In-Situ Detection for Atomic Density in the K-Rb-$^{21}$Ne Co-Magnetometer via an Optical Heterodyne Interferometry

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Abstract: The low-frequency fluctuations of the atomic density within the cell can induce the long-term drift of the K-Rb-$^{21}$Ne spin-exchange relaxation-free (SERF) co-magnetometer output, such that the accurate measurement of in situ atomic density is of great significance for improving the performance of co-magnetometer. In this paper, the complex refractive index model of the spin ensembles under the hybrid optical pumping condition is established first, according to which the relation between atomic density and its complex refractive index is revealed and an optical heterodyne-based scheme for atomic density detection is proposed. The dependence of the atomic density on the demodulated phase signal from the optical heterodyne-based scheme is provided by numerical simulations. After that, a dual acousto-optics frequency shifter (AOFS)-based optical heterodyne interferometry is constructed with a noise level below 1 mrad/$\sqrt{\text{Hz}}$ for frequencies $> 1$ Hz, and a compact SERF co-magnetometer is implemented as the testing medium, by which the atomic density detection with resolution of 0.40 K @ 473 K is reached and the experimental results agree well with theoretical simulations. Moreover, the detection scheme proposed in this paper has the properties of high detection sensitivity and immunity to laser power fluctuation, which are also proved experimentally.

Keywords: SERF co-magnetometer; atomic density; optically thick; optical heterodyne

1. Introduction
Polarized atomic vapors can be used to detect fields interacting with a spin. Based on this basic principle, quantum precision measurement instruments have been developed in recent years, including optical pumping magnetometers (OPM) [1], SERF magnetometers [2], SERF co-magnetometers [3], nuclear magnetic resonance gyroscopes (NMRG) [4], etc., which can be widely applicable to inertial navigation [5–7] and detection of the weak magnetic field [8,9]. Even in the frontiers of physics, they also have the potential to pursue spin-dependent force searching [10–13] and studies of CPT and Lorentz violation [14–16].

The SERF co-magnetometers rely on the high density of polarized alkali metal vapors to hyperpolarize the nuclear spins of the noble gas, thereby enabling a highly sensitive spin detection capability [17–19], yet the high atomic density leads to a large optical depth of the entire cell, in which the pump laser intensity attenuates gradually with its propagation that results in a spatially non-uniform distribution of the electronic spin polarization [20] and the inevitable decreasing of pumping rate [21]. To overcome this issue, a hybrid optical pumping approach is proposed that applies a low density (optically thin) alkali atoms spin to polarize another high density (optically thick) one by spin-exchange collisions [22], thus achieving a more uniform polarization distribution and a higher sensitivity [23,24].

With the implementation of hybrid optical pumping technology, the presence of different alkali metal species makes their density ratio a rather critical factor affecting the performance of SERF co-magnetometers. Roughly speaking, the optical pumping efficiency and uniform distribution of polarization are primarily determined by the optically thin and...
thick alkali atoms, respectively [25]. Hence, the fluctuation of atomic density can cause an error in the SERF co-magnetometer output signal and a decrease in sensitivity, where the optically thick one is the more noteworthy factor due to its quantitative dominance in the mixed alkali metal atoms.

A prerequisite for suppressing the atomic density fluctuation is to measure it accurately. The laser absorption spectroscopy method [26] is feasible and reliable for the low atomic density co-magnetometer cell, whereas the absorption profile distortion occurs due to the strong attenuation near the resonant frequency under a saturated atomic density or high temperature condition, such that no effective atomic density can be obtained by fitting the broaden linewidth [27,28]. The Faraday rotation measurement method provides the possibility to monitor density variation in optically thick alkali atoms [29–31], but the strong field (∼1.2 T) required for generating an apparent Faraday effect is in opposition to the operating conditions, a weak field (<1 nT), of SERF co-magnetometer cell. In recent years, the magnetic resonance linewidth measurement method [32] and phase-frequency analysis method [33] are suggested, based on a feature that the atomic spin-exchange relaxation rate is proportional to the atomic density. However, neither of them can be employed in the hybrid optical pumping condition with a fast spin-exchange collision. In order to be suitable for hybrid alkali cells, an density ratio measurement method according to the spin-exchange collision mixing of the K and Rb light-shifts is presented [25], since the light-shift information is obtained from the steady-state response errors under a longitudinal field modulation, thus it is hard to obtain the atomic density online or in real-time under normal gyro signal output mode. Additionally, a common problem with all these above methods is the indirect detection of atomic density through the fitting process, which may limit the accuracy and sensitivity of the experimental results.

