                   |        |             |             |       |         |
| Pseudoscalar |                  |                       |        |             |             |       |         |
| $\pi$        | $130.58 \pm 15$  | $139.5704 \pm 0.0002$ | 5.65   | 13.49       | 68.30       | 5.50  | 0.047   |
| $K$          | $496.90 \pm 4$   | $493.68 \pm 0.016$    | 5.65   | 0.0         | 42.0        | 2.3   | 0.648   |
| $\eta_N$     | $1458.91 \pm 21$ | $1475 \pm 4$          | 5.65   | 13.49       | 68.30       | 7.60  | 1.103   |
| $\eta_S$     | $1477.94 \pm 21$ | $1475 \pm 4$          | 5.65   | 13.49       | 68.30       | 1.25  | 0.199   |
| Scalar       |                  |                       |        |             |             |       |         |
| $a_0$        | $1482.64 \pm 15$ | $1474 \pm 19$         | 5.65   | 4.0         | 68.30       | 1.0   | 0.583   |
| $K_0^*$      | $893.40 \pm 4$   | $891.66 \pm 0.26$     | 5.65   | 1.0         | 25.0        | 1.8   | 0.195   |
| $\sigma_N$   | $1206.77 \pm 2$  | 1200–1500             | 5.65   | 1.0         | 55.0        | 7.60  | 0.561   |
| $\sigma_S$   | $1720.46 \pm 3$  | $1720 \pm 20$         | 5.65   | 1.0         | 46.0        | 7.60  | 0.027   |
| Vector       |                  |                       |        |             |             |       |         |
| $\omega_N$   | $784.42 \pm 6$   | $782.65 \pm 0.12$     | −21.5  | 13.49       | 68.30       | 7.60  | 0.225   |
| $\omega_S$   | $1020.26 \pm 5$  | $1019.46 \pm 0.11$    | −16.21 | 13.49       | 68.30       | 7.60  | 0.078   |
| $K^*$        | $1390.64 \pm 20$ | $1418 \pm 15$         | 5.65   | 13.49       | 68.30       | 7.60  | 1.967   |
| $\rho$       | $784.42 \pm 14$  | $775.5 \pm 38.8$      | −21.5  | 13.49       | 68.49       | 7.60  | 1.137   |
| Axialvector  |                  |                       |        |             |             |       |         |
| $a_1$        | $1289.17 \pm 21$ | $1230 \pm 30$         | 1.0    | 1.8         | 25          | 1.8   | 4.589   |
| $1f$         | $1427.26 \pm 2$  | $1426.4 \pm 0.9$      | 5.65   | 1.8         | 25          | 2.3   | 0.061   |
| $K_1$        | $1400.61 \pm 5$  | $1403 \pm 7$          | 5.65   | 1.8         | 25          | 2.67  | 0.171   |
| $1N$         | $1450.66 \pm 5$  | $1453.4 \pm 27$       | 5.65   | 1.8         | 25          | 3.5   | 0.189   |

**Table 2.** Masses of thirteen charmed mesons compared with the recent compilation of the particle data group (PDG) [32].

