

Review

Understanding Transversely Polarized Quarks Inside Longitudinally Polarized Nucleons

Hui Li ¹ , Xiaoyu Wang ^{2,*}  and Zhun Lu ^{3,*} 
¹ School of Physics and Electronic Engineering, Shanxi Normal University, Taiyuan 030031, China; lihui@sxnu.edu.cn

² School of Physics, Zhengzhou University, Zhengzhou 450001, China

³ School of Physics, Southeast University, Nanjing 211189, China

* Correspondence: xiaoyuwang@zzu.edu.cn (X.W.); zhunlu@seu.edu.cn (Z.L.)

Abstract

We review the current understanding of transversely polarized quarks inside longitudinally polarized nucleons from the perspective of the longi-transversity distribution function and transverse-momentum-dependent (TMD) factorization. Employing the recently developed TMD evolution formalism for the unpolarized and longi-transversity distribution functions, we present the theoretical formalism for the relevant physical observables, namely the $A_{LL}^{\cos 2\phi}$ asymmetry in the Drell–Yan process and the $A_{UL}^{\sin 2\phi_h}$ asymmetry in the semi-inclusive deep inelastic scattering process. The phenomenological results are presented numerically.

Keywords: longi-transversity distribution function; TMD factorization; SIDIS process; Drell–Yan process

1. Introduction

In Quantum Chromodynamics (QCD) [1–3], nucleons arise as relativistic, strongly interacting bound states composed of quarks and gluons—collectively referred to as partons. Characterizing the quark–gluon composition of hadrons and investigating how hadrons emerge from their underlying quark and gluon constituents pose intellectual challenges due to color confinement. The established collinear QCD factorization framework captures one-dimensional snapshots of the longitudinal momentum distributions of quarks and gluons within a colliding nucleon and represents them through universal parton distribution functions (PDFs). Over decades of deep inelastic scattering (DIS) [4,5] experiments, these collinear PDFs have provided essential insights into the longitudinal momentum distributions of partons within nucleons [6–8].

The confined motion and spatial distribution of quarks and gluons within bound nucleons are key features of the nucleon’s three-dimensional internal structure, arising directly from the QCD dynamics. To explore the three-dimensional internal structure of the nucleons, one needs a new kind of two-scale probe that is distinct from the DIS process. These are physical observables defined by a large momentum transfer, $Q_1 \gg \Lambda_{\text{QCD}} \sim 1/R$, which effectively localizes the probe and allows it to resolve the particle-like nature of quarks and gluons. Moreover, a distinct soft momentum scale, $Q_2 \ll Q_1$, is needed, which is particularly sensitive to the details of a hadron’s internal structure. For instance, the cross section in the Drell–Yan process [9,10] serves as a typical two-scale observable when the differential cross section is expressed in terms of the transverse momentum of the lepton pair, q_T , in addition to its invariant mass, Q . This is because the production



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rate is mainly controlled by the kinematic domain where $q_T \ll Q$. The semi-inclusive deep inelastic scattering (SIDIS) process [11–14] provides another well-defined two-scale process, characterized by a hard scale $Q \gg \Lambda_{\text{QCD}}$ and a soft scale given by the transverse momentum P_{hT} of the detected final-state hadron in the photon-hadron frame, where the exchanged virtual photon and the initial hadron define the z -axis. In this virtual photon-hadron frame, the distribution of P_{hT} in the lepton-hadron SIDIS process, in the region $P_{hT}/z \ll Q$, thus constitutes an additional two-scale observable.

In recent years, theoretical progress has led to the establishment of the transverse momentum dependent (TMD) factorization framework [15]. Over the last two decades, TMD factorization has become a key method for investigating the three-dimensional structure of the nucleon and has been extensively applied to various high-energy reactions. The TMD factorization theorem states that, in the region of small transverse momentum, $P_{hT}/z \ll Q$ for SIDIS or $q_T \ll Q$ for Drell–Yan, the differential cross section can be written as a convolution of short-distance hard scattering coefficients with long-distance dynamics, the latter being encoded in well-defined TMD parton distribution functions (PDFs) or fragmentation functions (FFs). This framework provides access to the three-dimensional motion of quarks and gluons inside a nucleon participating in a collision and permits the extraction of universal TMD PDFs [13–25]. TMD PDFs [11,26–28] provide essential information on the transverse-momentum structure of nucleons, allowing for a comprehensive characterization of the momentum space of nucleon constituents.

Formally, the quark–quark correlator of a nucleon characterized by momentum P and spin vector S can be written as

$$\Phi_{ij}(x, k_T; S) = \int \frac{d\bar{\xi}^- d^2\bar{\xi}_T}{(2\pi)^3} e^{ik \cdot \bar{\xi}} \langle P, S | \bar{\psi}_j(0) \mathcal{U}_{[0, \bar{\xi}]} \psi_i(\bar{\xi}) | P, S \rangle \Big|_{\bar{\xi}^+ = 0}, \quad (1)$$

where $\mathcal{U}_{[0, \bar{\xi}]}$ represents the gauge link that guarantees color gauge invariance. By performing the Dirac-structure decomposition [26,29], one obtains eight leading-twist TMD PDFs, each associated with a different type of spin–momentum correlation. Among these, three TMD PDFs—the unpolarized distribution $f_1(x, k_T)$, the helicity distribution $g_1(x, k_T)$, and the transversity distribution $h_1(x, k_T)$ —reduce to their corresponding collinear PDFs after integrating over the transverse momentum. The other five TMD PDFs, which depend on the polarization of the parent proton and/or the quarks, describe genuinely transverse-momentum-dependent phenomena.

Among the eight leading-twist TMD PDFs, the distribution function $h_{1L}^\perp$ is arguably the least studied, as it encodes the correlation between a longitudinally polarized nucleon and transversely polarized quarks via the matrix element

$$h_{1L}^\perp \propto \frac{S_L p_T^\alpha}{M} \langle \bar{\psi} i \sigma^{\alpha+} \gamma_5 \psi \rangle, \quad (2)$$

where S_L is a scalar representing the longitudinal spin of the nucleon. Consequently, $h_{1L}^\perp$ is commonly known as longitudinal transversity (often shortened to longi-transversity). A further distinctive feature of $h_{1L}^\perp$ is its chiral-odd character, which requires it to be accompanied by other chiral-odd functions (such as the Collins fragmentation function [30,31]) to be probed experimentally through the single-spin asymmetry $A_{UL}^{\sin 2\phi}$ in SIDIS processes $l + N \rightarrow l + h + X$. Predictions for this asymmetry have been presented in Refs. [32–37], and it has been measured in fixed-target experiments such as HERMES and CLAS [38–40]. These studies indicate that the magnitude of the asymmetry is at the level of a few percent, though the associated systematic uncertainties are still sizable. The distribution $h_{1L}^\perp$ can also be probed via the Drell–Yan process. As shown in Refs. [10,41], the convolution of the $h_{1L}^\perp$ distributions from the two colliding protons induces a $\cos 2\phi$ azimuthal modulation in

the doubly longitudinally polarized Drell–Yan process, where ϕ is the azimuthal angle of the dilepton pair with respect to the hadronic plane. This provides a theoretically clean observable that is especially sensitive to sea-quark contributions to $h_{1L}^\perp$ [10,13–25,41,42].

Two complementary processes, doubly polarized Drell–Yan and SIDIS with a longitudinally polarized target, serve as the primary experimental avenues for accessing the longi-transversity $h_{1L}^\perp$. Recent comprehensive studies have provided detailed phenomenological predictions for the corresponding azimuthal asymmetries ($A_{LL}^{\cos 2\phi}$ and $A_{UL}^{\sin 2\phi_h}$) within TMD factorization frameworks [20,22].

The theoretical treatment of these azimuthal asymmetries fundamentally depends on TMD factorization, which is valid in the kinematic regions where the transverse momentum is much smaller than the hard scale of the process ($q_T \ll Q$). A key element of this framework is the QCD evolution of TMDs, which is dictated by the Collins–Soper equation [15–17,43] and realized via the Sudakov form factor [13,15,17,44]. The evolution incorporates both perturbatively calculable components and non-perturbative elements that must be constrained by experimental data. The development of various parametrizations (such as the EIKV [20] and BDPRS approaches [22]) reflects ongoing efforts to understand the universal aspects of TMD evolution while accounting for process-dependent characteristics [13,15,17,18,20,22,45–55].

Phenomenological analyses face significant challenges due to limited experimental constraints on $h_{1L}^\perp$. The Wandzura–Wilczek approximation provides a practical solution by relating $h_{1L}^\perp$ to the better-understood transversity distribution h_1 , although this approach introduces theoretical uncertainties that require careful evaluation. Recent comprehensive studies have advanced our understanding through detailed investigations of both $\sin 2\phi_h$ asymmetries of pion production in SIDIS and $\cos 2\phi$ asymmetries in double-polarized Drell–Yan processes. These studies have produced valuable comparisons with available experimental data and have also generated predictions at facilities like RHIC and NICA. The findings reveal a marked sensitivity to sea quark contributions, with forecasts indicating potentially observable effects in specific conditions.

This review synthesizes significant developments, examining the theoretical foundations, phenomenological results, and future prospects for understanding $h_{1L}^\perp$ through measurements of azimuthal asymmetry. By comparing findings from SIDIS and Drell–Yan processes, we aim to highlight the complementary nature of these experimental approaches and identify promising directions for future research into the three-dimensional structure of the nucleon.

The rest of this review is structured as follows. Section 2 derives the main physical observables within TMD factorization, focusing on the $\cos 2\phi$ asymmetry in polarized Drell–Yan and the $\sin 2\phi_h$ asymmetry in SIDIS. In Section 3, we introduce the TMD evolution framework for distribution functions, covering the Collins–Soper equation and the Sudakov form factor. Section 4 offers an extensive overview of the numerical results, confronting theoretical predictions with experimental measurements and presenting novel projections for experiments. Finally, in Section 5, we provide a summary and discuss future perspectives.

2. Physical Observables in Drell–Yan and SIDIS in the Framework of TMD Factorization

In this section, we establish the formalism for the physical observables in polarized Drell–Yan and SIDIS processes that investigate the longi-transversity, $h_{1L}^\perp$, within the TMD factorization approach.

2.1. The $\cos 2\phi$ Asymmetry in the Double-Polarized Proton–Proton Drell–Yan Process

We first consider the doubly longitudinally polarized pp Drell–Yan process [10]:

$$p^{\rightarrow}(P_1, S_1) + p^{\rightarrow}(P_2, S_2) \rightarrow \gamma^*(q) + X \rightarrow l^+(\ell) + l^-(\ell') + X. \quad (3)$$

Here, $P_{1(2)}$ and $S_{1(2)}$ denote the four-momenta and spin vectors of the incoming protons, respectively. The virtual photon carries momentum $q = (\ell + \ell')$, which is time-like, and $Q^2 = q^2 = m_{ll}^2$ represents the invariant mass squared of the final-state dilepton pair. The arrow $\rightarrow$ indicates that the proton is longitudinally polarized.

As usual, the kinematics of the process are described by the squared center-of-mass energy $s = (P_1 + P_2)^2$, the longitudinal momentum fractions $x_{1(2)} = \frac{Q^2}{2P_{1(2)} \cdot q}$, the Feynman variable $x_F = 2q_L/s = x_1 - x_2$, and the dilepton rapidity $y = \frac{1}{2} \ln \frac{q^+}{q^-} = \frac{1}{2} \ln \frac{x_1}{x_2}$. Moreover, $x_{1/2}$ can be written in terms of x_F and $\tau = Q^2/s = x_1 x_2$, or alternatively in terms of y and τ , as

$$x_{1/2} = \frac{\pm x_F + \sqrt{x_F^2 + 4\tau}}{2}, \quad x_{1/2} = \sqrt{\tau} e^{\pm y}. \quad (4)$$

The Drell–Yan mechanism serves as a prototypical “two-scale” process, and its observables can be written within the framework of TMD factorization in the kinematic region where the transverse momentum of the dilepton pair fulfills $q_T \ll Q$ [13,15,20,22]. In this regime, the leading-twist differential cross section takes the form [41]

$$\begin{aligned} & \frac{d\sigma}{dx_1 dx_2 d^2q_T d\Omega} \\ &= \frac{\alpha_{em}^2}{4Q^2} \left\{ \left(1 + \cos^2 \theta\right) F_{UU} + S_{1L} S_{2L} \sin^2 \theta \cos 2\phi F_{LL}^{\cos 2\phi} \right. \\ & \quad + |S_{1T}| S_{2L} \left(1 + \cos^2 \theta\right) \cos \phi_1 F_{TL}^{\cos \phi_1} \\ & \quad + S_{1L} |S_{2T}| \sin^2 \theta \left[\cos(2\phi + \phi_2) F_{LT}^{\cos(2\phi + \phi_2)} \right. \\ & \quad \left. + \cos(2\phi - \phi_2) F_{LT}^{\cos(2\phi - \phi_2)} \right] \\ & \quad + |S_{1T}| |S_{2T}| \left(1 + \cos^2 \theta\right) \left[\cos(\phi_1 + \phi_2) F_{TT}^{\cos(\phi_1 + \phi_2)} \right. \\ & \quad \left. + \cos(\phi_1 - \phi_2) F_{TT}^{\cos(\phi_1 - \phi_2)} \right] + \dots \left. \right\}. \end{aligned} \quad (5)$$

Here, $\phi_{1/2}$ denotes the azimuthal angle of the transverse spin vector $S_{(1/2)T}$ with respect to the lepton plane, while ϕ and θ represent the azimuthal and polar angles, respectively, of the lepton momentum in the Collins–Soper frame. The solid angle Ω specifies the spatial orientation of the dilepton system.

In addition, the subscripts in the structure functions $P = UU$, $P = LL$, $P = TL$, $P = LT$, $P = TT$ specify the polarization configuration of the incoming protons (U for unpolarized, T for transverse polarization, and L for longitudinal polarization). The spin-dependent observables are associated with azimuthal asymmetries and can be written as the ratio of the spin-dependent structure function to the spin-averaged structure function, namely

$$A_P^{f(\phi)} = \frac{F_P^{f(\phi)}}{F_{UU}}, \quad (6)$$

which could be accessed through measurements in doubly longitudinally polarized Drell–Yan processes.