It is noted that an empirical formula popularly used in practical engineering can deduce the saturated alkali vapor from the in situ cell temperature [27,34,35], which implies that the atomic density measurement is equivalent to acquire in situ cell temperature information. Besides the thermistor, for instance, the Pt-resistance is an effectively miniatuirized temperature measurement tool, and can be attached at the cell tip or wall of the co-magnetometer cell. In fact, not only can the field disturbance generated by the thermistor (∼70 nT) [36] can reduce the sensitivity and stability, but the in situ cell temperature is also not consistent with the cell wall temperature given by thermistor (∼10 K) [32], nor able to suppress its fluctuation.

The optics-based temperature measurement scheme is always according to the relation between the optical radiation or absorption capacity of the medium and its temperature, and has the nature of magnetization-free and high-precision. A notable idea is to directly exploit the optical depth to reflect the in situ cell temperature under thermal equilibrium [37]. According to the Lambert–Beer law [38], the optical depth, i.e., the logarithm of transmission of the cell is jointly determined by the laser wavelength, electron polarization, and atomic density, it follows that the main error source is the measurement error of transmission. Yet, the strong transmission attenuation near the resonance still cannot be overcome and the stability of incident laser power is a challenge to be guaranteed [39]. Nevertheless, studies on the transverse pumping effect of the optical rotation detection system in recent years have revealed that the polarization rate error is a non-negligible term, such that the stability of transmission will also be affected by polarization fluctuation even with a linear polarized laser [40,41]. Therefore, there is an urgent need to develop a method for optics-based in situ co-magnetometer cell temperature (i.e., atomic density) measurement, which is not influenced by fluctuations in laser intensity.

In essence, the above optical depth measurement is an evaluation for photon energy dissipation of these spin ensembles within the co-magnetometer cell, i.e., the imaginary component of the complex refractive index of the spin ensembles. One can be inspired by the Kramers–Kronig relation [42], which indicates that the real component must also be related to the in situ cell temperature and atomic density. Indeed, in this paper, the optical heterodyne technology is selected to perform the validation of this idea, which
offers attractive disturbance resistibility [43], and with the characteristics of high resolution and sensitivity in application [44]. The optical heterodyne scheme is widely implemented in the fields of precision measurement [45–47], optical communication [48,49], optical imaging [50], etc. When compared to laser absorption spectroscopy method, the optical heterodyne-based scheme proposed in this paper is immune to laser power fluctuations, and more sensitive to in situ atomic density. Hence, the optical heterodyne-based scheme is more suitable for atomic density measurement in SERF co-magnetometer.

In this paper, the complex refractive index model of the spin ensembles under the hybrid optical pumping condition is established firstly, and an optical heterodyne-based scheme for atomic density detection is proposed, according to these theoretical bases the relations between the atomic density and demodulated phase from the optical heterodyne-based scheme under different conditions are provided through numerical simulations. Then, a dual AOFS-based optical heterodyne interferometry is constructed with a compact SERF co-magnetometer as the testing medium, by which the atomic density detection is achieved and the experimental results are in well agreement with theoretical simulations. In addition, the advantages of the high detection sensitivity and unaffected by laser intensity fluctuation are experimentally demonstrated.

The structure of this paper is as follows. Section 2 establishes a complex refractive index model of the spin ensembles under the hybrid optical pumping condition firstly, and then proposes an optical heterodyne-based scheme for atomic density detection. According to these theoretical bases, the relations between the atomic density and demodulated phase from the optical heterodyne-based scheme under different conditions are provided through numerical simulations. Section 3 describes the experimental setups of the SERF co-magnetometer and a dual AOFS-based optical heterodyne interferometry. Section 4 demonstrates the experimental verification and related conclusions for the atomic density detection method.