| Meson        | Mass [MeV]       |                     | $h_1$ | $\lambda_1$ | $\lambda_2$ | $g_1$ | % Error |
|--------------|------------------|---------------------|-------|-------------|-------------|-------|---------|
|              | This Work        | PDG                 |       |             |             |       |         |
| Pseudoscalar |                  |                     |       |             |             |       |         |
| $D$          | $1868.92 \pm 2$  | $1869.6 \pm 0.15$   | 5.65  | 4.2         | 68.30       | 1.0   | 0.036   |
| $\eta_s$     | $2977.46 \pm 3$  | $2981 \pm 1.1$      | 5.65  | 13.49       | 68.30       | 2.9   | 0.119   |
| $D_s$        | $1965.99 \pm 21$ | $1968.49 \pm 0.32$  | 5.65  | 3.6         | 68.30       | 1.0   | 0.127   |
| Scalar       |                  |                     |       |             |             |       |         |
| $\chi_{c0}$  | $3448.23 \pm 5$  | $3414.75 \pm 0.3$   | 5.65  | 13.49       | 68.30       | 7.60  | 0.971   |
| $D_0^*$      | $2421.82 \pm 7$  | $2403.38 \pm 14$    | 5.65  | 1.0         | 68.30       | 2.0   | 0.761   |
| $D_0^{*0}$   | $2332.02 \pm 8$  | $2317.8 \pm 0.06$   | 5.65  | 1.0         | 62.0        | 1.0   | 0.610   |
| $D_{s0}^*$   | $2121.6 \pm 6$   | $2112.3 \pm 0.3$    | 5.65  | 1.0         | 52.0        | 1.0   | 0.438   |
| Vector       |                  |                     |       |             |             |       |         |
| $D^0$        | $2016.07 \pm 3$  | $2010.28 \pm 0.13$  | 5.65  | 13.49       | 68.30       | 26.5  | 0.278   |
| $J/\psi$     | $3096.15 \pm 12$ | $3096.92 \pm 0.011$ | 183.0 | 13.49       | 68.30       | 7.6   | 0.025   |
| $D_s^*$      | $2109.98 \pm 11$ | $2112.3 \pm 0.5$    | 63.0  | 13.49       | 68.30       | 7.6   | 0.120   |
| Axialvector  |                  |                     |       |             |             |       |         |
| $D_{s1}$     | $3001.98 \pm 6$  | $2997.4 \pm 0.17$   | 98.0  | 13.49       | 68.30       | 7.6   | 0.152   |
| $D_1$        | $2846.77 \pm 6$  | $2841.8 \pm 0.48$   | 98.0  | 13.49       | 68.30       | 7.6   | 0.174   |
| $\chi_{sc1}$ | $3510.57 \pm 5$  | $3510.60 \pm 0.7$   | 170.0 | 13.49       | 68.30       | 7.6   | 0.001   |

Our calculations for the masses of various meson states are summarized in Tables 1 and 2. Our calculations, in which  $h_1$ ,  $\lambda_1$ ,  $\lambda_2$ , and  $g_1$  are the only free parameters, are in a good agreement with the recent compilation of the particle data group (PDG) [32]. The percent error is listed in the last column. It measures how different the two values, namely, the calculated and measured masses, are in a form of a percentage of their corresponding average value. Concretely, the percent error quantifies how close the calculated value is to that measured value. Thus, the percent error is conjectured to provide enough information for the certainty assessment. The reason why some of these free parameters play crucial roles in some meson states is to be found in the corresponding analytical expressions of such meson states.

The section that follows is devoted to the final conclusions and outlook.

#### 4. Conclusions and Outlook

We have constructed the meson states  $\langle \bar{q}q \rangle = \langle \bar{q}_\ell q_r - \bar{q}_\ell q_r \rangle \neq 0$  with and without charm quarks from the effective Lagrangian of the extended linear-sigma model. With respect to their quantum numbers, orbital angular momenta  $J$ , parity  $P$ , and charge conjugates  $C$ , the meson states could be classified into pseudoscalar  $J^{PC} = 0^{-+}$ , scalar  $J^{PC} = 0^{++}$ , vector  $J^{PC} = 1^{--}$ , and axialvector  $J^{PC} = 1^{+-}$ . We have introduced analytical expressions for the mass spectrum of sixteen noncharmed and thirteen charmed meson states, including pseudoscalar  $J^{PC} = 0^{-+}$ , scalar  $J^{PC} = 0^{++}$ , vector  $J^{PC} = 1^{--}$ , and axialvector  $J^{PC} = 1^{+-}$ . The Appendices give further details needed for the analytical analysis. The numerical analysis and the corresponding free parameters are summarized in Table 1 and 2, in which the calculations are confronted with the recent compilation of PDG. We conclude that the masses of the sixteen noncharmed and thirteen charmed meson states are in excellent agreement with PDG. Therefore, we also conclude that the eLSM with its set of parameters represents a suitable theoretical approach for the mass spectrum of noncharmed and charmed meson states, at vanishing temperature. The temperature dependence of these meson states represents the natural outlook to be accomplished elsewhere. Also, the in-medium modifications of these meson masses either in dense or magnetic or electric medium shall be derived elsewhere.