The spin-averaged/unpolarized structure function F_{UU} can be represented as a convolution of the unpolarized parton distribution functions from each proton in the initial-state [10,41]

$$F_{UU} = C \left[f_1^{q/p} f_1^{\bar{q}/p} \right], \quad (7)$$

whereas the spin-dependent structure function $F_{LL}^{\cos 2\phi}$ is given by the convolution of the longi-transversity distributions of each of the incoming protons [10,41]

$$F_{LL}^{\cos 2\phi} = C \left[\frac{2(\hat{h} \cdot \mathbf{k}_{1T})(\hat{h} \cdot \mathbf{k}_{2T}) - \mathbf{k}_{1T} \cdot \mathbf{k}_{2T}}{M^2} h_{1L}^{\perp, q/p} h_{1L}^{\perp, \bar{q}/p} \right], \quad (8)$$

where the unit vector $\hat{h}$ is given by $\hat{h} \equiv \mathbf{q}_\perp / |\mathbf{q}_\perp|$ and M denotes the proton mass [10,56]. The vectors $\mathbf{k}_{1\perp}$ and $\mathbf{k}_{2\perp}$ correspond to the transverse momenta of the quark and antiquark inside the initial protons, respectively. The convolution of the TMDPDFs in transverse-momentum space is given by [10]

$$\begin{aligned} & C[w(\mathbf{k}_{1T}, \mathbf{k}_{2T}) f_1 f_2] \\ &= \frac{1}{N_c} \sum_q e_q^2 \int d^2 \mathbf{k}_{1T} d^2 \mathbf{k}_{2T} \delta^{(2)}(\mathbf{q}_T - \mathbf{k}_{1T} - \mathbf{k}_{2T}) \\ & \times \omega(\mathbf{k}_{1T}, \mathbf{k}_{2T}) f_1^q(x_1, \mathbf{k}_{1T}^2) f_2^{\bar{q}}(x_2, \mathbf{k}_{2T}^2). \end{aligned} \quad (9)$$

Here, $N_c = 3$ denotes the number of colors, and $\mathbf{q}_T$ stands for the transverse momentum of the lepton pair. $\omega(\mathbf{k}_{1T}, \mathbf{k}_{2T})$ is a function that depends on $\mathbf{k}_{1T}$ and $\mathbf{k}_{2T}$.

In practice, a more practical way to analyze the structure function is to work in b_T space, where the convolution of TMDs reduces to an ordinary product of b_T -dependent TMDs. The corresponding physical observables are then obtained by performing an inverse Fourier transform from b_T space back to k_T space.

By employing the Fourier transform of the δ -function

$$\delta^2(\mathbf{q}_T - \mathbf{k}_{1T} - \mathbf{k}_{2T}) = \frac{1}{(2\pi)^2} \int d^2 \mathbf{b}_T e^{i\mathbf{b}_T \cdot (\mathbf{q}_T - \mathbf{k}_{1T} - \mathbf{k}_{2T})}, \quad (10)$$

one can derive the spin-dependent structure function $F_{LL}^{\cos 2\phi}$, which takes the form

$$\begin{aligned} F_{LL}^{\cos 2\phi} &= \frac{1}{N_c} \sum_q e_q^2 \int d^2 \mathbf{k}_{1T} d^2 \mathbf{k}_{2T} \int \frac{d^2 \mathbf{b}_\perp}{(2\pi)^2} e^{i(\mathbf{q}_\perp - \mathbf{k}_{1T} - \mathbf{k}_{2T}) \cdot \mathbf{b}_T} \\ & \times \left[\frac{2(\mathbf{h} \cdot \mathbf{k}_{1T})(\mathbf{h} \cdot \mathbf{k}_{2T}) - (\mathbf{k}_{1T} \cdot \mathbf{k}_{2T})}{M^2} \right] h_{1L}^{\perp, q/p}(x_1, \mathbf{k}_{1T}^2, Q) h_{1L}^{\perp, \bar{q}/p}(x_2, \mathbf{k}_{2T}^2, Q) \\ &= \frac{1}{N_c} \sum_q e_q^2 \int \frac{d^2 \mathbf{b}_T}{(2\pi)^2} e^{i\mathbf{q}_T \cdot \mathbf{b}_T} (2h_\alpha h_\beta - g_{\alpha\beta}) \cdot \tilde{h}_{1L}^{\perp, q/p, \alpha}(x_1, \mathbf{b}_T, Q) \tilde{h}_{1L}^{\perp, \bar{q}/p, \beta}(x_2, \mathbf{b}_T, Q). \end{aligned} \quad (11)$$

Here, the quantities with a tilde denote their counterparts in b_T -space. The longi-transversity of the incoming protons in b_T -space is defined as

$$\tilde{h}_{1L}^{\perp, q/p, \alpha}(x_1, \mathbf{b}_T; Q) = \int d^2 \mathbf{k}_{1T} e^{-i\mathbf{k}_{1T} \cdot \mathbf{b}_T} \frac{k_{1T}^\alpha}{M} h_{1L}^{\perp, q/p}(x_1, \mathbf{k}_{1T}^2; Q), \quad (12)$$

$$\tilde{h}_{1L}^{\perp, \bar{q}/p, \beta}(x_2, \mathbf{b}_T; Q) = \int d^2 \mathbf{k}_{2T} e^{-i\mathbf{k}_{2T} \cdot \mathbf{b}_T} \frac{k_{2T}^\beta}{M} h_{1L}^{\perp, \bar{q}/p}(x_2, \mathbf{k}_{2T}^2; Q). \quad (13)$$

Similarly, the spin-averaged structure function F_{UU} may be written as

$$\begin{aligned} F_{UU} &= \frac{1}{N_c} \sum_q e_q^2 \int d^2 \mathbf{k}_{1T} d^2 \mathbf{k}_{2T} \int \frac{d^2 \mathbf{b}_T}{(2\pi)^2} e^{i(\mathbf{q}_T - \mathbf{k}_{1T} - \mathbf{k}_{2T}) \cdot \mathbf{b}_T} \\ &\times f_1^{q/p}(x_1, \mathbf{k}_{1T}^2, Q) f_1^{\bar{q}/p}(x_2, \mathbf{k}_{2T}^2, Q) \\ &= \frac{1}{N_c} \sum_q e_q^2 \int \frac{d^2 \mathbf{b}_T}{(2\pi)^2} e^{i\mathbf{q}_T \cdot \mathbf{b}_T} \tilde{f}_1^{q/p}(x_1, \mathbf{b}_T, Q) \tilde{f}_1^{\bar{q}/p}(x_2, \mathbf{b}_T, Q), \end{aligned} \quad (14)$$

where the unpolarized distribution functions for each proton in $b_\perp$ -space are defined as follows

$$\tilde{f}_1^{q/p}(x_1, \mathbf{b}_T, Q) = \int d^2 \mathbf{k}_{1T} e^{-i\mathbf{k}_{1T} \cdot \mathbf{b}_T} f_1^{q/p}(x_1, \mathbf{k}_{1T}^2; Q), \quad (15)$$

$$\tilde{f}_1^{\bar{q}/p}(x_2, \mathbf{b}_T, Q) = \int d^2 \mathbf{k}_{2T} e^{-i\mathbf{k}_{2T} \cdot \mathbf{b}_T} f_1^{\bar{q}/p}(x_2, \mathbf{k}_{2T}^2; Q). \quad (16)$$

2.2. The $\sin 2\phi_h$ Asymmetry of Pion Production in SIDIS

Next, we consider the SIDIS process in which a lepton scatters from a longitudinally polarized nucleon target and produces an observed pion in the final state:

$$l(\ell) + N^\rightarrow(P) \rightarrow l(\ell') + \pi(P_\pi) + X, \quad (17)$$

where a lepton beam with momentum ℓ interacts with a longitudinally polarized nucleon N carrying momentum P . In the final state, the momentum ℓ' of the scattered lepton is detected together with an unpolarized hadron h (in this work, h is taken to be a π meson). We introduce the spacelike momentum transfer $q = \ell - \ell'$ and define the following kinematic invariants

$$x = \frac{Q^2}{2P \cdot q}, \quad y = \frac{P \cdot q}{P \cdot \ell} = \frac{Q^2}{xs}, \quad z = \frac{P \cdot P_\pi}{P \cdot q}, \quad Q^2 = -q^2, \quad s = (P + \ell)^2. \quad (18)$$

Here, s denotes the squared total center-of-mass energy, x represents the Bjorken scaling variable, y is the inelasticity, and z corresponds to the momentum fraction of the parent parton carried by the hadron in the final state.

To the leading order in $1/Q$, the SIDIS cross section (for a nucleon target with longitudinal polarization) can be written as [11]

$$\frac{d^5 \sigma}{dx dy dz d\phi_h dP_{\pi T}^2} = \frac{2\pi\alpha^2}{x_B y Q^2} \times \left\{ (1 - y + \frac{1}{2}y^2) F_{UU,T} + (1 - y) S_L \sin 2\phi_h F_{UL}^{\sin 2\phi_h} \right\}. \quad (19)$$

The relevant reference frame for the process is illustrated in Figure 1. In this frame, the momentum q of the virtual photon defines the z -axis; the hadron plane is spanned by the z -axis and the momentum direction of the detected final-state hadron, while the lepton plane is spanned by the momentum directions of ℓ and ℓ' . Accordingly, ϕ_h denotes the azimuthal angle of the produced hadron measured with respect to the lepton plane; $P_{\pi T}$ is the component of P_π transverse to q , with $P_{\pi T} = -zq_T$ [57]. TMD factorization for the SIDIS process is valid only in the regime where $P_{\pi T}^2/z^2 \ll Q^2$ [15]. Here, $F_{UU,T}$ and $F_{UL}^{\sin 2\phi_h}$ denote the spin-averaged and spin-dependent structure functions, respectively.

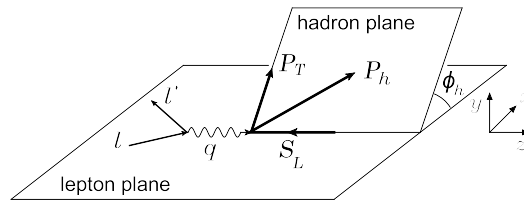


Figure 1. Kinematical setup of the SIDIS process. The momenta of the incoming and scattered leptons determine the lepton plane (the x – z plane), while the momentum of the observed hadron together with the z axis defines the hadron production plane. The nucleon’s longitudinal spin is oriented along the negative z direction.

The $\sin 2\phi_h$ azimuthal asymmetry can be written as

$$A_{UL}^{\sin 2\phi_h}(x, y, z, P_{\pi T}) = \frac{\frac{1}{xyQ^2}(1-y)F_{UL}^{\sin 2\phi_h}}{\frac{1}{xyQ^2}(1-y+\frac{1}{2}y^2)F_{UU,T}}. \quad (20)$$

Within the TMD factorization framework, the structure functions $F_{UU,T}$ and $F_{UL}^{\sin 2\phi_h}$ are expressed through an integral that convolves the transverse parton momentum in the distribution function with that in the fragmentation function [30,31] as

$$F_{UU,T}(x, z, Q; P_{\pi T}^2) = x \sum_q e_q^2 \int d^2 p_T d^2 k_T \delta^2(p_T - k_T - P_{\pi T}/z) f_1^{q/N}(x, p_T^2; Q) D_1^{\pi/q}(z, k_T^2; Q), \quad (21)$$

$$F_{UL}^{\sin 2\phi_h}(x, z, Q; P_{\pi T}) = x \sum_q e_q^2 \int d^2 p_T d^2 k_T \delta^2(p_T - k_T - P_{\pi T}/z) \left[-\frac{2(\hat{h} \cdot k_T)(\hat{h} \cdot p_T) - k_T \cdot p_T}{M_N M_\pi} \right] \\ \times h_{1L}^{\perp q/N}(x, p_T^2; Q) H_1^{\perp \pi/q}(z, k_T^2; Q), \quad (22)$$

where the unit vector $\hat{h}$ is given by $\hat{h} = P_{\pi T}/P_{\pi T}$ and the transverse momentum k_T is connected to the transverse momentum of the produced hadron with respect to the quark via the relation $K_T = -zk_T$.

Transforming to b -space yields a more tractable form. The structure functions become

$$F_{UU,T}(x, z, Q; P_{\pi T}) \\ = x \frac{1}{z^2} \sum_q e_q^2 \int \frac{d^2 b}{(2\pi)^2} e^{iP_{\pi T} \cdot b/z} \tilde{f}_1^{q/N}(x, b; Q) \tilde{D}_1^{\pi/q}(z, b; Q) \\ = x \frac{1}{z^2} \sum_q e_q^2 \int \frac{db b}{2\pi} J_0(|P_{\pi T}| |b|/z) \tilde{f}_1^{q/N}(x, b; Q) \tilde{D}_1^{\pi/q}(z, b; Q) \quad (23)$$

In a similar manner, the spin-dependent structure function $F_{UL}^{\sin(2\phi_h)}$ can be expressed as

$$F_{UL}^{\sin 2\phi_h}(x, z, Q; P_{\pi T}) \\ = x \frac{1}{z^3} \sum_q e_q^2 \int d^2 p_T d^2 K_T \int \frac{d^2 b}{(2\pi)^2} e^{-i(p_T + K_T/z - P_{\pi T}/z) \cdot b} [2(\hat{h} \cdot K_T)(\hat{h} \cdot p_T) - K_T \cdot p_T] \\ \times \frac{h_{1L}^{\perp q/N}(x, p_T^2; Q) H_1^{\perp \pi/q}(z, K_T^2; Q)}{M_N M_\pi} \\ = x \frac{1}{z^3} \sum_q e_q^2 \int \frac{d^2 b}{(2\pi)^2} e^{iP_{\pi T} \cdot b/z} (2\hat{h}_\alpha \hat{h}_\beta - g_{\alpha\beta}) \tilde{h}_{1L}^{\perp \beta, q/N}(x, b; Q) \tilde{H}_1^{\perp \alpha, \pi/q}(z, b; Q) \quad (24)$$

Therefore, we arrive at the evolved expressions for the structure functions $F_{UU,T}(x, z, Q; P_{\pi T})$ and $F_{UL}^{\sin 2\phi_h}(x, z, Q; P_{\pi T})$.

3. TMD Evolution of Parton Distribution and Fragmentation Functions

In this section, we discuss the evolution formalism for the unpolarized PDF f_1 and the longitudinal transversity PDF $h_{1L}^\perp$ of the nucleon, as well as for the unpolarized FF D_1 and the Collins FF $H_1^\perp$ of the pion, all within the framework of TMD factorization.

TMD evolution is most commonly carried out in the conjugate coordinate b -space, because in this representation the cross section takes a simpler, factorized form than in transverse-momentum k_T space (the b -space is related to k_T -space through a Fourier transform) [15,17]. Within TMD factorization formulated in various schemes (for example, the CS-81 [16], JMY [12,58], and Collins-11 schemes [15]), the TMDs $\tilde{F}(x, b; \mu, \zeta_F)$ and $\tilde{D}(z, b; \mu, \zeta_D)$ are functions of two distinct energy scales [13,15–17,50,59]. One scale is the renormalization scale μ , and the other is the energy scale ζ_F (or ζ_D), which acts as a cutoff to regularize the light-cone singularity appearing in the operator definition of the TMDs. These two types of energy dependence are governed by distinct evolution equations. The dependence of the TMD PDFs (or FFs) on ζ_F (or ζ_D) is described by the Collins–Soper (CS) equation [16]:

$$\frac{\partial \ln \tilde{F}(x, b; \mu, \zeta_F)}{\partial \sqrt{\zeta_F}} = \frac{\partial \ln \tilde{D}(z, b; \mu, \zeta_D)}{\partial \sqrt{\zeta_D}} = \tilde{K}(b; \mu), \quad (25)$$

Here, $\tilde{K}$ denotes the Collins–Soper (CS) evolution kernel, which for small values of b can be calculated perturbatively (up to order α_s) as

$$\tilde{K}(b; \mu) = -\frac{\alpha_s C_F}{\pi} \left[\ln(\mu^2 b^2) - \ln 4 + 2\gamma_E \right] + \mathcal{O}(\alpha_s^2), \quad (26)$$

where $\gamma_E \approx 0.577$ is Euler’s constant [16].