2. Theoretical Analysis
2.1. Atomic Spin Ensembles Refractive Index Model

Consider a linearly polarized laser beam propagating along the y-axis as a plane-wave in a medium with complex refractive index $\hat{n} = n_r + in_i$ whose initial amplitude and angular frequency of the excited mode are $\vec{A}$ and $\omega$, respectively.

$$\vec{E}(y,t) = \vec{A} \exp\left[ i \left( \frac{\omega}{c} y - \omega t \right) \right] = \vec{A} \exp\left( -\omega n_i \frac{y}{c} \right) \exp\left[ i \left( \frac{\omega n_r y}{c} - \omega t \right) \right]$$

(1)

The real component $n_r$ affects the spatial phase factor such that the effective wavelength is determined, and the imaginary component $n_i$ can induce the attenuation of amplitude, i.e., the optical absorption effect. Therefore, the relation between laser power $I$ and propagation distance $l$ in the medium $\hat{n}$ can be described by a simplified Beer’s law [38].

$$I(l) = \| \vec{E}(l, \frac{n_l}{c}) \|^2 = \| \vec{A} \|^2 \exp\left( -\frac{2\omega n_i l}{c} \right) = I(0) \exp\left( -\frac{2\omega n_i l}{c} \right),$$

(2)

where $\| \cdot \|$ denotes the spectral norm.

For the hybrid optical pumping-based SERF co-magnetometers, whose cell with diameter $L$ containing an optically thin (potassium, in our setup) and another optically thick (rubidium, in our setup) alkali metal species, their densities are denoted as $N_K$ and $N_Rb$ in this paper, respectively. The total alkali vapor density is dominated by rubidium atoms due to the relatively large density ratio of $N_Rb : N_K$, whose saturated density at a temperature $T$ in Kelvin can be given by an empirical formula [27,34].

$$N_{Rb} = 10^{26.178 - (4040 K)/T} \frac{(1 K^{-1}) T}{I}$$

(3)
Hence, the absorption of linearly polarized light with frequency $\nu$ over the entire propagation length $L$ can be interpreted in terms of the optical depth $\OD$ [38].

\[
\frac{I_{\text{out}}}{I_{\text{in}}} = \prod_j K_{\text{Rb}} D_j \exp(-\OD_j) = \prod_j K_{\text{Rb}} \exp(\nu_j \sigma_j(v)L)
\]

\[
\sigma_j(v) = cr_e \sum_k f_{jk} L_{jk}(v) = cr_e \sum_k f_{jk} \frac{\gamma_{jk}/2}{(\nu - \nu_{jk})^2 + \left(\gamma_{jk}/2\right)^2}
\]

(4)

where $c$ is the speed of light, $r_e$ is the classical electron radius, $f_{jk}$ is the oscillator strength of the transition, $\nu_{jk}$ and $\gamma_{jk}$ are the central frequency of the corresponding transition and its full-width at half maximum. The absorption cross section $\sigma_j(v)$ satisfies a Lorentzian profile.

It is noted that the optical depth of a co-magnetometer cell simultaneously meets the relation from Equations (2) and (4), by which the imaginary component $n_i(v)$ of the atomic spin ensembles refractive index can be obtained as follows, and one can note that the solution to the principal value integral of a Lorentzian has a standard dispersion shape $D(v)$.

\[
n_i(v) = \frac{c^2 r_e K_{\text{Rb}} D_1 D_2}{4\pi \nu} \sum_j \sum_k N_j f_{jk} \frac{\gamma_{jk}/2}{(\nu - \nu_{jk})^2 + \left(\gamma_{jk}/2\right)^2}
\]

(5)

Since the atomic spin ensembles within the cell can be seen as a kind of nonmagnetic medium with the relative permeability of $\mu_r = 1$, the refractive index is determined by relative permittivity $\bar{\epsilon} = \sqrt{\epsilon_r}$, and the electrical susceptibility $\chi_e$ can be further derived as below.

\[
\chi_e(v) = \hbar^2 (v) - 1 = 2n_e(v) - 2 + 2i n_i(v) + o[n_i(v)],
\]

(6)

where the higher order infinitesimal term $o[n_i(v)]$ can be neglected for simplification, and one can employ the Kramers–Kronig relation [42] to obtain the real component $\chi_r = \Re(\chi_e)$ of electrical susceptibility.

\[
\chi_r(v) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \chi_e(v') \frac{dv'}{\nu - v'}
\]

(7)

where $P \int_{-\infty}^{\infty} \frac{dv'}{\nu - v'}$ implies the Cauchy principal value of the integral.