The heavy nonet of pseudoscalar states, which includes  $\pi(1300)$ ,  $K$ , and  $\eta(1475)$ , is of great theoretical interest. These states seem to challenge the established principles of quark and hadron models [45], which typically involve  $\pi$ ,  $K$ ,  $\eta(547)$ , and  $\eta'(958)$ . Future research will involve comparing the eLSM with other approaches that consider radially excited mesons like  $\pi(1300)$ ,  $K(1460)$ ,  $\eta(1295)$ , and  $\eta(1405-1475)$  [46,47]. The applicability of  $U(4)$  to all types of interactions remains uncertain. Analysis of decays as reported in the literature [9] indicates that  $U(4)$  may hold for dominant interactions at large- $N_c$ . Further exploration using eLSM to study decay channels of various meson states is recommended.

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## Appendix A. Gell–Mann Matrices in SU(4)

The special unitary group, SU(4), can be represented as [48]

$$\text{SU}(4) = \left\{ A = 4 \times 4 \text{ complex matrix} \mid A^\dagger A = 1, \det(A) = 1 \right\}. \quad (\text{A1})$$

Analogous to the Pauli, SU(2) [49], and Gell–Mann matrices, SU(3) [1], the hermitian matrix generators of SU(4) are defined as

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \lambda_5 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_8 &= -\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \lambda_9 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{10} &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\ \lambda_{11} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \lambda_{12} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\ \lambda_{13} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \lambda_{14} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \\ \lambda_{15} &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \end{aligned}$$

that they are orthogonal and satisfy  $\text{Tr}(\lambda_A)^2 = 2$ , with  $A = 1, 2, \dots, 15$ .

### Appendix B. $4 \times 4$ $\Phi$ Fields

$$\begin{aligned}
 \Phi &= T_0\sigma_0 + T_8\sigma_8 + T_{15}\sigma_{15} = \frac{1}{2}\lambda_0\sigma_0 + \frac{1}{2}\lambda_8\sigma_8 + \frac{1}{2}\lambda_{15}\sigma_{15} \\
 &= \frac{1}{2}\sqrt{\frac{1}{2}} \begin{pmatrix} \sigma_0 & 0 & 0 & 0 \\ 0 & \sigma_0 & 0 & 0 \\ 0 & 0 & \sigma_0 & 0 \\ 0 & 0 & 0 & \sigma_0 \end{pmatrix} + \frac{1}{2}\sqrt{\frac{1}{3}} \begin{pmatrix} \sigma_8 & 0 & 0 & 0 \\ 0 & \sigma_8 & 0 & 0 \\ 0 & 0 & -2\sigma_8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 &+ \frac{1}{2}\sqrt{\frac{1}{6}} \begin{pmatrix} \sigma_{15} & 0 & 0 & 0 \\ 0 & \sigma_{15} & 0 & 0 \\ 0 & 0 & -2\sigma_{15} & 0 \\ 0 & 0 & 0 & -3\sigma_{15} \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} \frac{\sigma_0}{\sqrt{2}} + \frac{\sigma_8}{\sqrt{3}} + \frac{\sigma_{15}}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & \frac{\sigma_0}{\sqrt{2}} + \frac{\sigma_8}{\sqrt{3}} + \frac{\sigma_{15}}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & \frac{\sigma_0}{\sqrt{2}} - \frac{2\sigma_8}{\sqrt{3}} + \frac{\sigma_{15}}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & \frac{\sigma_0}{\sqrt{2}} - \sqrt{\frac{3}{2}}\sigma_{15} \end{pmatrix}. \tag{A2}
 \end{aligned}$$