The μ dependence of the TMDs is obtained from the renormalization group equation

$$\frac{d \tilde{K}}{d \ln \mu} = -\gamma_K(\alpha_s(\mu)), \quad (27)$$

$$\frac{d \ln \tilde{F}(x, b; \mu, \zeta_F)}{d \ln \mu} = \gamma_F(\alpha_s(\mu); \frac{\zeta_F^2}{\mu^2}), \quad (28)$$

$$\frac{d \ln \tilde{D}(z, b; \mu, \zeta_D)}{d \ln \mu} = \gamma_D(\alpha_s(\mu); \frac{\zeta_D^2}{\mu^2}), \quad (29)$$

where γ_K , γ_F , and γ_D denote the anomalous dimensions associated with $\tilde{K}$, $\tilde{F}$, and $\tilde{D}$, respectively,

$$\gamma_K = 2\frac{\alpha_s C_F}{\pi} + \mathcal{O}(\alpha_s^2), \quad (30)$$

$$\gamma_D = \gamma_F = \alpha_s \frac{C_F}{\pi} \left(\frac{3}{2} - \ln \left(\frac{\zeta_F}{\mu^2} \right) \right) + \mathcal{O}(\alpha_s^2). \quad (31)$$

Solving Equations (27)–(29), one can derive the general expression describing how $\tilde{F}$ (or $\tilde{D}$) depends on the energy [12,15–17,43,52,58]:

$$\tilde{F}(x, b; Q) = \mathcal{F} \times e^{-S(Q, b)} \times \tilde{F}(x, b; \mu_i), \quad (32)$$

$$\tilde{D}(z, b; Q) = \mathcal{D} \times e^{-S(Q, b)} \times \tilde{D}(z, b; \mu_i), \quad (33)$$

where $\mathcal{F}$ and $\mathcal{D}$ denote the factors associated with the hard scattering, and $S(Q, b)$ is the Sudakov form factor. In what follows, we choose $\mu = \sqrt{\zeta_F} = \sqrt{\zeta_D} = Q$, and, for brevity, we will write $\tilde{F}(x, b; \mu = Q, \zeta_F = Q^2)$ (or $\tilde{D}(z, b; \mu = Q, \zeta_D = Q^2)$) simply as $\tilde{F}(x, b; Q)$ (or $\tilde{D}(z, b; Q)$). Equation (32) (or Equation (33)) shows that the distribution $\tilde{F}$ (or $\tilde{D}$) at an

arbitrary scale Q is obtained from the corresponding distribution at an initial scale μ_i via the evolution in the exponential factor $\exp(-S(Q, b))$.

More precisely, the exponential factor $\exp(-S(Q, b))$ can be written in the following explicit form (using the expression for $\tilde{F}$ as an example)

$$\exp(-S(Q, b)) = \exp\left\{\ln\frac{Q}{\mu}\tilde{K}(b_*; \mu) + \int_{\mu_i}^{\mu} \frac{d\bar{\mu}}{\bar{\mu}} \left[\gamma_F(g(\bar{\mu}); 1) - \ln\left(\frac{Q}{\bar{\mu}}\right) \gamma_K(g(\bar{\mu})) \right]\right\} \\ \times \exp\left\{g_{j/P}(x, b) + g_K(b) \ln\frac{Q}{Q_0}\right\}. \quad (34)$$

By introducing the parameter $b_{\max}$ as the separation between the perturbative and nonperturbative regions, one obtains the full expression for the Sudakov form factor that appears in Equations (32) and (33) as

$$S(Q, b) = S_P(Q, b_*) + S_{NP}(Q, b). \quad (35)$$

Specifically, the exponential factor in the first line of Equation (34) is obtained from the solutions of Equations (25), (27) and (28) in the perturbative domain, i.e., the small- b region ($b \ll 1/\Lambda$). This factor includes $\tilde{K}(b_*; \mu)$, the CS evolution kernel valid at small b , and the anomalous dimensions γ_F and γ_K . In contrast, in the nonperturbative regime (the large- b region), the evolution kernel $\tilde{K}(b; \mu)$ cannot be calculated perturbatively. To incorporate the contributions from this large- b region, the exponential appearing in the second line of Equation (34) is introduced. In this term, the function $g_{j/P}(x, b)$ encodes the nonperturbative information of the intrinsic transverse momentum distribution usually defined as the large b_T correction of the perturbative matching of the TMDs onto the PDFs at small b_T , whereas the universal function g_K describes the nonperturbative large- b behavior of the evolution kernel $\tilde{K}(b; \mu)$.

The parameter $b_{\max}$ can ensure a smooth transition of b from the perturbative regime to the nonperturbative one and to avoid encountering the Landau pole. A typical choice for $b_{\max}$ is around 1 GeV^{-1} , which guarantees that b_* remains within the perturbative domain. One may further define a b -dependent function $b_*(b)$ such that $b_* \approx b$ for small b and $b_* \approx b_{\max}$ for large b . Several functional forms for $b_*(b)$ have been proposed in the literature [17,22,60]. A commonly adopted choice is the Collins–Soper–Sterman (CSS) prescription [17]:

$$b_* = b / \sqrt{1 + b^2/b_{\max}^2}, \quad b_{\max} < 1/\Lambda_{\text{QCD}}. \quad (36)$$

The perturbative contribution, $S_P(Q, b_*)$, has been investigated in detail in Refs. [20,45,46,50,59] and takes the following form:

$$S_P(Q, b_*) = \int_{\mu_b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[A(\alpha_s(\bar{\mu})) \ln\frac{Q^2}{\bar{\mu}^2} + B(\alpha_s(\bar{\mu})) \right], \quad (37)$$

which remains valid for various types of PDFs and FFs; in other words, S_P does not depend on spin. Furthermore, the coefficients A and B in Equation (37) may be expressed as a series of α_s/π :

$$A = \sum_{n=1}^{\infty} A^{(n)} \left(\frac{\alpha_s}{\pi}\right)^n, \quad (38)$$

$$B = \sum_{n=1}^{\infty} B^{(n)} \left(\frac{\alpha_s}{\pi}\right)^n. \quad (39)$$

In this analysis, we include the coefficients $A^{(n)}$ up to $A^{(2)}$ and $B^{(n)}$ up to $B^{(1)}$ at next-to-leading logarithmic (NLL) accuracy [13,17,45,47,59,61]:

$$A^{(1)} = C_F, \quad (40)$$

$$A^{(2)} = \frac{C_F}{2} \left[C_A \left(\frac{67}{18} - \frac{\pi^2}{6} \right) - \frac{10}{9} T_R n_f \right], \quad (41)$$

$$B^{(1)} = -\frac{3}{2} C_F, \quad (42)$$

with $C_F = 4/3$, $C_A = 3$ and $T_R = 1/2$.

The non-perturbative contribution S_{NP} in Equation (35) cannot be obtained using perturbative methods; instead, it is typically modeled and its parameters are determined from experimental measurements. Although more recent extractions of non-perturbative Sudakov factors exist [62–66], we adopt the Echevarria–Idilbi–Kang–Vitev (EIKV) [20] and Bacchetta–Delcarro–Pisano–Radici–Signori (BDPRS) [22] parametrizations as representative examples of two distinct approaches to modeling the non-perturbative TMD evolution. These two schemes have been widely used in the literature and allow a direct comparison with our previous studies [37]. Future updates using the latest parametrizations are left for subsequent work.

The EIKV non-perturbative Sudakov factor S_{NP} for unpolarized TMD PDFs (or TMD FFs) is given by

$$S_{\text{NP}}^{\text{pdf}}(b, Q) = b^2 (g_1^{\text{pdf}} + \frac{g_2}{2} \ln \frac{Q}{Q_0}), \quad (43)$$

$$S_{\text{NP}}^{\text{ff}}(b, Q) = b^2 (g_1^{\text{ff}} + \frac{g_2}{2} \ln \frac{Q}{Q_0}). \quad (44)$$

Here, $S_{\text{NP}}^{\text{pdf}}(b, Q)$ and $S_{\text{NP}}^{\text{ff}}(b, Q)$ denote the non-perturbative parts of the Sudakov form factor for the TMD PDF and TMD FF, respectively. g_2 characterizes the behavior of the evolution kernel $\tilde{K}$ at large b values ($g_K(b) = g_2 b^2$). This function is universal across different kinds of TMDs and is independent of the specific process, which is a key outcome of QCD factorization theorems for TMDs [13,15,20,21]. The parameter g_1 encodes the intrinsic nonperturbative transverse motion of the bound partons. It may vary with the type of TMD and can be viewed as the intrinsic transverse-momentum width of the corresponding TMDs at the initial scale Q_0 [13,51,61,67,68]. In addition, g_1^{pdf} and g_1^{ff} are specified as

$$g_1^{\text{pdf}} = \frac{\langle k_T^2 \rangle_{Q_0}}{4}, \quad (45)$$

$$g_1^{\text{ff}} = \frac{\langle p_T^2 \rangle_{Q_0}}{4z^2}, \quad (46)$$

where $\langle k_T^2 \rangle_{Q_0}$ and $\langle p_T^2 \rangle_{Q_0}$ denote the average intrinsic squared transverse momenta associated with the TMD PDFs and FFs at the initial scale Q_0 , respectively. In Ref. [20], the authors adjusted the empirically extracted ranges of the three parameters $\langle k_T^2 \rangle_{Q_0}$, $\langle p_T^2 \rangle_{Q_0}$, and g_2 , using $Q_0 = \sqrt{2.4} \text{ GeV}$ as in Refs. [69–71], and showed that the following fixed parameter values provide a reasonable simultaneous description of SIDIS data, Drell–Yan lepton-pair production, and W/Z boson production data:

$$\langle k_T^2 \rangle_{Q_0} = 0.38 \text{ GeV}^2, \quad \langle p_T^2 \rangle_{Q_0} = 0.19 \text{ GeV}^2, \quad g_2 = 0.16 \text{ GeV}^2, \quad b_{\text{max}} = 1.5 \text{ GeV}^{-1}. \quad (47)$$

In addition to the Sudakov form factors given in Equations (32) and (33), a further key ingredient in these equations is the TMDs evaluated at a fixed scale μ . In the small- b region, $\tilde{F}(x, b; \mu)$ and $\tilde{D}(z, b; \mu)$ at a fixed scale μ can be written as convolutions of

the perturbatively computable coefficients C with the corresponding collinear functions $F_{i/H}(\xi, \mu)$ (or $D_{H/j}(\xi, \mu)$),

$$\tilde{F}(x, b; \mu) = \sum_i \int_x^1 \frac{d\xi}{\xi} C_{q \leftarrow i}(x/\xi, b; \mu) F_{i/H}(\xi, \mu), \quad (48)$$

$$\tilde{D}(z, b; \mu) = \sum_j \int_z^1 \frac{d\xi}{\xi} C_{j \leftarrow q}(z/\xi, b; \mu) D_{H/j}(\xi, \mu), \quad (49)$$

Here, μ denotes a dynamic scale connected to b_* via $\mu = c/b_*$, where $c = 2e^{-\gamma_E}$. The functions $C_{q \leftarrow i}(x/\xi, b; \mu) = \sum_{n=0}^{\infty} C_{q \leftarrow i}^{(n)}(\alpha_s/\pi)^n$ and $C_{j \leftarrow q}(z/\xi, b; \mu) = \sum_{n=0}^{\infty} C_{j \leftarrow q}^{(n)}(\alpha_s/\pi)^n$ represent perturbatively calculable coefficient functions.

After obtaining the solutions of the evolution equations and including the Sudakov form factor, the scale-dependent TMDs in b space can be expressed as

$$\tilde{F}_{q/H}(x, b; Q) = e^{-\frac{1}{2}S_P(Q, b_*) - S_{\text{NP}}^{F_{q/H}}(Q, b)} F_{q/H}(x, \mu), \quad (50)$$

$$\tilde{D}_{H/q}(z, b; Q) = e^{-\frac{1}{2}S_P(Q, b_*) - S_{\text{NP}}^{D_{H/q}}(Q, b)} D_{H/q}(z, \mu) \quad (51)$$

The prefactor $\frac{1}{2}$ in front of S_P arises because S_P is shared equally between the initial-state quark and the final-state quark [72]. In this study, we use the leading-order (LO) expressions for the hard coefficients C , $\mathcal{F}$ and $\mathcal{D}$ associated with f_1 , $h_{1L}^\perp$, D_1 , and $H_1^\perp$, namely $C_{q \leftarrow i}^{(0)} = \delta_{iq}\delta(1-x)$, $C_{j \leftarrow q}^{(0)} = \delta_{qj}\delta(1-z)$, $\mathcal{F} = 1$, and $\mathcal{D} = 1$.

With all the above components specified, we can derive the unpolarized nucleon PDFs $\tilde{f}_1^{q/p}$ (and, analogously, the pion FFs $\tilde{D}_1^{\pi/q}$) in b space

$$\tilde{f}_1^{q/p}(x, b; Q) = e^{-\frac{1}{2}S_P(Q, b_*) - S_{\text{NP}}^{pdf}(Q, b)} f_1^{q/p}(x, \mu), \quad (52)$$

$$\tilde{D}_1^{\pi/q}(z, b; Q) = e^{-\frac{1}{2}S_P(Q, b_*) - S_{\text{NP}}^{ff}(Q, b)} D_1^{\pi/q}(z, \mu). \quad (53)$$

By carrying out the Fourier transform, we can derive the unpolarized nucleon PDFs $f_1^{q/N}$ (or, analogously, the pion FFs $D_1^{\pi/q}$) in transverse-momentum space

$$f_1^{q/N}(x, \mathbf{p}_T; Q) = \int_0^\infty \frac{dbb}{2\pi} J_0(|\mathbf{p}_T|b) e^{-\frac{1}{2}S_P(Q, b_*) - S_{\text{NP}}^{pdf}(Q, b)} f_1^{q/p}(x, \mu), \quad (54)$$

$$D_1^{\pi/q}(z, \mathbf{K}_T; Q) = \int_0^\infty \frac{dbb}{2\pi} J_0(|\mathbf{K}_T|b/z) e^{-\frac{1}{2}S_P(Q, b_*) - S_{\text{NP}}^{ff}(Q, b)} D_1^{\pi/q}(z, \mu). \quad (55)$$

The TMD distributions are constructed using the standard small- b matching formalism. The coefficient functions $C_{q \leftarrow i}$ ensure the matching between TMD and collinear factorization. At leading order, the perturbative matching coefficients reduce to delta functions, so that the longitudinal momentum dependence is entirely inherited from the corresponding collinear distributions, while the transverse-momentum dependence is generated through perturbative and nonperturbative Sudakov evolution. In principle, integrating the transverse momentum of the TMD, one would obtain the collinear PDF, which has been demonstrated in the recent study of the helicity distribution function g_1 [73]. Consequently, possible intrinsic $x - k_\perp$ correlations beyond this factorized approximation are not included. There are extractions of TMD distribution functions [22,73] that can parametrize the initial TMD functional form as the function of x and b , which can generate the transverse momentum dependence through Fourier Transformation.