Therefore, by substituting Equation (6) into Equation (7), the real component $n_r(v)$ of the atomic spin ensembles refractive index can be obtained as follows, and one can note that the solution to the principal value integral of a Lorentzian has a standard dispersion shape $D(v)$.

\[
n_r(v) = 1 - \frac{1}{\pi} P \int_{-\infty}^{\infty} n_i(v') \frac{dv'}{\nu - v'} = 1 - \frac{c^2 r_e K_{\text{Rb}} D_1 D_2}{4\pi \nu} \sum_j \sum_k N_j f_{jk} \int_{-\infty}^{\infty} \frac{L_{jk}(v')}{\nu - v'} dv' = 1 - \frac{c^2 r_e K_{\text{Rb}} D_1 D_2}{4\pi \nu} \sum_j \sum_k N_j f_{jk} \int_{-\infty}^{\infty} D_{jk}(v) = 1 - \frac{c^2 r_e K_{\text{Rb}} D_1 D_2}{4\pi \nu} \sum_j \sum_k N_j f_{jk} \frac{\nu - \nu_{jk}}{(\nu - \nu_{jk})^2 + \left(\gamma_{jk}/2\right)^2}
\]

(8)

In our setup, the detection light operates near the resonance of the D1 transition in rubidium, such that the real component of the refractive index is linearly dependence on the atomic density as shown in Figure 1b, while detection scale factor $\partial n_r/\partial N_{\text{Rb}}$ is correlated...
with the detuning of the detection light, which satisfies a dispersion lineshape as depicted in Figure 1a.

![Figure 1a](image1.png)

**Figure 1.** The numerical simulations indicate that (a) the real component $n_r$ of the refractive index versus the laser detuning at different atomic densities both satisfy the dispersion profile, and (b) the real component $n_r$ of the refractive index is linear with respect to the density of rubidium $N_{Rb}$, and the slopes of lines are determined by the different detuning of laser from the center frequency of the D1 transition of rubidium.

2.2. Optical Heterodyne Technique-Based Detection Scheme

The real component of the refractive index of the atomic spin ensembles within the co-magnetometers varies significantly with the alkali vapor densities, which can be effectively detected by the interference-based measurement methods such that the in situ information of atomic density can be obtained directly. The optical heterodyne interference arrangement is the one selected to be implemented in our experiment, which has a strong immunity to disturbance as compared to others in practical engineering due to its synchronous acquisition of common-mode reference and detection signals.

The reference and detection signals are generated by the optical heterodyne configuration, as Figure 2 shows. Each signal is combined by two linearly polarized beams with a fixed angular frequency difference $\Delta \omega = \omega_2 - \omega_1$. For simplicity, the amplitudes of the electric vectors of these two beams are assumed to be of equal magnitude, and their vibration direction can be aligned for maximum signal-to-noise ratio (SNR) at the photodiode.

\[
E_1 = A \cos(k_1 z - \omega_1 t), \quad E_2 = A \cos(k_2 z - \omega_2 t), \quad \text{where} \quad \omega_2 - \omega_1 = 2\pi \Delta F.
\]

![Figure 2](image2.png)

**Figure 2.** Schematic diagram of the optical heterodyne configuration for obtaining a reference (detection) signal. $\omega_{1,2}$ is the angular frequency of the excited mode of the beam with electric vector of $E_{1,2}$. 