Similarly, we obtain

$$\Phi^\dagger = \frac{1}{2} \begin{pmatrix} \frac{\sigma_0}{\sqrt{2}} + \frac{\sigma_8}{\sqrt{3}} + \frac{\sigma_{15}}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & \frac{\sigma_0}{\sqrt{2}} + \frac{\sigma_8}{\sqrt{3}} + \frac{\sigma_{15}}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & \sqrt{2}\left(\frac{\sigma_0}{2} - \sqrt{\frac{2}{3}}\sigma_8 + \frac{\sigma_{15}}{2\sqrt{3}}\right) & 0 \\ 0 & 0 & 0 & \sqrt{2}\left(\frac{\sigma_0}{2} - \frac{\sqrt{3}}{2}\sigma_{15}\right) \end{pmatrix} \tag{A3}$$

so that

$$\Phi\Phi^\dagger = \frac{1}{4} \begin{pmatrix} \left(\frac{\sigma_0}{\sqrt{2}} + \frac{\sigma_8}{\sqrt{3}} + \frac{\sigma_{15}}{\sqrt{6}}\right)^2 & 0 & 0 & 0 \\ 0 & \left(\frac{\sigma_0}{\sqrt{2}} + \frac{\sigma_8}{\sqrt{3}} + \frac{\sigma_{15}}{\sqrt{6}}\right)^2 & 0 & 0 \\ 0 & 0 & 2\left(\frac{\sigma_0}{2} - \sqrt{\frac{2}{3}}\sigma_8 + \frac{\sigma_{15}}{2\sqrt{3}}\right)^2 & 0 \\ 0 & 0 & 0 & 2\left(\frac{\sigma_0}{2} - \frac{\sqrt{3}}{2}\sigma_{15}\right)^2 \end{pmatrix}. \tag{A4}$$

### Appendix C. Explicit Chiral Symmetry Breaking

The chiral symmetry in explicitly broken

$$\begin{aligned}
 H &= \sum_{a=0}^{15} h_a T_a = h_0 T_0 + h_8 T_8 + h_{15} T_{15} \\
 &= \frac{1}{2} \begin{pmatrix} \frac{h_0}{\sqrt{2}} + h_3 + \frac{h_8}{\sqrt{3}} + \frac{h_{15}}{\sqrt{6}} & h_1 - ih_2 & h_4 - ih_5 & h_9 - ih_{10} \\ h_1 + ih_2 & \frac{h_0}{\sqrt{2}} - h_3 + \frac{h_8}{\sqrt{3}} + \frac{h_{15}}{\sqrt{6}} & h_6 - ih_7 & h_{11} - ih_{12} \\ h_4 + ih_5 & h_6 + ih_7 & \frac{h_0}{\sqrt{2}} + 2\frac{h_8}{\sqrt{3}} + \frac{h_{15}}{\sqrt{6}} & h_{13} - ih_{14} \\ h_9 - ih_{10} & h_{11} + ih_{12} & h_{13} + ih_{14} & \frac{h_0}{\sqrt{2}} + \sqrt{\frac{2}{3}}h_{15} \end{pmatrix}. \tag{A5}
 \end{aligned}$$

### Appendix D. Wavefunction Renormalization Factors

Here, we summarize the wavefunction renormalization factors needed for different meson masses,

$$Z_\pi = Z_\eta = \frac{m_{a_1}}{\sqrt{m_{a_1}^2 - g_1^2 \sigma_x^2}}, \quad Z_k = \frac{2m_{k_1}}{\sqrt{4m_{k_1}^2 - g_1^2 [\sigma_x + \sqrt{2}\sigma_y]^2}}, \quad (\text{A6})$$

$$Z_{k_s} = \frac{2m_K}{\sqrt{4m_K^2 - g_1^2 [\sigma_x + \sqrt{2}\sigma_y]^2}}, \quad Z_{\eta_s} = \frac{m_{f_{1s}}}{\sqrt{m_{f_{1s}}^2 - 2g_1^2 \sigma_y^2}}, \quad (\text{A7})$$

$$Z_{\eta_c} = \frac{m_{\chi_{c1}}}{\sqrt{m_{\chi_{c1}}^2 - 2g_1^2 \sigma_c^2}}, \quad Z_D = \frac{2m_{D_1}}{\sqrt{4m_{D_1}^2 - g_1^2 (\sigma_c + \sqrt{2}\sigma_c)^2}}, \quad (\text{A8})$$