According to Equations (48) and (49), in the small- b region, the longi-transversity PDF of the nucleon target, $\tilde{h}_{1L}^\perp$, and the pion Collins FF, $\tilde{H}_1^\perp$, at a fixed energy scale μ , can

likewise be represented in terms of perturbatively calculable coefficient functions and the associated collinear correlation functions:

$$\tilde{h}_{1L}^{\perp(\beta)q/N}(x, b; \mu) = ib^\beta M_N \tilde{h}_{1L}^{\perp(1)}(x, \mu), \quad (56)$$

$$\tilde{H}_1^{\perp(\alpha)\pi/q}(z, b; \mu) = \frac{ib^\alpha}{2} \hat{H}^{(3)}(z, z, \mu), \quad (57)$$

where the hard coefficients are evaluated at leading order (LO) and the nucleon target longi-transversity together with the pion Collins fragmentation function in b -space are defined as

$$\begin{aligned} \tilde{h}_{1L}^{\perp\beta,q/N}(x, b; \mu) &= \int d^2 p_T e^{-ip_T \cdot b} \frac{p_T^\beta}{M_N} h_{1L}^{\perp q/N}(x, p_T^2; \mu), \\ \tilde{H}_1^{\perp\alpha,\pi/q}(z, b; \mu) &= \int d^2 K_T e^{-iK_T \cdot b/z} \frac{K_T^\alpha}{M_\pi} H_1^{\perp\pi/q}(z, K_T^2; \mu). \end{aligned} \quad (58)$$

The collinear function $\hat{H}^{(3)}(z, z, \mu)$ denotes the twist-3 quark–gluon–quark correlation function, which is connected to the first transverse-momentum moment of the Collins fragmentation function $H_1^{\perp(1)}$ as shown in Ref. [74]:

$$\hat{H}^{(3)}(z, z, \mu) = \int d^2 K_T \frac{|K_T^2|}{M_\Lambda} H_1^{\perp\pi/q}(z, K_T^2, \mu) = 2M_\pi \tilde{H}_1^{\perp(1)}(z, \mu). \quad (59)$$

For the non-perturbative Sudakov factors associated with $h_{1L}^\perp$ and $H_1^\perp$, we assume they are the same as those for the unpolarized TMD PDF and FF, respectively, i.e., $S_{\text{NP}}^{\text{h}_{1L}^\perp} = S_{\text{NP}}^{\text{pdf}}$ and $S_{\text{NP}}^{H_1^\perp} = S_{\text{NP}}^{\text{ff}}$. This is a common simplifying assumption in the absence of direct extractions. Under this assumption, we can express the longi-transversity and the Collins function in b -space as

$$\tilde{h}_{1L}^{\perp(\beta)q/N}(x, b; Q) = ib^\beta M_N e^{-\frac{1}{2}S_P(Q, b_*) - S_{\text{NP}}^{\text{pdf}}(Q, b)} \tilde{h}_{1L}^{\perp(1)}(x, \mu), \quad (60)$$

$$\tilde{H}_1^{\perp(\alpha)\pi/q}(z, b; Q) = \frac{ib^\alpha}{2} e^{-\frac{1}{2}S_P(Q, b_*) - S_{\text{NP}}^{\text{ff}}(Q, b)} \hat{H}^{(3)}(z, z, \mu). \quad (61)$$

By applying a Fourier transform, the TMDs can be expressed in transverse momentum space

$$\frac{p_\perp^\beta}{M_N} \tilde{h}_{1L}^{\perp q/N(\beta)}(x, p_T; Q) = M_N \int_0^\infty \frac{db b^2}{2\pi} J_1(|\mathbf{p}_T|b) e^{-\frac{1}{2}S_P(Q, b_*) - S_{\text{NP}}^{\text{pdf}}(Q, b)} \tilde{h}_{1L}^{\perp(1)}(x, \mu), \quad (62)$$

$$\frac{K_T^\alpha}{M_\pi} \tilde{H}_1^{\perp\pi/q(\alpha)}(z, K_T; Q) = \int_0^\infty \frac{db b^2}{4\pi} J_1(|\mathbf{K}_T|b/z) e^{-\frac{1}{2}S_P(Q, b_*) - S_{\text{NP}}^{\text{ff}}(Q, b)} \hat{H}^{(3)}(z, z, \mu). \quad (63)$$

In addition to the EIKV parametrization for the non-perturbative factor S_{NP} discussed above, we also employ the BDPRS parametrization of S_{NP} for unpolarized TMDs, which is given by

$$\tilde{f}_1^a(x, b; Q) = f_1^a(x; \mu^2) e^{-S(\mu, Q)} e^{\frac{1}{2}g_K(b) \ln(Q^2/Q_0^2)} \tilde{f}_{1\text{NP}}^a(x, b), \quad (64)$$

$$\tilde{D}_1^{a \rightarrow h}(z, b; Q) = D_1^{a \rightarrow h}(z; \mu^2) e^{-S(\mu, Q)} e^{\frac{1}{2}g_K(b) \ln(Q^2/Q_0^2)} \tilde{D}_{1\text{NP}}^{a \rightarrow h}(z, b), \quad (65)$$

where $g_K = -g_2 b^2/2$, consistent with the convention adopted in Refs. [47,53,55]. The functions $\tilde{f}_{1\text{NP}}^a(x, b^2)$ and $\tilde{D}_{1\text{NP}}^{a \rightarrow h}(z, b^2)$ denote the intrinsic nonperturbative contributions to the PDFs and FFs, respectively, and are modeled as

$$\tilde{f}_{1\text{NP}}^a(x, b^2) = \frac{1}{2\pi} e^{-g_1 \frac{b^2}{4}} \left(1 - \frac{\lambda g_1^2}{1 + \lambda g_1} \frac{b^2}{4} \right), \quad (66)$$

$$\tilde{D}_{1\text{NP}}^{a \rightarrow h}(z, b^2) = \frac{g_3 e^{-g_3 \frac{b^2}{4z^2}} + \left(\frac{\lambda_F}{z^2} \right) g_4^2 (1 - g_4 \frac{b^2}{4z^2}) e^{-g_4 \frac{b^2}{4z^2}}}{2\pi z^2 (g_3 + \left(\frac{\lambda_F}{z^2} \right) g_4^2)}, \quad (67)$$

with

$$g_1(x) = N_1 \frac{(1-x)^\alpha x^\sigma}{(1-\hat{x})^\alpha \hat{x}^\sigma}, \quad (68)$$

$$g_{3,4}(z) = N_{3,4} \frac{(z^\beta + \delta)(1-z)^\gamma}{(\hat{z}^\beta + \delta)(1-\hat{z})^\gamma}. \quad (69)$$

Here, $\hat{x} = 0.1$ and $\hat{z} = 0.5$ are kept fixed, while $\alpha, \sigma, \beta, \gamma, \delta, N_1 \equiv g_1(\hat{x})$, and $N_{3,4} \equiv g_{3,4}(\hat{z})$ are free parameters that are determined by fitting to the available data from SIDIS, Drell–Yan, and W/Z boson production processes. In addition to the $b_*(b)$ prescription of the original CSS formalism [17], several alternative functional forms for $b_*(b)$ have been introduced in the literature [22,60]. In Ref. [22], a novel b_* prescription, distinct from Equation (36), was proposed as

$$b_* = b_{\text{max}} \left(\frac{1 - e^{-b^4/b_{\text{max}}^4}}{1 - e^{-b^4/b_{\text{min}}^4}} \right)^{1/4}. \quad (70)$$

Here, b_{max} again denotes the separation between the nonperturbative and perturbative domains in b space, and is fixed to $b_{\text{max}} = 2e^{-\gamma_E} \text{ GeV}^{-1} \approx 1.123 \text{ GeV}^{-1}$.

The introduction of the parameter b_{min} (or, more generally, the prescription regulating the small- b_T region) is not arbitrary but is closely related to the treatment of logarithmic contributions in the perturbative Sudakov exponent. In transverse-momentum resummation, the logarithmic expansion parameter has the form of $\ln(Q^2 b^2/b_0^2)$, with $b_0 = 2e^{-\gamma_E}$. In the limit $b \rightarrow 0$, the expansion parameter logarithms become unphysically large. To reduce the impact of unjustified higher-order contributions in the large- q_T (small b) region, Ref. [75] introduces a modified logarithmic structure $L \equiv \ln\left(1 + \frac{Q^2 b_T^2}{b_0^2}\right)$ that effectively freezes the argument of the logarithm in the ultraviolet region. In addition, the authors of Ref. [22] chose to saturate b_* at a minimum value $b_{\text{min}} \propto 2e^{-\gamma_E}/Q$ that corresponds to modifying the resummation logarithms.

4. Numerical Evaluation of Physical Observables in Drell–Yan and SIDIS Processes

4.1. Common Input: The Longitudinal Transversity Distribution and Its Evolution

Both the $\cos 2\phi$ asymmetry in the Drell–Yan process and the $\sin 2\phi_h$ asymmetry in SIDIS depend on the collinear function of the $h_{1L}^\perp$.

Since $h_{1L}^\perp$ has not yet been extracted from experimental measurements, we determine $h_{1L}^{\perp(1)}(x, \mu)$ by using the WW-type approximation [76,77]. This approximation is based on the assumption [76] that the quark–quark–gluon correlations (i.e., the genuine twist-3 contributions) and the current-quark mass terms appearing in the operator decomposition of certain distributions can be neglected. It has been shown that the WW approximation provides a good description of the data for the twist-3 structure function $g_2(x)$ (equivalently, the distribution $g_T(x)$) in DIS [78–85]. This result is further corroborated by lattice QCD

computations [86,87] and by theoretical analyses based on the instanton model [88,89]. Within the WW-type approximation, one can express $h_L(x)$ in the following way [77]

$$h_L(x) = 2x \int_x^1 \frac{dy}{y^2} h_1(y) + \tilde{h}_L(x) \stackrel{WW-type}{\approx} 2x \int_x^1 \frac{dy}{y^2} h_1^a(y), \quad (71)$$

which contains the twist-2 component $h_1(y)$ and the genuine twist-3 term $\tilde{h}_L(x)$ originating from quark–quark–gluon correlations; the latter is discarded in the WW approximation. At the same time, it should be emphasized that the WW approximation can be inadequate in certain situations. As discussed in Ref. [90], the genuine twist-3 contributions can violate the WW relation for the structure function g_2 (or $g_T(x)$) by about 15–40% of the magnitude of g_2 . Using both lattice data and data from global fits [91], the twist-3 distribution function $g_T(x)$ has been found to be consistent with its WW approximation for $x \lesssim 0.4$, but within uncertainties, one cannot exclude its violation at levels of up to 40%. Therefore, the WW approximation of $h_{1L}^\perp$ can be seen as a practical estimate, while the further inclusion of the genuine the higher-twist effects may induce uncertainties in the prediction of the $\sin 2\phi_h$ asymmetry.

Furthermore, one can similarly derive the following relation

$$xh_L(x, p_T^2) \stackrel{WW-type}{\approx} \frac{p_T^2}{M^2} h_{1L}^\perp(x, p_T^2). \quad (72)$$

Consequently, $h_{1L}^{\perp(1)}(x)$ is related to the transversity distribution through the relation [85]

$$h_{1L}^{\perp(1)}(x) \stackrel{WW-type}{\approx} -x^2 \int_x^1 \frac{dy}{y^2} h_1(y). \quad (73)$$

where h_1 denotes the transversity distribution. For its valence-quark component, we adopt the parametrization obtained in Ref. [21], which was derived within the TMD evolution framework. At the initial scale $Q_0 = \sqrt{2.4} \text{ GeV}$, this parametrization reads

$$h_1^q(x, Q_0) = N_q^h x^{a_q} (1-x)^{b_q} \frac{(a_q + b_q)^{a_q + b_q}}{a_q^{a_q} b_q^{b_q}} \times \frac{1}{2} \left(f_1^q(x, Q_0) + g_1^q(x, Q_0) \right). \quad (74)$$

Here, g_1^q denotes the helicity distribution function [92], while f_1^q represents the unpolarized distribution function. For the sea-quark contribution to the transversity distribution, we adopt the following assumption:

$$h_1^{sea}(x, Q_0) = N \frac{1}{2} \left(f_1^q(x, Q_0) + g_1^q(x, Q_0) \right), \quad (75)$$

where N denotes the fraction of the proton's contribution arising from sea quarks. In our analysis, we examine three possible values for this sea-quark contribution ratio in the proton: $N = 0$, $N = 0.5$, and $N = 1$.

The scale evolution of $h_{1L}^{\perp(1)}$ has been investigated in detail in Ref. [93], where it was shown to take a rather intricate form. Specifically, the corresponding evolution kernel is composed of a homogeneous and an inhomogeneous contribution. The former is associated with the diagonal component of the relevant operator, while the latter is determined by the nondiagonal component. Since, to date, neither a theoretical calculation nor a phenomenological extraction of the nondiagonal part of the operator for $h_{1L}^{\perp(1)}$ is available, we adopt the same simplifying assumption as in Refs. [21,24], retaining only the diagonal contribution in our phenomenological treatment of TMD evolution. Clearly, this constitutes an approximation to the full dynamics and introduces additional uncertainty in the predicted

asymmetry, originating from the neglected nondiagonal part. In existing studies, only the nondiagonal component of the Qiu–Sterman function $T_{q,F}$ has been analyzed [94] within a model framework, which indicates that the nondiagonal terms in the evolution equation for $T_{q,F}(x, x)$ substantially affect the evolution of these correlation functions, particularly in the small- x region. The inhomogeneous term of the evolution kernel of $h_{1L}^{\perp(1)}$ may also affect the evolution behavior in the small- x region. We anticipate that an analogous behavior may arise in the evolution of $h_{1L}^{\perp(1)}$. A quantitative estimate of the uncertainty due to the nondiagonal contribution is deferred to future work.

Accordingly, we adopt a comparable strategy and focus on the homogeneous contributions to the evolution kernel of $h_{1L}^{\perp(1)}$:

$$P_{qq}^h = C_F \left[\frac{2x}{(1-x)_+} + 2\delta(1-x) \right] - \frac{C_A}{2} \frac{2x}{1-x}. \quad (76)$$

For the evolution kernel of the twist-3 fragmentation function $\hat{H}^{(3)}$, we use the expression given in Ref. [21]:

$$P_{qq}^H = C_F \left[\frac{2x}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right]. \quad (77)$$

The DGLAP equations are solved numerically using the QCDNUM evolution package [95]. We have modified the original QCDNUM code to incorporate the evolution kernel for $h_{1L}^{\perp(1)}$. In Figure 2, we present $h_{1L}^{\perp(1)}(x, Q^2)$ (multiplied by x) as a function of x for light quark flavors, comparing the initial scale $Q^2 = 2.4 \text{ GeV}^2$ with the evolved scale $Q^2 = 100 \text{ GeV}^2$. The left and right panels display the results for the up and down quarks, respectively. The figure indicates that the absolute size of $h_{1L}^{\perp(1)}(x, Q^2)$ is larger for the up quark than for the down quark, with the two having opposite signs. Furthermore, the evolution from the lower to the higher scale shifts the peak of the distribution toward smaller values of x . More complete assessment of theoretical uncertainties, such as those obtained from Monte Carlo replica methods for TMDs, would be highly desirable in future dedicated phenomenological studies. Such an analysis would improve the quantitative assessment of the reliability of the predictions and their direct comparison with experimental measurements.