Then, they interfere with each other at a balanced (50-50) beamsplitter (BS), and one of the output beam from the BS is collected by a photodiode (PD), whose nominal power is

\[ I_n = \left| \sum_j E_j \right|^2 = 4A^2 \cos^2 \left( \frac{\pi\Delta F}{c} z - \pi\Delta F l \right) \cos^2 \left( \frac{k_1 + k_2}{2} z - \frac{\omega_1 + \omega_2}{2} l \right) \]

\[ = \sum_m A_m \cos(k_m z - \omega_m l), \]

(10)

it follows that the nominal power \( I_n \) contains \( m = 5 \) different frequency components, ordered from low to high as \( \omega_1 = 0 \) (DC), \( \omega_2 = 2\pi\Delta F \), \( \omega_3 = \sum_j \omega_j - 2\pi\Delta F \), \( \omega_4 = \sum_j \omega_j \) and \( \omega_5 = \sum_j \omega_j + 2\pi\Delta F \). It is noted that the bandwidth of a photodiode is far less than the laser frequency, such that only the lower frequency components (\( \omega_{1,2} \)) can be reserved. Then, one can further diminishes the DC bias component through a highpass filter or operating the photodiode in the AC-coupled mode (in our setup). In this case, the reference signal can be derived.

\[ S_{\text{ref}} = K_{PD} \langle I_n \rangle = K_{PD} \left( \sum_m A_m \cos(k_m z - \omega_m l) \right) \]

\[ = K_{PD} A^2 + K_{PD} A^2 \cos \left( \frac{2\pi\Delta F}{c} z - 2\pi\Delta F l \right) \]

\[ \xrightarrow{\text{Highpass}} K_{PD} A^2 \cos \left( \frac{2\pi\Delta F}{c} z - 2\pi\Delta F l \right) \]

(11)

Then, in terms of concern about the detection signal, based on the configuration in Figure 2, one can enable one of the beam, for instance \( E_2 \), to pass through a co-magnetometer cell with diameter \( L \) before reaching the BS. Thus, the detection signal can be expressed as follows, whose frequency as same as the reference one and phase can reveals the information of the real component of refractive index of the entire co-magnetometer cell.

\[ S_{\text{det}} = K_{PD} A^2 \cos \left( \frac{2\pi\Delta F}{c} z' - 2\pi\Delta F l - k_2 n_r L \right) \]

(12)

From Equations (11) and (12), their phase difference denoted, \( \phi \), is merely dependent on their optical path difference, but not on the magnitude of the frequency difference \( \Delta F \), which can be demodulated by feeding the reference and detection signals into a lock-in amplifier (LIA).

\[ \phi = \frac{2\pi\Delta F}{c} (z - z') + k_2 n_r L \]

\[ = \frac{2\pi\Delta F}{c} (z - z') + \frac{\omega_2 L}{c} - \frac{cr_2 L}{2} \sum_j \sum_k N_{jk} D_1 D_2 \frac{v - v_{jk}}{\nu - v_{jk}^2} \frac{v - v_{jk}}{(\nu - v_{jk})^2 + (\gamma_{jk}/2)^2} \]

(13)

It is noteworthy that the first two terms of Equation (13) become constants after the complete implementation of our experiment setup such that the evolution of \( \phi \) can reflect the variation in atomic density (majorly rubidium atoms \( N_{Rb} \)) under the stabilization of the detection laser frequency. Through simulation analysis, the plots of phase difference \( \phi \) versus in situ cell temperature can be obtained as shown in Figure 3, and the nonlinearity of the curves indicates the exponential increment of the atomic density with temperature.
Figure 3. Phase difference $\phi$ versus in situ cell temperature at different laser frequencies, where $\nu_{\text{Rb,D1}}$ denotes the D1 transition of rubidium 87 (about 377.1075 THz), and the distinction in the incremental rate of these profiles depends on the dispersion nature of the cell.