$$Z_{\eta_s} = \frac{m_{f_{1s}}}{\sqrt{m_{f_{1s}}^2 - 2g_1^2 \sigma_y^2}}, \quad Z_{D_s} = \frac{\sqrt{2}m_{D_{s1}}}{\sqrt{2m_{D_{s1}}^2 - g_1^2 (\sigma_y^2 + \sigma_c)^2}}, \quad (\text{A9})$$

$$Z_{D_0^*} = \frac{2m_{D^*}}{\sqrt{4m_{D^*}^2 - g_1^2 (\sigma_x^2 - \sqrt{2}\sigma_c)^2}}, \quad Z_{D_0^{*0}} = \frac{2m_{D^{*0}}}{\sqrt{4m_{D^{*0}}^2 - g_1^2 (\sigma_x - \sqrt{2}\sigma_c)^2}}, \quad (\text{A10})$$

$$Z_{D_{s0}^*} = \frac{\sqrt{2}m_{D_s^*}}{\sqrt{2m_{D_s^*}^2 - g_1^2 (\sigma_y - \sigma_c)^2}}, \quad (\text{A11})$$

where

$$g_1^2 = \frac{m_{a_1}^2}{f_\pi^2 Z_\pi^2} \left(1 - \frac{1}{Z_\pi^2}\right). \quad (\text{A12})$$

#### Mesonic Potential of SU(4) Linear-Sigma Model

We start with the mesonic potential of the SU(3) linear-sigma model,

$$\begin{aligned} U_m^{SU(3)}(\Phi) &= m^2 \text{Tr}(\Phi^\dagger \Phi) + \lambda_1 [\text{Tr}(\Phi^\dagger \Phi)]^2 + \lambda_2 [\text{Tr}(\Phi^\dagger \Phi)]^3 \\ &- C [\det(\Phi) + \det(\Phi^\dagger)] - \text{Tr}[H(\Phi + \Phi^\dagger)]. \end{aligned} \quad (\text{A13})$$

It should be noted that Equation (A13) refers to a different anomaly term to the one in the main text. For the current purpose, this is just a fine detail. The mesonic potential of SU(4) has a similar analogous form to that of SU(3). For a precise estimation of the mass spectrum, a new term must be added, namely,  $-2 \text{Tr}[\epsilon \Phi^\dagger \Phi]$ , so that

$$\begin{aligned} U_m^{SU(4)}(\Phi) &= \frac{m^2}{2} (\sigma_x^2 + \sigma_y^2 + \sigma_{15}^2)^2 \\ &+ \lambda_1 \left[ 4 \left( \sigma_x + \frac{\sigma_{15}}{\sqrt{6}} \right)^4 + \left( \sqrt{2}\sigma_y + \frac{\sigma_{15}}{\sqrt{6}} \right)^4 + \left( \sqrt{\frac{2}{3}}\sigma_0 - \sqrt{\frac{3}{2}}\sigma_{15} \right)^4 \right. \\ &\quad + 4 \left( \sigma_x + \frac{\sigma_{15}}{\sqrt{6}} \right)^2 \left( \sqrt{2}\sigma_y + \frac{\sigma_{15}}{\sqrt{6}} \right)^2 + 4 \left( \sigma_x + \frac{\sigma_{15}}{\sqrt{6}} \right)^2 \left( \sqrt{\frac{3}{2}}\sigma_0 - \sqrt{\frac{3}{2}}\sigma_{15} \right)^2 \\ &\quad \left. + 2 \left( \sqrt{2}\sigma_y + \frac{\sigma_{15}}{\sqrt{6}} \right)^2 \left( \sqrt{\frac{2}{3}}\sigma_0 - \sqrt{\frac{3}{2}}\sigma_{15} \right)^2 \right] \\ &+ \lambda_2 \left[ 2 \left( \sigma_x + \frac{\sigma_{15}}{\sqrt{6}} \right)^4 + \left( \sqrt{2}\sigma_y + \frac{\sigma_{15}}{\sqrt{6}} \right)^4 + \left( \sqrt{\frac{2}{3}}\sigma_0 - \sqrt{\frac{3}{2}}\sigma_{15} \right)^4 \right] \\ &- \frac{c}{8} \left[ \frac{2}{3} \sigma_x^2 \sigma_y \sigma_0 + \frac{\sigma_y \sigma_{15}^2 \sigma_0}{3\sqrt{3}} + \frac{2\sqrt{2}}{3} \sigma_x \sigma_y \sigma_{15} \sigma_0 + \frac{1}{3} \sigma_x^2 \sigma_{15} \sigma_0 + \frac{\sigma_{15}^3 \sigma_0}{8} \right. \\ &\quad \left. + \frac{\sqrt{2}\sigma_x \sigma_{15} \sigma_0^2}{3\sqrt{3}} - \sqrt{3}\sigma_x^2 \sigma_y \sigma_{15} - \frac{\sigma_{15}^3 \sigma_y}{2\sqrt{3}} - \sqrt{2}\sigma_x \sigma_y \sigma_{15}^2 - \frac{\sigma_x^2 \sigma_{15}^2}{2} - \frac{\sigma_{15}^4}{12} - \frac{\sigma_x \sigma_{15}^3}{\sqrt{6}} \right]. \end{aligned} \quad (\text{A14})$$