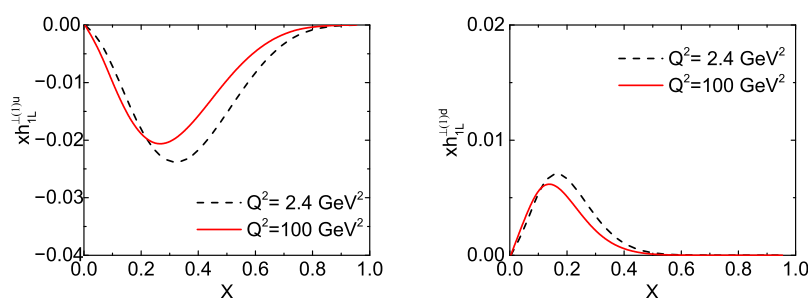


Figure 2. (Left panel): $x h_{1L}^{\perp(1)}(x, Q^2)$ of the proton longitudinal transversity PDF $h_{1L}^{\perp}$ for the up quark at $Q^2 = 2.41 \text{ GeV}^2$ and $Q^2 = 10 \text{ GeV}^2$. (Right panel): same as in the (left panel), but shown for the down quark.

4.2. Drell–Yan Process: Predictions for the $\cos 2\phi$ Asymmetry

In this section, we carry out a detailed numerical study of the $\cos 2\phi$ azimuthal asymmetry in double-longitudinally polarized proton–proton Drell–Yan processes based on the theoretical framework developed in the preceding sections. Our computations are

performed for the kinematic regions at both RHIC and NICA, with particular attention given to the impact of sea quark contributions.

The computation uses the collinear functions $h_{1L}^{\perp(1)}(x, \mu)$ and $f_1(x, \mu)$ as input. For the unpolarized distribution $f_1(x, \mu)$, we utilize the NLO CT10 parametrization (central PDF set) from Ref. [96]. For the longitudinally polarized transversity distribution $h_{1L}^{\perp(1)}(x, \mu)$, we employ the Wandzura–Wilczek approximation described in Section 4.1.

We also use the QCDNUM package [95] to perform the scale evolution of f_1 from the initial value $Q_0 = \sqrt{2.4}$ GeV to higher energies.

The Drell–Yan experiment at RHIC employed two colliding proton beams with center-of-mass energies of either $\sqrt{s} = 200$ GeV or 500 GeV [97]. In this work, we evaluate the $A_{LL}^{\cos 2\phi}$ asymmetry within the RHIC kinematic range [42]:

$$4 \text{ GeV} < Q < 9 \text{ GeV}, \quad 0 < q_T < 1 \text{ GeV}, \quad -1 < y < 2.$$

We further evaluate the $A_{LL}^{\cos 2\phi}$ asymmetry within the kinematic ranges relevant for NICA [42]:

$$\begin{aligned} \sqrt{s} &= 27 \text{ GeV}, & 4 \text{ GeV} < Q < 9 \text{ GeV}, \\ 0 < q_T < 1 \text{ GeV}, & 0.1 < x < 0.8. \end{aligned}$$

Figures 3 and 4 show our numerical predictions for the $\cos 2\phi$ azimuthal asymmetry in pp Drell–Yan processes under RHIC kinematic conditions. The left, middle, and right panels illustrate, respectively, how the asymmetry varies with rapidity (y), transverse momentum (q_T), and invariant mass (Q). Figure 3 presents the results for $\sqrt{s} = 200$ GeV, while Figure 4 displays the corresponding predictions for $\sqrt{s} = 500$ GeV. The dashed, solid, and dotted curves denote the asymmetry for sea-quark contribution parameters $N = 0$, $N = 0.5$, and $N = 1$, respectively.

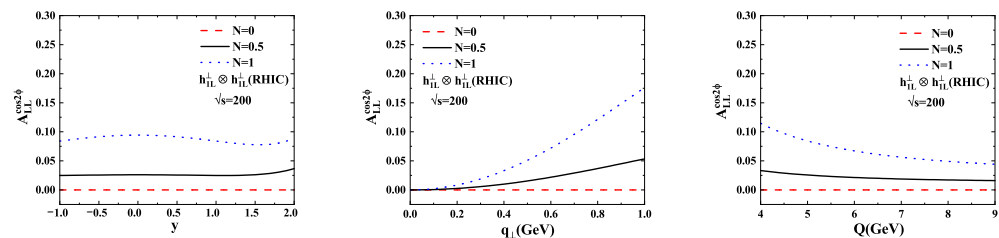


Figure 3. The asymmetry $A_{LL}^{\cos(2\phi)}$ for the double-longitudinally polarized pp Drell–Yan process at RHIC kinematics with $\sqrt{s} = 200$ GeV, shown as functions of y (left panel), q_T (middle panel), and Q (right panel). The kinematic cuts are $4 \text{ GeV} < Q < 9 \text{ GeV}$, $0 < q_T < 1 \text{ GeV}$, and $-1 < y < 2$. The dashed, solid, and dotted curves correspond to $N = 0, 0.5$, and 1 , respectively.

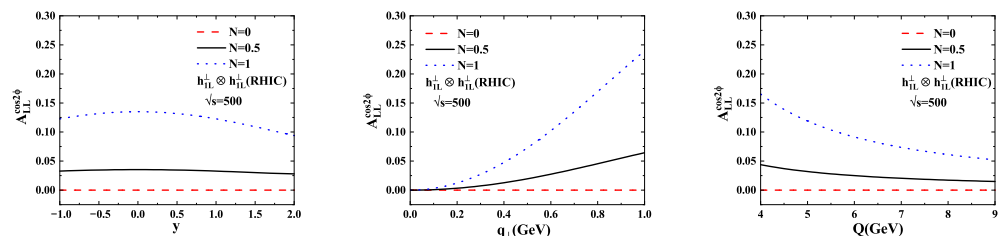


Figure 4. Same as Figure 3, but for asymmetries at $\sqrt{s} = 500$ GeV as functions of y (left panel), q_T (middle panel), and Q (right panel). The kinematic cuts are the same as in Figure 3. The dashed, solid, and dotted curves correspond to $N = 0, 0.5$, and 1 , respectively.

As displayed in Figure 3, for all values of N other than $N = 0$ (for which the asymmetry vanishes), our calculation yields a positive $\cos 2\phi$ azimuthal asymmetry in the doubly

longitudinally polarized proton–proton Drell–Yan process. Our results further indicate that the asymmetry exhibits only a mild variation with y . For $N = 0.5$, the y -dependent asymmetry lies between 2% and 5%, whereas for $N = 1$, it ranges from 8% to 10%. Regarding the q_T -dependent asymmetry, we observe a pronounced increase with q_T for $N = 1$, reaching about 17.5% at $q_T \approx 1$ GeV. For $N = 0.5$, the asymmetry shows only a moderate rise with q_T , attaining a maximum of roughly 5%. For the Q dependence, $A_{LL}^{\cos 2\phi}$ decreases moderately from 12% to 5% as Q grows from 4 to 9 GeV when $N = 1$. The Q dependence is weaker for $N = 0.5$, with the asymmetry varying from about 4% to 2%. Similar qualitative behaviors appear in Figure 2, but with larger absolute values (up to about 25%).

The NICA predictions ($\sqrt{s} = 27$ GeV) shown in Figure 5 display the same qualitative trends as the RHIC results but differ in their quantitative characteristics. Regarding the Bjorken- x dependence, we find a non-monotonic asymmetry for $N = 1$, with a peak value of about 22%. The asymmetry first decreases in the interval $0.1 < x < 0.2$, and then gradually increases up to $x \approx 0.7$. This pattern reflects the intricate balance between valence and sea quark contributions in the longitudinally polarized proton.

For the TMD dependence, the q_T behavior at NICA follows trends similar to those at RHIC, albeit with smaller magnitudes. The asymmetry reaches a maximum of 17% for $N = 1$, indicating the sensitivity of this observable to the transverse momentum distribution of quarks inside the proton.

The Q dependence at NICA is also analogous to that observed at RHIC, with the asymmetry slowly decreasing as Q increases. The maximum asymmetry is about 15% for $N = 1$, in line with expectations from TMD factorization. Collectively, these findings underscore the significance of measuring the asymmetry over a broad kinematic range to tightly constrain sea quark contributions and their variation with the hard scale Q .

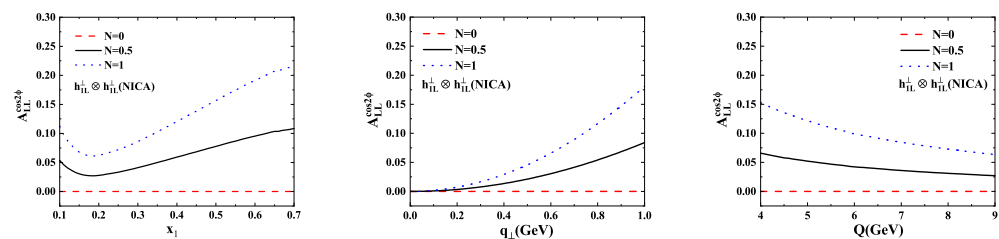


Figure 5. Same as Figure 3, but for asymmetries at NICA kinematics with $\sqrt{s} = 27$ GeV as functions of x_1 (left panel), q_T (middle panel), and Q (right panel). The kinematic cuts are the same as in Figure 3. The dashed, solid, and dotted curves correspond to $N = 0, 0.5$, and 1 , respectively. The kinematic cuts are $4 \text{ GeV} < Q < 9 \text{ GeV}$, $0 < q_T < 1 \text{ GeV}$, and $0.1 < x < 0.8$.

We additionally estimate the Drell–Yan event rate expected at NICA. Given the projected luminosity of $\mathcal{L} = 10^{32} \text{ cm}^{-2} \text{ s}^{-1}$ [98] and an assumed annual running time of $3 \times 10^7 \text{ s}$, the corresponding integrated luminosity is

$$\mathcal{L}_{\text{int}} = \mathcal{L} \times t \approx 3 \times 10^{39} \text{ cm}^{-2}. \quad (78)$$

The unpolarized cross section σ^{DY} obtained in our framework (Equation (5)) is first expressed in natural units (GeV^{-2}). By converting to experimental units (cm^2) using $1 \text{ GeV}^{-2} \approx 3.894 \times 10^{-28} \text{ cm}^2$, we find $\sigma^{\text{DY}} \approx 5 \times 10^{-37} \text{ cm}^2$. The corresponding expected number of events per year can then be estimated as

$$N_{\text{events}} = \sigma^{\text{DY}} \times \mathcal{L}_{\text{int}} \approx 1600. \quad (79)$$

This result indicates that the proposed asymmetry $A_{LL}^{\cos 2\phi}$ should be within experimental reach at NICA.

In conclusion, as illustrated in Figures 3–5, the $\cos 2\phi$ azimuthal asymmetries in doubly longitudinally polarized proton–proton Drell–Yan scattering are positive in all scenarios except for $N = 0$, where the asymmetry disappears. In particular, the asymmetry attains a peak value of about 25% for $N = 1$ and up to 10% for $N = 0.5$, indicating a pronounced sensitivity to the longitudinally polarized transversity distributions of sea quarks. These findings emphasize that high-precision measurements at RHIC and NICA can substantially deepen our knowledge of nucleon structure and QCD dynamics. Furthermore, the doubly polarized $p \rightarrow p \rightarrow$ Drell–Yan reaction stands out as a powerful tool for probing the contribution of sea quarks to the proton’s internal structure, offering key insights into the nonperturbative domain of the strong interaction.

4.3. SIDIS Process: Calculations and Experimental Comparison for the $\sin 2\phi_h$ Asymmetry

The most direct constraints on the longi-transversity distribution come from HERMES (proton and deuteron targets) and CLAS (proton target) measurements of the $\sin 2\phi_h$ asymmetry in SIDIS. For the Drell–Yan process, no double-polarized measurement has been performed to date, which motivates the predictions for facilities such as NICA presented in Section 4.2.

In this section, based on the framework established above, we provide the numerical evaluation of the $\sin 2\phi_h$ azimuthal asymmetry for π production in the process $lN \rightarrow l\pi X$ involving a longitudinally polarized nucleon. We compute the asymmetry for the kinematic settings of HERMES, CLAS, and CLAS12, and compare our results with the latest experimental data [38–40,99–105]. For this analysis, we require collinear functions, as input for the evolution, that enter Equations (23) and (24). For the nucleon unpolarized PDF $f_1(x, \mu)$, we use the NLO central set of the CT10 parametrization [96]. For the pion unpolarized FF $D_1(z, \mu)$, we employ the NLO set of the de Florian, Sassot, and Stratmann (DSS) fragmentation functions [106]. The twist-3 Collins FF $\hat{H}^{(3)}(z, z, \mu)$ of the pion is taken from the parametrization proposed in Ref. [21]. For the longitudinally polarized transversity distribution $h_{1L}^{\perp(1)}(x, \mu)$, we also employ the Wandzura–Wilczek approximation described in Section 4.1.

The $A_{UL}^{\sin 2\phi_h}$ asymmetry in pion production was determined at HERMES using both a proton target [38,99] and a deuteron target [39], within the kinematic region

$$1 \text{ GeV}^2 < Q^2 < 15 \text{ GeV}^2, \quad W > 2 \text{ GeV}, \quad 0.023 < x < 0.4, \quad 0.2 < y < 0.85, \quad 0.2 < z < 0.7, \quad (80)$$

while it was likewise determined for pion production from a proton target by CLAS [40,100] within the kinematic range

$$1 \text{ GeV}^2 < Q^2 < 5.4 \text{ GeV}^2, \quad W > 2 \text{ GeV}, \quad 0.12 < x < 0.48, \quad y < 0.85, \quad 0.4 < z < 0.7, \quad (81)$$

where $W^2 = (P + q)^2 \approx \frac{1-x}{x} Q^2$ denotes the invariant mass squared of the virtual photon–nucleon system.

In Figures 6 and 7, we present numerical predictions for the $\sin 2\phi_h$ azimuthal asymmetry in SIDIS off longitudinally polarized proton and deuteron targets at HERMES kinematics [38,39,82,99,107]. These results are obtained within the TMD factorization framework specified in Equations (20), (23) and (24). The solid squares denote the HERMES experimental measurements [38,39,107], with the error bars indicating the corresponding statistical uncertainties. To maintain the applicability of TMD factorization, the transverse momentum of the produced pion is integrated only over the range $P_{\pi T}/z < 0.5 \text{ GeV}$. In our computation, we employ the EIKV parametrization (dark dashed curve) and the BDPRS

parametrization (red solid curve) for the nonperturbative component. The yellow, green, and cyan shaded bands around the solid curves represent, respectively, the uncertainties originating from the BDPRS parametrization parameters, from the transversity distribution $h_1(x)$ used to determine $h_{1L}^{\perp(1)}(x)$, and from the twist-3 Collins FF $\hat{H}^{(3)}$. For the dashed curves, only two sources—namely the transversity distribution $h_1(x)$ entering $h_{1L}^{\perp(1)}(x)$ and the twist-3 Collins FF $\hat{H}^{(3)}$ —contribute to the total uncertainty, so their uncertainty bands are expected to be narrower than those associated with the solid curves.