3. Experimental Setup

In order to verify the validity and theoretical conclusions of the detection scheme for atomic density presented in Section 2, a series of experiments are performed in a compact SERF co-magnetometer as shown in Figure 4a. There are about 3 amagat of $^{21}\text{Ne}$, 50 Torr of nitrogen as quenching gas, and a mixture of alkali metals with density ratio of $N_{\text{K}} : N_{\text{Rb}} = 1 : 130$ in the co-magnetometer cell with diameter of 8 mm. The cell is fixed to the inside of the oven base that is made of boron nitride, while the flexible heating resistance wire covers the outside surface of the oven uniformly for heating the air between the inner wall of the oven and cell to heat atoms inner the cell indirectly. Then, the whole oven is wrapped in the three-axis magnetic field coils, which can compensate the field inner the triple layer shields to a desired compensation point. It is worth noting that the SERF co-magnetometers is based on the precise measurement of transverse polarization to achieve inertial sensitivity, such that even small magnetic disturbances can cause errors in the measurement results as well as a decrease in sensitivity. To address this issue, besides applying magnetic shields, we also need to adjust the field to the “compensation point” by coils, at which it satisfies $B_z^e = -B_z^n - B_z^e$, where $B_z^e$ and $B_z^n$ is the magnetic field induced by electron and nuclear spins, respectively. This can effectively reduce the interference of nuclear spin-induced magnetic field on electron spin. A left-handed circularly polarized laser with a center frequency of potassium D1 transition is produced by the pump optical path and incident into the cell for polarizing alkali vapor, and the noble gas can be further hyperpolarized by spin exchange interact in several hours [51]. Then, the off-resonant linearly polarized probe laser can reflect the alkali metal polarization $P_x$ along the $x$-axis through the optical rotation angle, which is detected by the balanced polarimeter [41]. Hence, the high sensitivity for rotation sensing can be realized by this SERF co-magnetometer.

Then, an optical path, as shown in Figure 4b, is constructed based on the structure of the compact SERF co-magnetometer to achieve optical heterodyne detection of atomic density fluctuation within the co-magnetometer cell. The detection laser is operated at the frequency $\nu$ near the frequency of rubidium D1 transition, which is similar to the probe laser and has the potential for integration in the future. Besides this, the power of the detection laser is 10 mW with a spot diameter of 1 mm, to guarantee a sufficient SNR of heterodyne signal after the loss of power, which is brought by those BS. A coupler (10:90) samples the detection laser and connects to a wavelength meter (WS7-30, HighFinesse)
such that the phase error induced by the frequency fluctuation of a detuned detection laser can be suppressed. Two AOFSs (M1205-P80L, Isomet) are driven by the radio frequency (RF) excitation with the frequencies of $f_1 = 80$ MHz and $f_2 = 81$ MHz from a dual channel RF driver (iSPA-SF2, Isomet) concurrently and then operated in Bragg condition, thus the +1 order diffracted beam is with the frequency of $\nu + f_{1,2}$ and the zero order transmission is useless in our setup such that not shown. Moreover, both these half-wave plates (HWP) and short focal length convex lenses (CL) are implemented for the achievement of maximum SNR, and the only long focal length convex lens is installed to cancel the divergence induced by the spherical cell, by which the beam intensity can be preserved during the subsequent propagation path. At the end of the optical path, ACPD$_1$ or ACPD$_2$, there are two beams with a fixed frequency difference of 1 MHz that interfere with each other, such that the 1 MHz co-frequency signals $S_{\text{ref}}$ and $S_{\text{det}}$ can be obtained. Then, they through a bandpass filter and a +20 dB amplifier, and are inputted into an LIA to demodulate the amplitude $R$ and phase difference $\phi$, where $S_{\text{ref}}$ and $S_{\text{det}}$ are employed as the reference and detection signals, respectively. In this way, the phase signal $\phi$ containing information on atomic density $N_{\text{Rb}}$ fluctuation is extracted.

Moreover, the blue-colored loop represents the previous temperature control scheme, i.e., one can utilize the thermistor (PT1000) at the cell tip to sense the cell wall temperature (CWT) and stabilize it.

**Figure 4.** (a) Schematic diagram of a compact SERF co-magnetometer. PBS, polarization beam splitter; LCVR, liquid crystal variable retarder; PD, photodiode; PL, planoconvex lenses; HWP, half-wave plate; M, mirror; QWP, quarter-wave plate; PD, photodiode. (b) Schematic diagram of the optical heterodyne-based detection scheme for atomic density. PM, single-mode polarization-maintaining fiber; BS, 50-50 plate beamsplitter; AOFS, acousto-optics frequency shifter; NE, noise eater; CL, convex lens; BPF, bandpass filter; LPF, lowpass filter; TM, thermistor; AM, amplitude modulator; ACPD, AC-coupled photodiode; LIA, lock-in amplifier.