### Appendix E. Parameters of the SU(4) Linear-Sigma Model

We start with the potential of the SU(4) linear-sigma model, Equation (A15). The global minimum, i.e., vanishing partial derivatives with respect to  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_c$ , leads to

$$h_x = m^2\sigma_x - \frac{c}{2}\sigma_x\sigma_y\sigma_c + \lambda_1\sigma_x\sigma_y^2 + \lambda_2\sigma_x\sigma_c^2 + \frac{1}{2}(2\lambda_1 + \lambda_2)\sigma_x^2, \quad (\text{A15})$$

$$h_y = m^2\sigma_y - \frac{c}{2}\sigma_x^2\sigma_c + \lambda_1\sigma_x^2\sigma_y + \lambda_2\sigma_y\sigma_c^2 + (\lambda_1 + \lambda_2)\sigma_y^2, \quad (\text{A16})$$

$$h_c = m^2\sigma_c - \frac{c}{2}\sigma_x^2\sigma_y + \lambda_1\sigma_x^2\sigma_c + \lambda_2\sigma_y^2\sigma_c + (\lambda_1 + \lambda_2)\sigma_c^2, \quad (\text{A17})$$

where  $\lambda_1$  and  $\lambda_2$  are, respectively, defined as

$$\lambda_1 = \frac{m_\sigma^2 - m_\pi^2 - m_{a_0}^2 + m_\eta^2}{3f_\pi^2}, \quad (\text{A18})$$

$$\lambda_2 = \frac{3(2f_K - f_\pi)m_K^2 - 3(2f_K - f_\pi)m_\pi^2 - 2(f_K - f_\pi)(m_{\eta'}^2 + m_\eta^2)}{(f_K - f_\pi)(3f_\pi^2 + 8f_K(f_K - f_\pi))}, \quad (\text{A19})$$

where  $f_K$  and  $f_\pi$  are the decay constants of  $K$  and  $\pi$  mesons which can be taken from the recent compilation of the particle data group [32].

The  $U(1)_A$  anomaly breaking term  $C$  is fixed by  $\lambda_2$  and the difference in pion and Kaon masses,

$$C = \frac{m_K^2 - m_\pi^2}{f_K - f_\pi} - \lambda_2(2f_K - f_\pi). \quad (\text{A20})$$

The external field  $\Delta$  could be expressed from the Lagrangian term  $\text{Tr}[\Delta(L^{\mu\nu} + L^{\mu\nu})]$

$$\Delta = \begin{pmatrix} \delta_u & 0 & 0 & 0 \\ 0 & \delta_d & 0 & 0 \\ 0 & 0 & \delta_s & 0 \\ 0 & 0 & 0 & \delta_c \end{pmatrix}, \quad (\text{A21})$$

from which we deduce that

$$\begin{pmatrix} \delta_u \\ \delta_d \\ \delta_s \\ \delta_c \end{pmatrix} = \begin{pmatrix} m_u^2 \\ m_d^2 \\ m_s^2 \\ m_c^2 \end{pmatrix}. \quad (\text{A22})$$