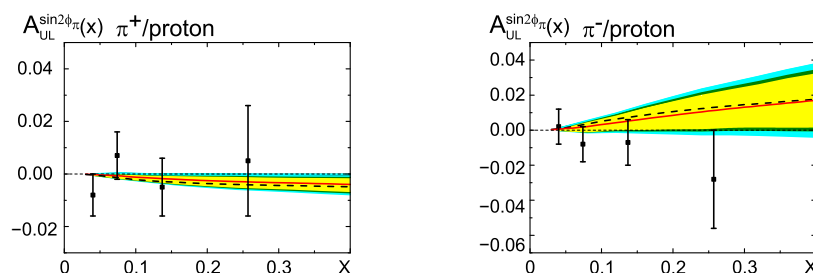


Figure 6. The $A_{UL}^{\sin 2\phi_h}$ asymmetry for π^+ (left) and π^- (right) production on a proton target is shown as a function of x for the HERMES experiment. The solid curves represent the predictions obtained using the BDPRS parametrization [22] of the nonperturbative form factor, while the dashed curves indicate the results based on the EIKV parametrization [20] of the nonperturbative form factor. The yellow, green, and cyan shaded bands around the solid curves denote, respectively, the uncertainties arising from the parameters of the BDPRS parametrization, from the transversity distribution $h_1(x)$ entering $h_{1L}^{\perp(1)}(x)$, and from the twist-3 Collins fragmentation function $\hat{H}^{(3)}$. The kinematic cuts are $1 \text{ GeV}^2 < Q^2 < 15 \text{ GeV}^2$, $W > 2 \text{ GeV}$, $0.023 < x < 0.4$, $0.2 < y < 0.85$, $0.2 < z < 0.7$, and $P_{\pi T}/z < 0.5 \text{ GeV}$. For comparison, the solid squares with error bars show the HERMES measurements [38,107].

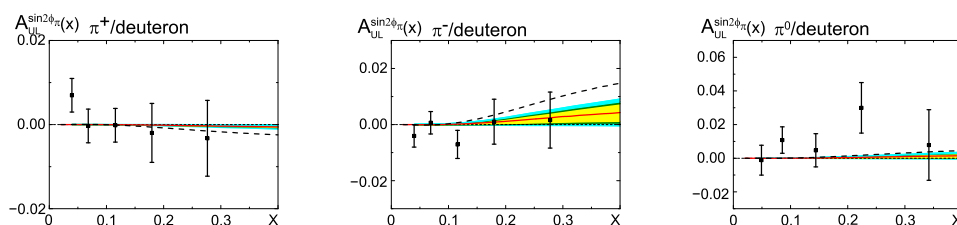


Figure 7. Same as Figure 6, but now for π^+ (left), π^- (middle), and π^0 (right) production on a deuteron target. The kinematic cuts are the same as in Figure 6. The data are taken from Ref. [39].

As illustrated in Figures 6 and 7, the $\sin 2\phi_h$ azimuthal asymmetries in SIDIS do not exceed roughly 2% in any of the considered cases. The predicted $\sin 2\phi_h$ asymmetry for π^+ is negative, whereas those for π^- and π^0 are positive, with the magnitude of the π^- asymmetry exceeding that of both π^+ and π^0 . Furthermore, our results indicate that the magnitude of the asymmetries grows with increasing x . For a proton target, the predictions based on the EIKV parametrization closely match those obtained with the BDPRS parametrization; however, for a deuteron target, the two parametrizations differ quantitatively, with the EIKV-based results being about twice as large as those from BDPRS. By comparing with the HERMES measurements [38,39,82,99,107], we observe that our calculations are compatible with the data within the experimental uncertainties for all channels, except for π^- production on a proton target. In this latter case, the predicted asymmetry disagrees with the HERMES result, most notably in the opposite sign. A plausible explanation for this discrepancy is the presence of sizable uncertainties that have not yet been incorporated into our current analysis. We will discuss the origin of these uncertainties at the end of this section. The nonvanishing $\sin 2\phi_h$ azimuthal asymmetries

predicted for $\pi^\pm$ suggest that the spin–orbit correlation of transversely polarized quarks inside a longitudinally polarized nucleon could be sizable.

In Figure 8, we present our numerical results for the $\sin 2\phi_h$ azimuthal asymmetry on a proton target, evaluated at CLAS kinematics [40,100]. The top panels of Figure 8 display the x -dependent asymmetries for π^+ (left) and π^- (right) production, while the bottom panels show the x -dependence (left) and the $P_{\pi T}$ -dependence (right) for π^0 production. For clarity, each panel is magnified to better highlight the trends. Despite the relatively large experimental uncertainties at CLAS, the data suggest that the asymmetries for both π^+ and π^- are predominantly negative. Consequently, our calculation does not reproduce the measured x -dependent asymmetry for π^- production. For π^0 production, we obtain a much smaller $\sin 2\phi$ azimuthal asymmetry at CLAS, as well as at HERMES for the deuteron target, consistent with the experimental observations. This suppression arises because π^0 production involves the sum of favored and unfavored Collins functions, which largely cancel each other. As before, the yellow, green, and cyan shaded regions surrounding the solid curves represent the uncertainty bands associated with the BDPRS parametrization parameters, the transversity distribution $h_1(x)$ entering the evaluation of $h_{1L}^{\perp(1)}(x)$, and the twist-3 Collins fragmentation function $H^{(3)}$, respectively.

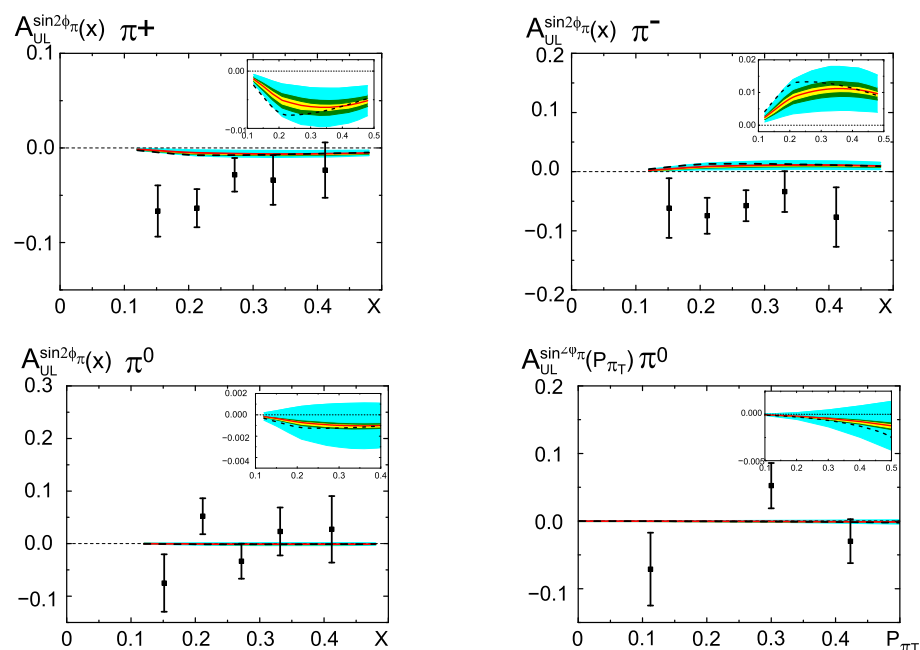


Figure 8. Same as Figure 6, but now showing $A_{UL}^{\sin 2\phi_h}$ as a function of x and $P_{\pi T}$ for CLAS kinematics. The (**upper** panels) display the x -dependent asymmetries for π^+ (**left**) and π^- (**right**) production, respectively, whereas the (**lower** panels) present the x dependence (**left**) and the $P_{\pi T}$ dependence (**right**) of the asymmetries for π^0 production. The kinematic cuts are $1 \text{ GeV}^2 < Q^2 < 5.4 \text{ GeV}^2$, $W > 2 \text{ GeV}$, $0.12 < x < 0.48$, $y < 0.85$, $0.4 < z < 0.7$, and $P_{\pi T}/z < 0.5 \text{ GeV}$. The solid curves represent the predictions obtained using the BDPRS parametrization [22] of the nonperturbative form factor, while the dashed curves indicate the results based on the EIKV parametrization [20] of the nonperturbative form factor. The yellow, green, and cyan shaded bands around the solid curves denote, respectively, the uncertainties arising from the parameters of the BDPRS parametrization, from the transversity distribution $h_1(x)$ entering $h_{1L}^{\perp(1)}(x)$, and from the twist-3 Collins fragmentation function $\hat{H}^{(3)}$. The solid squares indicate the CLAS data with error bars [40,100] included for comparison. In each panel, we zoom in to better highlight the observed trends.

Although the $A_{UL}^{\sin 2\phi_h}$ asymmetries on longitudinally polarized targets have been investigated in HERMES (proton, deuteron) [38,39,99] and CLAS (proton) [40,100] kinematics over the past two decades, a coherent picture of how each individual contribution enters

the total structure function is still lacking, likely because earlier measurements suffered from limited statistics and restricted kinematic coverage. Accordingly, we additionally evaluate the $A_{UL}^{\sin 2\phi_h}$ asymmetries on a proton target for pion production under the kinematic conditions of the CLAS12 experiment [101]

$$1 \text{ GeV}^2 < Q^2 < 7 \text{ GeV}^2, \quad W > 2 \text{ GeV}, \quad 0.13 < x < 0.52, \quad y < 0.75, \quad 0.18 < z < 0.7. \quad (82)$$

The estimated asymmetries for π^+ (upper panel), π^- (middle panel), and π^0 (lower panel) production as functions of x , $P_{\pi T}$, and z are shown in Figure 9, respectively. From Figure 9, we infer that the magnitude of the $\sin 2\phi_h$ azimuthal asymmetries at CLAS12 exceeds that observed at CLAS. Consequently, CLAS12 offers an enhanced opportunity to measure the $\sin 2\phi_h$ asymmetry. Furthermore, we note that the $P_{\pi T}$ -dependent asymmetries obtained from the EIKV parametrization and the BDPRS parametrization differ to some extent. These differences should be examined further with upcoming CLAS12 data on a proton target.

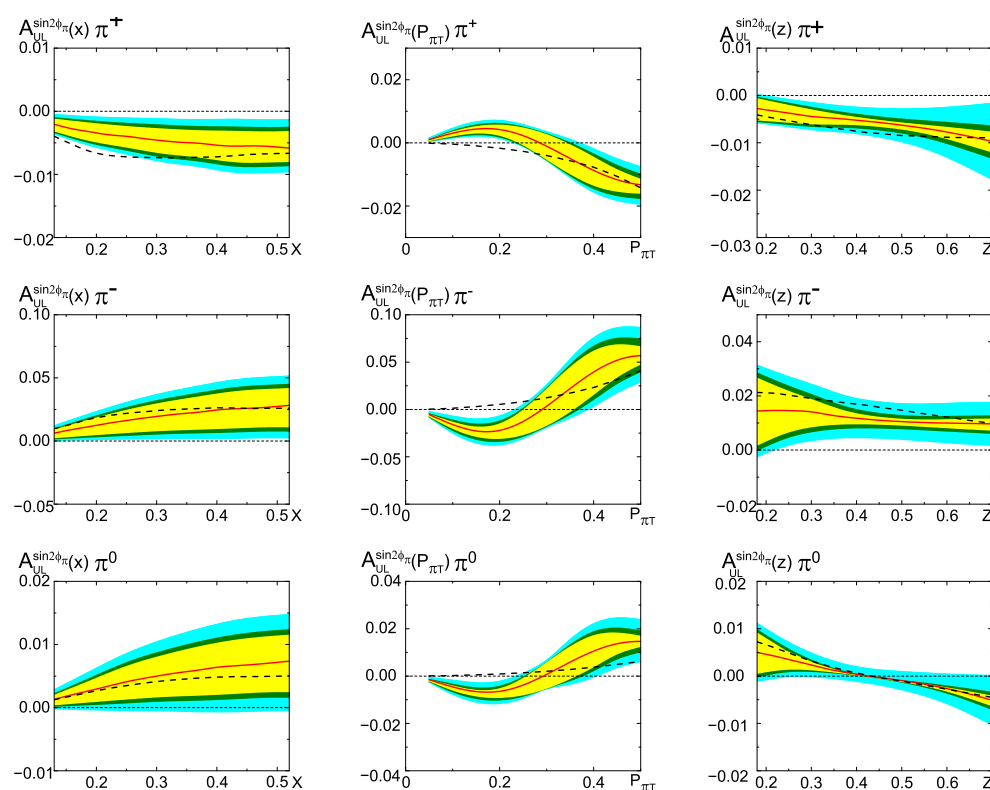


Figure 9. The $A_{UL}^{\sin 2\phi_h}$ asymmetry as a function of x , $P_{\pi T}$, and z for π^+ (top panel), π^- (middle panel), and π^0 (bottom panel) production under CLAS12 kinematic conditions. The solid curves represent the predictions obtained using the BDPRS parametrization [22] of the nonperturbative form factor, while the dashed curves indicate the results based on the EIKV parametrization [20] of the nonperturbative form factor. The yellow, green, and cyan shaded bands around the solid curves denote, respectively, the uncertainties arising from the parameters of the BDPRS parametrization, from the transversity distribution $h_1(x)$ entering $h_{1L}^{(1)}(x)$, and from the twist-3 Collins fragmentation function $\hat{H}^{(3)}$. The kinematic cuts are $1 \text{ GeV}^2 < Q^2 < 7 \text{ GeV}^2$, $W > 2 \text{ GeV}$, $0.13 < x < 0.52$, $y < 0.75$, $0.18 < z < 0.7$, and $P_{\pi T}/z < 0.5 \text{ GeV}$.

Finally, we comment on the theoretical sources of uncertainty. First, one should be careful about the applicability of TMD factorization in the HERMES and CLAS kinematic ranges, where Q^2 is relatively low. In our analysis, we restrict ourselves to $P_{\pi T}/z < 0.5 \text{ GeV}$, but this assumption needs to be examined with more precise data in the future. Second,

since the knowledge of $h_{1L}^\perp$ is still quite limited, we rely on the WW approximation to determine $h_{1L}^\perp$ from the underlying function $h_1(x)$; therefore, contributions beyond the WW approximation, which are not included in our calculation, may also play a role. Although experimental data and lattice results indicate that the WW approximation works reasonably well for the structure function g_2 (or $g_T(x)$), the analysis in Ref. [90] suggests that the violation of the WW relation may reach 15–40% of the magnitude of g_2 . This implies that, for $h_{1L}^\perp$, higher-twist effects could likewise be significant, and a rough estimate is that the associated uncertainty might be as large as about 40% of the central asymmetry value. Third, for $\hat{H}^{(3)}$ and $h_{1L}^{\perp(1)}$, we retain only the homogeneous terms in the evolution kernel. The inhomogeneous terms are highly nontrivial and are therefore omitted to keep the analysis tractable. This is, of course, an approximation to the actual situation. Fourth, in our computation, we use the NLL expression for the perturbative Sudakov form factor. Higher-order QCD corrections, which we do not include, may further modify the results and thus influence the overall precision. If all these sources of uncertainty were properly incorporated, the error bands in Figures 3–6 would be considerably broader. Additional investigations are required to resolve these issues.