**4. Results and Discussion**

The noise level of the ACPD (ET-2030A, EOT) is technically critical for the sensitivity of the whole experimental setup. The voltage of reference signal is fed into a spectrum analyzer (N9322C, Keysight) to measure the signal power. A typical heterodyne beat signal at 1 MHz observed by the analyzer is shown in Figure 5, which reveals that the reference signal is with a well SNR and the flat floor level is always close to dark noise. It follows that
the selected beat frequency at 1 MHz is reasonable due to the favorable noise environment in this band.

In order to exclude as much as possible the errors induced by optical elements to enable reliable detection of phase variation caused by atoms, one can compare the phase responses $\phi$ to the laser frequency ramp in the cases of with and without a co-magnetometer cell at room temperature. The test results as shown in Figure 6a demonstrate that other elements in the optical path respond slightly to the laser frequency at operating temperature, which is speculated to be possibly related to their coatings characteristics, yet even under an unheated condition, the atoms within the cell still exhibit a sharp dispersion. Another feature worth considering is the high-temperature-induced thermal expansion effect of the cell wall (Aluminosilicate glass); theoretically, under the temperature range of 420–470 K, the thermal expansion-effect-induced thickness variation in the cell wall is in the order of 0.1 $\mu m/K$ according to its specification sheet, which is more less than the order of 10 $\mu m/K$ induced by atomic density increment based on Figure 1. Furthermore, through comparing the phase detection sensitivity under different conditions as shown in Figure 6b, it is observed that the variation in atomic density is manifested clearly in the low-frequency band, and the noise level of this interferometry is below 1 mrad/$\sqrt{Hz}$ for frequencies $> 1$ Hz.

Then, the most intuitive way in general is to calibrate the phase signal $\phi$ versus the atomic density $N_{Rb}$. However, since the absence of an online method for measuring saturated rubidium atomic density, the cell wall temperature denoted $T_{wall}$ acquired by a thermistor at the cell tip, which is different from the concept of in situ cell temperature and carried out the synchronized acquisition with phase signal $\phi$. As illustrated in Figure 7a, under the stabilization of detection laser wavelength at 795.105 nm by the wavelength meter, the steady-state phase signal demonstrates an exponential increasing with CWT, which is consistent with the theoretical empirical formula. Moreover, the steady-state phase signal versus CWT is further investigated under six different detection laser wavelength around the D1 transition of rubidium, the related data are recorded in Table 1, and plotted the relation between the phase increment $\Delta \phi$ and CWT $T_{wall}$ starting at 423 K in Figure 7b, which is agree well with the theoretical model of Equation (13). The experimental results indicate that the characteristics of detuning from the D1 transition frequency of rubidium are highly correlated with the phase increment $\Delta \phi$, in which the trend of $\Delta \phi$ is determined by the direction of blue- or red-shift from $\nu_{RB,D1}$, and the detection scale factor $\partial \phi / \partial N_{RB}$ is dominated by the dispersion function $D(\nu)$, i.e., the magnitude of detuning.
Figure 6. (a) Comparison of the phase responses to the detection laser frequency ramp (red solid line) in the cases of with (blue solid line) and without (blue dashed line) a co-magnetometer cell at the room temperature, where the response in the case of without cell is amplified 20 times for easier viewing. (b) The sensitivity of the phase difference signal in the cases of without cell (blue line), with an unheated cell (yellow line), and with a heated cell (red line), respectively. The purple and back dash lines represent the LIA background noise and white noise, respectively.

Figure 7. (a) Variation curve of modulated phase signal $\phi$ (blue line) with respect to the CWT (red dash line). (b) Steady-state phase increment versus CWT under different detection laser wavelength. (c) Optical absorption profile of a co-magnetometer cell at the CWT of 473 K, the transmission in the range of 794.852–795.105 nm (gray covered area) is less than 1%. the detection scale factor (blue error bar) at different wavelength is calculated from Table 1.