In the isospin-symmetric limit, it is possible to set  $\delta_u = \delta_d = 0$ . In this case, for  $\delta_x$ ,  $\delta_y$ , and  $\delta_c$ , we might use the mass equations of vector mesons, for example,  $m_{\omega_N}^2$ ,  $m_{\omega_S}^2$ , and  $m_{\chi_{c1}}^2$ . Then, we obtain

$$\delta_x = \frac{1}{2} \left[ m_{\omega_N}^2 - m_1^2 + m_0^2 - \frac{\sigma_x^2}{2}(h_1 + h_2 + h_3) - \frac{h_1}{2}(\sigma_y^2 + \sigma_c^2) \right], \quad (\text{A23})$$

$$\delta_y = \frac{1}{2} \left[ m_{\omega_S}^2 - m_1^2 + m_0^2 - \frac{\sigma_y^2}{2} \left( \frac{h_1}{2} + h_2 + h_3 \right) - \frac{h_1}{2}(\sigma_x^2 + \sigma_c^2) \right], \quad (\text{A24})$$

$$\delta_c = \frac{1}{2} \left[ m_{\chi_{c1}}^2 - m_1^2 + m_0^2 - 2g_1^2\sigma_c^2 - \sigma_c^2 \left( \frac{h_1}{2} + h_2 - h_3 \right) - \frac{h_1}{2}(\sigma_y^2 + \sigma_y^2) \right]. \quad (\text{A25})$$

For the mass parameters,

$$m^2 = m_\pi^2 - \frac{f_\pi^2}{2} \lambda_2 + \frac{c}{2} (2f_K - f_\phi) - \lambda_1 \left( \frac{1}{2} (2f_K - f_\pi)^2 \right), \quad (\text{A26})$$

$$m_0^2 = \frac{1}{2} \left[ m_{a_0}^2 + m_{\sigma_s}^2 - \lambda_2 \left( \frac{3}{2} \sigma_x^2 + 3\sigma_y^2 \right) \right], \quad (\text{A27})$$

$$m_1^2 = m_{\omega_s}^2 - \left( \frac{h_1}{2} + h_2 + h_3 \right) \sigma_s^2 - \frac{h_1}{2} \sigma_x^2 - 2\delta_s. \quad (\text{A28})$$

Finally, for the parameters  $h_1$ ,  $h_2$ , and  $h_3$ , we recall the potential of the SU(3) linear-sigma model, Equation (A13), where  $\sigma_u$ ,  $\sigma_d$ , and  $\sigma_s$  can be related to  $\sigma_0$ ,  $\sigma_3$ , and  $\sigma_8$ ,

$$\sigma_u = \sqrt{2}\sigma_0 + \sigma_3 + \sigma_8, \quad (\text{A29})$$

$$\sigma_d = \sqrt{2}\sigma_0 - \sigma_3 + \sigma_8, \quad (\text{A30})$$

$$\sigma_s = \sigma_0 - \sqrt{2}\sigma_8. \quad (\text{A31})$$

Then,  $h_0$ ,  $h_3$ , and  $h_8$  become

$$h_0 = \frac{1}{\sqrt{6}} \left[ f_\pi m_\pi^2 + 2f_K m_K^2 \right], \quad (\text{A32})$$

$$h_3 = \left[ m^2 + \frac{c}{\sqrt{6}} \sigma_0 - \frac{c}{\sqrt{6}} \sigma_8 + \lambda_1 (\sigma_0^2 + \sigma_3^2 + \sigma_8^2) + \lambda_2 \left( \sigma_0^2 + \frac{\sigma_3^2}{2} + \frac{\sigma_8^2}{2} + \sqrt{2}\sigma_0\sigma_8 \right) \right] \sigma_3, \quad (\text{A33})$$

$$h_8 = \frac{2}{3} \left[ f_\pi m_\pi^2 - 2f_K m_K^2 \right]. \quad (\text{A34})$$

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