5. Summary and Prospects

This review synthesized recent phenomenological advances in the study of the longi-transversity distribution $h_{1L}^\perp$ within the framework of TMD factorization. By examining the complementary roles of doubly polarized Drell–Yan scattering and SIDIS, we have demonstrated how specific azimuthal asymmetries serve as unique and sensitive probes of the correlation between a longitudinally polarized nucleon and transversely polarized quarks—a cornerstone of the three-dimensional spin structure of the nucleon.

The numerical studies presented here, which consistently incorporate TMD evolution at next-to-leading logarithmic accuracy, lead to several robust conclusions. In polarized proton–proton Drell–Yan collisions, the $\cos 2\phi$ asymmetry is predicted to be substantial, potentially reaching 25% under maximal sea quark contributions at RHIC kinematics. This pronounced signal establishes the asymmetry as a powerful direct probe of polarized sea-quark dynamics. Event yield estimates further confirm the practical measurability of this observable at forthcoming facilities such as NICA. In contrast, in SIDIS off longitudinally polarized targets, the $\sin 2\phi_h$ asymmetry is typically smaller, at the level of a few percent, and exhibits clear dependencies on final-state hadron charge and target type. Theoretical predictions, relying on the Wandzura–Wilczek approximation for $h_{1L}^\perp$ and employing two distinct non-perturbative Sudakov parameterizations, show broad consistency with existing HERMES and CLAS data, although certain tensions, notably for π^- production, highlight areas for further scrutiny. The projected larger asymmetries at CLAS12 promise enhanced precision for future constraints. The application of different TMD evolution schemes underscores the overall stability of the predictions while simultaneously revealing that specific choices, particularly for the non-perturbative sector, can influence outcomes for deuteron targets and the detailed P_T dependence of observables.

These investigations firmly establish $h_{1L}^\perp$ as a pivotal element in the quest for a complete tomographic portrait of the nucleon. They also delineate clear paths for future theoretical and experimental progress. On the theoretical front, moving beyond the Wandzura–Wilczek approximation and incorporating the full inhomogeneous terms in evolution kernels are critical steps towards reducing current uncertainties and achieving precision. A global QCD analysis that simultaneously fits $h_{1L}^\perp$ from Drell–Yan and SIDIS data within a unified TMD framework will be essential to minimize model dependencies and rigorously test the universality of TMD functions. Experimentally, the future is bright. Upcoming and planned measurements at facilities such as the upgraded NICA, CLAS12, and EIC will

deliver high-statistics, multi-dimensional data across a vast kinematic landscape. This new generation of data will be indispensable for disentangling the contributions of valence and sea quarks, providing stringent tests of TMD evolution, and ultimately yielding a definitive understanding of longitudinal transversity and its integral role in the nucleon spin puzzle.

In summary, the exploration of azimuthal asymmetries through the lens of TMD factorization constitutes a dynamic and rapidly advancing frontier in hadron physics. The theoretical formalism and phenomenological results summarized in this work provide an overview of the current status of the field and offer a framework for understanding transversely polarized quarks inside longitudinally polarized nucleons, while also highlighting opportunities for future studies at complementary high-energy facilities.

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Abbreviations

The following abbreviations are used in this manuscript:

TMD	transverse momentum dependent
SIDIS	semi-inclusive deep inelastic scattering
QCD	Quantum Chromodynamics
PDFs	parton distribution functions
FFs	fragmentation functions
CSS	Collins–Soper–Sterman

References

1. Gross, D.J.; Wilczek, F. Ultraviolet Behavior of Non-Abelian Gauge Theories. *Phys. Rev. Lett.* **1973**, *30*, 1343–1346.
2. Politzer, H.D. Reliable Perturbative Results for Strong Interactions? *Phys. Rev. Lett.* **1973**, *30*, 1346–1349.
3. Gross, D.J.; Wilczek, F. Asymptotically Free Gauge Theories-I. *Phys. Rev. D* **1973**, *8*, 3633–3652.
4. Drell, S.D.; Levy, D.J.; Yan, T.M. A Field-Theoretic Model for electron-Nucleon Deep Inelastic Scattering. *Phys. Rev. Lett.* **1969**, *22*, 744–748.
5. Drell, S.D.; Levy, D.J.; Yan, T.M. Theory of Deep-Inelastic Lepton-Nucleon Scattering and Lepton-Pair Annihilation Processes. I. *Phys. Rev.* **1969**, *187*, 2159–2171.
6. Ball, R.D.; Carrazza, S.; Cruz-Martinez, J.; Debbio, L.D.; Forte, S.; Giani, T.; Iranipour, S.; Kassabov, Z.; Latorre, J.I.; Nocera, E.R.; et al. The path to proton structure at 1% accuracy: NNPDF Collaboration. *Eur. Phys. J. C* **2022**, *82*, 428.
7. Bailey, S.; Cridge, T.; Harland-Lang, L.A.; Martin, A.D.; Thorne, R.S. Parton distributions from LHC, HERA, Tevatron and fixed target data: MSHT20 PDFs. *Eur. Phys. J. C* **2021**, *81*, 341.
8. Hou, T.J.; Gao, J.; Hobbs, T.J.; Xie, K.; Dulat, S.; Guzzi, M.; Huston, J.; Nadolsky, P.; Pumplin, J.; Schmidt, C., et al. New CTEQ global analysis of quantum chromodynamics with high-precision data from the LHC. *Phys. Rev. D* **2021**, *103*, 014013.
9. Boer, D.; Vogelsang, W. Drell-Yan lepton angular distribution at small transverse momentum. *Phys. Rev. D* **2006**, *74*, 014004.
10. Arnold, S.; Metz, A.; Schlegel, M. Dilepton production from polarized hadron hadron collisions. *Phys. Rev. D* **2009**, *79*, 034005.
11. Bacchetta, A.; Diehl, M.; Goeke, K.; Metz, A.; Mulders, P.J.; Schlegel, M. Semi-inclusive deep inelastic scattering at small transverse momentum. *J. High Energy Phys.* **2007**, *02*, 093.

12. Ji, X.D.; Ma, J.P.; Yuan, F. QCD factorization for semi-inclusive deep-inelastic scattering at low transverse momentum. *Phys. Rev. D* **2005**, *71*, 034005.
13. Aybat, S.M.; Rogers, T.C. Transverse momentum dependent parton distribution and fragmentation functions with QCD evolution. *Phys. Rev. D* **2011**, *83*, 114042.
14. Collins, J.C.; Rogers, T.C. Equality of Two Definitions for Transverse Momentum Dependent Parton Distribution Functions. *Phys. Rev. D* **2013**, *87*, 034018.
15. Collins, J. *Foundations of Perturbative QCD*; Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology; Cambridge University Press: Cambridge, UK, 2011; Volume 32, pp. 1–624.
16. Collins, J.C.; Soper, D.E. Back-To-Back Jets in QCD. *Nucl. Phys. B* **1981**, *193*, 381–443. Erratum in *Nucl. Phys. B* **1983**, *213*, 545.
17. Collins, J.C.; Soper, D.E.; Sterman, G.F. Transverse Momentum Distribution in Drell-Yan Pair and W and Z Boson Production. *Nucl. Phys. B* **1985**, *250*, 199–224.
18. Echevarría, M.G.; Idilbi, A.; Scimemi, I. Soft and Collinear Factorization and Transverse Momentum Dependent Parton Distribution Functions. *Phys. Lett. B* **2013**, *726*, 795–801.
19. Pitonyak, D.; Schlegel, M.; Metz, A. Polarized hadron pair production from electron-positron annihilation. *Phys. Rev. D* **2014**, *89*, 054032.
20. Echevarria, M.G.; Idilbi, A.; Kang, Z.B.; Vitev, I. QCD Evolution of the Sivers Asymmetry. *Phys. Rev. D* **2014**, *89*, 074013.
21. Kang, Z.B.; Prokudin, A.; Sun, P.; Yuan, F. Extraction of Quark Transversity Distribution and Collins Fragmentation Functions with QCD Evolution. *Phys. Rev. D* **2016**, *93*, 014009.
22. Bacchetta, A.; Delcarro, F.; Pisano, C.; Radici, M.; Signori, A. Extraction of partonic transverse momentum distributions from semi-inclusive deep-inelastic scattering, Drell-Yan and Z-boson production. *J. High Energy Phys.* **2017**, *2017*, 81.
23. Wang, X.; Lu, Z.; Schmidt, I. Transverse momentum spectrum of dilepton pair in the unpolarized $\pi^- N$ Drell-Yan process within TMD factorization. *J. High Energy Phys.* **2017**, *2017*, 137.
24. Wang, X.; Lu, Z. Sivers asymmetry in the pion induced Drell-Yan process at COMPASS within transverse momentum dependent factorization. *Phys. Rev. D* **2018**, *97*, 054005.
25. Li, H.; Wang, X.; Lu, Z. $\sin(2\phi - \phi_S)$ azimuthal asymmetry in the pion induced Drell-Yan process within TMD factorization. *Phys. Rev. D* **2020**, *101*, 054013.
26. Mulders, P.J.; Tangerman, R.D. The Complete tree-level result up to order $1/Q$ for polarized deep-inelastic leptonproduction. *Nucl. Phys. B* **1996**, *461*, 197–237.
27. Barone, V.; Drago, A.; Ratcliffe, P.G. Transverse polarisation of quarks in hadrons. *Phys. Rep.* **2002**, *359*, 1–168.
28. Barone, V.; Bradamante, F.; Martin, A. Transverse-spin and transverse-momentum effects in high-energy processes. *Prog. Part. Nucl. Phys.* **2010**, *65*, 267–333.
29. Boer, D.; Mulders, P.J. Time-reversal odd distribution functions in leptonproduction. *Phys. Rev. D* **1998**, *57*, 5780–5786.
30. Kotzinian, A. New quark distributions and semi-inclusive electroproduction on polarized nucleons. *Nucl. Phys. B* **1995**, *441*, 234–256.
31. Kotzinian, A.M.; Mulders, P.J. Probing transverse quark polarization via azimuthal asymmetries in leptonproduction. *Phys. Lett. B* **1997**, *406*, 373–380.
32. Lu, Z.; Ma, B.Q.; She, J. $\sin 2\phi$ azimuthal asymmetry in single longitudinally polarized πN Drell-Yan process. *Phys. Rev. D* **2011**, *84*, 034010.
33. Zhu, J.; Ma, B.Q. Proposal for measuring new transverse momentum dependent parton distributions g_{1T} and $h_{1L}^\perp$ through semi-inclusive deep inelastic scattering. *Phys. Lett. B* **2011**, *696*, 246–251.
34. Boffi, S.; Efremov, A.V.; Pasquini, B.; Schweitzer, P. Azimuthal spin asymmetries in light-cone constituent quark models. *Phys. Rev. D* **2009**, *79*, 094012.
35. Ma, B.Q.; Schmidt, I.; Yang, J.J. Nucleon transversity distribution from azimuthal spin asymmetry in pion electroproduction. *Phys. Rev. D* **2001**, *63*, 037501.
36. Ma, B.Q.; Schmidt, I.; Yang, J.J. Azimuthal spin asymmetries of pion electroproduction. *Phys. Rev. D* **2002**, *65*, 034010.
37. Li, H.; Lu, Z. The $\sin 2\phi_h$ azimuthal asymmetry of pion production in SIDIS within TMD factorization. *Eur. Phys. J. C* **2022**, *82*, 668.
38. Airapetian, A.; Akopov, N.; Amarian, M.; Aschenauer, E.C.; Avakian, H.; Avakian, R.; Avetissian, A.; Avetissian, E.; Bains, B.; Baumgarten, C.; et al. Evidence for a Single-Spin Azimuthal Asymmetry in Semi-inclusive Pion Electroproduction. *Phys. Rev. Lett.* **2000**, *84*, 4047–4051.
39. Airapetian, A.; Akopov, N.; Akopov, Z.; Amarian, M.; Ammosov, V.V.; Andrus, A.; Aschenauer, E.C.; Augustyniak, W.; Avakian, H.; Avakian, R.; et al. Measurement of single-spin azimuthal asymmetries in semi-inclusive electroproduction of pions and kaons on a longitudinally polarised deuterium target. *Phys. Lett. B* **2003**, *562*, 182–192.