As shown in Figure 7c, the detection scale factor $\partial \phi / \partial N_{Rb}$ under the respective stabilization of those six detection laser wavelengths above is calculated and depicted, and by
measuring the optical absorption effect of the co-magnetometer cell, one can note that the valid atomic density cannot be obtained from the fitting of a flat-bottomed Lorentzian profile, and in which the transmission in the range of 794.852–795.105 nm (gray covered area) is even less than 1% due to the strong absorption of the near-resonant laser by saturated rubidium atoms and can lead to a low SNR beat with larger errors.

Table 1. The steady-state phase data versus cell wall temperature at different detection laser wavelengths.

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<td>795.148</td>
<td>−92.50</td>
<td>−90.90</td>
<td>−78.30</td>
<td>−61.48</td>
<td>−14.43</td>
<td>93.69</td>
</tr>
<tr>
<td>795.190</td>
<td>−131.83</td>
<td>−130.62</td>
<td>−122.32</td>
<td>−108.48</td>
<td>−69.76</td>
<td>16.51</td>
</tr>
</tbody>
</table>

One advantage of the optical heterodyne-based scheme compared with the previous laser absorption spectroscopy method is that it is not impacted by the laser power fluctuation as shown in Figure 8a, when the laser power incident to the cell is reduced by regulating NE, the demodulated amplitude signal obtained from LIA is decreased, whereas the demodulated phase difference is immune to this change. Another one appears in its sensitive in situ measurement capability when compared with the thermistor; to prove this, the co-magnetometer cell is heated to near $T_{\text{wall}} = 463$ K through a open loop constant heating power within a long time, then one can applied a power step in the incident pump laser of the compact SERF co-magnetometer by regulating the control voltage of the LCVR, as plotted in Figure 8b. In agreement with expectations, the phase increment $\Delta \phi$ clearly captures the decrease in atomic density caused by the heating-effect attenuation of pump laser, while the open loop thermistor data collected only displays a suspiciously related signal at the second power step, thus it is weakly correlated with the variation in pump power $I_{\text{pump}}$ over the entire time period. In this case, a roughly estimation results of the influence on rubidium atomic density fluctuations from pump light power variation in our setup is obtained by a linear fitting is of $\sim 1.90 \times 10^{16} \text{m}^{-3}/(\text{mW/cm}^2)$, and is of $\sim 734.3 \text{µK}/(\text{mW/cm}^2)$ around 463 K CWT. Theoretical calculation shows that when alkali metals in the cell satisfy the saturated vapor density, the ratio of the heating-effect-caused atomic density to the total one is about 1.8%. Furthermore, according to the noise band of phase difference $\Delta \phi$, it can be inferred that the measurement resolution for in situ cell temperature detection reaches 0.40 K @ 473 K by our scheme, which is an order of magnitude improvement over the one based on the laser absorption spectroscopy method.

Figure 8. The principal advantages of the optical heterodyne-based scheme are (a) its immunity to detection laser power fluctuation and (b) it is more sensitive to in situ atomic density variation than CWT.
5. Conclusions

In conclusion, in this paper, we firstly establish a complex refractive index model of the spin ensembles under the hybrid optical pumping condition, and analyze the relation between atomic density and its complex refractive index. Then, we propose an optical heterodyne-based scheme for atomic density detection, and reveal the dependence of the atomic density on the demodulated phase signal from this scheme through numerical simulations. After that, we construct a dual AOFS-based optical heterodyne interferometry with a noise level below 1 mrad/√Hz for frequencies > 1 Hz, and implement a compact SERF co-magnetometer as the testing medium to realize the detection for atomic density, and the experimental results agree well with theoretical simulations. In addition, we demonstrate the high detection sensitivity and immunity to laser power fluctuation of our detection scheme through the regulations in pump and detection laser intensity, respectively. Eventually, we achieve the measurement resolution of 0.40 K @ 473 K for in situ cell temperature detection.

Our future work will focus on the improvement of phase detection sensitivity and stability to suppress the long-term fluctuations of atomic density in the co-magnetometer, and miniaturization of the interferometer in practical engineering. Moreover, the optical-heterodyne-based detection method of atomic density (i.e., in situ cell temperature) provided in this paper also has the potential to be implemented in other atomic sensors.

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Conflicts of Interest: The authors declare no conflict of interest.

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