40. Avakian, H.; Bosted, P.; Burkert, V.D.; Elouadrhiri, L.; Adhikari, K.P.; Aghasyan, M.; Amarian, M.; Anghinolfi, M.; Baghdasaryan, H.; Ball, J.; et al. Measurement of Single- and Double-Spin Asymmetries in Deep Inelastic Pion Electroproduction with a Longitudinally Polarized Target. *Phys. Rev. Lett.* **2010**, *105*, 262002.
41. Zhu, J.; Ma, B.Q. Three-dimensional parton distribution functions g_{1T} and $h_{1L}^\perp$ in the polarized proton-antiproton Drell-Yan process. *Eur. Phys. J. C* **2011**, *71*, 1807.
42. Lu, Z.; Ma, B.Q.; Zhu, J. Azimuthal asymmetries in single polarized proton-proton Drell-Yan processes. *Phys. Rev. D* **2011**, *84*, 074036.
43. Idilbi, A.; Ji, X.; Ma, J.P.; Yuan, F. Collins-Soper equation for the energy evolution of transverse-momentum and spin dependent parton distributions. *Phys. Rev. D* **2004**, *70*, 074021.
44. Collins, J.C.; Hautmann, F. Infrared divergences and non-lightlike eikonal lines in Sudakov processes. *Phys. Lett. B* **2000**, *472*, 129–134.
45. Kang, Z.B.; Xiao, B.W.; Yuan, F. QCD Resummation for Single Spin Asymmetries. *Phys. Rev. Lett.* **2011**, *107*, 152002.
46. Echevarria, M.G.; Idilbi, A.; Scimemi, I. Unified treatment of the QCD evolution of all (un-)polarized transverse momentum dependent functions: Collins function as a study case. *Phys. Rev. D* **2014**, *90*, 014003.
47. Landry, F.; Brock, R.; Nadolsky, P.M.; Yuan, C.P. Fermilab Tevatron Run-1 Z boson data and the Collins-Soper-Sterman resummation formalism. *Phys. Rev. D* **2003**, *67*, 073016.
48. Davies, C.T.H.; Webber, B.R.; Stirling, W.J. Drell-Yan Cross Sections at Small Transverse Momentum. *Nucl. Phys. B* **1985**, *256*, 413–433.
49. Ellis, R.K.; Ross, D.A.; Veseli, S. Vector boson production in hadronic collisions. *Nucl. Phys. B* **1997**, *503*, 309–338.
50. Aybat, S.M.; Collins, J.C.; Qiu, J.W.; Rogers, T.C. QCD Evolution of the Sivers Function. *Phys. Rev. D* **2012**, *85*, 034043.
51. Sun, P.; Isaacson, J.; Yuan, C.P.; Yuan, F. Nonperturbative functions for SIDIS and Drell-Yan processes. *Int. J. Mod. Phys. A* **2018**, *33*, 1841006.
52. Collins, J.; Rogers, T. Understanding the large-distance behavior of transverse-momentum-dependent parton densities and the Collins-Soper evolution kernel. *Phys. Rev. D* **2015**, *91*, 074020.
53. Nadolsky, P.M.; Stump, D.R.; Yuan, C.P. Semi-inclusive hadron production at DESY HERA: The Effect of QCD gluon resummation. *Phys. Rev. D* **2000**, *61*, 014003. Erratum in *Phys. Rev. D* **2001**, *64*, 059903.
54. Aidala, C.A.; Field, B.; Gamberg, L.P.; Rogers, T.C. Limits on transverse momentum dependent evolution from semi-inclusive deep inelastic scattering at moderate Q . *Phys. Rev. D* **2014**, *89*, 094002.
55. Konychev, A.V.; Nadolsky, P.M. Universality of the Collins-Soper-Sterman nonperturbative function in vector boson production. *Phys. Lett. B* **2006**, *633*, 710–714.
56. Boer, D. Investigating the origins of transverse spin asymmetries at BNL RHIC. *Phys. Rev. D* **1999**, *60*, 014012.
57. Boer, D.; Diehl, M.; Milner, R.; Venugopalan, R.; Vogelsang, W.; Accardi, A.; Aschenauer, E.; Burkardt, M.; Ent, R.; Guzey, V.; et al. Gluons and the quark sea at high energies: Distributions, polarization, tomography. *arXiv* **2011**, arXiv:1108.1713.
58. Ji, X.; Ma, J.P.; Yuan, F. QCD factorization for spin-dependent cross sections in DIS and Drell-Yan processes at low transverse momentum. *Phys. Lett. B* **2004**, *597*, 299–308.
59. Echevarria, M.G.; Idilbi, A.; Schäfer, A.; Scimemi, I. Model Independent Evolution of Transverse Momentum Dependent Distribution Functions (TMDs) at NNLL. *Eur. Phys. J. C* **2013**, *73*, 2636.
60. Collins, J.; Gamberg, L.; Prokudin, A.; Rogers, T.C.; Sato, N.; Wang, B. Relating transverse-momentum-dependent and collinear factorization theorems in a generalized formalism. *Phys. Rev. D* **2016**, *94*, 034014.
61. Qiu, J.; Zhang, X. QCD Prediction for Heavy Boson Transverse Momentum Distributions. *Phys. Rev. Lett.* **2001**, *86*, 2724–2727.
62. The MAP Collaboration; Bacchetta, A.; Bertone, V.; Bissolotti, C.; Bozzi, G.; Cerutti, M.; Piacenza, F.; Radici, M.; Signori, A. Unpolarized transverse momentum distributions from a global fit of Drell-Yan and semi-inclusive deep-inelastic scattering data. *J. High Energy Phys.* **2022**, *2022*, 127.
63. The MAP collaboration; Bacchetta, A.; Bertone, V.; Bissolotti, C.; Bozzi, G.; Cerutti, M.; Delcarro, F.; Radici, M.; Rossi, L.; Signori, A. Flavor dependence of unpolarized quark transverse momentum distributions from a global fit. *J. High Energy Phys.* **2024**, *2024*, 232.
64. Bacchetta, A.; Bertone, V.; Bissolotti, C.; Cerutti, M.; Radici, M.; Rodini, S.; Rossi, L. Neural-Network Extraction of Unpolarized Transverse-Momentum-Dependent Distributions. *Phys. Rev. Lett.* **2025**, *135*, 021904.
65. Scimemi, I.; Vladimirov, A. Non-perturbative structure of semi-inclusive deep-inelastic and Drell-Yan scattering at small transverse momentum. *J. High Energy Phys.* **2020**, *2020*, 137.
66. Moos, V.; Scimemi, I.; Vladimirov, A.; Zurita, P. Determination of unpolarized TMD distributions from the fit of Drell-Yan and SIDIS data at N⁴LL. *J. High Energy Phys.* **2025**, *2025*, 134.
67. Qiu, J.; Zhang, X. Role of the nonperturbative input in QCD resummed Drell-Yan Q_T distributions. *Phys. Rev. D* **2001**, *63*, 114011.
68. Anselmino, M.; Boglione, M.; Melis, S. Strategy towards the extraction of the Sivers function with transverse momentum dependent evolution. *Phys. Rev. D* **2012**, *86*, 014028.

69. Anselmino, M.; Boglione, M.; D'Alesio, U.; Kotzinian, A.; Murgia, F.; Prokudin, A. Role of Cahn and Sivers effects in deep inelastic scattering. *Phys. Rev. D* **2005**, *71*, 074006.
70. Collins, J.C.; Efremov, A.V.; Goeke, K.; Menzel, S.; Metz, A.; Schweitzer, P. Sivers effect in semiinclusive deeply inelastic scattering. *Phys. Rev. D* **2006**, *73*, 014021.
71. Schweitzer, P.; Teckentrup, T.; Metz, A. Intrinsic transverse parton momenta in deeply inelastic reactions. *Phys. Rev. D* **2010**, *81*, 094019.
72. Prokudin, A.; Sun, P.; Yuan, F. Scheme dependence and transverse momentum distribution interpretation of Collins–Soper–Sterman resummation. *Phys. Lett. B* **2015**, *750*, 533–538.
73. Yang, K.; Liu, T.; Sun, P.; Zhao, Y.; Ma, B.Q. First Extraction of Transverse-Momentum Dependent Helicity Distributions. *Phys. Rev. Lett.* **2025**, *134*, 121902.
74. Yuan, F.; Zhou, J. Collins Function and the Single Transverse Spin Asymmetry. *Phys. Rev. Lett.* **2009**, *103*, 052001.
75. Bozzi, G.; Catani, S.; Ferrera, G.; de Florian, D.; Grazzini, M. Production of Drell-Yan lepton pairs in hadron collisions: Transverse-momentum resummation at next-to-next-to-leading logarithmic accuracy. *Phys. Lett. B* **2011**, *696*, 207–213.
76. Wandzura, S.; Wilczek, F. Sum Rules for Spin-Dependent Electroproduction: Test of Relativistic Constituent Quarks. *Phys. Lett. B* **1977**, *72*, 195–198.
77. Jaffe, R.L.; Ji, X.D. Chiral-odd parton distributions and Drell-Yan processes. *Nucl. Phys. B* **1992**, *375*, 527–560.
78. Abe, K.; Akagi, T.; Anthony, P.L.; Antonov, R.; Arnold, R.G.; Averett, T.; Band, H.R.; Bauer, J.M.; Borel, H.; Bosted, P.E.; et al. Measurements of the proton and deuteron spin structure functions g_1 and g_2 . *Phys. Rev. D* **1998**, *58*, 112003.
79. E155 Collaboration; Anthony, P.L.; Arnold, R.G.; Averett, T.; Band, H.R.; Benmouna, N.; Boeglin, W.; Borel, H.; Bosted, P.E.; Bültmann, S.L.; et al. Precision measurement of the proton and deuteron spin structure functions g_2 and asymmetries A_2 . *Phys. Lett. B* **2003**, *553*, 18–24.
80. The HERMES Collaboration; Airapetian, A.; Akopov, N.; Akopov, Z.; Aschenauer, E.C.; Augustyniak, W.; Avakian, R.; Avetissian, A.; Avetisyan, E.; Belostotski, S.; et al. Measurement of the virtual-photon asymmetry A_2 and the spin-structure function g_2 of the proton. *Eur. Phys. J. C* **2012**, *72*, 1921.
81. Kotzinian, A.; Parsamyan, B.; Prokudin, A. Predictions for double spin asymmetry A_{LT} in semiinclusive DIS. *Phys. Rev. D* **2006**, *73*, 114017.
82. Avakian, H.; Efremov, A.V.; Goeke, K.; Metz, A.; Schweitzer, P.; Teckentrup, T. Are there approximate relations among transverse momentum dependent distribution functions? *Phys. Rev. D* **2008**, *77*, 014023.
83. Metz, A.; Schweitzer, P.; Teckentrup, T. Lorentz invariance relations between parton distributions and the Wandzura-Wilczek approximation. *Phys. Lett. B* **2009**, *680*, 141–147.
84. Teckentrup, T.; Metz, A.; Schweitzer, P. Lorentz invariance relations and Wandzura-Wilczek approximation. *Mod. Phys. Lett. A* **2009**, *24*, 2950–2959.
85. Bastami, S.; Avakian, H.; Efremov, A.V.; Kotzinian, A.; Musch, B.U.; Parsamyan, B.; Prokudin, A.; Schlegel, M.; Schnell, G.; Schweitzer, P.; et al. Semi-Inclusive Deep-Inelastic Scattering in Wandzura-Wilczek-type approximation. *J. High Energy Phys.* **2019**, *2019*, 7.
86. Gockeler, M.; Horsley, R.; Kurzinger, W.; Oelrich, H.; Pleiter, D.; Rakow, P.E.L.; Schafer, A.; Schierholz, G. Lattice calculation of the nucleon's spin-dependent structure function g_2 reexamined. *Phys. Rev. D* **2001**, *63*, 074506.
87. Gockeler, M.; Horsley, R.; Pleiter, D.; Rakow, P.E.L.; Schafer, A.; Schierholz, G.; Stuben, H.; Zanotti, J.M. Investigation of the second moment of the nucleon's g_1 and g_2 structure functions in two-flavor lattice QCD. *Phys. Rev. D* **2005**, *72*, 054507.
88. Balla, J.; Polyakov, M.V.; Weiss, C. Nucleon matrix elements of higher-twist operators from the instanton vacuum. *Nucl. Phys. B* **1998**, *510*, 327–364.
89. Dressler, B.; Polyakov, M.V. Twist-3 contribution to h_L in the instanton vacuum. *Phys. Rev. D* **2000**, *61*, 097501.
90. Accardi, A.; Bacchetta, A.; Melnitchouk, W.; Schlegel, M. What can break the Wandzura-Wilczek relation? *J. High Energy Phys.* **2009**, *11*, 093.
91. Bhattacharya, S.; Cichy, K.; Constantinou, M.; Metz, A.; Scapellato, A.; Steffens, F. Insights on proton structure from lattice QCD: The twist-3 parton distribution function $g_T(x)$. *Phys. Rev. D* **2020**, *102*, 111501.
92. de Florian, D.; Sassot, R.; Stratmann, M.; Vogelsang, W. Extraction of Spin-Dependent Parton Densities and Their Uncertainties. *Phys. Rev. D* **2009**, *80*, 034030.
93. Zhou, J.; Yuan, F.; Liang, Z.T. QCD Evolution of the Transverse Momentum Dependent Correlations. *Phys. Rev. D* **2009**, *79*, 114022.
94. Kang, Z.B.; Qiu, J.W. Evolution of twist-3 multiparton correlation functions relevant to single transverse-spin asymmetry. *Phys. Rev. D* **2009**, *79*, 016003.
95. Botje, M. QCDNUM: Fast QCD Evolution and Convolution. *Comput. Phys. Commun.* **2011**, *182*, 490–532.
96. Lai, H.L.; Guzzi, M.; Huston, J.; Li, Z.; Nadolsky, P.M.; Pumplin, J.; Yuan, C.-P. New parton distributions for collider physics. *Phys. Rev. D* **2010**, *82*, 074024.
97. Bunce, G.; Saito, N.; Soffer, J.; Vogelsang, W. Prospects for spin physics at RHIC. *Annu. Rev. Nucl. Part. Sci.* **2000**, *50*, 525–575.

98. Abazov, V.M.; Abramov, V.; Afanasyev, L.G.; Akhunzyanov, R.R.; Akindinov, A.V.; Akopov, N.; Alekseev, I.G.; Aleshko, A.M.; Alexakhin, V.Y.; Alexeev, G.D.; et al. Conceptual design of the Spin Physics Detector. *arXiv* **2021**, arXiv: 2102.00442.
99. Airapetian, A.; Akopov, N.; Akopov, Z.; Amarian, M.; Aschenauer, E.C.; Avakian, H.; Avakian, R.; Avetissian, A.; Avetissian, E.; Bailey, P.; et al. Single-spin azimuthal asymmetries in electroproduction of neutral pions in semi-inclusive deep-inelastic scattering. *Phys. Rev. D* **2001**, *64*, 097101.
100. Jawalkar, S.; Koirala, S.; Avakian, H.; Bosted, P.; Griffioen, K.A.; Keith, C.; Kuhn, S.E.; Adhikari, K.P.; Adhikari, S.; Adikaram, D.; et al. Semi-Inclusive π^0 target and beam-target asymmetries from 6 GeV electron scattering with CLAS. *Phys. Lett. B* **2018**, *782*, 662–667.
101. Diehl, S.; Kim, A.; Angelini, G.; Joo, K.; Adhikari, S.; Amarian, M.; Arratia, M.; Atac, H.; Avakian, H.; Ayerbe Gayoso, C.; et al. Multidimensional, High Precision Measurements of Beam Single Spin Asymmetries in Semi-inclusive π^+ Electroproduction off Protons in the Valence Region. *Phys. Rev. Lett.* **2022**, *128*, 062005.
102. Parsamyan, B. Measurement of longitudinal target polarization dependent azimuthal asymmetries in SIDIS at COMPASS experiment. In Proceedings of the XXV International Workshop on Deep-Inelastic Scattering and Related Subjects, Birmingham, UK, 3–7 April 2017; p. PoS(DIS2017)259.
103. Parsamyan, B. Measurement of target-polarization dependent azimuthal asymmetries in SIDIS and Drell-Yan processes at COMPASS experiment. In Proceedings of the QCD Evolution 2017, Newport News, VA, USA, 22–26 May 2017; p. PoS(QCDEV2017)042.
104. Adolph, C.; Aghasyan, M.; Akhunzyanov, R.; Alexeev, M.G.; Alexeev, G.D.; Amoroso, A.; Andrieux, V.; Anfimov, N.V.; Anosov, V.; Augsten, K.; et al. Azimuthal asymmetries of charged hadrons produced in high-energy muon scattering off longitudinally polarised deuterons. *Eur. Phys. J. C* **2018**, *78*, 952. Erratum in *Eur. Phys. J. C* **2020**, *80*, 298.
105. The COMPASS Collaboration; Alekseev, M.G.; Alexakhin, V.Y.; Alexandrov, Y.; Alexeev, G.D.; Amoroso, A.; Austregesilo, A.; Badelek, B.; Balestra, F.; Barth, J.; et al. Azimuthal asymmetries of charged hadrons produced by high-energy muons scattered off longitudinally polarised deuterons. *Eur. Phys. J. C* **2010**, *70*, 39–49.
106. de Florian, D.; Sassot, R.; Stratmann, M. Global analysis of fragmentation functions for pions and kaons and their uncertainties. *Phys. Rev. D* **2007**, *75*, 114010.
107. Avakian, H. Azimuthal single-spin asymmetries in semi-inclusive DIS from HERMES. *Nucl. Phys. B—Proc. Suppl.* **1999**, *79*, 523–525.

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